To Bubble or Not to Bubble: Asset Price Dynamics and Optimality in OLG Economies

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Abstract

We study an overlapping generations (OLG) exchange economy with an asset that yields dividends. First, we derive general conditions, based on exogenous parameters, that give rise to three distinct scenarios: (1) only bubbleless equilibria exist, (2) a bubbleless equilibrium coexists with a continuum of bubbly equilibria, and (3) all equilibria are bubbly. Under stationary endowments and standard assumptions, we provide a complete characterization of the equilibrium set and the associated asset price dynamics. In this setting, a bubbly equilibrium exists if and only if the interest rate in the economy without the asset is strictly lower than the population growth rate and the sum of per capita dividends is finite. Second, we establish necessary and sufficient conditions for Pareto optimality. Finally, we investigate the relationship between asset price behaviors and the optimality of equilibria.

Keywords: exchange economy, overlapping generations, asset price bubble, fundamental value, low interest rate, Pareto optimal.
JEL Classifications: C6, D5, D61, E4, G12.

1 Introduction

The asset valuation and its effects on welfare, either negative or positive, are long-standing questions in economics. The seminar paper of Tirole (1985) studies the price formation of asset yielding non-negative dividends and shows its impact on the Pareto optimality of equilibrium allocations.

According to the traditional literature (Tirole, 1982, 1985; Santos and Woodford, 1997), given a dividend-paying asset with positive supply, its rational bubble is said to exist if the asset's market price exceeds its fundamental value, typically defined as

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the sum of the discounted stream of future dividends.[1] An equilibrium is referred to be *bubbly (bubbleless)* if a bubble exists (does not exist), to be *asymptotically bubbly* if a bubble exists and its bubble component does not converge to zero over time.

Let n, G_d and R^* denote, respectively, the gross population growth rate, the gross dividend growth rate and the steady-state interest rate of the economy without asset. Assuming a constant dividend (i.e., $G_d = 1$), the main insights from Proposition 1 in Tirole (1985) can be summarized as follows.

1. Insight 1: No bubbly equilibrium exists if $1 < n < R^*$.[2]

2. Insight 2: A continuum of equilibria (including both bubbly and bubbleless equilibria) exists if $1 < R^* < n$.

3. Insight 3: Every equilibrium is bubbly if $R^* < 1 < n$.[3]

Furthermore, Proposition 2 in Tirole (1985) claims (without providing a formal proof)[4] that: if $R^* < n$, then the asymptotically bubbleless equilibria are not Pareto optimal and the asymptotically bubbly equilibrium is Pareto optimal.

Following Tirole (1985), much of the subsequent literature has focused on the case of pure bubble assets, i.e., assets that pay no dividends but nonetheless have strictly positive prices. These works have extended Tirole's framework across various economic settings.

A smaller body of research has explored rational bubbles on dividend-paying assets.[5] However, Pham and Toda (2025a,b) have raised critical concerns about the analytical foundations of Tirole (1985) and offered a fairly complete analysis of the model under general assumptions. It is worth noting that such models - featuring both dividend-paying assets and capital accumulation - give rise to non-autonomous two-dimensional dynamical system with infinitely many parameters (including, the sequence dividends). The issue, as Bosi et al. (2018b) and Pham and Toda (2025a) proved, is that when introducing a paying-dividend asset in Diamond's model, there may exist an equilibrium (with or without bubbles) where the capital path converges to zero (they refer this situation as a "*resource curse*").

Hirano and Toda (2025a)'s Section IV considers an OLG exchange economy with a constant population. Assuming the convergence of growth factors and the forward rate function,[6] they manage to prove point 3 of Tirole above, while leaving points 1

[1]See Brunnermeier and Oehmke (2013) and Miao (2014) for surveys of bubbles in general, Martin and Ventura (2018) and Hirano and Toda (2024a) for surveys of rational bubbles, and Hirano and Toda (2024b) for a survey of rational bubbles of assets with positive dividends.

[2]Our notation n corresponds to $1 + n$ in Tirole (1985).

[3]In this case, Tirole (1985), page 1506, mentioned that "... bubbles are necessary for the existence of an equilibrium in an economy in which there exists an (arbitrarily small) rent." See Araujo et al. (2011)'s Section 4.3 for a discussion of the necessity of bubbles for equilibrium implementation.

[4]We quote Tirole (1985) on page 1526: "By (a straightforward extension of) Theorem 5.6 in Balasko-Shell [3], the asymptotically bubbleless equilibria are inefficient and the asymptotically bubbly one is efficient."

[5]For example, Bosi et al. (2018b) extend Tirole (1985)'s model by incorporating non-stationary dividends and altruism, recovering modified versions of Insights 1 and 2. Section V.A of Hirano and Toda (2025a) considers non-stationary dividends under logarithmic utility and presents a version of Insight 3.

[6]See Assumptions 2 and 3 in Hirano and Toda (2025a). See also Section 4.3.1 below.

and 2 aside. Then, Hirano and Toda (2024b)'s Section 5 considered a more specific model (where endowments and dividends grow at constant rates and the utility is homogeneous of degree 1), and provided a fairly complete analysis.

To sum up, the main points in Tirole (1985) still hold in exchange economies under some additional assumptions on dividends, growth factors, and preferences.

Our article has two objectives: (1) to provide a big picture by reexamining these above insights in an OLG exchange economy under general assumptions[7] and (2) to understand the deep relationship between the asset pricing and the Pareto optimality.

Before that, we define the bubble in Section 2 using only the asset price equation and offer a novel point: There is a bubble if and only if the ratio of fundamental value to price decreases over time and converges to zero. From a practical standpoint, our finding suggests that if we observe a period in which this ratio decreases, it may be a period of speculative bubble.

In the first main part of our paper, we investigate the asset prices (with and without bubbles) and the characterization of the equilibrium set. Our results on asset price bubbles can be summarized by Table 1.

Result	**Description**	**Utility function**
Proposition 3	∃ bubbleless eq'm	Non-separable
Proposition 4	∃ bubbleless eq'm	Non-separable
Theorem 1	∃ continuum of eq'a (bubbly and bubbleless)	Non-separable
Theorem 2	Every equilibrium is bubbly	Non-separable
Theorem 3	∃ continuum of eq'a (bubbly and bubbleless)	Separable (Ass 4, 5)
Proposition 5	∃ ! eq'm and it is bubbleless	Separable (Ass 4, 5)
Theorem 4	Full characterization	Separable (Ass 4, 5) and stationary endowment

Table 1: Summary of results on asset price bubbles. Abbreviations and symbols stand for ∃: existence of, !: uniqueness of, eq'm: equilibrium, eq'a: equilibria, Ass: Assumptions

Proposition 3 shows that there is no bubbly equilibrium if the dividend growth rate or the *benchmark interest rates* (i.e. the interest rates of the economy without asset) is higher than the growth rate of the aggregate endowments of young people. Proposition 4 proves that there always exists a bubbleless equilibrium if the present discounted value of dividends computed using the benchmark interest rates is finite.

Next, Theorems 1 and 3 provide conditions under which there exists a continuum of equilibria where bubbleless and bubbly equilibria co-exist. A key condition in Theorems 1 and 3 is that the ratio of the benchmark interest rate to the dividend growth rate remains bounded away from one. This ensures that the asset dividends are quite low with respect to the interest rates, making the fundamental value of the asset quite low. Another crucial condition is that the population growth rate is higher than the interest

[7]See, Santos and Woodford (1997), Kocherlakota (1992), Huang and Werner (2000), Araujo et al. (2011), Werner (2014), Bosi et al. (2022) among others for rational bubbles of dividend-paying assets in models with infinitely-lived agents. See Pham (2024) and the references therein for a discussion of the connections between OLG models and models with infinitely lived agents.

rate of the economy where the agent invests some positive amount for the asset. This condition ensure that the interest rate of the economy is bounded by the population growth rate and households spend a positive faction of their income for buying the asset, which makes the asset price higher than some threshold. By combining two conditions, the asset price may be higher than the fundamental value, i.e., there is a bubble.

Our Theorems 1 and 3 are novel in that they are constructive and do not rely on the convergence arguments used in Tirole (1985) and Pham and Toda (2025a), nor on the assumption of constant growth rates of endowments and dividends as in Hirano and Toda (2024b)'s Section 5, or asymptotically constant growth rates of endowments as in Hirano and Toda (2025a)'s Section IV.

In Theorem 2, we give conditions under which every equilibrium is bubbly (in other words, there is no bubbleless equilibrium). Our key conditions are that (1) the dividend growths are quite low (with respect to the endowment growths) and (2) the dividend growths are not too low with respect to the benchmark interest rates which are lower than the endowment growth rates. The latter condition ensures that the saving rate is bounded away from zero, and hence people always spend a significant amount to buy the asset, which implies that the asset price is quite high. The former condition ensures that the fundamental value of asset is quite low. Then, the asset price is always higher than the fundamental value, i.e., bubbles arise. Our Theorem 2 has a similar conclusion as Theorem 2 in Hirano and Toda (2025a) but our approach is different and our assumptions are weaker in the sense that Hirano and Toda (2025a) require the convergence of growth rates of endowments and the convergence of the forward rate function while we do not (see Section 4.3.1 for a detailed discussion).

In Theorem 4, we offer a full characterization of the equilibrium set as well as the long run properties of asset price in the economy where the utility function is separable, endowments are time-independent but dividends are time-dependent. The literature and our above results show some necessary conditions and sufficient conditions for rational bubbles but do not have a necessary and sufficient condition. In this setup, we manage to provide a novel result, namely a necessary and sufficient condition for the existence of a bubbly equilibrium: there is a bubbly equilibrium if and only if (1) the interest rate of the economy without asset is strictly lower than the population growth rate and (2) the sum of dividends (per capita) is finite.

The second part of our paper studies the Pareto optimality and deepens the relationship between asset price bubbles and Pareto optimality. We provide necessary and sufficient conditions for the Pareto optimality (see Lemma 6 and Theorem 5). We refine and extend the work Okuno and Zilcha (1980) and Balasko and Shell (1980) to our framework where we introduce a dividend-paying asset and allow for unbounded growth.

Okuno and Zilcha (1980) and Balasko and Shell (1980)'s models have L consumption goods. The assumptions on the strictness and smoothness in Okuno and Zilcha (1980) or on the Gaussian curvature of consumers' indifferent surfaces in Balasko and Shell (1980) are quite implicit. We introduce explicit assumptions that can be checked by using elementary calculations.

Our analyses of the asset price bubble and Pareto optimality allow us to bridge these two concepts.

First, our Proposition 7 indicates that every equilibrium is bubbleless and Pareto optimal if the benchmark interest rates are higher than the growth rate of aggregate endowment of young people or the dividend growth rate is higher than the growth rate of the aggregate supply of goods.

Second, in Proposition 8, we demonstrate that an equilibrium is Pareto optimal if it satisfies the uniform strictness condition and the asset value is significant (in the sense that the ratio of the asset value (in terms of good) to the aggregate supply of goods does not converge to zero). This happens regardless the type of equilibrium, whether it is bubbly or bubbleless.

Third, under mild assumptions, there exists a continuum of equilibria and we can rank the households' welfare generated by these equilibria by using its initial asset value (see Proposition 9): The higher the initial value of the asset purchased by young people, the higher the welfare of households. Therefore, there exists a continuum of bubbly equilibrium that are not Pareto optimal.

Fourth, under stationary endowment, as mentioned above, we can fully characterize the equilibrium set (see Theorem 6). Let us focus here on the case of low interest rate, i.e., when the benchmark interest rate is strictly lower than the population growth rate. If the present value of dividends (discounted by using the benchmark interest rates) is finite, there exists a continuum of equilibria. In this case, the maximum equilibrium (which is asymptotically bubbly) is Pareto optimal; we prove this point by extending the works of Okuno and Zilcha (1980), Balasko and Shell (1980). The other equilibria (which can be bubbly or bubbleless) are not optimal because they are strictly Pareto-dominated by the maximum equilibrium; our proof of this point is new and differs from the approach taken by Cass (1972), Okuno and Zilcha (1980), Balasko and Shell (1980).

If the dividend growth rate is lower than the population growth rate but higher than the benchmark interest rate, then there exists a unique equilibrium. Furthermore, this equilibrium is asymptotically bubbly and Pareto optimal.

Finally, it should be noticed that Tirole (1985)'s Proposition 2 also claims (without providing a formal proof) that in the low interest rate case, only the asymptotically bubbly equilibrium is Pareto optimal. We prove this conjecture when the dividends are low. We then contribute by arguing that when the benchmark interest rate is low, an equilibrium which is Pareto optimal may be bubbleless or asymptotically bubbly (see Proposition 8's point 1 and Proposition 10). The reason behind the Pareto optimality in this case does not rely on the fact that the equilibrium is bubbly or bubbleless (because both phenomena - formation of asset bubble and Pareto optimality are endogenous), but on the fact that households have a strong incentive for saving and the asset allows them to do so, making the equilibrium allocation Pareto optimal.

The remainder of our paper is organized as follows. Section 2 introduces a formal definition of asset price bubbles and offers a new insight. Section 3 describes an OLG exchange economy while Section 4 explores the issues of asset prices bubbles. Section 5 studies the Pareto optimality. Section 6 shows the interplay between asset price bubbles and Pareto optimality. Section 7 concludes. Technical proofs are presented in Appendices.

2 Asset price bubble: definition and new insight

We present the notion of asset price bubble and offer a new insight. The exposition here only depends on the following asset pricing equation. In other words, our results in this section apply for any model generating this asset pricing equation.

Definition 1. *Consider an asset with the sequences of prices* (q_t) *and dividends* $(\mathcal{D}_t)$. *We define discount factors* $(R_t)_t$ *by*

$$q_t = \frac{q_{t+1} + \mathcal{D}_{t+1}}{R_{t+1}} \tag{1}$$

It means that the market value of 1 unit of asset at date t, i.e., q_t, equals the discounted value of 1 unit of the same asset at date $t+1$, i.e., q_{t+1}/R_{t+1} plus the dividend $\mathcal{D}_{t+1}/R_{t+1}$.[8] By iterating the asset pricing equation $q_t = (q_{t+1} + \mathcal{D}_{t+1})/R_{t+1}$, we have

$$q_0 = \sum_{s=1}^{T} Q_s \mathcal{D}_s + Q_T q_T, \qquad q_t = \sum_{s=t+1}^{T} \frac{Q_s}{Q_t}\mathcal{D}_s + \frac{Q_T}{Q_t} q_T, \ \forall T > t \geq 1, \tag{2}$$
$$\text{where we denote } Q_t \equiv \frac{1}{R_1 \cdots R_t}, \quad Q_0 \equiv 1.$$

This leads to the traditional definition of fundamental value and bubble.[9]

Definition 2. *Given the sequences of prices* (q_t) *and dividends* $(\mathcal{D}_t)$. *The fundamental value* F_t *and the bubble component* B_t *of the asset at date* $t \geq 0$ *are*

$$F_t \equiv \sum_{s=1}^{\infty} \frac{\mathcal{D}_{t+s}}{R_{t+1} \cdots R_{t+s}}, \qquad B_t = q_t - F_t = \lim_{T \to \infty} \frac{q_T}{R_{t+1} \cdots R_T} \ \forall t \geq 0. \tag{3a}$$

We say that there is an asset price bubble if the market price exceeds the fundamental value, i.e., $q_0 > F_0$. *In this case, this price is called bubbly. Otherwise, it is called bubbleless.*

According to (3a), we can easily check that

$$B_{t+1} = R_{t+1} B_t \text{ and } F_{t+1} + \mathcal{D}_{t+1} = R_{t+1} F_t. \tag{4}$$

So, there is a bubble at date 0 if and only if there is a bubble at date t.

The following result shows two simple tests for the existence of bubble as well as the relationship between the relative value of fundamental value and bubble with respect to the asset price.

[8] In this deterministic framework, the sequence of discount factors (R_t) is uniquely determined. The reader is referred to Santos and Woodford (1997), Araujo et al. (2011), Pascoa et al. (2011), Bosi et al. (2018a) among others for the notion of bubbles in stochastic economies where discount factors (and state price processes) are not necessarily uniquely determined. See Miao and Wang (2012, 2018) for the notion of bubble on the value of firm and Becker et al. (2015), Bosi et al. (2017a) for the notion of bubble on physical capital.

[9] See, among others, Tirole (1982) (page 1172), Tirole (1985) (footnote 8), Kocherlakota (1992) (pages 249-250), Santos and Woodford (1997) (pages 27-29), Huang and Werner (2000) (page 259), Bosi et al. (2018b)'s Section 4, Hirano and Toda (2025a)'s Section II.

Proposition 1. *Consider the asset pricing (1) with $q_t > 0$ for any t.*

1. *The following statements are equivalent.*

   (a) *There is an asset price bubble.*

   (b) *The sequence of fundamental value to price $(\frac{F_t}{q_t})$ is strictly decreasing and converges to 0.*

   (c) *The sequence $(\frac{B_t}{q_t})$ is strictly increasing and converges to 1.*

   (d) $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} < \infty$. *Moreover,* $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} \leq \frac{\frac{F_0}{q_0}}{1-\frac{F_0}{q_0}} < \infty$.

2. *The following statements are equivalent.*

   (a) *There does not exist an asset price bubble.*

   (b) $F_t = q_t$ *for any* $t \geq 0$.

   (c) $B_t = 0$ *for any* $t \geq 0$.

   (d) $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} = \infty$.

Proof. See Appendix A. □

Our novel point is points (1b) and (1c). This result is simple but to the best of our knowledge new with respect to the literature. According to point (1b), the existence of an asset price bubble means that the ratio of the fundamental value to the asset price $\frac{F_t}{q_t}$ is strictly decreasing vanishing in the long run, and the bubble ratio $\frac{B_t}{q_t}$ tends to 1. It means that the fundamental value is negligible with respect to the bubble component, $\lim_{t\to\infty} \frac{F_t}{B_t} = 0$.

From a practical point of view, our result (1b) suggests that if we observe a period where the ratio of the fundamental value to the asset price decreases, this may be a bubbly period.

Condition $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} < \infty$ is firstly presented in Proposition 7 in Montrucchio (2004).[10] Here, we contribute by offering a new proof and proving that $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} \leq \frac{\frac{F_0}{q_0}}{1-\frac{F_0}{q_0}}$.

By definition of the fundamental value F_t, we have $F_t \leq q_t < \infty$. So, by applying the criteria of d'Alembert and of Cauchy, we obtain a relationship between the interest rate R_{t+1} and the dividend growth rates which can be defined as $\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}$ or $\mathcal{D}_t^{\frac{1}{t}}$.

Remark 1 (interest rate versus dividend growth rate)**.** *Assume that dividends are strictly positive ($\mathcal{D}_t > 0$ for any t). Consider an equilibrium. We have*

$$\liminf_{t\to\infty} \left(\frac{1}{R_{t+1}} \frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}\right) \leq 1, \qquad \liminf_{t\to\infty} \left(\frac{\mathcal{D}_t}{R_1 \cdots R_t}\right)^{\frac{1}{t}} \leq 1. \tag{5}$$

If R_t converges to some positive value R, then we must have

$$\liminf_{t\to\infty} \frac{\mathcal{D}_{t+1}}{\mathcal{D}_t} \leq R \text{ and } \liminf_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} \leq R. \tag{6}$$

Note that these properties hold whatever there exists a bubble or not.

[10] This simple characterization is useful in some models (see, for instance, Le Van and Pham (2016), Bosi et al. (2018a,b, 2022), Hirano and Toda (2025a)).

3 An OLG exchange economy

We now study an exchange economy OLG model with the dividend-paying asset. Time is discrete ($t = 0, 1, 2, \ldots$) and there is a single consumption good.

Generations. There are N_t new individuals entering the economy at time $t \geq 0$. The growth factor of population is constant over time: $N_{t+1}/N_t = G_n = n > 0$ for any $t \geq 0$.

Households. Each agent born at date t lives for two periods (young and old) and has $e_t^y \geq 0$ units of consumption as endowments when young and $e_{t+1}^o \geq 0$ when old. Both e_t^y and e_{t+1}^o are exogenous.

Assume that preferences of households born at date t are rationalized by an utility function $U^t(c_t^y, c_{t+1}^o)$ where c_t^y and c_{t+1}^o denote the consumption demands when young and old of a household born at time t.

There is a long-lived asset - the Lucas' tree (Lucas, 1978). At period t, if households buy 1 unit of asset at price q_t, they will, in the next period, receive $\mathcal{D}_{t+1}$ units of consumption good as dividend and they will be able to resell the asset with price q_{t+1}.

Constraints of household born at date t are written

$$c_t^y + q_t z_t \leq e_t^y, \quad c_{t+1}^o \leq e_{t+1}^o + (q_{t+1} + \mathcal{D}_{t+1})z_t, \quad c_t^y, c_{t+1}^o \geq 0.$$

At the date 0, the households born at date -1 only consume: $c_0^o = e_0^o + (q_0 + \mathcal{D}_0)z_{-1}$ where $z_{-1} > 0$ is given.

3.1 Intertemporal equilibrium

Let us denote this two-period OLG economy by $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(U^t, (\mathcal{D}_t)_t, (e_t^y, e_t^o)_t)$.

Definition 3. *An intertemporal equilibrium of the two-period OLG economy is a list $(z_t, (c_t^y, c_t^o), q_t)_{t\geq 0}$ satisfying three conditions: (1) given (q_t, q_{t+1}), the allocation (z_t, c_t^y, c_{t+1}^o) is a solution to the household's problem, (2) markets clear:*

$$N_t z_t = N_{t+1} z_{t+1} \text{ for any } t \geq -1 \tag{7}$$

$$N_t c_t^y + N_{t-1} c_t^o = N_t e_t^y + N_{t-1} e_t^o + \mathcal{D}_t z_{t-1} N_{t-1} \text{ for any } t \geq 0 \tag{8}$$

and (3) $q_t \geq 0$ for any $t \geq 0$.

Balasko and Shell (1981) focus on a model with multiple commodities and no dividend $\mathcal{D}_t = 0$ for any t. The model in Weil (1990) is a particular case of our model where $n = 1, \mathcal{D}_t = 0$ for any $t \geq T$ where the time T is exogenous. The model in Hirano and Toda (2025a)'s Section IV corresponds to the case $n = 1$.

Without loss of generality, we can normalize as follows.

Assumption 1. *$z_{-1} > 0$, $z_{-1}N_{-1} = 1$, $N_t = n^t$ for any $t \geq 0$, where $n > 0$.*

In equilibrium, we have $z_t n^t = z_t N_t = z_{-1}N_{-1} = 1$. So, $z_t = 1/n^t$.

Denote the asset value $a_t = q_t z_t$ and dividend per capita by

$$a_t \equiv \frac{q_t}{n^t} \text{ and } d_t \equiv \frac{\mathcal{D}_t}{n^t}.$$

Observe that $\frac{nd_{t+1}}{d_t} = \frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}$ and the good market clearing conditions become $c_t^y + \frac{c_t^o}{n} = e_t^y + \frac{e_t^o}{n} + d_t$ for any $t \geq 0$.

The following assumptions, which are similar to Assumption 1 in Hirano and Toda (2025a), ensure the existence of an equilibrium (see, for instance, Balasko and Shell (1980) and Wilson (1981)).

Assumption 2. *Assume that $U^t : \mathbb{R}_+^2 \to \mathbb{R}$ is strictly increasing, strictly quasi-concave, continuously differentiable on $\mathbb{R}_{++}^2$, $U_1^t(0, x_2) = \infty$, $U_2^t(x_1, 0) = \infty$ $\forall x_1, x_2 > 0$, where U_i^t denotes the partial derivative of U^t with respect to the i^{th} component. The sequence of endowments satisfies $e_t^y > 0, e_{t+1}^o \geq 0$ for any $t \geq 0$.*

Since the utility function is strictly quasi-concave, the allocation (c_t^y, c_t^o, z_t) is the unique solution to the maximization problem of the household born at date t if and only if

$$c_t^y + q_t z_t = e_t^y, \quad c_{t+1}^o = e_{t+1}^o + (q_{t+1} + \mathcal{D}_{t+1}) z_t \tag{9a}$$

$$q_t U_1^t(c_t^y, c_{t+1}^o) = (q_{t+1} + \mathcal{D}_{t+1}) U_2^t(c_t^y, c_{t+1}^o) \tag{9b}$$

Definition 4. *(1) Consider an equilibrium. Define the interest rate R_{t+1} between dates t and $t+1$ by $R_{t+1} \equiv \frac{U_1^t(c_t^y, c_{t+1}^o)}{U_2^t(c_t^y, c_{t+1}^o)}$.*

(2) Define the benchmark interest rate (i.e., the interest rates of the economy without asset) R_{t+1}^* between dates t and $t+1$ by $R_{t+1}^* \equiv \frac{U_1^t(e_t^y, e_{t+1}^o)}{U_2^t(e_t^y, e_{t+1}^o)}$.

To obtain a relationship between R_t and R_t^*, we introduce an additional assumption.

Assumption 3. *$U_1^t(x_1, x_2)$ is decreasing in x_1 and increasing in x_2. $U_2^t(x_1, x_2)$ is decreasing in x_2 and increasing in x_1.*

According to the Euler condition (9b), we have

$$\frac{q_{t+1} + \mathcal{D}_{t+1}}{q_t} = \frac{U_1^t\Big(e_t^y - q_t z_t, e_{t+1}^o + (q_{t+1} + \mathcal{D}_{t+1}) z_t\Big)}{U_2^t\Big(e_t^y - q_t z_t, e_{t+1}^o + (q_{t+1} + \mathcal{D}_{t+1}) z_t\Big)}.$$

In equilibrium, we have $z_t > 0$. Therefore, we immediately obtain the following result.

Lemma 1. *Under Assumptions 1, 2, 3, in equilibrium, we have $R_t \geq R_t^*$ for any t.*

4 Asset price bubbles

Given an equilibrium, by definition of the sequence (R_t) and the Euler condition (9b), we have the asset pricing equation (1). So, all results in Section 2 apply.

By consequence, with the notations

$$a_t \equiv \frac{q_t}{n^t}, \qquad d_t \equiv \frac{\mathcal{D}_t}{n^t}, \qquad f_t = \frac{F_t}{n^t}, \qquad b_t = \frac{B_t}{n^t}, \tag{10}$$

we can restate Proposition 1 as follows:

Lemma 2. *In the case of strictly positive dividends ($\mathcal{D}_t > 0$ for any t), the following statements are equivalent.*[11]

1. *A bubble exists.*
2. $\lim_{t\to\infty} Q_t q_t > 0$, *i.e.* $\lim_{t\to\infty} \dfrac{n^t a_t}{R_1 \cdots R_t} > 0$.
3. *The sequence of fundamental value to asset value* $(\frac{f_t}{a_t}) = (\frac{F_t}{q_t})$ *is strictly decreasing and converges to* 0.
4. *The sequence of bubble to asset value* $(\frac{b_t}{a_t}) = (\frac{B_t}{q_t})$ *is strictly increasing and converges to* 1.
5. $\sum_{t=1}^{\infty} \mathcal{D}_t/q_t < \infty$, *i.e.,* $\sum_{t=1}^{\infty} d_t/a_t < +\infty$.

Lemma 2's point 5 leads to an interesting implication regarding the role of saving rate.

Proposition 2 (Role of saving rate on the existence of bubble)**.** *Consider the case of strictly positive dividends ($\mathcal{D}_t > 0$ for any t). Consider an equilibrium. If the saving rate of young people is bounded away from zero, i.e.,* $\liminf_{t\to\infty} \frac{q_t z_t}{e_t^y} > 0$, *and the dividends grow slower than the economy's endowment (i.e.,* $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty$*), then this equilibrium is bubbly.*

Proof. See Appendix B. □

Proposition 2 highlights the importance of the saving rate. The underlying intuition is that when individuals consistently allocate a portion of their income to invest in the asset, they continue to purchase it - even when its fundamental value is low (as is the case when dividends are low). This persistent demand contributes to the formation of a bubble.

Theorems 2 and 4 below provide a condition under which $\liminf_{t\to\infty} \frac{q_t z_t}{e_t^y} > 0$. However, as we will show in Theorem 4, the saving rate may converge to zero or to some strictly positive value. One key issue is understanding how the asset demand and asset prices evolve over time, which we will address.

4.1 Equilibrium without asset price bubbles

The following result provides conditions to ensure that every equilibrium is bubbleless.

Proposition 3 (no bubble conditions)**.** 1. *Let Assumptions 1, 2 be satisfied. Every equilibrium is bubbleless if*

$$\text{(Non-negligible dividend condition):} \quad \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} = \infty. \tag{11}$$

[11] Condition $\mathcal{D}_t > 0$ for any t is to ensure that $q_t > 0$ at any t, which is needed to define $\mathcal{D}_t/q_t$.

2. Let Assumptions 1, 2, 3 be satisfied. Every equilibrium is bubbleless if

$$\text{(High interest rate condition):} \quad \lim_{t\to\infty} \frac{n^t e_t^y}{R_1^* \cdots R_t^*} = 0. \tag{12}$$

Proof. See Appendix B. □

Condition (11) indicates that the existence of bubble requires that the dividend growth rate must be lower than the endowment growth rate.[12]

Condition (12) means that, if the benchmark interest rare R_t^* is quite high (higher than the product of population growth rate and the endowment growth rate), then there is no bubbly equilibrium. In other words, every equilibrium is bubbleless.

The insight of condition (12) is in line with the main result in Santos and Woodford (1997): there is no bubble if the sum of discounted values of aggregate outputs is finite. However, the condition in Santos and Woodford (1997) is based on endogenous variables. By contrast, our condition (12) is based on exogenous variables. Notice that the high interest rate condition (12) is also in line with Proposition 1.(a) in Tirole (1985), Proposition 2.1 in Bosi et al. (2018b), Proposition 4 in Bosi et al. (2022), Lemma 3.2 in Pham and Toda (2025a).

We now provide a condition under which a bubbleless equilibrium always exists.

Proposition 4 (Existence of bubbleless equilibrium)**.** *Let Assumptions 1, 2, 3 be satisfied. If*

$$\text{(Not-too-low interest rate condition)} \quad \sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1^* \cdots R_t^*} < \infty, \tag{13}$$

then there exists a bubbleless equilibrium.

Proof. See Appendix B. □

Proposition 4 states that there exists a bubbleless equilibrium if the present discounted value of dividends computed with the interest rates of the economy without asset is finite. Proposition 4 is similar to Proposition 3.1 in Pham and Toda (2025a). However, our assumption 3 is more explicit and slightly weaker than their Assumption 3.

According to Proposition 4, when dividends growth factors are lower than return factors of the economy without asset (R_t^*), an equilibrium without bubbles always exists. By the way, Proposition 4 is related to Lemma 1 in Tirole (1985) which claims the existence of bubbleless equilibrium. However, Tirole (1985)'s proof is not complete (because he did not prove that his function Γ is continuous) and implicitly requires, in page 1522, the assumption that the present discounted value of the rent (dividends) computed with the Diamond bubbleless and rentless interest rates is finite, which corresponds to our condition (13). Note also that, the method of our proof is different from Tirole (1985).

It should be noticed that (13) is only a sufficient condition for the existence of a bubbleless equilibrium. Conditions (11) and (12) are also sufficient for the existence of a bubbleless equilibrium since they rule out any bubbly equilibrium.

In the next sections, we study conditions under which there exists a bubbly equilibrium.

[12] This point is in line with Corollary 1 in Bosi et al. (2018b), Corollary 3 in Bosi et al. (2022), Lemma 3.1 in Pham and Toda (2025a).

4.2 A continuum of equilibria (with and without bubbles)

Since $z_tN_t = 1$ for any $t \geq 0$, the equilibrium $(z_t, (c_t^y, c_t^o), q_t)_{t\geq 0}$ is one-to-one represented by the sequence of prices $(q_t)_{t\geq 0}$ (or the sequence of asset value $(a_t) = (q_t z_t)$) which we also call an equilibrium.

Recall that $a_t \equiv \frac{q_t}{n^t}, d_t \equiv \frac{\mathcal{D}_t}{n^t}$. By consequence, (q_t) is an equilibrium if and only if the sequence $(a_t, R_{t+1})_{t\geq 0}$ satisfies the Euler and non-arbitrage conditions

$$U_1^t\left(e_t^y - a_t, e_{t+1}^o + R_{t+1}a_t\right) - R_{t+1}U_2^t\left(e_t^y - a_t, e_{t+1}^o + R_{t+1}a_t\right) = 0 \quad (14a)$$

$$a_{t+1} + d_{t+1} = a_t\frac{R_{t+1}}{n}, \quad 0 < a_t < e_t^y \text{ for any } t \geq 0. \quad (14b)$$

The Euler condition leads to the following definition.

Definition 5. *Let $t \geq 0$, $e_t^y > 0, e_{t+1}^o > 0$. Define the function $K_t : (0, e_t^y)\times[0,\infty) \to \mathbb{R}$ by*

$$K_t(a, R) \equiv U_1^t(e_t^y - a, e_{t+1}^o + Ra) - RU_2^t\left(e_t^y - a, e_{t+1}^o + Ra\right). \quad (15)$$

By Assumption 3, we have a direct consequence.

Lemma 3. *Let Assumption 3 be satisfied. The function $K_t(a, R)$ is increasing in a.*

We now provide conditions under which there exists a continuum of equilibria. The intuition is that with the initial asset value a_0, we construct the sequences of asset values (a_t) and interest rates (R_t) which satisfy the Euler and non-arbitrage conditions.

Theorem 1 (Continuum of equilibria: co-existence of bubbly and bubbleless equilibria)**.** *Let Assumptions 1, 2, 3 be satisfied and $e_t^o > 0$ for any t.*

Assume that there exists a sequence $(\epsilon_t)_{t\geq 0}$ such that, for any $t \geq 0$,

(i) $\epsilon_t \in (0, e_t^y)$, $\epsilon_t - d_{t+1} \leq \epsilon_{t+1}$.

(ii) $K_t(\epsilon_t, n) < 0$ *(this condition implies that there exists R_{t+1}^ϵ such that $0 < R_{t+1}^\epsilon < n$ and $K_t(\epsilon_t, R_t^\epsilon) = 0$).*

(iii) If R satisfies $K_t(\epsilon_t, R) \geq 0$ and $R < n$, then $R < R_{t+1}^\epsilon$.[13]

Assume also that there exist $\lambda > 0$ and $\gamma > 1 + \frac{1}{\lambda}$ satisfying $\lambda d_t < \epsilon_t$ and

$$\textit{(Not-too-low interest rate condition):} \quad R_{t+1}^* \geq \left(n\frac{d_{t+1}}{d_t}\right)\gamma \text{ for any } t \geq 0. \quad (16)$$

Then, there exists at least one bubbleless equilibrium and there exists a continuum of bubbly equilibria (where the asset values and interest rates satisfy $a_t \in [\lambda d_t, \epsilon_t]$ and $R_t^* \leq R_t \leq R_t^\epsilon$).

Proof. See Appendix B.1. □

[13]Notice that, in general, the function $K_t(\epsilon, \cdot) : [0,\infty) \to \mathbb{R}$ may not be monotonic on $[0,\infty)$.

The key intuition of Theorem 1 is that when the benchmark interest rates R_t^* is lower than the population growth rate n but higher than the dividend growth rate (condition (16)), then bubbly and bubbleless co-exist.

Theorem 1 is new with respect to the literature because it provides general conditions (with non-separable utility functions, non-stationary endowment, non-stationary dividend) under which bubbleless and bubbly equilibria co-exist. It should be noticed that Theorem 1 and its proof do not rely on any form of convergence of variables as required in some papers in the literature (Tirole, 1985; Farhi and Tirole, 2012; Hirano and Toda, 2025a; Pham and Toda, 2025a). Another added value of Theorem 1 is that it is constructive because it shows us how to construct equilibria with bubbles.[14]

Let us explain our constructive proof. We start from the initial asset value a_0. By using the Euler equation, condition (i) and (ii), we can find R_1 (such an interest rate may not necessarily be unique). Then, we determine a_1 by the non-arbitrage condition $a_1 + d_1 = a_0\frac{R_1}{n}$. Conditions (ii) and (iii) which are a kind of low interest rate conditions ensure that the equilibrium interest rate R_t is always lower than the (population) growth rate n. This low interest condition makes sure that the bubble and asset values do not explode. The not-too-low interest rate condition (16) guarantees that our sequence (a_t) satisfies

$$\frac{a_{t+1}}{d_{t+1}} \geq \gamma\frac{a_t}{d_t} - 1 > 0.$$

Since $\gamma > 1$, we have $\sum_{t\geq 1} d_t/a_t < \infty$, i.e., this equilibrium is bubbly (thanks to Lemma 2). Moreover, by Proposition 4, the not-too-low interest rate condition (16) also ensures the existence of a bubbleless equilibrium.

Naturally, we may ask whether conditions in Theorem 1 can be satisfied. Actually, they may hold under a large class of model.[15] Indeed, the following corollary provides an answer.

Corollary 1. *Assumptions in Theorem 1 are satisfied if the following conditions hold:*

1. *The utility is* $U^t(x_1, x_2) = \frac{x_1^{1-\sigma}}{1-\sigma} + \beta\frac{x_2^{1-\sigma}}{1-\sigma}$ *where* $\sigma > 0, \beta > 0$, *while endowments satisfy* $\frac{e^o_{t+1}}{e^y_t} = g_e > 0$ *and* $e^y_t \leq e^y_{t+1}$ *for any* t.

   In this case, the benchmark interest rate $R_t^* = R^* = \frac{g_e^{1/\sigma}}{\beta}$ $\forall t$.

2. *Dividend growth rate:* $d_t = d_0 d^t$, *where* $d_0, d > 0$, *for any* t *(i.e.,* $\mathcal{D}_t = d_0 n^t d^t$*).*

3. *Lower interest rate and low dividend conditions:* $nd < R^* < n$.

4. *Take* $\epsilon_t \equiv \epsilon e^y_t$, *where* ϵ *satisfies* $g_e n^{\frac{-1}{\sigma}} + \epsilon(n^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}} < 0$.

5. *Take* σ *so that* $\frac{g_e}{\epsilon(\sigma-1)} < n$. *Note that* σ *may be lower or higher than* 1.

6. *Take* $\gamma \in (0, \frac{R^*}{nd})$ *and* $\lambda > 0$ *so that* $\gamma > 1 + \frac{1}{\lambda}$ *and* $\lambda d_0 d^t < \epsilon_t \equiv \epsilon e^y_t$.

Proof. See Appendix B.1 □

[14] Bosi et al. (2022)'s Proposition 7 provide a condition to have a continuum of bubbly equilibria in an exchange economy with heterogeneous infinitely-lived agents and logarithmic utility functions. Their working paper (Bosi et al., 2021) gives several examples.

[15] In Section 4.4, we will present more explicit conditions in the case of separable utility functions for the existence of multiple equilibria.

4.3 Conditions under which every equilibrium is bubbly

We now provide conditions under which every equilibrium is bubbly (in other words, there is no bubbleless equilibrium). When this case happens, Tirole (1985) wrote in page 1506 that bubbles are necessary for the existence of an equilibrium.

The following observation, which is a direct consequence of Propositions 3 and 4, gives some intuitions.

Corollary 2. *Let Assumptions 1, 2. Assume that every equilibrium is bubbly. Then, we must have*

$$\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty \tag{17}$$

If we require, in addition, Assumption 3, then we have

$$\limsup_{t\to\infty} \frac{n^t e_t^y}{R_1^* \cdots R_t^*} > 0 \text{ and } \sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1^* \cdots R_t^*} = \infty. \tag{18a}$$

These conditions suggest that when there is no bubbleless equilibrium, the benchmark interest rates should be lower than the growth rates of aggregate good supply and the dividend growth rates. Moreover, the dividends should be low (i.e., $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty$) but not too low (i.e., $\sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1^* \cdots R_t^*} = \infty$).

To present our conditions (based on exogenous variables) which ensure that every equilibrium is bubbly, we use the Euler equation (14a) which is equivalent to

$$U_1^t\left(e_t^y\Big(1-\frac{a_t}{e_t^y}\Big), e_t^y\Big(\frac{e_{t+1}^o}{e_t^y}+R_{t+1}\frac{a_t}{e_t^y}\Big)\right) - R_{t+1}U_2^t\left(e_t^y\Big(1-\frac{a_t}{e_t^y}\Big), e_t^y\Big(\frac{e_{t+1}^o}{e_t^y}+R_{t+1}\frac{a_t}{e_t^y}\Big)\right) = 0.$$

This equation motivates us to define two functions V_1^t and V_2^t: $\mathbb{R}_+^2 \to \mathbb{R}_+$ by $V_1^t(x_1,x_2) \equiv U_1^t(e_t^y x_1, e_t^y x_2)$ and $V_2^t(x_1,x_2) \equiv U_2^t(e_t^y x_1, e_t^y x_2)$.

Denote $a_t^e \equiv \frac{a_t}{e_t^y}$ and $g_{e,t+1} \equiv \frac{e_{t+1}^o}{e_t^y}$ the saving rate and the endowment growth rate of household born at date t. The Euler equation becomes

$$V_1^t\left(1-a_t^e, g_{e,t+1}+R_{t+1}a_t^e\right) - R_{t+1}V_2^t\left(1-a_t^e, g_{e,t+1}+R_{t+1}a_t^e\right) = 0. \tag{19}$$

We are now ready to state our result.

Theorem 2. *Let Assumptions 1, 2 be satisfied.*

Assume the so-called Condition (B): there exist $\bar{\epsilon} \in (0,1)$, positive sequences (X_t) and $(\bar{X}_t)$, and a date T satisfying the following conditions:

1. *(Not-too-low dividend condition)* $\sum_{t=1}^{\infty} \dfrac{\mathcal{D}_t}{X_1 \cdots X_t} = \infty.$

2. *(Low interest rate conditions)*

   (a) $X_{t+1} \leq n\dfrac{e_{t+1}^y}{e_t^y}\ \forall t \geq T.$

(b) For any $t \geq T$, if $\epsilon \in (0, \bar{\epsilon})$ and $X \in [0, \bar{X}_{t+1}]$ satisfy

$$X = \frac{V_1^t(1-\epsilon, g_{e,t+1} + X\epsilon)}{V_2^t(1-\epsilon, g_{e,t+1} + X\epsilon)}, \tag{20}$$

then $X \leq X_{t+1}$.

Then, the following statements hold.

1. *For any equilibrium, the ratio of asset value to endowment is uniformly bounded away from zero (i.e., $\liminf_{t\to\infty} \frac{a_t}{e_t^y} > 0$).*

2. *If $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty$, then every equilibrium is bubbly.*

3. *If $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} = \infty$, then every equilibrium is bubbleless.*

4. *By consequence, any equilibrium is bubbly if and only if*

$$\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty \quad \textit{(low dividend condition)}. \tag{21}$$

Proof. See Appendix B.2. □

It should be noticed that condition $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty$ is necessary for the existence of a bubbly equilibrium (see Proposition 3's point 1). Here, we go further by showing that every equilibrium is bubbly if we add Condition (B) in Theorem 2. Let us now explain Condition (B). Looking at (20), the value X in (20) represents the expected interest rate when the saving rate equals ϵ. The intuition of (B2) is that when the saving rate $\frac{a_t}{e_t^y}$ is lower than the threshold $\bar{\epsilon}$, the expected interest rate R_{t+1} will be lower than X_{t+1}. Then, the present value of the asset $\sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1 \cdots R_t}$ will be higher than $\sum_{t=1}^{\infty} \dfrac{\mathcal{D}_t}{X_1 \cdots X_t} = \infty$, which is impossible.

To see better the link between Theorem 2 and our previous results, let us consider a particular case where Assumption 3 holds. In such a case, $R_t \geq R_t^*$ for any t. So, (B2) implies that $R_{t+1}^* \leq X_{t+1}$, i.e., the benchmark interest rate is low. Condition (B1), i.e., $\sum_{t=1}^{\infty} \dfrac{\mathcal{D}_t}{X_1 \cdots X_t} = \infty$, implies that

$$\limsup_{t\to\infty} \left(\frac{\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}}{X_{t+1}} \right) \geq 1. \tag{22}$$

Therefore, we get that

$$\limsup_{t\to\infty} \left(\frac{\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}}{R_{t+1}^*} \right) \geq 1. \tag{23}$$

This condition can be interpreted as the dividend growth rate $\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}$ is higher than the benchmark interest rate R^*_{t+1}. Note that (23) is opposed to the not-too-low interest rate conditions (13) or (16) which ensures the existence of a bubbleless equilibrium.

The following consequence of Theorem 2 helps us not only to better understand it and but also to show that assumptions in Theorem 2 can be satisfied in standard settings.

Corollary 3. *Assume that*

1. $U^t(x_1, x_2) = ln(x_1) + \beta ln(x_2)$ *where* $\beta > 0$. *In this case, the benchmark interest rate is* $R^*_{t+1} = \frac{e^o_{t+1}}{e^y_t}\frac{1}{\beta}$.

2. $\limsup_{t\to\infty} \dfrac{R^*_{t+1}}{n\frac{e^y_{t+1}}{e^y_t}} < 1$ *(or, equivalently,* $\limsup_{t\to\infty} \dfrac{e^o_t}{\beta n e^y_t} < 1$*).*

3. $\dfrac{\mathcal{D}_t}{n^t e^y_t} = \frac{1}{t^\alpha}$ *where* $\alpha > 1$.

Then, every equilibrium is bubbly and $\liminf_{t\to\infty} \frac{a_t}{e^y_t} > 0$ *(the saving rate is uniformly bounded away from zero).*

Proof. See Appendix B.2. □

In this corollary, we have $\sum_{t\geq 1} \dfrac{\mathcal{D}_t}{n^t e^y_t} < \infty$, which violates no-bubble condition (11). Moreover, we observe that

$$\limsup_{t\to\infty} \frac{R^*_{t+1}}{\frac{e^y_{t+1}}{e^y_t}} < n = \lim_{t\to\infty} \frac{\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t}}{\frac{e^y_{t+1}}{e^y_t}}. \tag{24}$$

By consequence, we have $\lim_{t\to\infty} \frac{n^t e^y_t}{R^*_1\cdots R^*_t} = \infty$ which violates condition (12). We also have $\sum_{t\geq 1} \frac{\mathcal{D}_t}{R^*_1\cdots R^*_t} = \infty$ which violates condition (13).

4.3.1 Comparison with the existing literature

Theorem 2 is related to Proposition 1.c of Tirole (1985) in an OLG production economy and Theorem 1 in Pham and Toda (2025b) where they claim that, under mild conditions, there exists a unique equilibrium and this is asymptotically bubbly (see Pham and Toda (2025b) for a review on this issue). However, Pham and Toda (2025a,b) raise some concerns in the proof of Tirole (1985) and Theorem 1 in Pham and Toda (2025b) restores Proposition 1.c of Tirole (1985). It should be also noticed that although we consider an OLG exchange economy, we work under non-stationary endowments and non-stationary dividends while Tirole (1985) considers an OLG production economy with constant dividend and stationary production function.

Theorem 2 in Hirano and Toda (2025a)'s Section IV also consider an OLG exchange economy like our model and proves, under some assumptions, that every equilibrium is bubbly. This is similar to the statement (2) in Theorem 2. We go further by establishing the statements (3) and (4) in Theorem 2.

Moreover, our assumptions are significantly different from those in Hirano and Toda (2025a). The following remarks explain the differences.

Remark 2. *Assumption 2 in Hirano and Toda (2025a) requires the convergence of grow factors in the long run (i.e.,* $\lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t} \in (0,\infty)$, $\lim_{t\to\infty} \frac{e^o_t}{e^y_t} \in [0,\infty)$ *and their Assumption 3 requires an uniform convergence condition of the so-called forward rate function* f_t *defined by*

$$f_t(x_1,x_2) \equiv \frac{U^t_1(e^y_t x_1, e^y_t x_2)}{U^t_2(e^y_t x_1, e^y_t x_2)} = \frac{V^t_1(x_1,x_2)}{V^t_2(x_1,x_2)},$$

We do not require these assumptions in our Theorem 2. For example, Corollary 3 does not require such convergences. Moreover, our proof of Theorem 2 seems to be simpler than the proof of Theorem 2 in Hirano and Toda (2025a).

Remark 3. *Assumptions 2, 3 and condition (20) in Theorem 2 in Hirano and Toda (2025a), which is*

$$\lim_{t\to\infty} \frac{V^t_1\left(1, \lim_{s\to\infty} \frac{e^0_{s+1}}{e^y_s}\right)}{V^t_2\left(1, \lim_{s\to\infty} \frac{e^0_{s+1}}{e^y_s}\right)} < G_d \equiv \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} < \lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t}, \tag{25}$$

are stronger than conditions in the statement 2 of our Theorem 2 (see Appendix B.2 for a formal proof of this claim). More precisely, the first inequality in (25) together with Assumptions 2 and 3 in Hirano and Toda (2025a) imply Condition (B) in our Theorem 2. Then, their assumption $\limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} < \lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t}$ *implies* (21).

Remark 4. *Hirano and Toda (2025a)'s Theorem 2 does not apply for the case* $\limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} = \lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t}$ *while our Theorem 2 can be used for this case. Indeed, in Corollary 3, take* $n = 1$ *and* $e^y_{t+1} = e^y_t \ \forall t$*, we have* $\mathcal{D}_t = \frac{1}{t^\alpha}$ *and* $\limsup_{t\to\infty} \left(\frac{1}{t^\alpha}\right)^{\frac{1}{t}} = 1 = \lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t}$*, which violates the second inequality in (25).*

4.4 Characterization of the equilibrium set

We have so far provided conditions to rule out bubbly equilibria and conditions to have bubbly equilibrium.

We now seek to provide a full characterization of the set of equilibrium in order to see a big picture. To do so, we focus on separable utility functions.

Assumption 4. *Assume that* $U^t(x_1,x_2) = u(x_1) + \beta v(x_2)$ *for any* t, x_1, x_2*. The functions* $u, v : \mathbb{R}_+ \to \mathbb{R}$ *are twice continuously differentiable, strictly increasing, and strictly concave with* $u'(0) = v'(0) = \infty$*. The endowments satisfy* $e^y_t > 0, e^o_{t+1} \geq 0$ *for any* $t \geq 0$.

4.4.1 Equilibrium system

Recall that $a_t \equiv \frac{q_t}{n^t}, d_t \equiv \frac{\mathcal{D}_t}{n^t}$. By consequence, (q_t) is an equilibrium if and only if the sequence $(a_t, R_{t+1})_{t\geq 0}$ satisfies the following conditions:

$$u'(e^y_t - a_t) = \beta R_{t+1} v'\left(e^o_{t+1} + R_{t+1}a_t\right) \tag{26a}$$

$$a_{t+1} + d_{t+1} = a_t \frac{R_{t+1}}{n}, \quad 0 < a_t < e^y_t \text{ for any } t \geq 0. \tag{26b}$$

Since $(R_{t+1})_{t\geq 0}$ is uniquely determined via (a_t) by $R_{t+1} = (a_{t+1} + d_{t+1})\frac{n}{a_t}$, we also call $(a_t)_{t\geq 0}$ an equilibrium.

We introduce an assumption allowing us to express R_{t+1}, and by consequence a_{t+1}, as a function of a_t.

Assumption 5. *The function $cv'(c)$ is increasing on $(0, \infty)$ and $e^o_t > 0$ $\forall t$.*

Lemma 4. *Let Assumptions 1, 4, 5 be satisfied. For $a \in (0, e^y_t)$, there exists a unique $R_{t+1} > 0$ satisfying $u'(e^y_t - a) = \beta R_{t+1} v'\left(e^o_{t+1} + R_{t+1}a\right)$ if and only if $au'(e^y_t - a) < \beta \lim_{c\to\infty} cv'(c)$.*[16]

So, we can define the function $g_t : \mathcal{D}_t \equiv \{a \in (0, e^y_t) : au'(e^y_t - a) < \beta \lim_{c\to\infty} cv'(c)\} \to \mathbb{R}_+$ by $g_t(a) = R_{t+1}$ where R_{t+1} is uniquely determined by $u'(e^y_t - a) = \beta R_{t+1} v'\left(e^o_{t+1} + R_{t+1}a\right)$. Note that g_t is increasing and

$$\lim_{a\to 0} g_t(a) = R^*_t \equiv \frac{u'(e^y_t)}{\beta v'(e^o_{t+1})}. \tag{27}$$

Proof. See Appendix B.3. □

According to Lemma 4, the sequence (a_t) is an equilibrium if and only if

$$u'(e^y_t - a_t) = \beta R_{t+1} v'(e^o_{t+1} + R_{t+1}a_t) \tag{28a}$$

$$R_{t+1} = g_t(a_t), \text{ where } g_t \text{ is defined by Lemma 4} \tag{28b}$$

$$a_{t+1} + d_{t+1} = a_t \frac{R_{t+1}}{n}, \quad 0 < a_t < e^y_t \text{ for any } t \geq 0. \tag{28c}$$

It is worth highlighting several points about the function g_t, which maps a_t to R_{t+1}.

- In some particular cases, we can explicitly compute R_{t+1}. For instance, if $u'(c) = v'(c) = 1/c$, then we have $R_{t+1}\left[\beta e^y_t - (1+\beta)a_t\right] = e^o_{t+1}$. If $u'(c) = v'(c) = c^{-\sigma}$ with $\sigma > 0$, we have $a_t\left(R_{t+1}^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\right) = \beta^{\frac{1}{\sigma}} e^y_t - \frac{e^o_{t+1}}{R_{t+1}^{\frac{1}{\sigma}}}$. So, when $\sigma < 1$, then R_{t+1} is increasing in a_t.

- If $e^o_t = 0$ for any t, the Euler condition becomes

$$u'(e^y_t - a_t) = \beta R_{t+1} v'(R_{t+1}a_t). \tag{29}$$

[16] Here, we also allow for the case where $\lim_{c\to\infty} cv'(c) = \infty$. If $v(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma \in (0,1)$, then $\lim_{c\to\infty} cv'(c) = \infty$. If $v(c) = ln(c) + Aln(B + c^\sigma)$, where $A \geq 0, B \geq 0, \sigma \in (0,1)$, then $\lim_{c\to\infty} cv'(c) < \infty$.

In some cases, this equation does not allow us to express R_{t+1} as a function of a_t and e^y_t (in this case, R_{t+1} must be determined by (1)).[17] Indeed, when $cv'(c)$ equals a constant $x > 0$ for any c, condition (29) implies that $a_t u'(e^y_t - a_t) = \beta x > 0$ for any t. Since $u'(0) = \infty$ and the function u' is decreasing, this equation has a unique solution a_t, denoted by $\gamma(e^y_t)$ (note that $\gamma(e^y_t)$ is increasing in e^y_t). It means that there exists a unique equilibrium. According to Lemma 2, this equilibrium is bubbly if and only if

$$\sum_{t\geq 1} \frac{\mathcal{D}_t}{n^t \gamma(e^y_t)} < \infty \text{ (i.e., low dividend condition).}$$

Note also that in this case, $R^*_t = 0$ for any t.

4.4.2 Basic properties of the equilibrium set

Following Tirole (1985), we redefine equilibrium by using its initial asset value.

Definition 6. *Denote $\mathcal{A}_0$ the equilibrium set of all values $a \geq 0$ such that there exists a sequence $(a_t)_{t\geq 0}$ satisfying (26) and the initial asset value equals $a_0 = a$.*

For each a in the equilibrium set $\mathcal{A}_0$ and the associated equilibrium sequence $(a_t)_{t\geq 0}$ with $a_0 = a$, we define the fundamental value $f_t = f_t(a)$ and the bubble component $b_t = b_t(a)$ by

$$f_t = f_t(a) \equiv \frac{F_t}{n^t} = \sum_{s=1}^{\infty} \frac{n}{R_{t+1}} \cdots \frac{n}{R_{t+s}} d_{t+s}, \quad b_t = b_t(a) \equiv a_t - f_t. \tag{30a}$$

We can verify that $f_{t+1} = \frac{R_{t+1}}{n} f_t - d_{t+1}, \quad b_{t+1} = b_t \frac{R_{t+1}}{n}$ for any $t \geq 0$. We can now redefine the notion of asset price bubble which is consistent with Tirole (1985).

Definition 7. *We say that an equilibrium $a \in \mathcal{A}_0$ is bubbleless if $a = f_0(a)$. An equilibrium a is bubbly if $a > f_0(a)$. A bubbly equilibrium is asymptotically bubbly (bubbleless) if $\limsup_{t\to\infty} b_t > 0$ ($\limsup_{t\to\infty} b_t = 0$).*

Following Tirole (1985), Bosi et al. (2018b), Bosi et al. (2022), we have the following result showing the form of the equilibrium set.

Lemma 5. *Let Assumptions 1, 4, 5 be satisfied.*

1. *The set $\mathcal{A}_0$ is a compact interval.*
2. *The fundamental value function $f_t(a_0)$ is decreasing in the initial value a_0 while the size of bubble $b_t(a_0)$ is strictly increasing.*
3. *There exists at most one bubbleless solution. Moreover, if there are two equilibria with initial asset values $a_{1,0} < a_{2,0}$, then any equilibrium with initial asset value $a_0 \in (a_{1,0}, a_{2,0}]$ is bubbly.*

Proof. See Appendix B. □

[17] See Footnote ?? below.

Proposition 3, Proposition 4 and Lemma 5's point 3 lead to an interesting result showing not only the uniqueness of bubbleless equilibrium and but also the uniqueness of equilibrium.

Proposition 5 (Uniqueness of equilibrium and of bubbleless equilibrium)**.** *Let Assumptions 1, 4, 5 be satisfied.*

1. *If condition (13) holds, there exists a unique bubbleless equilibrium.*
2. *If condition (11) or condition (12) holds, then there is a unique equilibrium and it is bubbleless.*

In the next section, we seek to understand when the equilibrium set is singleton or multiple-valued and the dynamics of asset prices.

4.4.3 A continuum of equilibria (with and without bubbles)

We now provide conditions under which the equilibrium set is a compact interval and there exists a continuum of bubbly equilibria.

Theorem 3 (Continuum of equilibria)**.** *Let Assumptions 1, 4, 5 be satisfied. Assume also that there exists a positive sequence* $(\epsilon_t)_{t\geq 0}$*,* $\lambda > 0$*, and* $\gamma > 1 + \frac{1}{\lambda}$ *such that*

$$\epsilon_t \in (0, e_t^y), \quad \epsilon_t - d_{t+1} \leq \epsilon_{t+1}, \quad \lambda d_t < \epsilon_t \tag{31a}$$

$$\epsilon_t u'(e_t^y - \epsilon_t) < \beta \lim_{c\to\infty} cv'(c) \tag{31b}$$

$$\textit{(Low Interest Rate Condition):} \quad g_t(\epsilon_t) \leq n \ \forall t \geq 0 \tag{31c}$$

$$\textit{(Not-Too-Low Interest Rate Condition):} \quad R^*_{t+1} \geq \left(n\frac{d_{t+1}}{d_t}\right)\gamma \ \forall t \geq 0. \tag{31d}$$

Then, there exists a continuum of equilibria. The set of equilibria is an interval $[\underline{a}, \bar{a}]$ *with* $[\lambda d_0, \epsilon_0] \subset [\underline{a}, \bar{a}]$*. Moreover,*

1. *For* $a_0 = \underline{a}$*, the equilibrium is bubbleless.*
2. *For* $a_0 > \underline{a}$*, the equilibrium is bubbly.*

Proof. See Appendix B.3.1. □

Basically, Theorem 3 is a consequence of Theorem 1 and Lemma 5. The difference is that we work under separable utility functions and $cv'(c)$ is increasing. This specification allows us to know that the equilibrium set is a compact interval, and weaken other assumptions in Theorem 1.

Remark 5. *We now explain how to choose models satisfying all assumptions in Theorem 3. First, we choose endowments (bounded away from zero and above) and* $R > 0$ *so that* $R^*_t \equiv \frac{u'(e_t^y)}{\beta v'(e_{t+1}^o)}$ *satisfy* $R^*_t < R < n$ *for any* t*.*

We can choose $\epsilon_t = \epsilon > 0$ *small enough so that* $\epsilon \in (0, e_t^y)$ *and* $\epsilon u'(e_t^y - \epsilon) < \beta \lim_{c\to\infty} cv'(c)$*. Actually, we can still choose* ϵ *small enough so that* $g_t(\epsilon) < n$ *because* $\lim_{a\to\infty} g_t(a) = R^*_t < R < n$*; see (27).*

Then, we choose the dividend sequence and G_d *so that* $\frac{\mathcal{D}_{t+1}}{\mathcal{D}_t} \equiv n\frac{d_{t+1}}{d_t} \leq G_d < R$ *(note that this ensures that* $\frac{d_{t+1}}{d_t} < \frac{R}{n} < 1$*). Now, we choose* $\gamma > 1$ *so that* $R \geq G_d\gamma$*. Next, we choose dividends (low enough) and choose* $\lambda > 0$ *so that* $\gamma > 1 + \frac{1}{\lambda}$ *and* $\lambda d_t < \epsilon$ *for any* t*. Therefore, all conditions in Theorem 3 hold.*

4.5 Full characterization under stationary endowments

When endowments are time-independent, we have the following result showing the full characterization of the equilibrium set.

Theorem 4. *Let Assumptions 1, 4, 5 be satisfied. Consider stationary endowments, i.e., $e^y_t = e^y > 0, e^o_t = e^o > 0$ for any t. Denote $R^* \equiv \frac{u'(e^y)}{\beta v'(e^o)}$ the interest rate in the economy without assets.*

1. *If $R^* > n$ (high interest rate condition) or $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} = \infty$ (not-too-low dividend condition), then there exists a unique equilibrium and this equilibrium is bubbleless.*

2. *If $R^* < n$ and $\sum_{t=1}^{\infty} d_t = \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$, then one of the following cases must hold.*

   (a) *There exists a continuum of equilibria. The set of equilibria is a compact interval $[\underline{a}, \bar{a}]$.*

      i. *For $a_0 \in (\underline{a}, \bar{a}]$, the equilibrium is bubbly.*

      ii. *For $a_0 \in [\underline{a}, \bar{a})$, the equilibrium satisfies $(a_t, b_t, R_t) \to (0, 0, R^*)$.*

      iii. *For $a_0 = \bar{a}$, the equilibrium satisfies $(a_t, b_t, R_t) \to (\hat{a}, \hat{a}, n)$, where $\hat{a} > 0$ is uniquely determined by $u'(e^y - \hat{a}) = \beta n v'(e^o + n\hat{a})$ (i.e., $n = g(\hat{a})$).*

   (b) *There exists a unique equilibrium. This equilibrium is bubbly and (a_t, b_t, R_t) converges to $(\hat{a}, \hat{a}, n)$ where $\hat{a} > 0$ is uniquely determined by $u'(e^y - \hat{a}) = \beta n v'(e^o + n\hat{a})$ (i.e., $n = g(\hat{a})$).*

   Moreover, the following claims hold.

   Claim 1: *If $R^* < n$ and $\sum_{t \geq 1} \frac{\mathcal{D}_t}{(R^*)^t} < \infty$, then the statement 2a is true.*

   Claim 2: *If $R^* < n$, $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$ and $R^* < \limsup_{t \to \infty} \mathcal{D}_t^{\frac{1}{t}}$, then the statement 2b is true.*

3. *If $R^* = n$ and $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$, then there exists a unique equilibrium. This equilibrium is bubbleless and $(a_t, b_t, R_t) \to (0, 0, n)$.*

Proof. See Appendix B.3.2. □

Theorem 4 explores the equilibrium set and the asymptotic properties of asset price bubbles in all possible cases. This is the added-value with respect to the literature and the previous results in the present paper. We observe that the equilibrium set depends on the interplay between the return of the economy without asset R^*, the population growth factor n and the dividend growth rates.

We now discuss how our Theorem 4 is related to the existing literature. First, Theorem 4 corresponds Proposition 1 in Tirole (1985), who studies the asset price an OLG model with dividend-paying asset and production. However, the proof of Proposition 1 in Tirole (1985) contains some concerns (see Pham and Toda (2025a) for a more detailed discussion). Our Theorem 4 provides a full characterization of the equilibrium set in an exchange economy with stationary endowment, non-stationary dividend. Note that Tirole (1985) assumes that $\mathcal{D}_t = 1$ and did not study the case $R^* = n$.

Since three cases in Theorem 4 are mutually exclusive, Theorem 4 leads to an important implication.

Corollary 4. *Let Assumptions 1, 4, 5 be satisfied. Consider the case of stationary endowments, i.e., $e^y_t = e^y > 0, e^o_t = e^o > 0$ for any t.*

There exists a bubbly equilibrium if and only if the two following conditions hold: (1) $R^* < n$, and (2) $\sum_{t=1}^{\infty} d_t = \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$.

When there exists a bubbly equilibrium, part 2 of Theorem 4 shows that either this exists a unique equilibrium (and this is bubly) or there exists a continuum of equilibria (bubbly and bubbleless equilibria co-exist). Then, Claims 1 and 2 of Theorem 4 provide conditions under which each case must happen.

Claim 2 in Theorem 4 is related to our Theorem 2 above, Proposition 1.c in Tirole (1985), Theorem 2 in Hirano and Toda (2025a), Theorem 1 in Pham and Toda (2025b). Here, the added value is to provide the uniqueness and asymptotic properties of equilibrium under general dividends but a stronger assumption (namely, stationary endowment).

Section 5 in Hirano and Toda (2024b) studies the case where the utility is homogeneous of degree 1 and $e^y_t = aG^t, e^o_t = bG^t, \mathcal{D}_t = DG^t_d$, where a, G, D, G_d are positive constant (see their Assumptions 1 and 2, page 16). Thanks to these assumptions, they obtained an autonomous dynamical system and, by the way, they could use the local stability of manifolds to provide a fairly complete analysis regarding long-run behavior of asset prices. However, their approach cannot be directly applied to our setting where we only impose very minimal conditions on the dividend sequence and our utility function is not necessarily homogeneous of degree 1. Moreover, Section 5 in Hirano and Toda (2024b) studies neither the case $R^* = n$ nor $\limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} = n$ while our Theorem 4 covers these cases.

An explicit model with asset bubbles

We now provide a model, where we can explicitly compute the equilibrium prices with bubbles and it completely fulfills Theorem 4. According to (26), the Euler condition becomes $u'(e^y_t - a_t) = \beta n \frac{a_{t+1}+d_{t+1}}{a_t} v'\Big(e^o_{t+1} + n(a_{t+1} + d_{t+1})\Big)$.

Let us consider a special setup where $u(c) = v(c) = ln(c)$ and assume that $e^o_t > 0$ for any t. We have the following non-autonomous system

$$a_{t+1} + d_{t+1} = \frac{a_t}{\frac{n\beta e^y_t}{e^o_{t+1}} - \frac{n(1+\beta)}{e^o_{t+1}} a_t}, \text{ or, equivalently, } \frac{1}{a_{t+1} + d_{t+1}} = \frac{n\beta e^y_t}{e^o_{t+1}} \frac{1}{a_t} - \frac{n(1+\beta)}{e^o_{t+1}}.$$

Assume a stationary endowment: $e^y_t = e^y > 0, e^o_t = e^o > 0$ for any t. Note that the interest rate $R^* = \frac{e^o}{\beta e^y}$.

Let the interest rate be lower than the population growth rate: $R^* < n$.

Let $x > 0$ be such that $\frac{x+1}{x}\frac{R^*}{n} > 1$, or, equivalently, $1 - x(\frac{n}{R^*} - 1) > 0$.

Denote $h \equiv \frac{n(1+\beta)}{e^o}$. Define the dividend sequence (d_t) by[18]

$$\frac{1}{d_t} - \frac{hx(1+x)}{1-x(\frac{n}{R^*}-1)} = \Big(\frac{x+1}{x}\frac{R^*}{n}\Big)^t\Big(\frac{1}{d_0} - \frac{hx(1+x)}{1-x(\frac{n}{R^*}-1)}\Big) \tag{32}$$

$$0 < d_0 < \frac{1-x(\frac{n}{R^*}-1)}{hx(1+x)}. \tag{33}$$

We can check that $\frac{1}{d_{t+1}} = \frac{x+1}{x}\frac{R^*}{n}\frac{1}{d_t} - \frac{(x+1)hR^*}{n}$. Moreover, $\lim_{t\to\infty} d_t^{\frac{1}{t}} = \frac{xn}{(x+1)R^*}$ which is, by our assumption, lower than 1.

In the economy with above specifications, we can check that the following sequence is an equilibrium

$$a_t = \Big(\frac{n}{R^*} - 1\Big)\frac{1}{h} + xd_t \text{ for any } t \geq 0. \tag{34}$$

Since $\frac{x+1}{x}\frac{R^*}{n} > 1$, we have $\sum_{t\geq 1} d_t < \infty$ and hence $\sum_{t\geq 1}\frac{d_t}{a_t} < \infty$. Therefore, this equilibrium price is bubbly. Moreover, we have $\lim_{t\to\infty} a_t = \Big(\frac{n}{R^*} - 1\Big)\frac{1}{h}$. According to Theorem 4's part 2, this is the unique equilibrium satisfying $\lim_{t\to\infty} a_t > 0$. By applying Claims 1 and 2 in Theorem 4, we see that:

- If $R^* > \lim_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} = n\lim_{t\to\infty} d_t^{\frac{1}{t}} = \frac{xn^2}{(x+1)R^*}$ (i.e., $\Big(\frac{e^o}{\beta e^y}\Big)^2\frac{x+1}{x} > n^2$), then Claim 1 in Theorem 4 holds. We have a continuum of equilibria and the maximal equilibrium is (a_t) defined by (34).
- If $R^* < \lim_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} = n\lim_{t\to\infty} d_t^{\frac{1}{t}} = \frac{xn^2}{(x+1)R^*}$ (i.e., $\Big(\frac{e^o}{\beta e^y}\Big)^2\frac{x+1}{x} < n^2$), then Claim 1 in Theorem 4 holds. There exists a unique equilibrium and the equilibrium asset value (a_t) is defined by (34).

5 Pareto optimality

In this section, we investigate the Pareto optimality. Let us start by providing a formal definition (see Balasko and Shell (1980) for instance).

Definition 8. *Let $c_{-1}^y > 0$ and $(d_t)_{t=0}^\infty$ be an exogenous non-negative sequence and $N_t = n^t > 0$ for any t.*

A feasible allocation path is a positive sequence $(c_t^y, c_t^o)_{t\geq 0}$ satisfying

$$N_t c_t^y + N_{t-1}c_t^o = N_t e_t^y + N_{t-1}e_t^o + \mathcal{D}_t \quad (i.e.,\ c_t^y + \frac{c_t^o}{n} = e_t^y + \frac{e_t^o}{n} + d_t) \text{ for any } t.$$

A feasible allocation path is said to be Pareto optimal if there is no other feasible allocation path $(c_t^{y\prime}, c_t^{o\prime})_t$ such that

$$U^t\left(c_t^{y\prime}, c_{t+1}^{o\prime}\right) \geq U^t\left(c_t^y, c_{t+1}^o\right) \text{ for any } t \geq -1$$

with strict inequality for some t.

[18] Our example here is based on Example 3 in Bosi et al. (2021).

Assumption 6. *The function U^t is strictly concave, continuously differentiable, strictly increasing in each component.*

As in Proposition 5.3 in Balasko and Shell (1980), we have the following result.

Lemma 6 (Sufficient conditions for Pareto optimality)**.** *Let Assumptions 1, 6 be satisfied. Consider a feasible allocation $(c_t^y, c_t^o)_{t\geq 0}$. Define the sequence $(R_t)_{t\geq 0}$ by*

$$R_{t+1} = \frac{U_1^t(c_t^y, c_{t+1}^o)}{U_2^t(c_t^y, c_{t+1}^o)} \text{ for any } t \geq 0. \tag{35}$$

The path $(c_t^y, c_t^o)_{t\geq 0}$ is Pareto optimal if $\liminf_{t\to\infty} \frac{n^t}{R_1\cdots R_t} c_t^y = 0$.

Proof. See Appendix C. □

To prove this result, the idea is to construct support prices (Malinvaud, 1953; Cass, 1972), which are $\frac{n^t}{R_1\cdots R_t}$ in our setting.

Corollary 5. *Let Assumptions 1, 6 be satisfied. Consider an equilibrium. Assume that*

$$\liminf_{t\to\infty} R_t > n \text{ and } \sup_{t\geq 0}(e_t^y + \frac{e_t^o}{n} + d_t) < \infty. \tag{36}$$

Then, this equilibrium is Pareto optimal.

Proof of Corollary 5. When $\liminf_{t\to\infty} R_t > n$, we have that $P_t \equiv \frac{n^t}{R_1\cdots R_t}$ converges to zero. Since $c_t^y \leq e_t^y + \frac{e_t^o}{n} + d_t$, which is uniformly bounded from above, we obtain that $P_t c_t^y$ converges to zero. So, we have the Pareto optimality. □

A natural question arises: can an equilibrium still be Pareto-optimal if the conditions stated in Lemma 6 or Corollary 5 are not satisfied? It is well known that (see, for instance, Okuno and Zilcha (1980), page 802) this question is, in general, difficult. To address this issue, we extend Okuno and Zilcha (1980) and Balasko and Shell (1980).

Consider an equilibrium allocation $(c_t^y, c_t^o)_t$. Denote, for each $t \geq 1$,

$$Q_t \equiv \frac{1}{R_1\cdots R_t}, \quad P_t \equiv \frac{n^t}{R_1\cdots R_t}. \tag{37}$$

In equilibrium, we observe that

$$Q_t c_t^y + Q_{t+1} c_{t+1}^y = Q_t e_t^y + Q_{t+1} e_{t+1}^y \tag{38}$$

$$\frac{U_1^t(c_t^y, c_{t+1}^o)}{U_2^t(c_t^y, c_{t+1}^o)} = R_{t+1} = \frac{Q_t}{Q_{t+1}}, \quad \frac{P_{t+1}}{P_t} = \frac{n}{R_{t+1}} \tag{39}$$

Denote

$$e_t \equiv e_t^y + \frac{e_t^o}{n} + d_t$$

the aggregate good supply per capita at date t.

We now introduce the notions of strictness and smoothness used by Benveniste (1976), Okuno and Zilcha (1980), which are closed to the notion of Gaussian curvature used by Balasko and Shell (1980).

Definition 9. *Given an equilibrium allocation* $(c_t^y, c_t^o)_t$*, the upper contour of the t-th generation is given by*

$$B_t(c) \equiv \Big\{(c_t^{y'}, c_{t+1}^{o'}) \in \mathbb{R}_+^2 : U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o)\Big\}.$$

1. *We say that* $B_t(c)$ *is strict at* c *with respect to the price* P_t *if there exist* $\mu_t > 0$ *such that*

$$P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o) + nP_t(c_t^{y'} - c_t^y) \geq \frac{\mu_t}{nP_t c_t^y}\Big(nP_t(c_t^{y'} - c_t^y)\Big)^2 \ \forall (c_t^{y'}, c_{t+1}^{o'}) \in B_t(c),\ c_t^{y'} < c_t^y. \tag{40}$$

We say that this allocation satisfies the so-called "uniform strictness condition" if there exist $h \in (0,1]$ *and* $\bar{\mu} > 0$ *such that, for any* t*,*

$$P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o) + nP_t(c_t^{y'} - c_t^y) \geq \frac{\mu}{nP_t c_t^y}\Big(nP_t(c_t^{y'} - c_t^y)\Big)^2 \tag{41}$$
$$\forall (c_t^{y'}, c_{t+1}^{o'}) \in B_t(c) \text{ satisfying } (1-h)c_t^y < c_t^{y'} < c_t^y, c_{t+1}^{o'} > c_{t+1}^o.$$

2. *We say that* $B_t(c)$ *is smooth at* c *with respect to the price* P_t *if there exist* $\theta_t > 0$ *and* x_t *satisfying the following condition:*

$$(c_t^{y'}, c_{t+1}^{o'}) \in B_t(c) \text{ if} \tag{42}$$
$$\begin{cases} x_t c_t^y < c_t^{y'} < c_t^y, c_{t+1}^{o'} > c_{t+1}^o, c_{t+1}^{o'} < ne_{t+1} \\ P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o) + nP_t(c_t^{y'} - c_t^y) \geq \frac{\theta_{2t}}{P_{t+1}c_{t+1}^o}\Big(P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o)\Big)^2 \\ \qquad + \frac{\theta_{1t}}{nP_t c_t^y}\Big(nP_t(c_t^{y'} - c_t^y)\Big)^2 \end{cases}$$

We say that this allocation satisfies the so-called "uniform smoothness condition" if for each $x > 0$*, there exists* $\theta_1(x), \theta_2(x) > 0$ *such that, for any* t*, if the couple* $(c_t^{y'}, c_{t+1}^{o'})$ *satisfies*

$$\begin{cases} xc_t^y < c_t^{y'} < c_t^y, c_{t+1}^o < c_{t+1}^{o'} < ne_{t+1} \\ P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o) + nP_t(c_t^{y'} - c_t^y) \geq \frac{\theta_2(x)}{P_{t+1}c_{t+1}^o}\Big(P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o)\Big)^2 \\ \qquad + \frac{\theta_1(x)}{nP_t c_t^y}\Big(nP_t(c_t^{y'} - c_t^y)\Big)^2 \end{cases} \tag{43}$$

then $(c_t^{y'}, c_{t+1}^{o'}) \in B_t(c)$*. i.e.,* $U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o)$*.*

The notion of strictness in Definition 9 is similar to (but weaker than) that in Okuno and Zilcha (1980)'s Definition 10. Our notion of smoothness is quite different from the smoothness in Definition 11 in Okuno and Zilcha (1980) (indeed, Definition 11 in Okuno and Zilcha (1980) corresponds to our case with $\theta_2(x) = 0$). Note that Okuno and Zilcha (1980) did not explicitly provide conditions to ensure the uniform strictness and smoothness.

Since the uniform strictness and smoothness conditions are quite implicit, a natural issue is to justify them. Observe that (41) is equivalent to

$$U_2^t(c_{t+1}^{o'} - c_{t+1}^o) + U_1^t(c_t^{y'} - c_t^y) \geq \frac{\mu}{U_1^t c_t^y}\Big(U_1^t(c_t^{y'} - c_t^y)\Big)^2 \ \forall (c_t^{y'}, c_{t+1}^{o'}) \in B_t(c),\ c_t^{y'} < c_t^y.$$

while the second inequality in (43) becomes

$$v'(c^o_{t+1})(c^{o'}_{t+1} - c^o_{t+1}) + u'(c^y_t)(c^{y'}_t - c^y_t) \geq \frac{\theta_2(x)}{v'(c^o_{t+1})c^o_{t+1}}\Big(v'(c^o_{t+1})(c^{o'}_{t+1} - c^o_{t+1})\Big)^2 + \frac{\theta_1(x)}{u'(c^y_t)c^y_t}\Big(u'(c^y_t)(c^{y'}_t - c^y_t)\Big)^2.$$

The following results justify the uniform strictness and smoothness conditions by proving that they can be satisfied in many cases.

Lemma 7 (checking the uniform strictness condition)**.** *Assume that* $U^t(x_1, x_2) = u_t(x_1) + v_t(x_2)$ *where the two functions* $u_t, v_t : \mathbb{R}_+ \to \mathbb{R}$ *are in* C^2*, strictly concave, strictly increasing.*

1. *Any allocation* $(c^y_t, c^o_t)_t$ *with* $c^y_t > 0, c^o_t > 0$ *for any* t*, satisfies the uniform strictness condition if there exists* $h \in (0, 1]$ *such that*

$$\inf_{t\geq 0}\Big\{\frac{c^y_t}{u'_t(c^y_t)} \inf_{x\in[(1-h)c^y_t, c^y_t]}\Big(-\frac{1}{2}u''_t(x)\Big)\Big\} > 0$$

2. *If* $u'_t(c) = c^{-\sigma}$ *with* $\sigma > 0$*, then any allocation* $(c^y_t, c^o_t)_t$ *with* $c^y_t > 0, c^o_t > 0$ *for any* t*, satisfies the uniform strictness condition.*

Lemma 8 (checking the uniform smoothness condition)**.** *Assume that* $U^t(x_1, x_2) = u_t(c_1) + v_t(x_2)$ *where the two functions* $u_t, v_t : \mathbb{R}_+ \to \mathbb{R}$ *are in* C^2*, strictly concave, strictly increasing.*

1. *Any allocation* $(c^y_t, c^o_t)_t$ *with* $c^y_t > 0, c^o_t > 0$ *for any* t*, satisfies the uniform smoothness condition if for each* $x \in (0, 1)$*, we have*

$$\bar{M}_1 \equiv \sup_{t\geq 0}\Big\{\frac{c^y_t}{u'_t(c^y_t)} \sup_{c\in[xc^y_t, c^y_t]}\Big(-\frac{1}{2}u''_t(c)\Big)\Big\} < \infty \tag{44}$$

$$\bar{M}_2 \equiv \sup_{t\geq 0}\Big\{\frac{c^o_{t+1}}{v'_t(c^o_{t+1})} \sup_{c\in[c^o_{t+1}, ne_{t+1}]}\Big(-\frac{1}{2}v''_t(c)\Big)\Big\} < \infty. \tag{45}$$

2. *Assume that* $u_t(c) = \frac{c^{1-\sigma}}{1-\sigma}$ *and* $v'_t(c) = \gamma_t c^{-\sigma}$ *with* $\sigma > 0, \gamma_t > 0$ *for any* t*. Then any allocation* $(c^y_t, c^o_t)_t$ *with* $c^y_t > 0, c^o_t > 0$ *for any* t*, satisfies the uniform smoothness condition.*

Proof. See Appendix C.1. □

The following result is similar to Theorem 3A and Theorem 3B in Okuno and Zilcha (1980) and Proposition 5.6 in Balasko and Shell (1980).

Theorem 5. *Let Assumptions 1 and 6 be satisfied. Consider an equilibrium with the allocation* $(c^y_t, c^o_t)_t$ *and the interest rates* $(R_t)_t$*. Denote, for each* $t \geq 1$*,*

$$Q_t \equiv \frac{1}{R_1 \cdots R_t}, \quad P_t \equiv \frac{n^t}{R_1 \cdots R_t}. \tag{46}$$

1. *Assume that the equilibrium allocation satisfies the uniform strictness condition in Definition 9. Then, this equilibrium allocation* $(c_t^y, c_t^o)_t$ *is Pareto optimal if*

$$\sum_{t\geq 1} \frac{1}{P_t e_t} = \infty \quad (i.e., \sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t e_t} = \infty). \tag{47}$$

2. *Assume that the equilibrium allocation* $(c_t^y, c_t^o)_t$ *is Pareto optimal and satisfies the uniform smoothness condition in Definition 9. Assume also that* $\liminf_{t\to\infty} \frac{c_t^y}{e_t} > 0$, $\limsup_{t\to\infty} \frac{c_t^o}{ne_t} < 1$, $\liminf_{t\to\infty} \frac{P_{t+1}c_{t+1}^o}{P_t e_t} > 0$. *Then, we have*

$$\sum_{t\geq 1} \frac{1}{P_t e_t} = \infty \quad (i.e., \sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t e_t} = \infty). \tag{48}$$

Proof. See Section C.2. □

Okuno and Zilcha (1980) present an example of Pareto inefficient equilibrium which satisfies condition (48) to show the importance of the uniform strictness condition in Theorem 5. Balasko and Shell (1980) introduce the so-called *properties (C) and (C')* which require the boundedness from above and away from zero of the Gaussian curvature (a fundamental concept in differential geometry) of households' indifferent surface through their equilibrium consumption at any date. Properties (C) and (C') are respectively related to the uniform smoothness and strictness conditions in Okuno and Zilcha (1980) and in Definition 9 above.[19]

Okuno and Zilcha (1980) and Balasko and Shell (1980)'s assumptions are different from ours. Indeed, households in their models consumer L goods at each date while we assume $L = 1$. They require the boundedness (above and away from zero) of endowments and while we do not require this assumption. Moreover, the uniform smoothness and strictness conditions in our paper seem to be more explicit and can be verified by using elementary calculus (see Lemmas 7 and 8) while the conditions in Okuno and Zilcha (1980) and Balasko and Shell (1980) are quite implicit. However, the most important difference is that we introduce the dividend-paying asset to study asset bubbles and this generates new insights that we will present.

Remark 6. *Part 1 of Theorem 5 still holds if we replace the uniform strictness condition by the so-called property (C').*

Property (C'). *We say that the allocation* (c_t^y, c_{t+1}^o) *satisfies the property (C') if there exists* $\alpha > 0$ *such that, for any* t, *if the couple* $(c_t^{y'}, c_{t+1}^{o'}) \in \mathbb{R}^2_{++}$ *satisfies*

$$U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o), \tag{49a}$$

$$\epsilon_t^y \equiv c_t^{y'} - c_t^y < 0, \epsilon_{t+1}^o \equiv c_{t+1}^{o'} - c_{t+1}^o > 0 \tag{49b}$$

$$\frac{c_{t+1}^{o'}}{n} < e_{t+1}^y + \frac{e_{t+1}^o}{n} + d_{t+1}, \tag{49c}$$

$$P_{t+1}\epsilon_{t+1}^o - P_t\epsilon_t^o > 0, \text{ where we denote } \epsilon_t^o \equiv -n\epsilon_t^y, \tag{49d}$$

then $(P_{t+1}\epsilon_{t+1}^o)^2 \leq \alpha P_{t+1}c_{t+1}^o \Big(P_{t+1}\epsilon_{t+1}^o - P_t\epsilon_t^o\Big)$.

[19] See Footnote 8 in Okuno and Zilcha (1980) and Footnote 9 in Balasko and Shell (1980).

Theorem 5 leads to an interesting consequence showing the important of the benchmark interest rate on the Pareto optimality.

Corollary 6. *Let Assumptions 1, 2, 3 be satisfied. Assume that*

$$\sum_{t\geq 1}\frac{R_1^*\cdots R_t^*}{n^t e_t}=\infty.$$

(This means that the benchmark interest rate, R_{t+1}^* is higher than the product of population growth rate (n) and the endowment growth rate e_{t+1}/e_t.)

Then every equilibrium satisfying the uniform strictness condition is Pareto optimal.

Proof. By Lemma 1, we have $R_t \geq R_t^*$ for any t. Then, for any equilibrium, we have

$$\sum_{t\geq 1}\frac{R_1\cdots R_t}{n^t e_t}\geq \sum_{t\geq 1}\frac{R_1^*\cdots R_t^*}{n^t e_t}=\infty. \tag{50}$$

Applying Theorem 5's part 1, any equilibrium satisfying the uniform smoothness condition is Pareto optimal. □

6 Asset price bubble and Pareto optimality

In this section, we investigate the interplay between asset price (with or without bubbles) and Pareto optimality. We should start by pointing out that the notions of asset bubbles and Pareto optimality are different. Indeed, the existence of bubble is equivalent to $\lim_{t\to\infty}\frac{n^t a_t}{R_1\cdots R_t} > 0$ while the Pareto optimality is, in many cases, equivalent to (48), i.e., $\sum_{t\geq 1}\frac{R_1\cdots R_t}{n^t e_t}=\infty$.

We now look at the bubbleless and Pareto optimal equilibrium.

Proposition 6. *Let Assumptions 1, 2 be satisfied.*

1. *An equilibrium is Pareto optimal and bubbleless if* $\liminf_{t\to\infty}\frac{n^t}{R_1\cdots R_t}e_t^y=0$.
2. *Every equilibrium is Pareto optimal and bubbleless if the dividends are significant in the sense that* $\limsup_{t\to\infty}\frac{d_t}{e_t^y}>0$.

Proof. See Appendix D. □

The insight in point 2 of Proposition 6 is that a significant level of dividends makes the market economy Pareto optimal. This is in line with Propositions 5 and 8 in Le Van and Pham (2016) in a model with infinitely live-agents. The difference is that we work under non-stationary OLG exchange economy and study the Pareto optimality while they consider a general equilibrium models with infinitely-lived agents and study the dynamically efficiency in the sense of Malinvaud (1953). Point 2 of Proposition 6 is also in line with Proposition 1 in Rhee (1991) in an OLG model with land, where he proves that an economy is dynamically efficient if the income share of land does not vanish.

By combining Proposition 3 and Theorem 5, we obtain the following result which deepens our understanding regarding the role of dividends and the benchmark interest rates.

Proposition 7. *1. Let Assumptions 1, 2 be satisfied. Any equilibrium, which satisfies the uniform strictness condition, is bubbleless and Pareto optimal if*

$$\text{(Non-negligible dividend condition):} \quad \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{e_t n^t} = \infty. \tag{51}$$

2. Let Assumptions 1, 2, 3 be satisfied. Every equilibrium is bubbleless and Pareto optimal if

$$\text{(High interest rate condition):} \quad \lim_{t\to\infty} \frac{n^t e_t^y}{R_1^* \cdots R_t^*} = 0. \tag{52}$$

If we add Assumptions 4, 5, then there exists a unique equilibrium. This is Pareto optimal and bubbleless.

Proof. See Appendix D. □

The following result shows the importance of the asset value on the Pareto optimality.

Proposition 8. *Let Assumptions 1 and 2 be satisfied.*

1. *An equilibrium is Pareto optimal if it satisfies the uniform strictness condition and the asset value is significant (in the sense that* $\limsup_{t\to\infty} \frac{a_t}{e_t} > 0$*).*
2. *An equilibrium is Pareto optimal if it is bubbleless and the saving rate is bounded away from zero (i.e.,* $\liminf_{t\to\infty} \frac{a_t}{e_t^y} > 0$*).*

Proof. See Appendix D. □

The condition that the asset value is significant is very important. Indeed, in Theorem 6's part 2a, we show some cases where the equilibrium is not optimal and $\lim_{t\to\infty} \frac{a_t}{e_t^y} = \lim_{t\to\infty} \frac{a_t}{e_t} = 0$ (by Claims 1 of Theorem 4).

So far, we have presented some sufficient conditions for the Pareto optimality. We now show how an equilibrium can be not Pareto optimal. We show that this may happen when there are continuum of equilibria with bubbles.

Proposition 9 (Equilibria are bubbly and not Pareto optimal)**.** *Let Assumptions 1, 4, 5 be satisfied. Assume that there exists a continuum of equilibria (this happens under Theorem 3 or Claim 1 in Theorem 4).*

Then, the utility of households born at any date is strictly increasing in the initial asset value. By consequence, any equilibrium a_0 *satisfying* $a_0 < \bar{a}_0 \equiv \max\{a \in \mathbf{A}_0\}$ *is not Pareto optimal. So, there exists a continuum of bubbly equilibrium which are not Pareto optimal.*

Proof. See Appendix D. □

Our proof of Proposition 9 is not based on Theorem 5. Moreover, it offers detailed information as it shows us the ranking of households' welfare generated by several equilibria.

In the case of stationary endowment, by combining Theorems 4 and Theorem 5, we have a fairly complete characterization.

Theorem 6. *Let Assumptions in Theorem 4 be satisfied.*

1. *If $R^* > n$, there exists a unique equilibrium. This equilibrium is bubbleless and Pareto optimal.*

2. *If $R^* < n$ and $\sum_{t\geq 1} \frac{\mathcal{D}_t}{(R^*)^t} < \infty$, then there exist a continuum of equilibria. The set of equilibria is a compact interval $[\underline{a}, \bar{a}]$.*

   (a) Any equilibrium with initial asset value $a_0 < \bar{a}$ is not Pareto optimal. In particular, the bubbleless equilibrium $a_0 = \underline{a}$ is not Pareto optimal.
   Any equilibrium with $a_0 \in (\underline{a}, \bar{a})$ is not Pareto optimal, bubbly but asymptotically bubbleless.

   (b) The maximal equilibrium $a_0 = \bar{a}$ is asymptotically bubbly and Pareto optimal.

3. *If $R^* < n$, $\sum_{t=1}^{\infty} \dfrac{\mathcal{D}_t}{n^t} < \infty$, and $R^* < \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}}$, there exists a unique equilibrium. This equilibrium is asymptotically bubbly and Pareto optimal.*

Proof. See Appendix D. □

According to Theorem 6, when the benchmark interest rate is low, i.e., $R^* < 1$ and the dividends are low (i.e., $\sum_t d_t < \infty$), only the asymptotically bubbly equilibrium can be Pareto optimal. This point is consistent with the traditional insight (see Proposition 2 in Tirole (1985), which claims that in the case of low interest rate ($R^* < n$) and $d_t = \frac{d_0}{n^t}$, only the asymptotically bubbly equilibrium is Pareto optimal). Tirole (1985) considers a specific form of dividend (i.e., $d_t = \frac{d_0}{n^t}$) and by consequence does not analyze the role of dividend growth. More importantly, Tirole (1985) did not provided a formal proof for his Proposition 2.

However, the following explicit model shows that when the interest rate in the economy without asset is low, an equilibrium which is Pareto optimal can be bubbly or bubbleless.

Proposition 10. *$U^t(x_1, x_2) = ln(x_1) + \beta ln(x_2)$ where $\gamma \in (0, 1)$, and $e_t^o = 0$ for any t. There exists a unique equilibrium, which is determined by $\frac{q_t}{n^t} = a_t = \frac{\beta}{1+\beta} e_t^y$.*[20]

This equilibrium is Pareto optimal.

1. *If $\sum_{t\geq 1} \frac{d_t}{e_t^y} < \infty$ (i.e., $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \infty$), this equilibrium is asymptotically bubbly.*

2. *If $\sum_{t\geq 1} \frac{d_t}{e_t^y} = \infty$ (i.e., $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} = \infty$), this equilibrium is bubbleless.*

Proof. See Appendix D. □

In Proposition 10, the interest rate in the economy without asset equals zero, i.e., $R_t^* = 0$ for any t. However, the equilibrium in Proposition 10 can be bubbly or bubbleless, depending on the growth rates of dividend and endowments. When $\sum_{t\geq 1} \frac{d_t}{e_t^y} = \infty$ ($< \infty$), the equilibrium is bubbleless (bubbly).[21] However, in both cases, it is Pareto optimal. This insight is consistent with Proposition 8's point 1 and complements the main point in Tirole (1985)'s Proposition 2.

[20] This equilibrium is similar to that in Section 5.1.1 in Bosi et al. (2017b) or Proposition 1 in Hirano and Toda (2025a). See Bosi et al. (2021)'s Section 4 for more explicit models with bubbles.

[21] According to the proof of Corollary 3, the model in Proposition 10 satisfies Condition (B) in Theorem 2

7 Conclusion

We have proved that a sequence of asset prices has a bubble if and only if the ratio of fundamental value to price decreases over time and converges to zero. Then, we have explored the formation of the asset bubbles in an OLG exchange economy under general assumptions. The asset price bubble, which is a phenomenon in equilibrium, and the Pareto optimality are outcomes of interplay between the interest rate of the economy without assets, growth rate of endowments and dividend, and behavior of households.

We have also studied the interplay between asset price bubble and Pareto optimality. Although both the existence of asset bubble and the non-optimality of equilibrium allocation often happen under similar conditions, we have shown that the link between them is not very strong. Indeed, a bubbly equilibrium may be optimal or non-optimal while a bubbleless equilibrium may also be optimal or non-optimal.

A Proof of Section 2

Proof of Proposition 1. We only present a proof for part 1 because part 2 is a direct consequence of part 1 and the definition of B_t.

By definition of F_t and B_t, we have $q_t = F_t + B_t$. So, (1*b*) $\Leftrightarrow$ (1*c*).

(1*a*) $\Leftrightarrow$ (1*b*). We can check, from (1), that

$$\frac{F_t}{q_t} - \frac{F_{t+1}}{q_{t+1}} = \Big(1 - \frac{F_t}{q_t}\Big)\frac{\mathcal{D}_{t+1}}{q_{t+1}} \ \forall t \geq 0. \tag{53}$$

So, we see that (1b) implies (1a) because $\frac{F_t}{q_t} > \frac{F_{t+1}}{q_{t+1}}$ implies that $q_t > F_t$. We now prove that (1a) implies (1b). Assume that there is a bubble. By (53), the sequence $(\frac{F_t}{q_t})$ is strictly decreasing. Moreover, the existence of bubble means that $\lim_{t\to 0} Q_t q_t = \lim_{t\to 0} \frac{q_t}{R_1\cdots R_t} > 0$. Then, there exists $x > 0$ and $t_0 > 0$ such that $\frac{q_t}{R_1\cdots R_t} > x$ for any $t \geq t_0$.

Take $t \geq t_0$. We have

$$\frac{F_t}{q_t} = \frac{1}{q_t}\sum_{s\geq 1}\frac{\mathcal{D}_{t+s}}{R_{t+1}\cdots R_{t+s}} = \frac{R_1\cdots R_t}{q_t}\sum_{s\geq 1}\frac{\mathcal{D}_{t+s}}{R_1\cdots R_{t+s}} < \frac{1}{x}\sum_{s\geq 1}\frac{\mathcal{D}_{t+s}}{R_1\cdots R_{t+s}}.$$

because $\frac{q_t}{R_1\cdots R_t} > x$ for any $t \geq t_0$.

Recall that $q_0 \geq F_0 = \sum_{t\geq 1}\frac{\mathcal{D}_t}{R_1\cdots R_t}$. It means that the series $\sum_{t\geq 1}\frac{\mathcal{D}_t}{R_1\cdots R_t}$ converges, which implies that $\sum_{s\geq 1}\frac{\mathcal{D}_{t+s}}{R_1\cdots R_{t+s}}$ converges to zero. By consequence, $\lim_{t\to\infty}\frac{F_t}{q_t} = 0$.

(1*a*) $\Leftrightarrow$ (1*d*). This is Proposition 7 in Montrucchio (2004). For a pedagogical purpose, we give a simple proof. The asset pricing equation $q_t = \frac{q_{t+1}+\mathcal{D}_{t+1}}{R_{t+1}}$ implies that $q_t Q_t = q_{t+1}Q_{t+1}(1 + \frac{\mathcal{D}_{t+1}}{q_{t+1}})$. By iterating, we get that $q_0 = q_T Q_T \prod_{t=1}^{T}(1 + \frac{\mathcal{D}_t}{q_t})$. There exists a bubble (i.e., $\lim_{t\to 0} Q_t q_t > 0$) if and only if $\lim_{T\to\infty}\prod_{t=1}^{T}(1 + \frac{\mathcal{D}_t}{q_t}) < \infty$ which is equivalent to $\sum_{t\geq 1}\mathcal{D}_t/q_t < \infty$.

As we want prove that $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} \leq \frac{\frac{F_0}{q_0}}{1-\frac{F_0}{q_0}}$, we now present a new proof. We have, as in (53),

$$\frac{F_t}{q_t} - \frac{F_{t+1}}{q_{t+1}} = \left(1 - \frac{F_t}{q_t}\right)\frac{\mathcal{D}_{t+1}}{q_{t+1}} \ \forall t \geq 0. \tag{54}$$

If there is a bubble, we have $q_t > F_t$ for any t and hence F_t/q_t is strictly decreasing. In particular, we have $F_t/q_t < F_0/q_0$. From (53), we get that

$$\frac{F_t}{q_t} - \frac{F_{t+1}}{q_{t+1}} = \left(1 - \frac{F_t}{q_t}\right)\frac{\mathcal{D}_{t+1}}{q_{t+1}} \geq \left(1 - \frac{F_0}{q_0}\right)\frac{\mathcal{D}_{t+1}}{q_{t+1}} \text{ for any } t \geq 0. \tag{55}$$

Taking the sum over t, we have

$$\frac{F_0}{q_0} > \frac{F_0}{q_0} - \frac{F_T}{q_T} \geq \left(1 - \frac{F_0}{q_0}\right)\sum_{t=1}^{T}\frac{\mathcal{D}_t}{q_t}. \tag{56}$$

Let T tend to infinity, we have $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{q_t} \leq \frac{\frac{F_0}{q_0}}{1-\frac{F_0}{q_0}} < \infty$.

□

B Appendix: Proofs for Section 4

Proof of Proposition 2. If $\liminf_{t\to\infty} \frac{q_t z_t}{e_t^y} > 0$, then there exists a constant $x > 0$ and t_0 such that $\frac{q_t z_t}{e_t^y} > x$ for any $t \geq t_0$. Therefore,

$$\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{q_t} < \sum_{t=1}^{T_0-1}\frac{\mathcal{D}_t}{q_t} + \sum_{t=T_0}^{\infty}\frac{\mathcal{D}_t}{x n^t e_t^y} < \infty.$$

So, Lemma 2 implies that there is a bubble. □

Proof of Proposition 3. (1) If there exists a bubbly equilibrium, then, by Lemma 2, we have $\sum_{t=1}^{\infty} \mathcal{D}_t/q_t < \infty$. Since $q_t z_t < e_t^y$ and $z_t = 1/n^t$, we get that $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t^y} < \sum_{t=1}^{\infty} \mathcal{D}_t/q_t < \infty$, a contradiction.

(2) According to point 2 of Lemma 2, there is no bubble if and only if $\lim_{t\to\infty} \dfrac{a_t n^t}{R_1 \cdots R_t} = 0$. Since $a_t \leq e_t^y$ and $R_t \geq R_t^*$ (by Lemma 1) for any t, we have

$$\frac{a_t n^t}{R_1 \cdots R_t} < \frac{n^t e_t^y}{R_1^* \cdots R_t^*} \text{ for any } t.$$

By our assumption (12), there is no bubble. In other words, every equilibrium is bubbleless. □

Proof of Proposition 4. Consider the T-truncated economy which is defined as the economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(U^t, (\mathcal{D}_t)_t, (e_t^y, e_t^o)_t)$ except that there is no activity from date $T+1$ on. i.e., households born at date T only consume $c_T^y = e_T^y$ and the budget constraints of household born at date $T-1$ are $c^y_{T-1} + q_{T-1}z_{T-1} \leq e^y_{T-1}, c^o_T \leq e^o_T + \mathcal{D}_T z_{T-1}$, and $q_T = 0$, $z_T = 0$.

By the standard argument, there is an equilibrium $(a^T_t)_{t\leq T}$ for the T-truncated economy.

Let now T tend to infinity and consider the product topology, there exists a sub-sequence (t_n) such that $\lim_{n\to\infty} a^{t_n}_t = a_t$ for any t. It is easy to see that $(a_t)_{t\geq 0}$ satisfies $a_{t+1} = a_t \frac{R_{t+1}}{n} - d_{t+1}$.

By Lemma 1, we have $R^T_t \geq R^*_t$ for any $t < T$.

Fix t. We will prove that $\sum_{t=1}^{\infty} \frac{n^t}{R_{s+1}\cdots R_{s+t}} d_{s+t} < \infty$. Let $t_n > t$. We have

$$a^{t_n}_{s+1} = a^{t_n}_s \frac{R^{t_n}_{s+1}}{n} - d_{s+1} \text{ for any } s \geq t.$$

From this, we have

$$a^{t_n}_s = \frac{n}{R^{t_n}_{s+1}} d_{s+1} + \cdots + \frac{n^{t_n - s}}{R^{t_n}_{s+1}\cdots R^{t_n}_{t_n}} d_{t_n} + \frac{n^{t_n-s}}{R^{t_n}_{s+1}\cdots R^{t_n}_{t_n}} a^{t_n}_{t_n} \tag{57}$$

$$= \frac{n}{R^{t_n}_{s+1}} d_{s+1} + \cdots + \frac{n^{t_n - s}}{R^{t_n}_{s+1}\cdots R^{t_n}_{t_n}} d_{t_n} \text{ (because } a^{t_n}_{t_n} = 0) \tag{58}$$

$$\leq \frac{n}{R^*_{s+1}} d_{s+1} + \cdots + \frac{n^{t_n - s}}{R^*_{s+1}\cdots R^*_{t_n}} d_{t_n} \tag{59}$$

for any $t \leq s \leq t_n - 1$. Thanks to our assumption that $\sum_{t\geq 1} \frac{n^t}{R^*_1\cdots R^*_t} d_t = \sum_{t\geq 1} \frac{\mathcal{D}_t}{R^*_1\cdots R^*_t} < \infty$ and the dominated convergence theorem, the series

$$a^{t_n}_t = \frac{n}{R^{t_n}_{t+1}} d_{t+1} + \cdots + \frac{n^{t_n - s}}{R^{t_n}_{t+1}\cdots R^{t_n}_{t_n}} d_{t_n}$$

converges to

$$a_t = \sum_{\tau=1}^{\infty} \frac{n^\tau}{R_{t+1}\cdots R_{t+\tau}} d_{t+\tau} < \infty.$$

when $n \to \infty$.

Now, recall, since the non-arbitrage condition,

$$a_t = \frac{n}{R_{t+1}}(a_{t+1} + d_{t+1}) = \cdots = \sum_{\tau=1}^{T} \frac{n^\tau}{R_{t+1}\cdots R_{t+\tau}} d_{t+\tau} + \frac{n^T}{R_{t+1}\cdots R_{t+T}} a_{t+T} \tag{60}$$

for any $T \geq 1$. Let T tend to infinity and note that $a_t = \sum_{\tau=1}^{\infty} \frac{n^\tau}{R_{t+1}\cdots R_{t+\tau}} d_{t+\tau}$, we have

$$\lim_{t\to\infty} \frac{n^t}{R_{s+1}\cdots R_{s+t}} a_{s+t} = 0.$$

According to Lemma 2, there is no bubble in this equilibrium. □

B.1 Proof of Section 4.2

Our proof requires intermediate steps.

Lemma 9. *Condition (13) is satisfied if there exists T such that $\frac{R^*_{t+1}}{n}\frac{d_t}{d_{t+1}} \geq \gamma > 1$ for any $t \geq T$.*

Proof of Lemma 9. We have

$$\frac{R^*_{t+1}\cdots R^*_{t+s}}{n^s}\frac{d_t}{d_{t+s}} \geq \gamma^s \Rightarrow \sum_{s\geq 1}\frac{n^s}{R^*_{t+1}\cdots R^*_{t+s}}d_{t+s} \leq d_t\sum_{s\geq 1}\frac{1}{\gamma^s} < \infty \text{ for any } t \geq T.$$

By consequence, we obtain (13). □

Lemma 10. *(i) If $a_t \in (0, e^y_t)$ satisfies*

$$H_t(a_t) \equiv \lim_{x\to\infty}\Big(a_t U^t_1(e^y_t - a_t, x) - xU^t_2(e^y_t - a_t, x)\Big) < 0, \tag{61}$$

then there exists $R_{t+1} > 0$ such that $K_t(a_t, R_{t+1}) = 0$.

(ii) If $a_t \in (0, e^y_t)$ satisfies

$$K_t(a_t, n) \equiv U^t_1(e^y_t - a_t, e^o_{t+1} + na_t) - nU^t_2\left(e^y_t - a_t, e^o_{t+1} + na_t\right) < 0, \tag{62}$$

then there exists $R_{t+1} \in (0, n)$ such that $K_t(a_t, R_{t+1}) = 0$.

Proof of Lemma 10. Observe that, since $e^o_{t+1} > 0$, $K_t(a_t, 0) = U^t_1(e^y_t - a_t, e^o_{t+1}) > 0$. So, the second statement of our lemma is obvious.

We now look at $\lim_{x\to\infty} K_t(a, x)$. We write

$$K_t(a_t, R) = U^t_1(e^y_t - a_t, e^o_{t+1} + Ra_t) - \frac{1}{a_t}\frac{Ra_t}{e^o_{t+1} + Ra_t}(e^o_{t+1} + Ra_t)U^t_2\left(e^y_t - a_t, e^o_{t+1} + Ra_t\right).$$

By consequence, we have

$$\lim_{R\to\infty} K_t(a_t, R) = \lim_{x\to\infty}\Big(U^t_1(e^y_t - a_t, x) - \frac{1}{a_t}xU^t_2(e^y_t - a_t, x)\Big) < 0$$

thanks to the assumption $\lim_{x\to\infty}\Big(a_t U^t_1(e^y_t - a_t, x) - xU^t_2(e^y_t - a_t, x)\Big) < 0$. Therefore, there exists $R_{t+1} > 0$ such that $K_t(a_t, R_{t+1}) = 0$.

□

Proof of Theorem 1. Lemma 9 and Proposition 4 imply that there exists a bubbleless equilibrium,

We now show that we can construct a continuum of bubbly equilibria.

Let a_0 be such that $\lambda d_0 \leq a_0 \leq \epsilon_0$.

Since the function K_0 is increasing in the first component (Lemma 3) and $a_0 \leq \epsilon_0$, we have $K_0(a_0, n) \leq K_0(\epsilon_0, n) < 0$. According to Lemma 10, there exists $R_1 \in (0, n)$ such that $K_0(a_0, R_1) = 0$, i.e.,

$$K_0(a_0, R_1) \equiv U^0_1(e^y_0 - a_0, e^o_1 + R_1 a_0) - R_1 U^0_2(e^y_0 - a_0, e^o_1 + R_1 a_0) = 0$$

Then, since $\epsilon_0 \geq a_0$, we have $K_0(\epsilon_0, R_1) \geq K_0(a_0, R_1) = 0$. So, $K_0(\epsilon_0, R_1) \geq 0$. By condition (iii) in Theorem 1, we have $R_1 \leq R_1^{\epsilon} < n$.

Then, we determine a_1 by

$$a_1 = \frac{R_1}{n} a_0 - d_1. \tag{63}$$

We observe that

$$a_1 = \frac{R_1}{n} a_0 - d_1 \leq \frac{n}{n} \epsilon_0 - d_1 = \epsilon_0 - d_1 \leq \epsilon_1 \text{ (by condition (i) above).} \tag{64}$$

We now give a lower bound of a_1. By using our assumption (31d) (i.e., $R_1^* \geq \gamma \frac{\mathcal{D}_1}{\mathcal{D}_0}$, or equivalently, $\frac{R_1^*}{n} \frac{d_0}{d_1} \geq \gamma$) and definition of a_1, we have

$$\frac{a_1}{d_1} = \frac{R_1}{n} \frac{d_0}{d_1} \frac{a_0}{d_0} - 1 \geq \frac{R_1^*}{n} \frac{d_0}{d_1} \frac{a_0}{d_0} - 1 \geq \gamma \frac{a_0}{d_0} - 1 \geq \gamma\lambda - 1 > \lambda.$$

where we use $R_t \geq R_t^*$ (see Lemma 1). To sum up, we have

$$a_1 \leq \epsilon_1 \tag{65}$$
$$a_1 \geq \lambda d_1 \tag{66}$$
$$\frac{a_1}{d_1} \geq \gamma \frac{a_0}{d_0} - 1. \tag{67}$$

Suppose that we can construct $(a_0, a_1, \dots, a_t)$ with

$$\lambda d_s \leq a_s \leq \epsilon_s, \quad \frac{a_s}{d_s} \geq \gamma \frac{a_{s-1}}{d_{s-1}} - 1$$

for any $s \leq t$.

Let us look at date $t+1$. Since $a_t \leq \epsilon_t$, we have $K_t(a_t, n) \leq K_t(\epsilon_t, n) < 0$. Then, according to Lemma 10, there exists R_{t+1} such that $K_t(a_t, R_{t+1}) = 0$. By using the same argument, we have $R_{t+1} \leq R_{t+1}^{\epsilon} < n$ and

$$a_{t+1} = \frac{R_{t+1}}{n} a_t - d_{t+1} \leq \frac{n}{n} \epsilon_t - d_{t+1} = \epsilon_t - d_{t+1} \leq \epsilon_{t+1} \tag{68}$$
$$\frac{a_{t+1}}{d_{t+1}} = \frac{R_{t+1}}{n} \frac{d_t}{d_{t+1}} \frac{a_t}{d_t} - 1 \geq \left(\frac{R_{t+1}^*}{n} \frac{d_t}{d_{t+1}} \right) \frac{a_t}{d_t} - 1 \geq \gamma \frac{a_t}{d_t} - 1 \tag{69}$$

We have constructed an equilibrium (a_t) with $a_t \in (0, \epsilon_t) \subset (0, e_t^y)$ with $\frac{a_{t+1}}{d_{t+1}} \geq \gamma \frac{a_t}{d_t} - 1$ and $a_t/d_t \geq \lambda > 0$ for any t.

We now prove that this equilibrium is bubbly. Define x by $(\gamma - 1)x = 1$. We have

$$\frac{a_{t+1}}{d_{t+1}} - x \geq \gamma \left(\frac{a_t}{d_t} - x \right) \text{ for any } t \geq 0, \tag{70}$$
$$\Rightarrow \frac{a_t}{d_t} - x \geq \gamma^t \left(\frac{a_0}{d_0} - x \right) \text{ for any } t. \tag{71}$$

Note that $\frac{a_0}{d_0} > \lambda > \frac{1}{\gamma - 1} = x$. Hence, $\frac{a_0}{d_0} - x > 0$.

By consequence, we have

$$\frac{a_t}{d_t} \geq x + \gamma^t(\frac{a_0}{d_0} - x) > \gamma^t(\frac{a_0}{d_0} - x)$$
$$\sum_{t\geq 1}\frac{d_t}{a_t} \leq \sum_{t\geq 0}\frac{1}{\gamma^t(\frac{a_0}{d_0} - x)} = \frac{1}{\frac{a_0}{d_0} - x}\frac{1}{1-\frac{1}{\gamma}}$$

Therefore, $\sum_{t\geq 1}\frac{d_t}{a_t} < \infty$. It means that this equilibrium is bubbly.

□

Proof of Corollary 1. Assume that $U^t(x_1, x_2) = \frac{x_1^{1-\sigma}}{1-\sigma} + \beta\frac{x_2^{1-\sigma}}{1-\sigma}$ where $\sigma > 0, \beta > 0$. The Euler condition becomes $(e_t^y - a_t)^{-\sigma} = \beta R_{t+1}(e_{t+1}^o + R_{t+1}a_t)^{-\sigma}$ and the function $K_t(a, R) = (e_t^y - a)^{-\sigma} - \beta R(e_{t+1}^o + Ra)^{-\sigma}$. Observe that

$$K_t(a, R) \lesseqgtr 0 \Leftrightarrow H_t(a, R) \equiv \frac{e_{t+1}^o}{e_t^y}R^{\frac{-1}{\sigma}} + \frac{a}{e_t^y}\Big(R^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}\Big) - \beta^{\frac{1}{\sigma}} \lesseqgtr 0. \tag{72}$$

Neither K_t nor H_t depends on dividends.

We have $\frac{\partial H_t}{\partial R}(a, R) = \frac{R^{-\frac{1}{\sigma}-1}}{\sigma}\Big(-\frac{e_{t+1}^o}{e_t^y} - (1-\sigma)\frac{a}{e_t^y}R\Big)$. So, when $\sigma \in (0, 1)$, the function H_t is decreasing in the second component. When $\sigma > 1$, given $a > 0$, the function $H_t(a, R)$ is decreasing in R on the interval $(0, \frac{e_{t+1}^o}{(\sigma-1)a})$.

We now explain how to choose parameters so that conditions in Theorem 1 holds. To simplify, assume that the endowment growth of each household is constant: $\frac{e_{t+1}^o}{e_t^y} = g_e > 0$ for any t. The benchmark interest rate $R_t^* = R^*$ determined by $R^* = \frac{g_e^{1/\sigma}}{\beta}$.

Assume that $d_t = d_0 d^t$ for any t (i.e., $\mathcal{D}_t = d_0 n^t d^t$).

Let $nd < R^* < n$.

Let $R^* < n$. Then, we can choose $\epsilon_t = \epsilon > 0$ for any t, where

$$H_t(a, R) \equiv g_e R^{\frac{-1}{\sigma}} + \frac{a}{e_t^y}(R^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}}. \tag{73}$$

Since $R^* < n$, we have $g_e n^{\frac{-1}{\sigma}} - \beta^{\frac{1}{\sigma}} < 0$, and, hence, we can choose $\epsilon > 0$ such that $g_e n^{\frac{-1}{\sigma}} + \epsilon(n^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}} < 0$.

Define $\epsilon_t \equiv \epsilon e_t^y$. We have $H_t(\epsilon_t, n) < 0$. Then, we can take R^ϵ such that $H_t(\epsilon_t, R^\epsilon) = 0$, i.e., $g_e(R^\epsilon)^{\frac{-1}{\sigma}} + \epsilon((R^\epsilon)^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}} = 0$.

We have $H_t(\epsilon_t, n) < 0 = H_t(\epsilon_t, R^\epsilon)$.

Define $R_t^\epsilon = R^\epsilon$.

We look at conditions (i), (ii), (iii) in Theorem 1.

Condition (i) becomes $\epsilon e_t^y < \epsilon e_{t+1}^y + d_{t+1}$. This is satisfied if $e_t^y \leq e_{t+1}^y$ for any t.

Condition (ii) becomes $H_t(\epsilon_t, n) < 0$.

Condition (iii) states that: If $R \in (0, n)$ and $g_e R^{\frac{-1}{\sigma}} + \epsilon(R^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}} > 0$, then $R < R^*$.

Consider the function $H(R) \equiv g_e R^{\frac{-1}{\sigma}} + \epsilon(R^{1-\frac{1}{\sigma}} + \beta^{\frac{1}{\sigma}}) - \beta^{\frac{1}{\sigma}}$. We have

$$H'(R) = \frac{R^{-\frac{1}{\sigma}-1}}{\sigma}\Big(-g_e - (1-\sigma)\epsilon R\Big) \tag{74}$$

If $\sigma < 1$, then $H'(R) < 0$. Condition (iii) is satisfied.

If $\sigma > 1$, then we can choose parameters so that $\frac{g_e}{\epsilon(\sigma-1)} < n$ (actually, we can choose σ high enough). In this case, condition (iii) is satisfied.

Lastly, we verify the not-too-low interest rate condition (16). With the above settings, condition (16) becomes

$$R^* > nd\gamma. \tag{75}$$

This is satisfied if we take $\gamma \in (0, \frac{R^*}{nd})$. We choose λ and the dividend growth rate d small enough so that $\gamma > 1 + \frac{1}{\lambda}$ and $\lambda d_t < \epsilon_t \equiv \epsilon e_t^y$.

To sum up, all assumptions in Theorem 1 are satisfied under conditions in Corollary 1.

□

B.2 Proof of Section 4.3

Proof of Theorem 2. Part 1. We need to prove that $\liminf_{t\to\infty} \frac{a_t}{e_t^y} > 0$ for any equilibrium.

Let $\bar{\epsilon} \in (0,1)$, positive sequences (X_t) and $(\bar{X}_t)$, and a date T be in Condition (B).

Take an equilibrium. Suppose that $\liminf_{t\to\infty} \frac{a_t}{e_t^y} = 0$. Then there exists $t_0 \geq T$ such that $a_{t_0}^e \equiv \frac{a_{t_0}}{e_{t_0}^y} < \bar{\epsilon}$.

In equilibrium, we have

$$V_1^{t_0}\left(1 - a_{t_0}^e, g_{e,t_0+1} + R_{t_0+1}a_{t_0}^e\right) - R_{t_0+1}V_2^{t_0}\left(1 - a_{t_0}^e, g_{e,t_0+1} + R_{t_0+1}a_{t_0}^e\right) = 0. \tag{76}$$

By our condition (B2), we have $R_{t_0+1} \leq X_{t_0+1}$. Since $X_{t_0+1} \leq n\frac{e_{t_0+1}^y}{e_{t_0}^y}$, we have $R_{t_0+1} \leq n\frac{e_{t_0+1}^y}{e_{t_0}^y}$. Combining with the non-arbitrage condition $a_{t_0+1} + d_{t_0+1} = a_{t_0}\frac{R_{t_0+1}}{n}$, we get

$$a_{t_0+1} \leq a_{t_0}\frac{R_{t_0+1}}{n} \leq a_{t_0}\frac{e_{t_0+1}^y}{e_{t_0}^y} \tag{77}$$

$$\Rightarrow a_{t_0+1}^e \equiv \frac{a_{t_0+1}}{e_{t_0+1}^y} \leq \frac{a_{t_0}}{e_{t_0}^y} = a_{t_0}^e < \bar{\epsilon}. \tag{78}$$

Therefore, by induction, we have, for any $t \geq t_0$,

$$a_{t+1}^e \leq a_t^e < \bar{\epsilon} \tag{79}$$

$$R_{t+1} \leq X_{t+1}. \tag{80}$$

This implies that $R_{t_0+1}\cdots R_t \leq X_{t_0+1}\cdots X_t$ for any $t > t_0$. We now look at the fundamental value

$$F_0 = \sum_{t\geq 1}\frac{\mathcal{D}_t}{R_1\cdots R_t} = \sum_{t=1}^{t_0}\frac{\mathcal{D}_t}{R_1\cdots R_t} + \sum_{t=t_0+1}^{\infty}\frac{\mathcal{D}_t}{R_1\cdots R_{t_0}R_{t_0+1}\cdots R_t}. \tag{81}$$

Consider the second term $A_0 \equiv \sum_{t=t_0+1}^{\infty}\frac{\mathcal{D}_t}{R_1\cdots R_{t_0}R_{t_0+1}\cdots R_t}$. We have

$$A_0 = \frac{1}{R_1\cdots R_{t_0}}\sum_{t=t_0+1}^{\infty}\frac{\mathcal{D}_t}{R_{t_0+1}\cdots R_t} \geq \frac{1}{R_1\cdots R_{t_0}}\sum_{t=t_0+1}^{\infty}\frac{\mathcal{D}_t}{X_{t_0+1}\cdots X_t} \tag{82}$$

$$= \frac{X_1\cdots X_{t_0}}{R_1\cdots R_{t_0}}\sum_{t=t_0+1}^{\infty}\frac{\mathcal{D}_t}{X_1\cdots X_t} = \infty \tag{83}$$

because of our assumption (B1), i.e., $\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{X_1\cdots X_t}=\infty$.

This implies that $F_0=\infty$. Since $q_0\geq F_0$, we have $q_0=\infty$, a contradiction. We have finished our proof.

Parts 2 and 3. Let the condition in the first statement be satisfied. According to part 1 of Theorem 2, any equilibrium satisfies $\liminf_{t\to\infty}\frac{a_t}{e^y_t}>0$.

Proposition 2 and our assumption $\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{n^te^y_t}<\infty$ imply that this equilibrium is bubbly.

If our assumption in Part 3 is satisfied, i.e., $\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{n^te^y_t}=\infty$, then Proposition 3's point 1 implies that every equilibrium is bubbleless.

Part 4 is a direct consequence of Part 2. □

Proof of Corollary 3. The Euler equation now is $(e^y_t-a_t)^{-1}-\beta R_{t+1}(e^o_{t+1}+R_{t+1}a_t)^{-1}=0$ and equation (20) becomes

$$X(1-\frac{\epsilon}{\beta(1-\epsilon)})=\frac{g_{e,t+1}}{\beta(1-\epsilon)}. \tag{84}$$

Let us check Condition (B) in Theorem 2. Condition (B) holds if we can choose $\bar{\epsilon}$ small enough, the sequences (X_t), $(\bar{X}_t)$, and a date T such that (1) $\sum_{t=1}^{\infty}\dfrac{\mathcal{D}_t}{X_1\cdots X_t}=\infty$, (2) $X_{t+1}\leq n\dfrac{e^y_{t+1}}{e^y_t}$ $\forall t\geq T$, and (3) for any $t\geq T$, if $\epsilon\in(0,\bar{\epsilon})$, $X\in[0,\bar{X}_t]$ satisfy (84), then $X\leq X_{t+1}$

Since $\limsup_{t\to\infty}\frac{R^*_{t+1}}{n\frac{e^y_{t+1}}{e^y_t}}<1$, then we can choose $\bar{\epsilon}\in(0,1/2)$ small enough, δ close enough to 1 and a date T so that $\frac{R^*_{t+1}}{n\frac{e^y_{t+1}}{e^y_t}}<\frac{\beta-\bar{\epsilon}(1+\beta)}{\beta}\delta<1$ for any $t\geq T$.

We next define $X_{t+1}\equiv\delta n\frac{e^y_{t+1}}{e^y_t}$ and take $\bar{X}_t\geq X_t$ for any t. Then, we have $X_{t+1}\leq n\frac{e^y_{t+1}}{e^y_t}$.

Since $\frac{\mathcal{D}_t}{n^te^y_t}=\frac{1}{t^\alpha}$ where $\alpha>1$, and $X_{t+1}\equiv\delta n\frac{e^y_{t+1}}{e^y_t}$, where $\delta\in(0,1)$ we have

$$\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{X_1\cdots X_t}=\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{\delta^tn^te^y_t}e^y_0=e^y_0\sum_{t=1}^{\infty}\frac{1}{\delta^tt^\alpha}=\infty. \tag{85}$$

For any $\epsilon\in(0,\bar{\epsilon})$, we have

$$\frac{e^o_{t+1}}{e^y_t}\frac{1}{\beta-\epsilon(1+\beta)}\leq\frac{e^o_{t+1}}{e^y_t}\frac{1}{\beta-\bar{\epsilon}(1+\beta)}\leq\delta n\frac{e^y_{t+1}}{e^y_t}=X_{t+1}\ \forall t\geq T. \tag{86}$$

Let $\epsilon\in(0,\bar{\epsilon})$ and $X\in[0,\bar{X}_t]$ be satisfied (84), then we have, thanks to (86),

$$X=g_{e,t+1}\frac{1}{\beta-\epsilon(1+\beta)}=\frac{e^o_{t+1}}{e^y_t}\frac{1}{\beta-\epsilon(1+\beta)}\leq X_{t+1}.$$

So, Condition (B) is satisfied.

By construction $\frac{\mathcal{D}_t}{n^te^y_t}=\frac{1}{t^\alpha}$ where $\alpha>1$, we have

$$\sum_{t=1}^{\infty}\frac{\mathcal{D}_t}{n^te^y_t}=\sum_{t=1}^{\infty}\frac{1}{t^\alpha}<\infty.$$

Applying Theorem 2's point 2, every equilibrium is bubbly and $\liminf_{t\to\infty} \frac{a_t}{e_t^y} > 0$. $\square$

Proof of Remark 3. Let Assumptions 2, 3 and condition (20) in Theorem 2 in Hirano and Toda (2025a) be satisfied.

Assumption 2 in Hirano and Toda (2025a) requires that: $\lim_{t\to\infty} \frac{e_{t+1}^y}{e_t^y} = G \in (0,\infty)$, $\lim_{t\to\infty} \frac{e_t^o}{e_t^y} = w \in [0,\infty)$. Note that our notations e_t^y and e_t^o respectively correspond to a_t and b_t in Hirano and Toda (2025a). They define the forward rate between time t and $t+1$ when generation t consumes $(e_t^y x_1, e_t^y x_2)$ by the following

$$f_t(x_1,x_2) = \frac{U_1^t(e_t^y x_1, e_t^y x_2)}{U_2^t(e_t^y x_1, e_t^y x_2)}$$

where U_i^t is the partial derivative of the function U^t with respect to $x_i, i = 1,2$. Their Assumption 3 states that there exists a continuous function $f : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any nonempty compact set $K \subset \mathbb{R}_{++} \times \mathbb{R}_+$, we have $\lim_{t\to\infty} \sup_{(x_1,x_2)\in K} |f_t(x_1,x_2) - f(x_1,x_2)| = 0$.

Third, condition (20) in Hirano and Toda (2025a) writes

$$f(1,g_e) \equiv \lim_{t\to\infty} \frac{V_1^t\Big(1, \lim_{s\to\infty} \frac{e_{s+1}^0}{e_s^y}\Big)}{V_2^t\Big(1, \lim_{s\to\infty} \frac{e_{s+1}^0}{e_s^y}\Big)} < \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} < \lim_{t\to\infty} \frac{e_{t+1}^y}{e_t^y} \tag{87}$$

where $g_e \equiv \lim_{t\to\infty} \frac{e_{s+1}^0}{e_s^y} = Gw$.

Recall that Hirano and Toda (2025a) considers $n = 1$.

We will check (21) and Condition (B) in our Theorem 2 **for the case** $n = 1$.

Step 1. We prove that: $G_d \equiv \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} < G \equiv \lim_{t\to\infty} \frac{e_{t+1}^y}{e_t^y}$ implies our assumption (21), i.e., $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{e_t^y} < \infty$.

Let a, b be such that $G_d < a < b < G$. There exists a date t_0 such that $\mathcal{D}_t^{\frac{1}{t}} < a < b < \frac{e_{t+1}^y}{e_t^y}$ for any $t \geq t_0$. Therefore, for any $t > t_0$,

$$\frac{e_t^y}{e_{t_0}^y} = \frac{e_t^y}{e_{t-1}^y} \cdots \frac{e_{t_0+1}^y}{e_{t_0}^y} > b^{t-t_0}. \tag{88}$$

So, we have $\frac{\mathcal{D}_t}{e_t^y} < \frac{a^t}{e_{t_0}^y b^{t-t_0}}$. Since $a < b$, we have $\sum_{t>t_0} \frac{\mathcal{D}_t}{e_t^y} < \infty$.

Step 2. We prove Condition (B) in our Theorem 2. We need to find $\bar{\epsilon} \in (0,1)$, positive sequences (X_t) and $(\bar{X}_t)$, and a date T which satisfy Condition (B).

Take $\bar{X} > G \equiv \lim_{t\to\infty} \frac{e_{t+1}^y}{e_t^y}$. Define $\bar{X}_t = \bar{X}$ for any t.

Since $f(1,g_e) < G_d$ and the function f is continuous, there exist $\bar{\epsilon} > 0$, $\gamma > 0$, $G_1, G_2, G_3 \in (0, G_d)$ such that

$$f(1-\epsilon, g) < G_1 < G_2 < G_3 < G_d \tag{89}$$

for any $\epsilon \in [0,\bar{\epsilon}], 0 \leq g \leq g_e + \gamma$, and $\bar{X}\bar{\epsilon} < \frac{\gamma}{2}$.

Since $g_e \equiv \lim_{s\to\infty} \frac{e^0_{s+1}}{e^y_s}$, we can choose t_0 such that $\frac{e^0_{s+1}}{e^y_s} < g_e + \frac{\gamma}{2}$ for any $t \geq t_0$.

Take $\epsilon_f \in (0, G_2 - G_1)$.

By Assumption 3 in Hirano and Toda (2025a), there exists t_1 such that

$$\sup_{(\epsilon,g)\in[0,\bar{\epsilon}]\times[0,g_e+\gamma]} |f_t(1-\epsilon, g) - f(1-\epsilon, g)| < \epsilon_f \tag{90}$$

for any $t \geq t_1$.

Define $X_t = G_2$ for any t.

Since $G_2 < G_d < G \equiv \lim_{t\to\infty} \frac{e^y_{t+1}}{e^y_t}$, we can choose t_2 such that $G_2 < \frac{e^y_{t+1}}{e^y_t}$ for any $t \geq t_2$

Take $T > \max(t_0, t_1, t_2)$.

Now, let $t \geq T$, $\epsilon \in (0, \bar{\epsilon})$ and $X \in (0, \bar{X})$.

Suppose that X satisfies (20), i.e.

$$V_1^t(1-\epsilon, g_{e,t+1} + X\epsilon) - XV_2^t(1-\epsilon, g_{e,t+1} + X\epsilon) = 0. \tag{91}$$

Then

$$X = \frac{V_1^t(1-\epsilon, g_{e,t+1} + X\epsilon)}{V_2^t(1-\epsilon, g_{e,t+1} + X\epsilon)}. \tag{92}$$

We have

$$g_{e,t+1} + X\epsilon = \frac{e^0_{s+1}}{e^y_s} + X\epsilon \leq g_e + \frac{\gamma}{2} + \bar{X}\bar{\epsilon} < g_e + \frac{\gamma}{2} + \frac{\gamma}{2} = g_e + \gamma.$$

It implies that $\epsilon \in [0, \bar{\epsilon}]$ and $g_{e,t+1} + X\epsilon \in [0, g_e + \gamma]$. Then, condition (90) implies that

$$|X - f(1-\epsilon, g_{e,t+1} + X\epsilon)| \tag{93}$$

$$= \left|\frac{V_1^t(1-\epsilon, g_{e,t+1} + X\epsilon)}{V_2^t(1-\epsilon, g_{e,t+1} + X\epsilon)} - f(1-\epsilon, g_{e,t+1} + X\epsilon)\right| < \epsilon_f. \tag{94}$$

Therefore, we have

$$X < \epsilon_f + f(1-\epsilon, g_{e,t+1} + X\epsilon). \tag{95}$$

Since $g_{e,t+1} + X\epsilon \leq g_e + \gamma$, condition (89) implies that $f(1-\epsilon, g_{e,t+1} + X\epsilon) < G_1$. Combining with $\epsilon_f < G_2 - G_1$, we have

$$\begin{aligned} X &< \epsilon_f + f(1-\epsilon, g_{e,t+1} + X\epsilon) && (96)\\ &< (G_2 - G_1) + G_1 && (97)\\ &= G_2 = X_{t+1} \text{ (by definition of } X_t) && (98)\\ &\leq \frac{e^y_{t+1}}{e^y_t}. && (99) \end{aligned}$$

So, $X < X_{t+1} \leq \frac{e^y_{t+1}}{e^y_t}$.

The last step is to prove that $\sum_{t=1}^{\infty} \dfrac{\mathcal{D}_t}{X_1 \cdots X_t} = \infty$.

Since $G_d \equiv \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}} > G_3$, we can find an infinite and increasing sequence $(s_k)_{k\geq 1}$ with $s_1 > T$ such that $\mathcal{D}_{s_k}^{\frac{1}{s_k}} > G_3$ for any $k \geq 1$. Hence, $\mathcal{D}_{s_k} > G_3^{s_k}$. This implies that, for any s_k,

$$\frac{\mathcal{D}_{s_k}}{X_1\cdots X_{s_k}} = \frac{1}{X_1\cdots X_T}\frac{\mathcal{D}_{s_k}}{X_{T+1}\cdots X_{s_k}} > \frac{1}{X_1\cdots X_T}\frac{G_3^{s_k}}{G_2^{s_k-T}} = \frac{G_2^T}{X_1\cdots X_T}\frac{G_3^{s_k}}{G_2^{s_k}} \tag{100}$$

$$> \frac{G_2^T}{X_1\cdots X_T} \text{ (because } G_3 > G_2) \tag{101}$$

for any s_k. By consequence, we have

$$\sum_{t\geq 1}\frac{\mathcal{D}_t}{X_1\cdots X_t} \geq \sum_{k\geq 1}\frac{\mathcal{D}_{s_k}}{X_1\cdots X_{s_k}} > \sum_{k\geq 1}\frac{G_2^T}{X_1\cdots X_T} = \infty.$$

We have finished our proof. □

B.3 Proofs of Section 4.4

Proof of Lemma 4. Let $e_t^y > 0, e_{t+1}^o > 0$. Let $a_t \in (0, e_t^y)$. Consider the function $K : [0,\infty) \to \mathbb{R}$ defined by $K(R) \equiv u'(e_t^y - a_t) - \beta R v'\left(e_{t+1}^o + Ra_t\right)$ for any $R \in (0,\infty)$. We have

$$K'(R) = -\beta v'\left(e_{t+1}^o + Ra_t\right) - \beta R a_t v''\left(e_{t+1}^o + Ra_t\right).$$

Since $e_t^o > 0$ for any t, then we have

$$\begin{aligned} K'(R) &= -\beta v'\left(e_{t+1}^o + Ra_t\right) - \beta R a_t v''\left(e_{t+1}^o + Ra_t\right) \\ &< -\beta v'\left(e_{t+1}^o + Ra_t\right) - \beta (e_{t+1}^o + Ra_t) v''\left(e_{t+1}^o + Ra_t\right) \leq 0 \end{aligned}$$

because $cv'(c)$ is increasing in c.

Therefore, the function K is strictly decreasing on $(0,\infty)$. Now, observe that, since $e_{t+1}^o > 0$, $K(0) = u'(e_t^y - a_t) > 0$. We now look at $\lim_{x\to\infty} K(x)$. We have

$$\begin{aligned} \lim_{x\to\infty} xv'\left(e_{t+1}^o + xa_t\right) &= \lim_{x\to\infty}\frac{x}{e_{t+1}^o + xa_t}\left(e_{t+1}^o + xa_t\right)v'\left(e_{t+1}^o + xa_t\right) \\ &= \frac{1}{a_t}\lim_{x\to\infty}\left(e_{t+1}^o + xa_t\right)v'\left(e_{t+1}^o + xa_t\right) = \frac{1}{a_t}\lim_{c\to\infty} cv'(c) \end{aligned}$$

It means that

$$\lim_{x\to\infty} K(x) = u'(e_t^y - a_t) - \frac{\beta}{a_t}\lim_{c\to\infty} cv'(c).$$

Therefore, there exists a unique R_{t+1} satisfying $u'(e_t^y - a_t) = \beta R_{t+1} v'\left(e_{t+1}^o + R_{t+1}a_t\right)$ if and only if $a_t u'(e_t^y - a_t) < \beta \lim_{c\to\infty} cv'(c)$.

Taking the derivative of both sides of the equation $u'(e_t^y - a_t) = \beta R_{t+1} v'\left(e_{t+1}^o + R_{t+1}a_t\right)$ with respect to a_t, we get that

$$-u''(e_t^y - a_t) = \beta R_{t+1} v''\left(e_{t+1}^o + R_{t+1}a_t\right)\left(a_t\frac{\partial R_{t+1}}{\partial a_t} + R_{t+1}\right) + \beta\frac{\partial R_{t+1}}{\partial a_t} v'\left(e_{t+1}^o + R_{t+1}a_t\right)$$

This implies that

$$\beta\left[R_{t+1}a_t v''\left(e^o_{t+1}+R_{t+1}a_t\right)+v'\left(e^o_{t+1}+R_{t+1}a_t\right)\right]\frac{\partial R_{t+1}}{\partial a_t}$$
$$=-u''(e^y_t-a_t)-\beta R^2_{t+1}v''\left(e^o_{t+1}+R_{t+1}a_t\right)>0.$$

Again, by Assumption 5, we have

$$R_{t+1}a_t v''\left(e^o_{t+1}+R_{t+1}a_t\right)+v'\left(e^o_{t+1}+R_{t+1}a_t\right)=-\frac{K'(R_{t+1})}{\beta}>0$$

which implies that, $\frac{\partial R_{t+1}}{\partial a_t}>0$. Therefore, R_{t+1} is strictly increasing in a_t.

□

Proof of Lemma 5. We follow the strategy of in Tirole (1985), Bosi et al. (2018b, 2022).

Point 2. Let $a'_0>a_0$ be two elements in $\mathcal{A}_0$, and $(a'_t),(a_t)$ be two associated equilibrium sequences. We have $R'_1=g_1(a'_0)\geq g_1(a_0)=R_1$. Then, we have $a'_1=\frac{R'_1}{n}a'_0-d_1>\frac{R_1}{n}a_0-d_1=a_1$. By induction, we have $a'_t>a_t$ and $R'_t\geq R_t$ for any t. Thus, we can compare the fundamental values

$$f'_0=\sum_{s=1}^{\infty}\frac{n}{R'_1}\cdots\frac{n}{R'_s}d_s\leq\sum_{s=1}^{\infty}\frac{n}{R_1}\cdots\frac{n}{R_s}d_s=f_0$$
$$b'_0=a'_0-f'_0>a_0-f_0=b_0.$$

Point 3 is a direct consequence of the above proof and the fact that $\mathcal{A}_0$ is an interval.

Let us prove Point 1. Firstly, we prove that $\mathcal{A}_0$ is an interval. Let us consider two equilibria $(a_{1,t},R_{1,t+1})_{t\geq 0}$ and $(a_{2,t},R_{2,t+1})_{t\geq 0}$ with initial asset values $a_{1,0}<a_{2,0}$. Take $a_0=\lambda a_{1,0}+(1-\lambda)a_{2,0}\in(a_{1,0},a_{2,0})$ with $\lambda\in(0,1)$. We have to prove that there exists a sequence $(a_t)_{t\geq 0}$ satisfying (29). Clearly, $a_0\in(0,e^y_0)$. From a_0, we can define $R_1=g_1(a_0)$, thanks to Lemma 4 and the fact that $a_0u'(e^y_0-a_0)<a_{2,0}u'(e^y_0-a_{2,0})<\beta\lim_{c\to\infty}cv'(c)$.

Since $a_0\in(a_{1,0},a_{2,0})$, we have $R_1\in[R_{1,1},R_{2,1}]$. Then, we define a_1 by $a_1+d_1=\frac{R_1}{n}a_0$. We see that

$$a_{1,1}=\frac{R_{1,1}}{n}a_{1,0}-d_1<a_1=\frac{R_1}{n}a_0-d_1<\frac{R_{2,1}}{n}a_{2,0}-d_1=a_{2,1}.$$

By induction, we construct that the equilibrium sequence (a_t). So, $a_0\in\mathcal{A}_0$. It means that $\mathcal{A}_0$ is an interval.

It remains to prove that $\mathcal{A}_0$ is closed. This is a direct consequence of Lemmas 11 and 12 below.

Lemma 11. *The equilibrium set $\mathcal{A}_0$ in Definition (6) is closed on the right: if $(a^m_0)_{m\geq 1}$ is a strictly increasing sequence with $a^m_0\in\mathcal{A}_0$ for any $m\geq 1$, then $a_0\equiv\lim_{m\to\infty}a^m_0$ belongs to $\mathcal{A}_0$.*[22]

[22]This result can be viewed as an adapted version of Tirole (1985)'s Lemma 10.

Proof of Lemma 11. By definition, we have

$$u'(e_t^y - a_t^m) = \beta R_{t+1}^m v'\left(e_{t+1}^o + R_{t+1}^m a_t^m\right) \tag{102}$$
$$R_{t+1}^m = g_t(a_t^m) \tag{103}$$
$$a_{t+1}^m = \frac{R_{t+1}^m}{n} a_t^m - d_{t+1} \tag{104}$$
$$0 < a_t^m < e_t^y \text{ for any } t \geq 0 \tag{105}$$

Since the sequence $(a_0^m)_m$ is increasing in m, we have $R_1^m = g_1(a_0^m)$ is increasing in m. This implies that $a_1^m = \frac{R_1^m}{n} a_0^m - d_1$ is increasing in m. By induction, a_t^m and R_t^m are increasing in m. Define $a_t \equiv \lim_{m\to\infty} a_t^m, R_t \equiv \lim_{m\to\infty} R_t^m$. To prove that a_0 is in the set $\mathcal{A}_0$ in Definition (6), it remains to prove that $a_t \in (0, e_t^y)$ for any t.

It is easy to see that $a_t \geq a_t^m > 0$.

We now prove that $a_t < e_t^y$. We have

$$u'(e_t^y - a_t^m) = \beta R_{t+1}^m v'\left(e_{t+1}^o + R_{t+1}^m a_t^m\right) \leq \beta R_{t+1}^m v'\left(e_{t+1}^o\right).$$

If $\lim_{m\to\infty} a_t^m = e_t^y$, then $u'(e_t^y - a_t^m)$ converges to infinity. This implies that R_{t+1}^m converges to infinity.

For $m \geq 1$, we have

$$b_{t+1}^m = b_0^m \frac{R_1^m \cdots R_t^m R_{t+1}^m}{n^{t+1}} \geq b_0^1 \frac{R_1^* \cdots R_t^* R_{t+1}^m}{n^{t+1}}. \tag{106}$$

where R_t^* is the interest rate of the economy without assets.

Since $b_0^1 > 0$ and $\lim_{m\to\infty} R_{t+1}^m = \infty$, we obtain that $\lim_{m\to\infty} b_{t+1}^m = \infty$. However, this is impossible because $b_t^m \leq a_t^m < e_t^y$. □

Lemma 12. *The equilibrium set $\mathcal{A}_0$ in Definition (6) is closed on the left: if $(a_0^m)_{m\geq 1}$ is a strictly decreasing sequence with $a_0^m \in \mathcal{A}_0$ for any $m \geq 1$, then $a_0 \equiv \lim_{m\to\infty} a_0^m$ belongs to $\mathcal{A}_0$.*

Proof of Lemma 12. By definition, we have

$$R_{t+1}^m = g_t(a_t^m), \quad a_{t+1}^m = \frac{R_{t+1}^m}{n} a_t^m - d_{t+1}, \quad 0 < a_t^m < e_t^y \text{ for any } t \geq 0. \tag{107}$$

As in the proof of Lemma 11, we can define $a_t \equiv \lim_{m\to\infty} a_t^m, R_t \equiv \lim_{m\to\infty} R_t^m$. It is easy to see that $R_t^m \geq R_t \geq R_t^*$ for any m and for any t, where the sequence (R_t) corresponds to the initial condition a_0 and R_t^* is the return rate of the economy without assets.

It is obvious that $a_t \leq a_t^m < e_t^y$. So, it remains to prove that $a_t > 0$ for any $t \geq 0$.

Fix a date t. We have

$$a_t^m = \frac{n}{R_{t+1}^m}(a_{t+1}^m + d_{t+1}) \geq \frac{n}{R_{t+1}^m} d_{t+1}.$$

Let $m \to \infty$, we get that $a_t \geq \frac{n}{R_{t+1}} d_{t+1} > 0$. □

□

B.3.1 Proof of Theorem 3

Lemma 9 and Proposition 4 imply that there exists a bubbleless equilibrium, denoted by $\underline{a}$. By Lemma 5 and Lemma 12, this bubbleless equilibrium is the lowest equilibrium.

We now prove there exists a continuum of equilibria (then, Lemma 5 implies that the equilibrium set has the form $[\underline{a}, \bar{a}]$). We can do so by verifying all assumptions in Theorem 1 or using the argument used in the proof of Theorem 1.

B.3.2 Proof of Theorem 4

Here under stationary endowments, the function g_t defined by Lemma 4 does not depend on t. So, we write g instead of g_t. We summarize our equilibrium system.

$$u'(e^y - a_t) = \beta R_{t+1} v'(e^o + R_{t+1} a_t) \tag{108a}$$

$$R_{t+1} = g(a_t), \text{ where } g \text{ is defined by Lemma 4} \tag{108b}$$

$$a_{t+1} + d_{t+1} = a_t \frac{R_{t+1}}{n}, \quad 0 < a_t < e^y \text{ for any } t \geq 0 \tag{108c}$$

We need several steps. We are inspired by the strategy of Tirole (1985), Bosi et al. (2018b), Pham and Toda (2025a).

Lemma 13. *Consider a solution to the system (108). Only three mutually exclusive cases hold:*

Case A. *$R_t \geq R_{t-1}$ for any t.*

Case B: *There exists t such that $R_t < R_{t-1}$ and $R_t \leq n$.*

Case C: *There exists t such that $R_t < R_{t-1}$ and for any t_0 satisfying $R_{t_0} < R_{t_0-1}$, we have $R_{t_0} > n$.*

Proof of Lemma 13. The proof is immediate. □

We then have the following result which is related to Tirole (1985)'s Lemma 2.

Lemma 14. *Consider the system (108). Consider an equilibrium a_0. Assume that $R_t < R_{t-1}$ and $R_t \leq n$ for some t. Then R_t converges to R^*, a_t converges to zero, and*

$$R^* < n \tag{109}$$

$$\frac{n}{R^*} \liminf_{t\to\infty} \frac{d_{t+1}}{d_t} \leq 1, \quad \frac{n}{R^*} \liminf_{t\to\infty} d_t^{\frac{1}{t}} \leq 1 \tag{110}$$

Proof of Lemma 14. Since $R_t \leq n$, we have $a_t = \frac{R_t}{n} a_{t-1} - d_t < a_{t-1}$. This implies that

$$R_{t+1} = g(a_t) < g(a_{t-1}) = R_t \leq n.$$

Hence, $R_{t+1} < R_t$. It means that we have $R_{t+1} < R_t$ and $R_{t+1} < n$. By induction, we get

$$n \geq R_t > R_{t+1} > \cdots$$

which implies that (R_t, a_t) converge to (R, a) and $a(R - n) = 0$.

We have $R < n$ because $\lim_{s\to\infty} R_s \leq R_{t+1} < R_t \leq n$. This implies that $a = 0$. So, $R_{t+1} = g(a_t)$ converges to R^*.

Recall that

$$f_0 = \sum_{t=1}^{\infty} \frac{n^t d_t}{R_1 \cdots R_t} \leq a_0 < \infty \text{ for any } t \geq 0.$$

Conditions (110) are a direct consequence of d'Alembert criterion and Cauchy criterion. □

Lemma 15. *Consider the system (108). Consider an equilibrium a_0. If a_t converges to 0, then R_t converges to $g(0) = R^*$.*

Proof of Lemma 15. It is obvious because $R_t = g(a_{t-1})$ for any t. □

Lemma 16. *If $\liminf_{t\to\infty} R_t \geq n$ and $\limsup_{t\to\infty} d_t^{1/t} < 1$, then $f_t \equiv \sum_{s=1}^{\infty} \frac{n}{R_{t+1}} \cdots \frac{n}{R_{t+s}} d_{t+s}$ converges to zero.*

Proof of Lemma 16. Since $\limsup_{t\to\infty} d_t^{1/t} < 1$, we can choose $x \in (0,1)$ and t_0 such that $d_t^{1/t} < x$ for any $t \geq t_0$. So, $d_t < x^t$ for any $t \geq t_0$.

Since $\liminf_{t\to\infty} R_t \geq n$, we have, for any $\epsilon \in (0,n)$, $\liminf_{t\to\infty} R_t > n - \epsilon$. So, we can choose $y \in (x,1)$ and $t_1 \geq t_0$ such that $R_t > ny$ for any $t \geq t_1$. Then, we have, for any $t \geq t_1$,

$$f_t = \sum_{s=1}^{\infty} \frac{n}{R_{t+1}} \cdots \frac{n}{R_{t+s}} d_{t+s} \leq \sum_{s=1}^{\infty} (\frac{n}{ny})^s x^{t+s} = x^t \sum_{s=1}^{\infty} (\frac{x}{y})^s.$$

Since $x < y < 1$, we get $\lim_{t\to\infty} f_t = 0$. □

The following is similar to Tirole (1985)'s Lemma 3

Lemma 17. *Consider the system (108). Assume that*

$$\sum_{t\geq 1} d_t < \infty \tag{111}$$

Consider an equilibrium. One of the following cases must hold.

1. *The equilibrium is bubbly, (a_t, b_t, R_t) converges to $(0, 0, R^*)$ and $R^* < n$.*
2. *(a_t, b_t, R_t) converges to $(\hat{a}, \hat{b}, n)$ with $\hat{a} \geq \hat{b}$ and $\hat{a}$ is uniquely determined by $u'(e^y - \hat{a}) = \beta n v'(e^o + n\hat{a})$ (i.e., $n = g(\hat{a})$).*
3. *This equilibrium is bubbleless, (i.e., $b_t = 0$ for any t), and $(a_t, b_t, R_t) = (a_t, 0, R_t)$ converges to $(0, 0, R^*)$.*

Proof of Lemma 17. We consider three cases: A, B, C in Lemma 13.

Case A. If $R_t \geq R_{t-1}$ for any t. Then R_t converges.

Case A.1. If $R_{t_0} > n$ for some t_0, then $R_t \geq R_{t_0} > n$ for any $t \geq t_0$. Then, there is no bubble. Indeed, if $b_0 > 0$, then b_t converges to infinity (because $b_{t+1} = b_t \frac{R_{t+1}}{n}$ for any t), which is a contradiction.

So, there is no bubble. Since there is no bubble, we have

$$a_t = f_t = \sum_{s=1}^{\infty} \frac{n}{R_{t+1}} \cdots \frac{n}{R_{t+s}} d_{t+s} \leq \sum_{s=1}^{\infty} d_{t+s} \text{ for any } t \geq t_0.$$

Using condition (111), we obtain that a_t converges to 0. Hence, $R_t = g(a_{t-1})$ converges to $g(0) = R^*$. To sum up, we have

$$(a_t, b_t, R_t) = (a_t, 0, R_t) \to (0, 0, R^*) \text{ with } R^* > n.$$

We are in the case 3 of Lemma 17.

Case A.2: $R_t \le n$ for any t, then we have $R_t \ge R_{t-1}$ and $R_t \le n$ for any t. This implies that R_t converges to some value R with $0 \le R \le n$.

There are two subcases.

- Case A.2.1: $R < n$. Then, b_t converges to zero (because $b_{t+1} = \frac{R_{t+1}}{n} b_t$) and a_t converges to zero because $\lim_{t\to\infty} R_t < n$ and

$$a_{t+1} = \frac{R_{t+1}}{n} a_t - d_{t+1} < \frac{R_{t+1}}{n} a_t \text{ for any } t.$$

  So, we have

$$(a_t, b_t, R_t) \to (0, 0, R^*) \text{ with } R^* < n$$

  We are in the case 1 or the case 3 in Lemma 17.

- Case A.2.2: $R = n$. Since $b_{t+1} = b_t \frac{R_{t+1}}{n}$ and $R_t \le n$ for any t, the sequence b_t is decreasing and hence converges to some value $\hat{b}$.

  Look at the sequence (a_t).

  - If $R_t < n$ for any t. We have

$$a_{t+1} = \frac{R_{t+1}}{n} a_t - d_{t+1} \le \frac{R_{t+1}}{n} a_t < a_t \text{ for any } t.$$

    So, a_t must converge to some value $\hat{a}$ with $\hat{a} \ge \hat{b}$ because $a_t \ge b_t$ for any t.

    To sum up, we have $(a_t, b_t, R_t) \to (\hat{a}, \hat{b}, n)$. We are in the case 2 of Lemma 17.

    Note that if we add the assumption $\limsup_{t\to\infty} d_t^{\frac{1}{t}} < 1$, then Lemma 16 implies that f_t converges to zero and, hence, $\hat{a} = \hat{b}$.

  - If there exists T such that $R_T = n$ for some T, then $R_t = n \; \forall t \ge T$ (because (R_t) is increasing and $\lim_{t\to\infty} R_t = R = n$). Therefore, for any $t > T$, we have

$$f_t = \sum_{s=1}^{\infty} \frac{n^s d_{t+s}}{R_{t+1} \cdots R_{t+s}} = \sum_{s\ge 1} d_{t+s}.$$

    So, by assumption (111), f_t converges to zero. It means that $\lim_{t\to\infty} a_t = \lim_{t\to\infty} b_t$.

    To sum up, we have $(a_t, b_t, R_t) \to (\hat{a}, \hat{a}, n)$. We are in the case 2 of Lemma 17.

Case B: There exists t such that $R_t < R_{t-1}$ and $R_t \le n$. Then, by using Lemma 14, we have $\lim_{t\to\infty} R_t < n$ and $\lim_{t\to\infty} a_t = 0$. Therefore, $(a_t, b_t, R_t) \to$

$(0,0,R^*)$ with $R^* < n$. So, the equilibrium is either in the case 1 or in the case 3 of our proposition.

Case C: There exists t such that $R_t < R_{t-1}$, and for any t_0 satisfying $R_{t_0} < R_{t_0-1}$, we have $R_{t_0} > n$. In this case, we have $R_t > n$ since $R_t < R_{t-1}$. We claim $R_{t+1} > n$. Indeed, if $R_{t+1} \leq n$, we have $R_{t+1} \leq n < R_t$ which implies that $R_{t+1} > n$, a contradiction.

By induction, we have $R_{t+\tau} > n$, for any $\tau \geq 0$. This implies that $\liminf_{t\to\infty} R_t \geq n$.

Case C.1: $b_0 > 0$. Since $b_t \leq a_t \leq e^y$, the sequence (b_t) is uniformly bounded from above. $R_{t+\tau} > n$ for any $\tau \geq 0$ and $b_{t+1} = b_t\frac{R_{t+1}}{n}$, we must have $\limsup_{t\to\infty} R_t \leq n$ (otherwise, $\limsup_{t\to\infty} b_t = \infty$ which is impossible). So, $\lim_{t\to\infty} R_t = n$.

Again, since $R_{t+\tau} > n$ for any $\tau \geq 0$, we have, for any $T > t$

$$f_T = \sum_{s=1}^{\infty} \frac{n^s d_{T+s}}{R_{T+1}\cdots R_{T+s}} < \sum_{s\geq 1} d_{T+s}.$$

By assumption (111), we obtain that f_t converges to zero. Since $(b_{t+\tau})_\tau$ converges (because it is increasing thanks to $R_{t+\tau} > n$), we have $\lim_{t\to\infty} b_t > 0$. Then $\lim_{t\to\infty} a_t = \lim_{t\to\infty} b_t > 0$. Since $\lim_{t\to\infty} R_t = n$, we must have $\lim_{t\to\infty} a_t = \hat{a}$. Summing up, we have

$$(a_t, b_t, R_t) \to (\hat{a}, \hat{a}, n).$$

So, the equilibrium is in the case 2 of Lemma 17.

Case C.2: If $b_0 = 0$, then $b_h = 0$ for any $h \geq 0$. Since $R_{t+\tau} > n$ for any $\tau \geq 0$, we have

$$f_\tau = \sum_{s=1}^{\infty} \frac{n^s d_{\tau+s}}{R_{\tau+1}\cdots R_{\tau+s}} < \sum_{s\geq 1} d_{\tau+s}.$$

By combining with our condition (111), f_t converges to zero. Then $a_t = f_t + b_t$ also converges to zero. To sum up, we have $R_{t-1} > R_t$ and $R_{t+\tau} > n$ for any $\tau \geq 0$. Summing up, $(a_t, b_t, R_t) \to (0, 0, R)$ with $R \geq n$. The equilibrium is in the case 3 of Lemma 17. □

Lemma 18. *Assume that*

$$\sum_{t\geq 1} d_t < \infty.$$

Consider the system (108). There exists at most one bubbly equilibrium a_0 such that the interest rate R_t converges to n. So, there exists at most one asymptotically bubbly equilibrium.[23]

Proof of Lemma 18. Consider $a'_0 < a_0$ two bubbly equilibria.

Suppose that for both values, the interest rates R_t, R'_t converge to n. By Lemma 17, we have $\lim_{t\to\infty} a_t = \lim_{t\to\infty} a'_t = \hat{a}$. Moreover, $\hat{a} > 0$ because $R^* < n$. Since $a'_0 < a_0$, we have $R'_t < R_t, a'_t < a_t$ for any $t \geq 1$. Therefore, we have

$$\frac{a'_t}{a_t} = \frac{\frac{R'_t}{n}a'_{t-1} - d_t}{\frac{R_t}{n}a_{t-1} - d_t} < \frac{R'_t a'_{t-1}}{R_t a_{t-1}} < \frac{a'_{t-1}}{a_{t-1}} < \cdots < \frac{a'_0}{a_0} < 1.$$

So, $\frac{a'_t}{a_t}$ does not converge to 1. □

[23]This can be viewed as a generalized version of Tirole (1985)'s Lemma 5.

We continue with the following result corresponding to Tirole (1985)'s Lemma 7.

Lemma 19. *Consider the system (108). Assume that there exists an equilibrium* (a^b_t) *satisfying* $a^b_t \to a^b \in [0, e^y)$ *and* $R^b_t \to R^b < n$.

Then, there exists $\bar{b}_0$ *such that for any* $b_0 \in (0, \bar{b}_0)$*, the sequence* (a_t) *defined by*

$$a_0 = a^b_0 + b_0, \quad R_{t+1} = g(a_t), \quad a_{t+1} = \frac{R_{t+1}}{n}a_t - d_{t+1} \tag{112}$$

is a bubbly equilibrium, and we have $\lim_{t\to\infty} R_t < n$.

Proof of Lemma 19. Let (a^b_t) be an equilibrium satisfying $a^b_t \to a^b \in [0, e^y)$ with $R^b = \lim_{t\to\infty} g(a^b_t) < n$.

We have $u'(e^y - a^b_t) = \beta R^b_{t+1} v'(e^o + R^b_{t+1} a^b_t)$ and $u'(e^y - a^b) = \beta R^b v'(e^o + R^b a^b)$. By Lemma 4, we have $a^b u'(e^y - a^b) < \beta \lim_{c\to\infty} cv'(c)$ (Indeed, this is trivial if $a^b = 0$. If $a^b > 0$, we apply Lemma 4).

Therefore, there exists T, x, x_a such that $x \in (0, n)$, $x_a \in (0, e^y)$, $g(a^b_t) < x < n$, $a^b_t < x_a$ for any $t \geq T$, and $x_a u'(e^y - x_a) < \beta \lim_{c\to\infty} cv'(c)$

We can choose $b_0 > 0$ small enough and define the sequence $(a_t, R_t)_{t=0}^T$ by

$$a_0 = a^b_0 + b_0, \quad R_{t+1} = g(a_t), \quad a_{t+1} = \frac{R_{t+1}}{n}a_t - d_{t+1} \tag{113}$$

such that $a_T < x_a, g(a_T) < x$, $a_t u'(e^y - a_t) < \beta \lim_{c\to\infty} cv'(c)$ for any $t = 0, 1, \ldots, T$

Then, it is easy to see that $a_t > a^b_t, R_t > R^b_t$, for any $t = 0, 1, \ldots, T$.

We now define R_{t+1}, a_{T+1}. Since $a_T u'(e^y - a_T) < \beta \lim_{c\to\infty} cv'(c)$, Lemma 4 allows us to define $R_{T+1} = g(a_T)$ and then a_{T+1} as follows:

$$R_{T+1} = g(a_T) < x < n \tag{114}$$

$$a_{T+1} = \frac{R_{T+1}}{n}a_T - d_{T+1} \leq \frac{x}{n}a_T < a_T < x_a. \tag{115}$$

Since $a_{T+1} < x_a$ and $x_a u'(e^y - x_a) < \beta \lim_{c\to\infty} cv'(c)$, we have $a_{T+1} u'(e^y - a_{T+1}) < \beta \lim_{c\to\infty} cv'(c)$.

Then, by induction, we construct $(a_t, R_t)_{t\geq 0}$ such that $R_{T+s} < x < n, a_{T+s} < x_a$ for any $s \geq 1$. This implies that

$$a_{T+s+1} = \frac{R_{T+s+1}}{n}a_{T+s} - d_{T+s+1} < \frac{n}{n}a_{T+s} = a_{T+s}.$$

Hence, $R_{T+s} < R_{T+s-1}$ for any $s \geq 1$, which implies that $\lim_{t\to\infty} R_t < R_{T+1} < n$.

Summing up, the sequence (a_t) is an equilibrium because $0 < a^b_t < a_t < x_a < e^y$. According to Lemma 5's point 3, this is bubbly because $a_0 > a^b_0$. □

We are now ready to prove Theorem 4.

Part 1 is a direct consequence of Lemma 1 and Proposition 3.

Part 2. We consider two cases.

Case 1: The equilibrium set is singleton.

If a_t does not converge to $\hat{a}$ (note that $\hat{a} > 0$ because $R^* < n$), then by Lemma 17, a_t must converge to zero. Hence, $R_{t+1} = g(a_t)$ converges to $g(0) = R^*$. By our assumption $R^* < n$ and Lemma 19, we can construct another equilibrium. This is a contradiction.

Therefore, we have $a_t \to \hat{a}$. Since $\hat{a} > 0$, only case (2) in Lemma 17 holds. Since $a_t \to \hat{a} > 0$, we can take $x > 0$ and t_0 such that $a_t \geq x$ for any $t \geq t_0$. Then,

$$\sum_{t \geq t_0} \frac{d_t}{a_t} \leq \sum_{t \geq t_0} \frac{d_t}{x} < \infty.$$

So, Lemma 2 implies that this equilibrium is bubbly. Then, we have $b_t = \frac{R_1 \cdots R_t}{n^t} b_0 > 0$. We write

$$\frac{b_t}{a_t} = \frac{R_1 \cdots R_t}{n^t} b_0 \frac{1}{a_t} = \frac{b_0}{\frac{n^t a_t}{R_1 \cdots R_t}}$$

Recall that

$$a_0 = \sum_{t \geq 1} \frac{n^t}{R_1 \cdots R_t} d_t + \lim_{t \to \infty} \frac{n^t}{R_1 \cdots R_t} a_t = f_0 + b_0.$$

So, $\lim_{t\to\infty} b_t/a_t = \frac{b_0}{a_0 - f_0} = 1$. Hence $\lim_{t\to\infty} b_t = \lim_{t\to\infty} a_t = \hat{a}$.

Case 2: The equilibrium set is not singleton. By Lemma 5, the equilibrium set is a compact interval, denoted by $[\underline{a}, \bar{a}]$.

For any $a_0 > \underline{a}$, the equilibrium is bubbly (by point 2 of Lemma 5).

For $a_0 = \bar{a}$, the equilibrium is bubbly and $a_t \to \hat{a}$. Indeed, if a_t does not converge to $\hat{a}$, then by Lemma 17, a_t must converge to zero. By our assumption $R^* < n$ and Lemma 19, we can construct another equilibrium with $a_0' > a_0 = \bar{a}$. This is a contradiction. So, $\lim_{t\to\infty} a_t = \hat{a} > 0$. Then, we have $b_t = \frac{R_1 \cdots R_t}{n^t} b_0 > 0$. As above, we have

$$\frac{b_t}{a_t} = \frac{R_1 \cdots R_t}{n^t} b_0 \frac{1}{a_t} = \frac{b_0}{\frac{n^t a_t}{R_1 \cdots R_t}}$$

$$\lim_{t \to \infty} \frac{b_t}{a_t} = \frac{b_0}{a_0 - f_0} = 1.$$

Hence $\lim_{t\to\infty} b_t = \lim_{t\to\infty} a_t = \hat{a}$.

Then, by Lemma 18, $a_0 = \bar{a}$ is the unique bubbly equilibrium satisfying $\lim_{t\to\infty} R_t = n$. By consequence, for any bubbly equilibrium $a_0 \in (\underline{a}, \bar{a})$, we have $\lim_{t\to\infty} R_t \neq n$. Therefore, Lemma 17 implies that (a_t, b_t, R_t) converges to $(0, 0, R^*)$ for any $a_0 \in (\underline{a}, \bar{a})$.

Last, look at the minimal equilibrium $a_0 = \underline{a}$. Take an equilibrium a_0' with $a_0' \in (\underline{a}, \bar{a})$. According to the proof of point 2 of Lemma 5, we have $a_t' > a_t, R_t' > R_t$ for any t. Since we have proved that (a_t', b_t', R_t') converges to $(0, 0, R^*)$, we get that $(a_t, b_t, R_t) \to (0, 0, R^*)$.

Proof of Claim 1 in Theorem 4. Assume that $R^* < n$ and $\sum_{t\geq 1} \frac{\mathcal{D}_t}{(R^*)^t} < \infty$. Since $\sum_{t \geq 1} \frac{\mathcal{D}_t}{(R^*)^t} < \infty$, Proposition 4 implies that there exists a bubbleless equilibrium. So, case 2a must hold.

Proof of Claim 2 in Theorem 4. Assume that $R^* < n$, $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$, and $R^* < \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}}$. Recall that $R^* < \limsup_{t\to\infty} \mathcal{D}_t^{\frac{1}{t}}$ is equivalent to $R^* < n \limsup_{t\to\infty} d_t^{\frac{1}{t}}$.

We will prove that case 2a of Theorem 4 cannot happen. Suppose that case 2a of Theorem 4 happens. Then we take an equilibrium $a_0 \in (\underline{a}, \bar{a})$. This is a bubbly equilibrium with $a_t \to 0$ and $R_t \to R^*$. We have

$$\begin{aligned} a_t &= \frac{n}{R_{t+1}} a_{t+1} + \frac{n}{R_{t+1}} d_{t+1} \\ &= \frac{n}{R_{t+1}} \frac{n}{R_{t+2}} a_{t+2} + \frac{n}{R_{t+1}} \frac{n}{R_{t+2}} d_{t+2} + \frac{n}{R_{t+1}} d_{t+1} \\ &= \sum_{s \geq 1} \frac{n^s}{R_{t+1} \cdots R_{t+s}} d_{t+s} + \lim_{s \to \infty} \frac{n^s}{R_{t+1} \cdots R_{t+s}} a_{t+s} \end{aligned}$$

Take $\epsilon > 0$ and an infinite increasing sequence (t_k) such that $R^* + \epsilon_1 < n d_{t_k}^{\frac{1}{t_k}}$, i.e., $(R^* + \epsilon)^{t_k} < n^{t_k} d_{t_k}$.

Since $\lim_{t \to \infty} R_t = R^* < R^* + \epsilon$, we can take T such that $R_t < R^* + \epsilon$ for any $t \geq T$. Let t_{k_0} be such that $T < t_{k_0}$.

For any $t > t_{k_0}$, we have

$$a_t = \sum_{s \geq 1} \frac{n^s}{R_{t+1} \cdots R_{t+s}} d_{t+s} + \lim_{s \to \infty} \frac{n^s}{R_{t+1} \cdots R_{t+s}} a_{t+s} \geq \sum_{s \geq 1} \frac{n^s}{R_{t+1} \cdots R_{t+s}} d_{t+s} \tag{116}$$

$$\geq \sum_{k:\ t_k > t} \frac{n^{t_k - t}}{R_{t+1} \cdots R_{t_k}} d_{t_k} \geq \frac{1}{n^t} \sum_{k:\ t_k > t} \frac{n^{t_k} d_{t_k}}{(R^* + \epsilon)^{t_k - t}} = \frac{(R^* + \epsilon)^t}{n^t} \sum_{k:\ t_k > t} \frac{n^{t_k} d_{t_k}}{(R^* + \epsilon)^{t_k}}. \tag{117}$$

Since $n^{t_k} d_{t_k} > (R^* + \epsilon)^{t_k}$ for any k, we have

$$a_t \geq \frac{(R^* + \epsilon)^t}{n^t} \sum_{k:\ t_k > t} \frac{n^{t_k} d_{t_k}}{(R^* + \epsilon)^{t_k}} \geq \frac{(R^* + \epsilon)^t}{n^t} \sum_{i=1}^{\infty} 1 = \infty$$

a contradiction. It means that case 2a cannot happen. Therefore, case 2b holds.

Part 3 of Theorem 4. Conditions $R^* = n$ and $\sum_{t=1}^{\infty} d_t = \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t} < \infty$ imply that

$$\sum_{t=1}^{n} \frac{\mathcal{D}_t}{R_1^* \cdots R_t^*} = \sum_{t=1}^{n} \frac{\mathcal{D}_t}{(R^*)^t} = \sum_{t=1}^{n} \frac{\mathcal{D}_t}{n^t} < \infty.$$

Applying Proposition 4, there exists a bubbleless equilibrium.

Let (a_t) be an equilibrium. Since $\sum_{t=1}^{\infty} d_t < \infty$ and $R^* = n$, Lemma 17 indicates that only case 2 or case 3 in Lemma 17 happens. In both cases, we have $\lim_{t \to \infty} R_t = n$. If case 3 in Lemma 17 happens, we directly obtain $(a_t, b_t, R_t) \to (0, 0, R^*)$. If case 2 in Lemma 17 happens, we have $\lim_{t \to \infty} a_t = \hat{a}$. However, since $R^* = n$ and definition of $\hat{a}$, we have $\hat{a} = 0$, and, hence, $\lim_{t \to \infty} b_t = 0$.

Consider an equilibrium, by Lemma 1, we always have $R_t \geq R^* = n$. By consequence, the bubble component is

$$b_0 = \lim_{t \to \infty} \frac{n^t}{R_1 \cdots R_t} a_t \leq \lim_{t \to \infty} \frac{n^t}{n^t} a_t \leq \lim_{t \to \infty} a_t = 0. \tag{118}$$

So, there is no bubble. In other words, there is no bubbly equilibrium. Combining with Lemma 5's point 3, we get that there exists a unique equilibrium and this is bubbleless. We have finished our proof.

C Proofs for Section 5

Proof of Lemma 6. Let us consider a feasible allocation path $(c_t^{y\prime}, c_t^{o\prime})_t$. We have

$$c_t^{y\prime} + \frac{c_t^{o\prime}}{n} = e_t^y + \frac{e_t^o}{n} + d_t \text{ for any } t.$$

We follow the classical idea of support prices (Malinvaud, 1953; Cass, 1972).

Denote $U_t \equiv U^t\left(c_t^y, c_{t+1}^o\right)$ and $U_t' \equiv U^t\left(c_t^{y\prime}, c_{t+1}^{o\prime}\right)$. Since the function U^t is concave, we have that

$$U_t - U_t' \geq U_1^t(c_t^y, c_{t+1}^o)\left(c_t^y - c_t^{y\prime}\right) + U_2^t\left(c_t^y, c_{t+1}^o\right)\left(c_{t+1}^0 - c_{t+1}^{o\prime}\right) \text{ for any } t \geq -1 \quad (119)$$

By the feasibility of allocations, we have $c_t^o - c_t^{o\prime} = -n(c_t^y - c_t^{y\prime})$ for any $t \geq 0$. Combining with (119) and the no-arbitrage condition (35), we get

$$\begin{aligned}(U_t - U_t')\frac{1}{R_{t+1}U_2^t(c_t^y, c_{t+1}^o)} &\geq c_t^y - c_t^{y\prime} + \frac{1}{R_{t+1}}\left(c_{t+1}^0 - c_{t+1}^{o\prime}\right) &(120)\\ &= c_t^y - c_t^{y\prime} - \frac{n}{R_{t+1}}\left(c_{t+1}^y - c_{t+1}^{y\prime}\right) \text{ for any } t \geq 0 &(121)\end{aligned}$$

For the households born at date -1, we have

$$\begin{aligned}U_{-1} - U_{-1}' &\geq U_1^{-1}(c_{-1}^y, c_0^o)\left(c_{-1}^y - c_{-1}^{y\prime}\right) + U_2^{-1}(c_{-1}^y, c_0^o)\left(c_0^o - c_0^{o\prime}\right)\\ &= -nU_2^{-1}(c_{-1}^y, c_0^o)\left(c_0^y - c_0^{y\prime}\right) \text{ because } c_{-1}^y = c_{-1}^{y\prime},\\ \Rightarrow \frac{U_{-1} - U_{-1}'}{nU_2^{-1}(c_{-1}^y, c_0^o)} &\geq -\left(c_0^y - c_0^{y\prime}\right)\end{aligned}$$

Denote $X_t \equiv \frac{n^t}{R_1 \cdots R_{t+1} U_2^t(c_t^y, c_{t+1}^o)}$ for any $t \geq 0$ and $X_{-1} \equiv \frac{1}{nU_2^{-1}(c_{-1}^y, c_0^o)}$.

We denote $P_t \equiv \frac{n^t}{R_1 \cdots R_t}$ for any $t \geq 1$ and $P_0 = 1$. We have

$$X_t\left(U_t - U_t'\right) \geq P_t\left(c_t^y - c_t^{y\prime}\right) - P_{t+1}(c_{t+1}^y - c_{t+1}^{y\prime}) \text{ for any } t \geq 0 \quad (122)$$

Therefore, we get

$$\sum_{t=-1}^{T} X_t(U_t - U_t') \geq -P_{T+1}(c_{T+1}^y - c_{T+1}^{y\prime}) \geq -P_{T+1}c_{T+1}^y. \quad (123)$$

Combining with our assumption $\liminf_{t\to\infty} \frac{n^t}{R_1 \cdots R_t} c_t^y = 0$, we get

$$\limsup_{T\to\infty} \sum_{t=-1}^{T} X_t\left(U_t - U_t'\right) \geq \limsup_{T\to\infty}\Big(-P_{T+1}c_{T+1}^y\Big) = -\liminf_{T\to\infty} P_{T+1}c_{T+1}^y = 0.$$

So, $(c_t^y, c_t^o)_t$ is Pareto optimal. Indeed, take another feasible allocation $(c_t^{y'}, c_t^{o'})_t$. Suppose that $U_t' \geq U_t$ for any t and there exists t_0 such that $U_{t_0}' > U_{t_0}$. Then, $\sum_{t=-1}^{T} X_t(U_t - U_t') \leq P_{t_0}(U_{t_0} - U_{t_0}') < 0$ for any $t \geq t_0$. By consequence, $\limsup_{t\to\infty} \sum_{t=-1}^{T} X_t(U_t - U_t') \leq P_{t_0}(U_{t_0} - U_{t_0}') < 0$, a contradiction. Therefore, $(c_t^y, c_t^o)_t$ is Pareto optimal. □

C.1 Proofs of Lemmas 7 and 8

Proof of Lemma 7. The allocation $(c_t^y, c_t^o)_t$ satisfies the uniform strictness condition if there exists $\mu > 0$ such that

$$\frac{U_2^t(c_{t+1}^{o'} - c_{t+1}^o) + U_1^t(c_t^{y'} - c_t^y)}{\left(U_1^t(c_t^{y'} - c_t^y)\right)^2} U_1^t c_t^y \geq \mu \tag{124}$$

for any t and for any couple $(c_t^{y'}, c_{t+1}^{o'})$ satisfying

$$U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o), \quad 0 < (1-h)c_t^y < c_t^{y'} < c_t^y, \quad c_{t+1}^{o'} > c_t^o \tag{125}$$

By the Taylor's expansion, there exists $s_{yt}, s_{ot} \in (0,1)$ such that

$$U^t(c_t^{y'}, c_{t+1}^{o'}) - U^t(c_t^y, c_{t+1}^o) = u_t(c_t^{y'}) - u_t(c_t^y) + v_t(c_{t+1}^{o'}) - v_t(c_{t+1}^o) \tag{126}$$

$$= u_t'(c_t^y)(c_t^{y'} - c_t^y) + \frac{1}{2} u_t''\Big(c_t^y + s_{yt}(c_t^{y'} - c_t^y)\Big)(c_t^{y'} - c_t^y)^2 \tag{127}$$

$$+ v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o) + \frac{1}{2} v_t''\Big(c_{t+1}^o + s_{ot}(c_{t+1}^{o'} - c_{t+1}^o)\Big)(c_{t+1}^{o'} - c_{t+1}^o)^2. \tag{128}$$

Since $U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o)$, we have

$$u_t'(c_t^y)(c_t^{y'} - c_t^y) + \frac{1}{2} u_t''\Big(c_t^y + s_{yt}(c_t^{y'} - c_t^y)\Big)(c_t^{y'} - c_t^y)^2 \tag{129}$$

$$+ v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o) + \frac{1}{2} v_t''\Big(c_{t+1}^o + s_{ot}(c_{t+1}^{o'} - c_{t+1}^o)\Big)(c_{t+1}^{o'} - c_{t+1}^o)^2 \geq 0. \tag{130}$$

Since $v_t'' \leq 0$, we have

$$u_t'(c_t^y)(c_t^{y'} - c_t^y) + v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o) \geq -\frac{1}{2} u_t''\Big(c_t^y + s_{yt}(c_t^{y'} - c_t^y)\Big)(c_t^{y'} - c_t^y)^2. \tag{131}$$

By consequence,

$$\frac{U_2^t(c_{t+1}^{o'} - c_{t+1}^o) + U_1^t(c_t^{y'} - c_t^y)}{\left(U_1^t(c_t^{y'} - c_t^y)\right)^2} U_1^t c_t^y \geq \frac{-1}{2} u_t''\Big(c_t^y + s_{yt}(c_t^{y'} - c_t^y)\Big) \frac{c_t^y}{u_t'(c_t^y)} \tag{132}$$

$$\geq \frac{c_t^y}{u_t'(c_t^y)} \inf_{x \in [(1-h)c_t^y, c_t^y]} \frac{-1}{2} u_t''(x). \tag{133}$$

1. If $\bar{x} \equiv \inf_{t \geq 0} \left\{ \frac{c_t^y}{u_t'(c_t^y)} \inf_{x \in [(1-h)c_t^y, c_t^y]} \frac{-1}{2} u_t''(x) \right\} > 0$, we define $\mu \equiv \bar{x}$.
2. Consider the case where $u_t'(c) = c^{-\sigma}$. Then, $u_t''(c) = -\sigma c^{-(\sigma+1)}$. Therefore,

$$\frac{-1}{2} u_t''\Big(c_t^y + s_{yt}(c_t^{y'} - c_t^y)\Big) \frac{c_t^y}{u_t'(c_t^y)} = \frac{\sigma}{2} \Big(\frac{c_t^y}{c_t^y + s_{yt}(c_t^{y'} - c_t^y)}\Big)^{1+\sigma} > \frac{\sigma}{2}$$

because $c_t^{y'} - c_t^y < 0$. Therefore, we get

$$\frac{U_2^t(c_{t+1}^{o'} - c_{t+1}^o) + U_1^t(c_t^{y'} - c_t^y)}{\left(U_1^t(c_t^{y'} - c_t^y)\right)^2} U_1^t c_t^y > \frac{\sigma}{2}. \tag{134}$$

We have finished our proof.

□

Proof of Lemma 8. We will prove that, for each $x > 0$, there exists $\theta_1(x), \theta_2(x) > 0$ such that, for any t, if the couple $(c_t^{y'}, c_{t+1}^{o'})$ satisfies

$$\begin{cases} xc_t^y < c_t^{y'} < c_t^y, c_{t+1}^o < c_{t+1}^{o'} < ne_{t+1} \\ P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o) + nP_t(c_t^{y'} - c_t^y) \geq \frac{\theta_2(x)}{P_{t+1}c_{t+1}^o}\Big(P_{t+1}(c_{t+1}^{o'} - c_{t+1}^o)\Big)^2 \\ \quad + \frac{\theta_1(x)}{nP_t c_t^y}\Big(nP_t(c_t^{y'} - c_t^y)\Big)^2 \end{cases} \tag{135}$$

then $(c_t^{y'}, c_{t+1}^{o'}) \in B_t(c)$. i.e., $U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o)$.

By the Taylor's expansion, there exists $s_y, s_o \in (0,1)$ such that

$$\begin{aligned} U^{t'} - U^t \equiv & U^t(c_t^{y'}, c_{t+1}^{o'}) - U^t(c_t^y, c_{t+1}^o) = u_t(c_t^{y'}) - u_t(c_t^y) + v_t(c_{t+1}^{o'}) - v_t(c_{t+1}^o) \\ = & u_t'(c_t^y)(c_t^{y'} - c_t^y) + \frac{1}{2}u_t''(\tilde{c}_t^y)(c_t^{y'} - c_t^y)^2 \\ & + v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o) + \frac{1}{2}v_t''(\tilde{c}_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o)^2. \end{aligned}$$

where $\tilde{c}_t^y \equiv c_t^y + s_y(c_t^{y'} - c_t^y) \in (c_t^{y'}, c_t^y)$ and $\tilde{c}_{t+1}^o \equiv c_{t+1}^o + s_{ot}(c_{t+1}^{o'} - c_{t+1}^o) \in (c_{t+1}^o, c_{t+1}^{o'})$.

From (43) and $\frac{P_{t+1}}{P_t} = \frac{n}{R_{t+1}} = n\frac{U_2^t}{U_1^t} = n\frac{v_t'(c_{t+1}^o)}{u_t'(c_t^y)}$, we have

$$\begin{aligned} v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o) + u_t'(c_t^y)(c_t^{y'} - c_t^y) \geq & \frac{\theta_2(x)}{v_t'(c_{t+1}^o)c_{t+1}^o}\Big(v_t'(c_{t+1}^o)(c_{t+1}^{o'} - c_{t+1}^o)\Big)^2 \\ & + \frac{\theta_1(x)}{u_t'(c_t^y)c_t^y}\Big(u'(c_t^y)(c_t^{y'} - c_t^y)\Big)^2. \end{aligned}$$

Therefore

$$U^{t'} - U^t \geq \Big(\theta_2(x)\frac{v_t'(c_{t+1}^o)}{c_{t+1}^o} + \frac{1}{2}v_t''(\tilde{c}_{t+1}^o)\Big)(\epsilon_{t+1}^o)^2 + \Big(\theta_1(x)\frac{u_t'(c_t^y)}{c_t^y} + \frac{1}{2}u_t''(\tilde{c}_t^y)\Big)(\epsilon_t^y)^2.$$

Point 1 of Lemma 8 is clear since $xc_t^y < c_t^{y'} < c_t^y, c_{t+1}^o < c_{t+1}^{o'} < ne_{t+1}$. Indeed, if we choose $\theta_1(x) > \bar{M}_1$ and $\theta_2(x) > \bar{M}_2$, where $\bar{M}_1, \bar{M}_2$ are defined in Lemma 8, we have $U^{t'} - U^t > 0$.

Let us check point 2. In this case, we have

$$\begin{aligned} U^{t'} - U^t \geq & \Big(\theta_2(x)\frac{v_t'(c_{t+1}^o)}{c_{t+1}^o} + \frac{1}{2}v_t''(\tilde{c}_{t+1}^o)\Big)(\epsilon_{t+1}^o)^2 + \Big(\theta_1(x)\frac{u_t'(c_t^y)}{c_t^y} + \frac{1}{2}u_t''(\tilde{c}_t^y)\Big)(\epsilon_t^y)^2 \\ = & \gamma_t\Big(\frac{\theta_2(x)}{(c_{t+1}^o)^{1+\sigma}} - \frac{\sigma}{2(\tilde{c}_{t+1}^o)^{1+\sigma}}\Big)(\epsilon_{t+1}^o)^2 + \Big(\frac{\theta_1(x)}{(c_t^y)^{1+\sigma}} - \frac{\sigma}{2(\tilde{c}_t^y)^{1+\sigma}}\Big)(\epsilon_t^y)^2 \end{aligned}$$

Choose $\theta_2(x) > \frac{\sigma}{2}$, we have $\frac{\theta_2(x)}{(c_{t+1}^o)^{1+\sigma}} - \frac{\sigma}{2(\tilde{c}_{t+1}^o)^{1+\sigma}} > 0$ because $\tilde{c}_{t+1}^o > c_{t+1}^o$.

Choose $\theta_1(x)$ such that $\theta_1(x)x^{1+\sigma} - \frac{\sigma}{2} > 0$, we have

$$\frac{\theta_1(x)(\tilde{c}_t^y)^{1+\sigma}}{(c_t^y)^{1+\sigma}} - \frac{\sigma}{2} > \theta_1(x)x^{1+\sigma} - \frac{\sigma}{2} > 0.$$

Therefore, we have $U^{t'} - U^t > 0$. We have finished our proof. □

C.2 Proof of Theorem 5 and Remark 6

We need an intermediate step.

Lemma 20. *Let Assumptions 1, 6 be satisfied.*

Consider an equilibrium. Denote, for each $t \geq 1$,

$$Q_t \equiv \frac{1}{R_1 \cdots R_t}, \quad P_t \equiv \frac{n^t}{R_1 \cdots R_t}. \tag{136}$$

This equilibrium is not Pareto optimal if and only if there exist a feasible allocation $(c_t^{y'}, c_t^{o'})_t$ which Pareto dominates the allocation $(c_t^y, c_t^o)_t$ and a date t_0 such that

$$\epsilon_t^y \equiv c_t^{y'} - c_t^y < 0, \quad \epsilon_t^o \equiv c_t^{o'} - c_t^o > 0 \ \forall t \geq t_0 \tag{137a}$$
$$\epsilon_t^o = -n\epsilon_t^y \tag{137b}$$
$$\epsilon_t^y = \epsilon_t^o = 0 \ \forall t < t_0 \tag{137c}$$
$$Q_{t+1}\epsilon_{t+1}^o > \frac{1}{n} Q_t \epsilon_t^o \quad (i.e., P_{t+1}\epsilon_{t+1}^o > P_t\epsilon_t^o) \ \forall t \geq t_0 - 1. \tag{137d}$$

Proof. The sufficient condition ($\Leftarrow$) is obvious. We prove the necessary condition: ($\Rightarrow$).

Suppose that $(c_t^y, c_t^o)_t$ is not Pareto optimal. Then, there exists a feasible allocation $(c_t^{y'}, c_t^{o'})_t$ which Pareto dominates the allocation $(c_t^y, c_t^o)_t$. By consequence, there exists a date s such that $U^s(c_s^{y'}, c_{s+1}^{o'}) > U^s(c_s^y, c_{s+1}^o)$. This allows us to define

$$t_0 \equiv \min\{s : c_s^{o'} \neq c_s^o\}. \tag{138}$$

By definition of t_0, we have $c_t^{y'} = c_t^y, c_t^{o'} = c_t^o$ for any $t \geq t_0 - 1$.

Since $(c_t^{y'}, c_t^{o'})_t$ Pareto dominates the allocation $(c_t^y, c_t^o)_t$, we have

$$U^{t_0-1}(c_{t_0-1}^{y'}, c_{t_0}^{o'}) \geq U^{t_0-1}(c_{t_0-1}^y, c_{t_0}^o). \tag{139}$$

Recall that $c_{t_0-1}^{y'} = c_{t_0-1}^y$, $c_{t_0}^{o'} \neq c_{t_0}^{o'}$, and the function U^{t_0-1} is strictly increasing in each component, we have $c_{t_0}^{o'} > c_{t_0}^o$ and, hence, $c_{t_0}^{y'} < c_{t_0}^y$ because $c_t^y + \frac{c_t^o}{n} = c_t^{y'} + \frac{c_t^{o'}}{n} = e_t^y + \frac{e_t^o}{n} + d_t$.

Now, since $c_{t_0}^{y'} < c_{t_0}^y$ and $U^{t_0}(c_{t_0}^{y'}, c_{t_0+1}^{o'}) \geq U^{t_0}(c_{t_0}^y, c_{t_0+1}^o)$, we get that $c_{t_0+1}^{o'} > c_{t_0+1}^o$. By induction, we obtain conditions (137).

It remains to prove that $Q_{t+1}\epsilon_{t+1}^o > \frac{1}{n} Q_t \epsilon_t^o \ \forall t \geq t_0 - 1$.

Observe that, by the non-arbitrage condition $R_{t+1} = \frac{q_{t+1} + \mathcal{D}_{t+1}}{q_t}$, the budget constraint of households t can be rewritten as

$$Q_t c_t^y + Q_{t+1} c_{t+1}^o = Q_t e_t^y + Q_{t+1} e_{t+1}^o. \tag{140}$$

Therefore, we have

$$Q_t c_t^{y'} + Q_{t+1} c_{t+1}^{o'} = Q_t e_t^y + Q_{t+1} e_{t+1}^o + \Big(Q_t \epsilon_t^y + Q_{t+1} \epsilon_{t+1}^o\Big). \tag{141}$$

Consider $t \geq t_0 - 1$. We have $U^t(c_t^{y'}, c_{t+1}^{o'}) \geq U^t(c_t^y, c_{t+1}^o)$.

Suppose that $Q_t\epsilon_t^y + Q_{t+1}\epsilon_{t+1}^o = 0$, we have $Q_t c_t^{y'} + Q_{t+1}c_{t+1}^{o'} = Q_t e_t^y + Q_{t+1}e_{t+1}^o$. Recall that $U^t(c_t^y, c_{t+1}^o)$ is the maximum value of the maximization problem of household t. By consequence, $U^t(c_t^{y'}, c_{t+1}^{o'})$ is also the maximum value. This implies that $(c_t^{y'}, c_{t+1}^{o'})$ is a solution to the maximization problem of agent t. However, since the function U^t is strictly quasi-concave, the solution is unique. Therefore, we have $(c_t^{y'}, c_{t+1}^{o'}) = (c_t^y, c_{t+1}^o)$, a contradiction.

So, we get that $Q_t\epsilon_t^y + Q_{t+1}\epsilon_{t+1}^o > 0$. We have finished our proof. □

Proof of Theorem 5. Part 1 ("if" part). Suppose that $(c_t^y, c_t^o)_t$ is not Pareto optimal. Applying Lemma 20, there exist a feasible allocation $(c_t^{y'}, c_t^{o'})_t$ which Pareto dominates the allocation $(c_t^y, c_t^o)_t$ and a date t_0 such that

$$\epsilon_t^y \equiv c_t^{y'} - c_t^y < 0, \quad \epsilon_t^o \equiv c_t^{o'} - c_t^o > 0 \; \forall t \geq t_0 \tag{142a}$$

$$\epsilon_t^o = -n\epsilon_t^y \tag{142b}$$

$$\epsilon_t^y = \epsilon_t^o = 0 \; \forall t < t_0 \tag{142c}$$

$$Q_{t+1}\epsilon_{t+1}^o > \frac{1}{n}Q_t\epsilon_t^o \quad (i.e., P_{t+1}\epsilon_{t+1}^o > P_t\epsilon_t^o) \; \forall t \geq t_0 - 1. \tag{142d}$$

So, $(c_t^{y'}, c_{t+1}^{o'}) \in B_t(c)$ and $c_t^{y'} < c_t^y$.

Let $h \in (0,1)$ in Definition 9. We define the sequence $(x_t^y, x_t^o)_t$ by

$$x_t^y \equiv c_t^y + h\epsilon_t^y, \qquad x_t^o \equiv c_t^o + h\epsilon_t^o. \tag{143}$$

Then, we have $x_t^y = (1-h)c_t^y + hc_t^{y'} > (1-h)c_t^y$, and $x_t^o = (1-h)c_t^o + hc_t^{o'}$. Since the function U^t is strictly concave, we have

$$\begin{aligned} U^t(x_t^y, x_{t+1}^o) &= U^t\Big((1-h)c_t^y + hc_t^{y'}, (1-h)c_{t+1}^o + hc_{t+1}^{o'}\Big) \\ &> (1-h)U^t(c_t^y, c_{t+1}^o) + hU^t(c_t^y, c_{t+1}^o) \geq U^t(c_t^y, c_{t+1}^o). \end{aligned}$$

By the uniform strictness condition in Theorem 5, there exists $\bar{\mu}$ such that

$$P_{t+1}(x_{t+1}^o - c_{t+1}^o) + nP_t(x_t^y - c_t^y) \geq \frac{\bar{\mu}}{P_t c_t^y}\Big(nP_t(x_t^y - c_t^y)\Big)^2 \tag{144}$$

$$\Leftrightarrow P_{t+1}\epsilon_{t+1}^o \geq P_t\epsilon_t^o + \frac{h\bar{\mu}}{P_t c_t^y}\Big(P_t\epsilon_t^o\Big)^2 \tag{145}$$

$$P_{t+1}\epsilon_{t+1}^o \geq P_t\epsilon_t^o\Big(1 + h\bar{\mu}\frac{P_t\epsilon_t^o}{P_t e_t}\Big). \tag{146}$$

where recall that $e_t \equiv e_t^y + \frac{e_t^o}{n} + d_t > c_t^y$.

By consequence, we have

$$\begin{aligned} &\frac{1}{P_{t+1}\epsilon_{t+1}^o} \leq \frac{1}{P_t\epsilon_t^o\Big(1 + \frac{h\bar{\mu}P_t\epsilon_t^o}{P_t e_t}\Big)} = \frac{1}{P_t\epsilon_t^o}\Bigg(1 - \frac{\frac{h\bar{\mu}P_t\epsilon_t^o}{P_t e_t}}{1 + \frac{h\bar{\mu}P_t\epsilon_t^o}{P_t e_t}}\Bigg) \\ \Rightarrow &\frac{1}{P_t\epsilon_t^o} - \frac{1}{P_{t+1}\epsilon_{t+1}^o} \geq \frac{1}{P_t\epsilon_t^o}\frac{\frac{h\bar{\mu}P_t\epsilon_t^o}{P_t e_t}}{1 + \frac{h\bar{\mu}P_t\epsilon_t^o}{P_t e_t}} = \frac{h\bar{\mu}}{P_t e_t\Big(1 + h\bar{\mu}\frac{P_t\epsilon_t^o}{P_t e_t}\Big)}. \end{aligned}$$

Since $\epsilon_t^o \leq e_t^o \leq e_t$ for any t, we get that $\frac{P_t\epsilon_t^o}{P_t e_t} \leq 1$ and

$$\frac{1}{P_t\epsilon_t^o} - \frac{1}{P_{t+1}\epsilon_{t+1}^o} \geq \frac{1}{P_t e_t}\frac{h\bar{\mu}}{1+h\bar{\mu}}.$$

Taking the sum over t, we have $\sum_{t\geq 1}\frac{1}{P_t e_t} < \infty$, a contradiction. Therefore, the equilibrium allocation is Pareto optimal.

Part 2 ("only if" part). Let conditions in part 2 be satisfied. Then there exist $\underline{x} > 0, \bar{x} \in (0,1), y > 0$ such that

$$c_t^y > \underline{x}e_t, \qquad c_t^o < \bar{x}ne_t, \qquad P_{t+1}c_{t+1}^o > yP_te_t \tag{147}$$

Suppose that $\sum_{t\geq 1}\frac{1}{P_te_t} < \infty$. Then, there exists M such that $\sum_{t=1}^T \frac{1}{P_te_t} < M$ for any T.

For $h > 0$, define the sequence $(\epsilon_t)_t$ by

$$\epsilon_t = \frac{P_1e_1\epsilon_1}{P_te_t} + \frac{h}{P_te_t}\Big(\frac{1}{P_1e_1} + \cdots + \frac{1}{P_{t-1}e_{t-1}}\Big). \tag{148}$$

Since $\sum_{t\geq 1}\frac{1}{P_te_t} < \infty$, we have $\lim_{t\to\infty}\frac{1}{P_te_t} = 0$ and $\lim_{t\to\infty}\epsilon_t = 0$. So, we can choose ϵ_1 and h small enough and $x \in (0,\bar{x})$ so that

$$c_t^y - \frac{1}{n}\epsilon_te_t > xe_t, \quad c_t^o + \epsilon_te_t < ne_t \ \forall t. \tag{149}$$

Let $\lambda \in (0,1)$. Define

$$\epsilon_t^o \equiv \lambda\epsilon_te_t, \qquad c_t^{o'} \equiv c_t^o + \epsilon_t^o \tag{150}$$

$$\epsilon_t^y \equiv -\frac{1}{n}\epsilon_t^o, \qquad c_t^{y'} \equiv c_t^y + \epsilon_t^y. \tag{151}$$

It is clear that the allocation $(c_t^{y'}, c_t^{o'})_t$ is feasible. We have

$$c_t^{y'} = c_t^y + \epsilon_t^y = c_t^y - \frac{1}{n}\lambda\epsilon_te_t > xe_t > xc_t^y \ \forall t \tag{152}$$

$$c_t^{o'} \equiv c_t^o + \lambda\epsilon_te_t < c_t^o + \epsilon_te_t < ne_t \ \forall t. \tag{153}$$

By Definition (148) of ϵ_t, we have $P_t\epsilon_t^o - P_1\epsilon_1^o = \lambda h\Big(\frac{1}{P_1e_1} + \cdots + \frac{1}{P_{t-1}e_{t-1}}\Big)$. This implies that

$$P_{t+1}\epsilon_{t+1}^o - P_t\epsilon_t^o = \frac{\lambda h}{P_te_t} \tag{154}$$

$$\frac{P_{t+1}\epsilon_{t+1}^o - P_t\epsilon_t^o}{\frac{(P_t\epsilon_t^o)^2}{P_tc_t^y}} = \lambda h\frac{P_tc_t^y}{P_te_t}\frac{1}{(P_t\epsilon_t^o)^2}. \tag{155}$$

We have $\frac{P_tc_t^y}{P_te_t} = \frac{c_t^y}{e_t} \geq \underline{x}$ and

$$P_t\epsilon_t^o = \lambda P_1e_1\epsilon_1 + \lambda h\Big(\frac{1}{P_1e_1} + \cdots + \frac{1}{P_{t-1}e_{t-1}}\Big) < \lambda(P_1e_1\epsilon_1 + hM)$$

Therefore, we get that

$$\frac{1}{2}\frac{P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t}{\frac{(P_t\epsilon^o_t)^2}{P_tc^y_t}} \geq \frac{1}{2}\lambda h\underline{x}\frac{1}{(\lambda(P_1e_1\epsilon_1+hM))^2} \equiv \frac{1}{2\lambda}\frac{h\bar{x}}{(P_1e_1\epsilon_1+hM)^2}$$
$$\Rightarrow \frac{1}{2}\Big(P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t\Big) \geq \frac{1}{2\lambda}\frac{h\bar{x}}{(P_1e_1\epsilon_1+hM)^2}\frac{(P_t\epsilon^o_t)^2}{P_tc^y_t}\ \forall t. \tag{156}$$

We also have

$$\frac{1}{2}\frac{P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t}{\frac{(P_{t+1}\epsilon^o_{t+1})^2}{P_{t+1}c^o_{t+1}}} = \frac{1}{2}\lambda h\frac{P_{t+1}c^o_{t+1}}{P_te_t}\frac{1}{(P_{t+1}\epsilon^o_{t+1})^2} \geq \frac{1}{2}\lambda hy\frac{1}{\lambda^2(P_1e_1\epsilon_1+hM)^2}$$
$$\Rightarrow \frac{1}{2}\Big(P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t\Big) \geq \frac{1}{2\lambda}\frac{hy}{(P_1e_1\epsilon_1+hM)^2}\frac{(P_{t+1}\epsilon^o_{t+1})^2}{P_{t+1}c^o_{t+1}} \tag{157}$$

Since we can choose $\lambda > 0$ arbitrarily small, we choose λ so that

$$\frac{1}{2\lambda}\frac{hy}{(P_1e_1\epsilon_1+hM)^2} > \theta_2(x), \quad \frac{1}{2\lambda}\frac{h\bar{x}}{(P_1e_1\epsilon_1+hM)^2} > \theta_1(x). \tag{158}$$

where $\theta_1(x), \theta_2(x)$ are defined in Definition 9's part 2.

From (156) and (157), we have

$$P_{t+1}(c^{o'}_{t+1}-c^o_{t+1})+nP_t(c^{y'}_t-c^y_t) \geq \frac{\theta_2(x)}{P_{t+1}c^o_{t+1}}\Big(P_{t+1}(c^{o'}_{t+1}-c^o_{t+1})\Big)^2+\frac{\theta_1(x)}{nP_tc^y_t}\Big(nP_t(c^{y'}_t-c^y_t)\Big)^2.$$

Recall that $0 < xc^y_t < c^{y'}_t < c^y_t$ and $c^o_{t+1} < c^{o'}_{t+1} < ne_{t+1}$. By the uniform smoothness condition, we have $(c^{y'}_t, c^{o'}_{t+1}) \in B_t(c)$, i.e., $U^t(c^{y'}_t, c^{o'}_{t+1}) \geq U^t(c^y_t, c^o_{t+1})$ for any t.

For $\lambda \in (0,1)$, we define the allocation $(x^y_t, x^o_t)_t$ by $x^y_t \equiv \lambda c^{y'}_t + (1-\lambda)(c^y_t > 0$ and $x^o_{t+1} \equiv \lambda c^{o'}_{t+1} + (1-\lambda)c^o_{t+1} > 0$. Of course, (x^y_t, x^o_t) is feasible. Since the function U^t is strictly concave, we have

$$U^t(x^y_t, x^o_{t+1}) = U^t\Big(\lambda c^{y'}_t + (1-\lambda)(c^y_t, \lambda c^{o'}_{t+1} + (1-\lambda)c^o_{t+1}\Big) \tag{159}$$
$$> \lambda U^t(c^{y'}_t, c^{o'}_{t+1}) + (1-\lambda)U^t(c^y_t, c^o_{t+1}) \geq U^t(c^y_t, c^o_{t+1}). \tag{160}$$

Therefore, the allocation $(c^{y'}_t, c^{o'}_t)_t$ dominates $(c^y_t, c^o_t)_t$ in the sense of Pareto, a contradiction. So, we have $\sum_{t\geq 1}\frac{1}{P_te_t} = \infty$.

□

Proof of Remark 6. Suppose that $(c^y_t, c^o_t)_t$ is not Pareto optimal. We can take the allocation $(c^{y'}_t, c^{o'}_t)_t$ as in the proof of Theorem 5.

So, the couple $(c^{y'}_t, c^{o'}_t) \in \mathbb{R}^2_{++}$ satisfies Property (C'). Therefore, we have

$$(P_{t+1}\epsilon^o_{t+1})^2 \leq \alpha P_{t+1}c^o_{t+1}\Big(P_{t+1}\epsilon^o_{t+1} - P_t\epsilon^o_t\Big) \tag{161}$$

Denote $\mu_t \equiv P_{t+1}\epsilon^o_{t+1} - P_t\epsilon^o_t > 0$. Since $\mu_t = 0\ \forall t < t_0 - 1$, and $\mu_t > 0\ \forall t \geq t_0 - 1$, we have $\mu_{t_0-1} + \cdots + \mu_t = P_{t+1}\epsilon^o_{t+1}$. This implies that

$$\epsilon^o_{t+1} = \frac{\mu_{t_0-1} + \cdots + \mu_t}{P_{t+1}}\ \forall t \geq t_0 - 1. \tag{162}$$

We have

$$\frac{(P_{t+1}\epsilon^o_{t+1})^2}{P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t}=\frac{(P_{t+1}\epsilon^o_{t+1})^2}{\mu_t}=\frac{(\mu_{t_0-1}+\cdots+\mu_t)^2}{\mu_t}.$$

This implies that, for any $t\geq t_0$,

$$\begin{aligned}\frac{1}{P_{t+1}c^o_{t+1}}&\leq\alpha\frac{P_{t+1}\epsilon^o_{t+1}-P_t\epsilon^o_t}{(P_{t+1}\epsilon^o_{t+1})^2}\\&=\alpha\frac{\mu_t}{(\mu_{t_0-1}+\cdots+\mu_t)^2}\\&<\alpha\frac{\mu_t}{(\mu_{t_0-1}+\cdots+\mu_t)(\mu_{t_0-1}+\cdots+\mu_{t-1})}\\&=\alpha\Big(\frac{1}{\mu_{t_0-1}+\cdots+\mu_{t-1}}-\frac{1}{\mu_{t_0-1}+\cdots+\mu_t}\Big).\end{aligned}$$

By taking the sum over t from t_0-1 until $T-1$ of this inequality, we have

$$\frac{1}{P_{t_0}c^o_{t_0}}+\cdots+\frac{1}{P_Tc^o_T}\leq\alpha\Big(\frac{1}{\mu_{t_0-1}}-\frac{1}{\mu_{t_0-1}+\cdots+\mu_{T-1}}\Big)<\frac{\alpha}{\mu_{t_0-1}}\tag{163}$$

Therefore, we have $\sum_{t\geq1}\frac{1}{P_tc^o_t}<\infty$. Combining with $c^o_t\leq ne_t$ for any t, we get that $\sum_{t\geq1}\frac{1}{P_te_t}<\infty$, a contradiction. As a result, the equilibrium allocation $(c^y_t,c^o_t)_t$ is Pareto optimal.

□

D Proofs for Section 6

Proof of Proposition 6. Part 1. According to Proposition 2, an equilibrium is bubbly if and only if $\lim_{t\to\infty}\dfrac{n^ta_t}{R_1\cdots R_t}>0$. Since $a_t\leq e^y_t$, we have that

$$\begin{aligned}\frac{n^ta_t}{R_1\cdots R_t}&\leq\frac{n^te^y_t}{R_1\cdots R_t}\text{ for any }t\\\Rightarrow\lim_{t\to\infty}\frac{n^ta_t}{R_1\cdots R_t}&\leq\liminf_{t\to\infty}\frac{n^te^y_t}{R_1\cdots R_t}.\end{aligned}$$

Therefore, an equilibrium is bubbleless because $\liminf\limits_{t\to\infty}\frac{n^t}{R_1\cdots R_t}e^y_t=0$.

Since $c^y_t\leq e^y_t$, we have $\liminf\limits_{t\to\infty}\frac{n^t}{R_1\cdots R_t}c^y_t\leq\liminf\limits_{t\to\infty}\frac{n^t}{R_1\cdots R_t}e^y_t=0$. So, we have $\liminf\limits_{t\to\infty}\frac{n^t}{R_1\cdots R_t}c^y_t=0$. By Lemma 6, this equilibrium is Pareto optimal.

Part 2. Consider an equilibrium. Recall that $\sum_{t=1}^\infty\frac{n^td_t}{R_1\cdots R_t}=f_0\leq a_0<\infty$. This implies that $\lim_{t\to\infty}\frac{n^td_t}{R_1\cdots R_t}=0$. Our assumption $\limsup_{t\to\infty}\frac{d_t}{e^y_t}>0$ implies that there exists a sequence $(t_k)_{k\geq1}$ and $x>0$ such that $d_{t_k}\geq xe^y_{t_k}$ for any $k\geq1$.

We have

$$\frac{n^{t_k}e^y_{t_k}}{R_1\cdots R_{t_k}}=\frac{n^{t_k}d_{t_k}}{R_1\cdots R_{t_k}}\frac{e^y_{t_k}}{d_{t_k}}\leq\frac{n^{t_k}d_{t_k}}{R_1\cdots R_{t_k}}\frac{1}{x}\tag{164}$$

Since $\lim_{t\to\infty} \frac{n^t d_t}{R_1\cdots R_t} = 0$, we have $\lim_{k\to\infty} \frac{n^{t_k} e^y_{t_k}}{R_1\cdots R_{t_k}} = 0$. It means that $\liminf_{t\to\infty} \frac{n^t}{R_1\cdots R_t} e^y_t = 0$. According to Part 1, this equilibrium is Pareto optimal and bubbleless.

□

Proof of Proposition 7. 1. Note that

$$\Big(\sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1\cdots R_t}\Big)\Big(\sum_{t\geq 1} \frac{R_1\cdots R_t}{n^t e_t}\Big) \geq \sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e_t} = \infty. \tag{165}$$

Recall that we have $\sum_{t\geq 1} \frac{\mathcal{D}_t}{R_1\cdots R_t} \leq q_0 < \infty$. So, we have $\sum_{t\geq 1} \frac{R_1\cdots R_t}{n^t e_t} = \infty$. By Theorem 5, this equilibrium is Pareto optimal.

Moreover, condition $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{e_t n^t} = \infty$ implies that $\sum_{t=1}^{\infty} \frac{\mathcal{D}_t}{n^t e^y_t} = \infty$ because $e^y_t < e_t$. By Proposition 3, this equilibrium is bubbleless.

2. Take any equilibrium. By Lemma 1, we have $R_t \geq R^*_t$ for any t. So, condition $\lim_{t\to\infty} \frac{n^t e^y_t}{R^*_1\cdots R^*_t} = 0$ implies that $\lim_{t\to\infty} \frac{n^t e^y_t}{R_1\cdots R_t} = 0$. Proposition 6's part 1 implies that this equilibrium is Pareto optimal and bubbleless.

Since there is no bubbly equilibrium, Lemma 5 implies that there is a unique equilibrium. □

Proof of Proposition 8. Part 1. $\limsup_{t\to\infty} \frac{a_t}{e_t} > 0$ implies that there exist $x > 0$ and an infinite and increasing sequence of time $(t_k)_{k\geq 1}$ such that $\frac{a_{t_k}}{e_{t_k}} > x$ for any t.

Recall that $\infty > a_0 > \frac{n^t}{R_1\cdots R_t} a_t$. Hence, we have

$$\frac{R_1\cdots R_t}{n^t e_t} = \frac{a_t}{e_t}\frac{R_1\cdots R_t}{n^t a_t} > \frac{1}{a_0}\frac{a_t}{e_t}\ \forall t. \tag{166}$$

For any t_k, we have

$$\frac{R_1\cdots R_{t_k}}{n^{t_k} e_{t_k}} = \frac{1}{a_0}\frac{a_{t_k}}{e_{t_k}} > \frac{1}{a_0} x. \tag{167}$$

By consequence, $\sum_k \frac{R_1\cdots R_{t_k}}{n^{t_k} e_{t_k}} = \infty$. Therefore, $\sum_{t\geq 1} \frac{R_1\cdots R_t}{n^t e_t} = \infty$. Applying Theorem 5's part 1, this equilibrium is Pareto optimal.

Part 2. Consider an equilibrium. The no-bubble condition means that $\lim_{t\to\infty} \frac{n^t}{R_1\cdots R_t} a_t = 0$.

Since $\liminf_{t\to\infty} \frac{a_t}{e^y_t} > 0$, there exists $x > 0$ and t_0 such that $\frac{a_t}{e^y_t} > x$ for any $t \geq t_0$. Then, for any $t \geq t_0$, we have

$$\frac{n^t e^y_t}{R_1\cdots R_t} = \frac{n^t a_t}{R_1\cdots R_t}\frac{e^y_t}{a_t} < \frac{n^t a_t}{R_1\cdots R_t}\frac{1}{x}$$
$$\liminf_{t\to\infty} \frac{n^t e^y_t}{R_1\cdots R_t} \leq \lim_{t\to\infty} \frac{n^t a_t}{R_1\cdots R_t}\frac{1}{x}$$

Combining with $\lim_{t\to\infty} \frac{n^t}{R_1\cdots R_t} a_t = 0$, we get that $\lim_{t\to\infty} \frac{n^t}{R_1\cdots R_t} e^y_t = 0$. By Proposition 6, this equilibrium is Pareto optimal. □

Proof of Proposition 9. Proposition 9 is a direct consequence of the following result and Lemma 5.

Lemma 21 (Ranking welfares when there exists a continuum of equilibria)**.** *Let Assumptions 1, 4, 5 be satisfied.*

For two equilibria with initial asset values $a_0 > a_0'$, we denote U_t and U_t' the utility of households born at date t in the equilibrium a_0 and a_0' respectively.

Then we have $U_t > U_t'$ for any date $t \geq 0$. It means that the utility of each generation is increasing in the initial value of asset.

Let us prove this lemma. The utility of households born at date t is

$$U_t = u(e_t^y - a_t) + \beta v(e_{t+1}^o + R_{t+1}a_t). \tag{168}$$

Taking the derivative with respect to a_t, we have

$$\frac{\partial U_t}{\partial a_t} = -u'(e_t^y - a_t) + \beta R_{t+1}v'(e_{t+1}^o + R_{t+1}a_t) + \beta a_t \frac{\partial R_{t+1}}{\partial a_t} v'(e_{t+1}^o + R_{t+1}a_t) \tag{169}$$

$$= \beta a_t \frac{\partial R_{t+1}}{\partial a_t} v'(e_{t+1}^o + R_{t+1}a_t) > 0 \tag{170}$$

because, by Lemma 4, $\frac{\partial R_{t+1}}{\partial a_t} > 0$.

Since a_t is strictly increasing in a_0, the utility U_t is also strictly increasing in a_0. □

Proof of Theorem 6. 1. Point 1. According to Theorem 4, there exists a unique equilibrium and this equilibrium is bubbleless. By Lemma 1, we have, in equilibrium, $R_t \geq R_t^*$ for any t. So, $\liminf_{t\to\infty} R_t > n$. Therefore, Corollary 5 implies that this equilibrium is Pareto optimal.

2. Point (2a) is a consequence of Theorem 4 and Proposition 9.[24]

   Let us prove point (2b) by using Theorems 4, 5 and Lemma 7's part 1. First, by Theorem 4, this equilibrium satisfies (a_t, b_t, R_t) converges to $(\hat{a}, \hat{a}, n)$ where $\hat{a} > 0$ is uniquely determined by $u'(e^y - \hat{a}) = \beta n v'(e^o + n\hat{a})$ (i.e., $n = g(\hat{a})$).

   Since $b_0 = \lim_{t\to\infty} \frac{n^t}{R_1 \cdot R_t} a_t = \lim_{t\to\infty} \frac{n^t}{R_1 \cdot R_t}\hat{a} > 0$, we have $\lim_{t\to\infty} \frac{n^t}{R_1 \cdot R_t} = \frac{b_0}{\hat{a}} \in (0, \infty)$. Then, combining with $\lim_{t\to\infty} d_t = 0$, we have

$$\lim_{t\to\infty} \frac{R_1 \cdots R_t}{n^t(e^y + \frac{e_o}{n} + d_t)} = \frac{R_1 \cdots R_t}{n^t(e^y + \frac{e_o}{n})} = \frac{\hat{a}}{b_0(e^y + \frac{e_o}{n})} > 0.$$

   By consequence, we have $\sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t(e^y + \frac{e_o}{n} + d_t)} = \infty$.

[24] We can also use Theorem 5 to prove that any equilibrium $a_0 < \bar{a}$ is not Pareto optimal. Indeed, by Theorem 4, for this equilibrium, we have $\lim_{t\to\infty} R_t = R^* < n$ which implies that $\sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t} < \infty$. Note that $e_t = e^y + \frac{e^o}{n} + d_t$. Therefore, we have

$$\sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t e_t} \sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t \left(e^y + \frac{e_o}{n} + d_t\right)} \leq \frac{1}{e^y + \frac{e_o}{n}} \sum_{t\geq 1} \frac{R_1 \cdots R_t}{n^t} < \infty.$$

Since $c_t^y \to e^y > 0, c_t^o \to e^o > 0, d_t \to 0, R_t \to R^* > 0$, we can check that all conditions in Theorem 5's part 2 are satisfied (here, of course, we apply Lemma 8's point 1 to verify that this equilibrium allocation (c_t^y, c_t^o) satisfies the uniform smoothness condition). Therefore, this equilibrium is not Pareto optimal.

To conclude that this equilibrium is Pareto optimal, it suffices to verify the uniform strictness condition. We do so by using Lemma 7's part 1. Indeed, recall that $c_t^y > 0$ for any t and $\lim_{t\to\infty} c_t^y = e^y - \hat{a} > 0$. Since the function u is in C^2 and $u''(e^y - \hat{a}) \in (-\infty, 0)$, there exists $h > 0$ such that

$$\inf_{t\geq 0}\left\{\frac{c_t^y}{u'(c_t^y)}\inf_{x\in[(1-h)c_t^y, c_t^y]}\Big(-\frac{1}{2}u''(x)\Big)\right\} > 0.$$

So, by Lemma 7's part 1, the equilibrium allocation satisfies the uniform strictness condition. We have finished our proof.

3. By Claim 2 in Theorem 4, there exists a unique equilibrium. This is asymptotically bubbly and (a_t, b_t, R_t) converges to $(\hat{a}, \hat{a}, n)$. By using the same argument as above, this equilibrium is Pareto optimal.

□

Proof of Proposition 10. First, since $e_t^o = 0\ \forall t$, the benchmark interest rate equals zero, i.e., $R_t^* = 0$ for any t.

It is easy to see that there exists a unique equilibrium determined by $a_t = \frac{\beta}{1+\beta}e_t^y$. The interest rate sequence (R_t) is determined by

$$R_{t+1} = n\frac{a_{t+1} + d_{t+1}}{a_t} = n\frac{\frac{\beta}{1+\beta}e_{t+1}^y + d_{t+1}}{\frac{\beta}{1+\beta}e_t^y}. \tag{171}$$

According to Definition 4, $R_{t+1} \equiv \frac{1}{\beta}\frac{c_{t+1}^o}{c_t^y}$. Then, we can find the consumption allocation (c_t^y, c_t^o) by

$$c_t^y = \frac{1}{\beta}e_t^y, \quad c_{t+1}^o = n(\frac{\beta}{1+\beta}e_{t+1}^y + d_{t+1}). \tag{172}$$

Applying Lemma 7's point 2, the equilibrium allocation given by (172) satisfies the uniform strictness condition. According to Proposition 8's part 1, this equilibrium is Pareto optimal

Lemma 2's point 5, the equilibrium is bubbly if and only if $\sum_{t\geq 1}\frac{d_t}{a_t} < \infty$, which is equivalent to $\sum_{t\geq 1}\frac{d_t}{e_t^y} < \infty$. When it is bubbly, we have, by using Lemma 2's point 4, $\lim_{t\to\infty}\frac{b_t}{e_t^y} = \lim_{t\to\infty}\frac{b_t}{a_t}\frac{a_t}{e_t^y} = \gamma$. It means that the equilibrium is asymptotically bubbly.

□

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