Reduction analysis of hierarchical spatial economy: Trade strategy around Brexit

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Reduction Analysis of Hierarchical Spatial Economy: Trade Strategy around Brexit

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Abstract

This paper investigates how international trade competition influences cross-country migration by using a general equilibrium model of economic geography. We employ a global–local system to represent local places grouped into countries, which collectively form a global network. Through the place-to-country reduction analysis proposed herein, the governing equation at the place level are reduced to a country-level equation that efficiently describes each country's trade environment. We model and analyze international trade competition—including trade liberalization and protectionism—among the UK, France, and Germany, using the Helpman (1998) model. The recommended strategies for the UK and the EU include reducing domestic transportation costs, while tariffs and retaliatory tariffs act as a double-edged sword, potentially enhancing or undermining their trade positions.

Keywords: Brexit, economic geography model, global–local system, hierarchical spatial economy, reduction analysis, tariffs, trade liberalization, trade strategy.

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1 Introduction

Average tariff rates in France, the UK, and the US have declined over time (1790–2019; Irwin, 2020). Globalization has had mixed impacts worldwide. Brexit in 2020 cast a shadow over the EU's success in globalization and led to a decline in net migration from EU countries to the UK (Di Iasio and Wahba, 2023). Today, globalization is being undermined by the potential resurgence of tariffs, particularly in the US. There is an urgent need to evaluate the effects of these tariffs and identify the winners and losers in international trade competition.

This paper proposes a systematic method for investigating trade competition using a general equilibrium model of economic geography. Tariffs are modeled as international transportation costs, whereas reductions in domestic transportation costs represent improvements in each country's infrastructure. Given the progress of global trade liberalization—particularly within the EU—we analyze trade competition among countries with internationally mobile workers. A country is considered a winner (loser) if it gains (loses) population as a result of changes in tariff policies.

As a spatial platform, this paper employs a global–local system,[1] which encompasses various spatial structures with a two-level hierarchy. Figure 1(a) depicts a continuous network type, where distributed local places are grouped into several countries. For example, European countries belong to this type. The discrete network type, shown in Fig. 1(b), can represent the US highway system, high-speed railway networks, and air transportation networks. Figure 2 depicts a global–local model of the UK, France, and Germany. This model combines both continuous and discrete types: France and Germany are connected continuously, whereas the UK is linked to them through a discrete network.

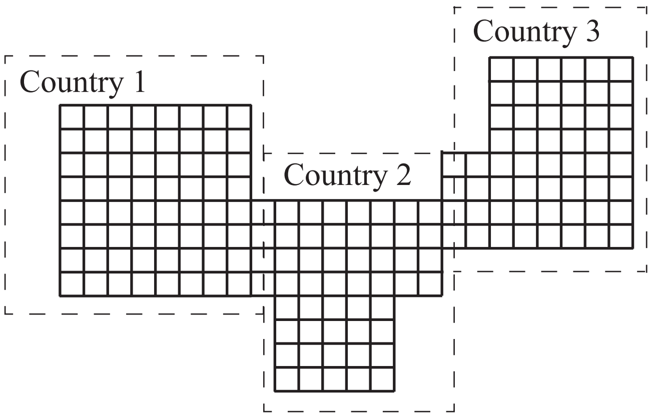
Many economic geography models employ the dynamics $\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ along with the corresponding static governing equation:[2]

$$\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \mathbf{0}, \tag{1}$$

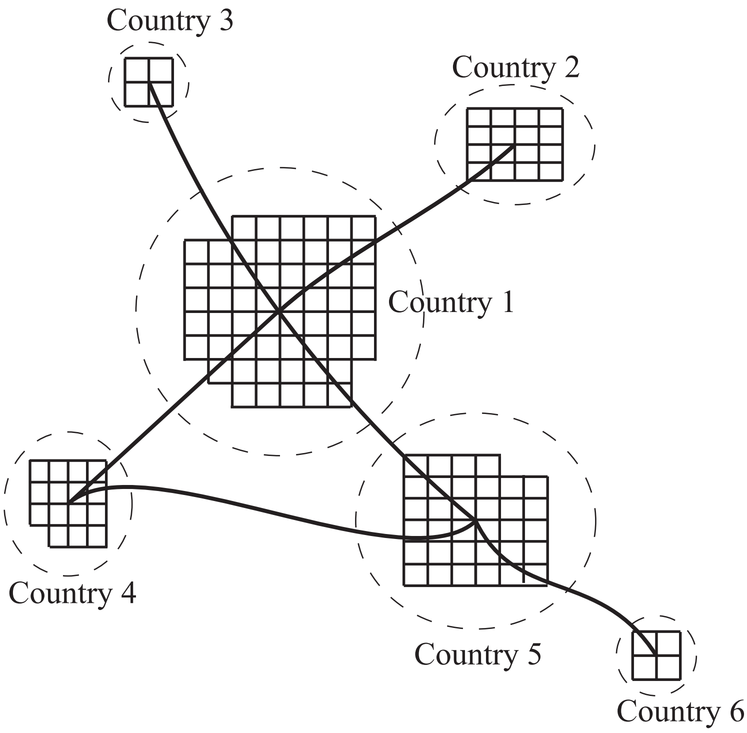
where $\boldsymbol{\lambda} \in \mathbb{R}^n$ represents a population vector; $\boldsymbol{\tau} \in \mathbb{R}^p$ denotes an economic parameter vector; and $\boldsymbol{F} \in \mathbb{R}^n$ is a nonlinear function. For example, the *replicator and logit dynamics* are used to investigate stable equilibria of an economic system. Direct analysis of Eq. (1) in large-scale settings generates vast amounts of data and requires extensive processing to yield economic insights.

[1]See Kogure and Ikeda (2022) and Ikeda and Takayama (2024) for a global–local system.

[2]For example, this form is identical with "N equations in N populations in each location" in Eq. 16 of Redding and Rossi-Hansberg (2017).



(a) Continuous network type



(b) Discrete network type

Figure 1: Global–local system comprising countries with local places

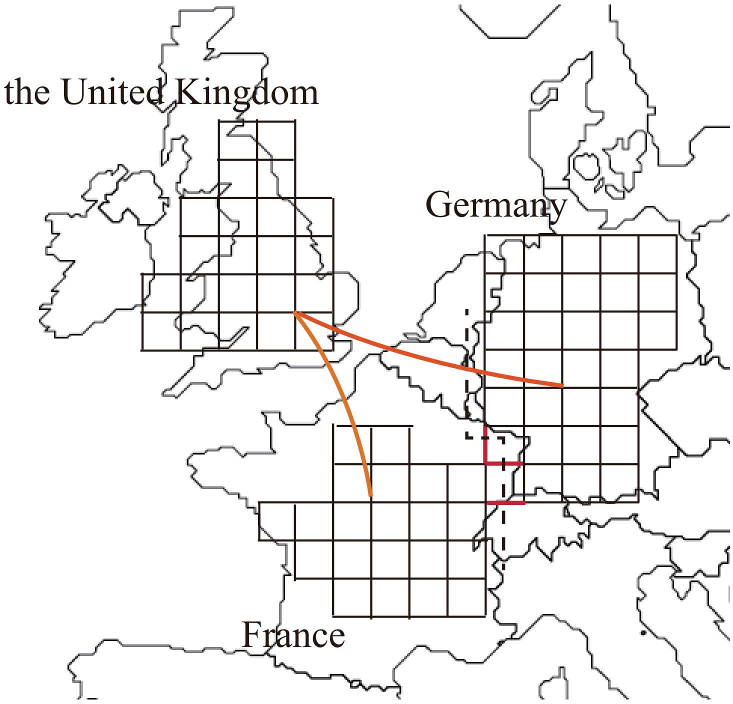


Figure 2: Spatial model of three countries: the UK, France, and Germany

We introduce a place-to-country reduction method[3] as a systematic approach to analyzing large-scale global–local systems for an economic geography model. Using this method, we transform the original place-level governing equation in (1) into a country-level equation. For example, in the spatial platform of three countries in Fig. 2, the original place-level governing equation, with 109 degrees of freedom, is reduced to a country-level equation with three degrees of freedom corresponding to the UK, France, and Germany. This equation is compatible with the analysis of the international trade competition. Moreover, we formulate the *population gradient matrix*, which captures the effects of economic parameters on a population of each country.

We apply the proposed theoretical framework to international trade competition around Brexit. This paper does not aim to provide an extensive review of the Brexit literature (see Section 2) but aims instead to analyze trade competition applying a general equilibrium approach to the global–local model[4] in Fig. 2. We employ an economic geography model à la Helpman[5] with the replicator dynamics. For the

[3]It is called Lyapunov-Schmidt reduction in nonlinear mathematics and is used on a different purpose: the bifurcation analysis of a symmetric system (e.g., Ikeda and Murota, 2019).

[4]The importance of the lattice analysis of the combination of France and Germany was suggested by Prof. J.-F. Thisse during the preparation of Takayama et al. (2020).

[5]This model is based on the multi-region version of the new economic geography model of Helpman (1998) (Redding and Strum, 2008). This multi-region version is similar to several quantitative spatial models (e.g., Allen and Arkolakis, 2014; Becker et al., 2021).

countries' local networks, we use irregularly shaped two-dimensional square lattices, where discrete places are located at grid points and goods are transported along the lattice.[6] We analyze how changes in tariffs affect population distributions across countries to elucidate how the three countries should implement trade policy to attract mobile workers. We examine the effects of changes in national and international trade freeness on the populations of the three countries. The population gradient matrix is used to systematically grasp these effects.

We examine the trade competition under several scenarios: (i) In the scenario of **pre-Brexit EU single market**, the three countries jointly change their national and international trade environment. Trade liberalization reduces the price index in all three countries. Ironically, the liberalization benefits the UK the most, even though it was the country that initiated economic disintegration through Brexit. The development of infrastructure at the same pace benefits France and Germany. (ii) In the scenario of the **UK's post-Brexit trade strategy**, a recommended strategy involves enhancing both national and international trade freeness, thereby reducing its domestic price index. After a significant reduction in the national trade freeness, the UK's trade position undergoes a phase shift, making trade liberalization unfavorable. (iii) In the scenario of **post-Brexit trade strategy of the EU** (France and Germany), infrastructure promotion always favors the EU, but changing the import tariff level is like a double-edged sword, as the EU may gain or lose population, depending on the UK's choice of tariff rates.

In summary, infrastructure development benefits the respective country in all three scenarios. In contrast, the effects of trade liberalization and protectionism depend on countries and scenarios.

The remainder of the paper is organized as follows. Section 2 reviews related studies. Section 3 presents the economic model. A place-to-country reduction for a global–local system is presented in Section 4. Section 5 introduces the further reduction to a simplex and presents an inverse analysis. The population gradient matrix is proposed in Section 6. The spatial and economic model of the UK, France, and Germany is presented, and their cooperative economic integration is analyzed in Section 7. The trade strategy is studied in Section 8 for the UK and in Section 9 for the EU. Section 10 concludes.

[6]Square road networks do exist worldwide. Chicago and Kyoto, for example, are well known for having such square networks.

2 Related Studies

Population distribution in hierarchical systems comprising countries and regions has been extensively studied in economic geography. The impact of reductions in tariffs or transportation costs between countries on within-country population distribution has been analyzed by Krugman and Elizondo (1996) and by Crozet and Soubeyran (2004). Behrens et al. (2006, 2007) show that both national transportation costs and international trade costs influence domestic population distribution in a two-country, four-region model. Gallego and Zofio (2018) examine how changes in transportation costs affect population distribution within countries.

Cross-country mobility of population and capital has also been studied: Zeng and Zhao (2010) qualitatively explore how reductions in trade costs affect the spatial distribution of capital across countries, using a two-country, four-region model. Persyn et al. (2023) quantitatively examine how reduced transportation costs influence population distribution in the EU and the UK, using a spatial dynamic general equilibrium model across 267 European NUTS-2 regions.

Several studies have examined the effects of uncertainty shocks triggered by the 2016 Brexit referendum on labor markets, investment, and trade, as reviewed by Dhingra and Sampson (2022). Brakman et al. (2023) note: "As part of its 'Global Britain' strategy and as a reaction to its Brexit decision, the UK government is pursuing a series of Free Trade Agreements with countries around the world, ..." De Lucio et al. (2024) show that the UK's exit reduced both Spanish exports to and imports from the UK. Freeman et al. (2025) find that both UK imports and exports declined following its exit from the EU single market. Among studies focusing on economic disintegration, such as Brexit, several have examined its effects on the spatial distribution of workers and firms across countries. Commendatore et al. (2021) and Saraiva and Gaspar (2023) explore how economic disintegration affects the spatial distribution of workers using a three-country model. Janeba and Schulz (2024) investigate how economic disintegration influences firm relocation and national tax policies, using a general equilibrium trade model.

An extensive body of research has emerged in quantitative spatial economics (QSE), as reviewed by Redding and Rossi-Hansberg (2017) and Allen and Arkolakis (2025). Key contributions include the following: Eaton and Kortum (2002) model international trade. Allen and Arkolakis (2014) estimate the topography of trade costs, productivity, and amenities in the US using an irregular lattice. Ahlfeldt et al. (2015) develop a model of internal city structure and apply it to data from thousands of city blocks in Berlin. Behrens et al. (2017) introduce a multi-city general equilibrium model to analyze the influence of spatial frictions. Desmet et al. (2018) propose a dynamic theory of spatial growth incorporating realistic geography. Behrens and Murata (2021) show that spatial equilibrium conditions in QSE can be derived from the McFadden model (1974) through comparison with the Helpman model (1998).

Recent studies have employed various spatial models of economic activity. Redding and Rossi-Hansberg (2017) use a 30×30 latitude–longitude grid divided into two countries. Allen and Arkolakis (2022) use an irregular mesh to simulate the US highway network and a regular mesh to model Seattle's road network. Fajgelbaum and Schaal (2020) use a 15×15 square grid and discretized road network models of France, Spain, and Western Europe. Ikeda and Murota (2014) use a hexagonal lattice to demonstrate the self-organization of hexagonal patterns.

3 Economic Modeling

We present a general equilibrium model of economic geography.

3.1 Dynamics and Governing Equation

We consider the dynamic equation $\dot{\boldsymbol{\lambda}} = \boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ along with its corresponding static governing equation:

$$\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \mathbf{0}, \tag{2}$$

where $\boldsymbol{\lambda} = (\lambda_i) \in \mathbb{R}^n$ denotes the vector of independent variables; $\boldsymbol{\tau} = (\tau_k) \in \mathbb{R}^p$ denotes the vector of economic parameters; and $\boldsymbol{F} = (F_i) \in \mathbb{R}^n$ is a sufficiently smooth nonlinear function. The specific forms of $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$, and $\boldsymbol{F}$ depend on the chosen economic model. In economic geography models, a typical example of τ_k is a transportation cost parameter, while λ_i denotes the population of mobile workers at place $i \in \{1, \ldots, n\}$. Hereafter, $(\cdot)_i$ refers to a variable at place i and $(\cdot)_k$ to the kth economic parameter.

To analyze how international trade competition influences migration between countries (cf. Sections 7–9), we employ the *replicator dynamics* (Taylor and Jonker, 1978; Sandholm, 2010):

$$F_i(\boldsymbol{\lambda}, \boldsymbol{\tau}) = (v_i(\boldsymbol{\lambda}, \boldsymbol{\tau}) - \bar{v}(\boldsymbol{\lambda}, \boldsymbol{\tau}))\lambda_i. \tag{3}$$

Each worker in place i is assigned an indirect utility v_i, defined by the specific economic model. The term $\bar{v} = (\sum_{i=1}^n \lambda_i v_i)/\sum_{i=1}^n \lambda_i$ denotes the weighted average utility. Mobile workers migrate across places in pursuit of higher utilities. We have the relation:

$$\sum_{i=1}^n F_i(\boldsymbol{\lambda}, \boldsymbol{\tau}) = 0. \tag{4}$$

By this relation, the n-dimensional governing equation can be reduced to a system defined on an $(n-1)$-dimensional simplex (cf. Section 5).

The governing equations of certain dynamics, such as the replicator and logit dynamics, satisfy the population conservation law:

$$\sum_{i=1}^n \lambda_i = 1. \tag{5}$$

3.2 Transportation Costs

We adopt the iceberg transportation cost: when one unit of goods is shipped from place i to place j, only $1/\tau_{ij}$ arrives. For a global–local system comprising m countries, indexed by $\alpha \in \{1, \ldots, m\}$, we consider two types of transportation cost parameters:

- τ_α: the national transportation cost within country α
- $\tau_{\alpha\to\beta}$: the trade cost of exporting from country α to country β

If places i and j are located in the same country α, there is no trade cost, and the transportation cost is $\tau_{ij} = \exp(L_{ij}\tau_\alpha)$, where L_{ij} denotes the road distance between i and j. When places i and j are located in different countries α and β, and place i^* in country α is directly connected to place j^* in country β, the total transportation and trade cost between places i and j is given by

$$\tau_{ij} = \exp(L_{i\,i^*}\tau_\alpha + L_{j\,j^*}\tau_\beta + \tau_{\alpha\to\beta}).$$

We use the *trade freeness* parameter to capture changes in transportation costs. In general economic geography models, this parameter is defined as

$$\phi_k = \exp[-(\sigma - 1)\tau_k] \qquad (k = \alpha, \alpha \to \beta;\ 0 < \phi_k < 1),$$

which is inversely related to the transportation cost. Here, σ is an economic parameter representing the constant elasticity of substitution (cf. Section 3.3).

3.3 The Helpman Model

We employ the Helpman model (1998) to analyze trade competition (cf. Sections 7–9). This model is outlined below; details are provided in Appendix A.

This model assumes homogeneous workers. The utility function of a worker in place i is given by

$$u_i = (Q_i/\mu)^\mu (h_i/(1-\mu))^{1-\mu} \qquad (0 < \mu < 1),$$

where Q_i denotes the consumption index over differentiated traded goods, h_i is the consumption of housing services, and μ is the expenditure share allocated to the consumption of differentiated goods. The consumption index Q_i is defined using a constant elasticity of substitution (CES) function:

$$Q_i = \left(\sum_{j=1}^{n} \int_0^{m_j} q_{ji}(\varphi)^{(\sigma-1)/\sigma} \, \mathrm{d}\varphi \right)^{\sigma/(\sigma-1)} \quad (\sigma > 1).$$

Here, m_j is the mass of varieties in place j; σ is the constant elasticity of substitution; and $q_{ji}(\varphi)$ is the consumption of the φth differentiated good produced in place j and consumed in place i.

The budget constraint is given by

$$\left(\sum_{j=1}^{n} \int_0^{m_j} p_{ji}(\varphi) q_{ji}(\varphi) \, \mathrm{d}\varphi \right) + r_i h_i = Y_i.$$

Here, $p_{ji}(\varphi)$ is the price of the φth differentiated good; r_i is the housing price; and Y_i is a worker's income. Each worker's income consists of wage earnings and income from housing services. Housing stock is identical across places and equally owned by all workers. Solving the utility maximization problem, we obtain $q_{ji}(\varphi)$, h_i, and indirect utility v_i:

$$q_{ji}(\varphi) = \frac{\mu Y_i}{p_{ji}(\varphi)} \left(\frac{p_{ji}(\varphi)}{P_i} \right)^{1-\sigma}, \quad h_i = \frac{(1-\mu) Y_i}{r_i}, \quad v_i = \frac{Y_i}{P_i^{\mu} r_i^{1-\mu}}. \tag{6}$$

Here,

$$P_i = \left(\sum_{j=1}^{n} \int_0^{m_j} p_{ji}(\varphi)^{1-\sigma} \, \mathrm{d}\varphi \right)^{1/(1-\sigma)} \tag{7}$$

is the price index that depends on the prices of the differentiated goods consumed by workers in place i.

Each place hosts a continuum of firms producing differentiated goods under monopolistic competition. Each firm produces a single type of differentiated good using only labor, which is supplied inelastically at the aggregate level λ_i. The labor market is perfectly competitive, and all firms treat wages as given.

4 Place-to-Country Reduction

We propose a place-to-country reduction for a global–local system in which local places are grouped into countries (cf. Fig. 1). The original n-dimensional place-level governing equation is reduced to an m-dimensional country-level equation, with $m < n$. This reduction decreases the number of independent variables from n to m and allows for a systematic formulation of the influence of economic parameters. The country-level equation is particularly useful for analyzing international trade (cf. Sections 7–9). Although we employ the Helpman model (1998) in this analysis, the proposed theoretical framework (cf. Sections 4–6) applies to a broader class of economic models.

4.1 Country-Level Variables in Global–Local System

To prepare for the place-to-country reduction in Sections 4.2–4.3, we introduce country-level variables. We consider a country α with n_α local places ($\alpha = 1, \dots, m$).

The n-dimensional place-level population vector and governing equation vector in the original equation (2) are constructed by assembling country-level vectors as

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_m \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} \boldsymbol{F}_1 \\ \vdots \\ \boldsymbol{F}_m \end{pmatrix}.$$

The country-level vectors for country α are defined as

$$\boldsymbol{\lambda}_\alpha = (\lambda_\alpha^1, \dots, \lambda_\alpha^{n_\alpha})^\top, \quad \boldsymbol{F}_\alpha = (F_\alpha^1, \dots, F_\alpha^{n_\alpha})^\top \qquad (\alpha = 1, \dots m).$$

We define the population share a_α of country α and the sum A_α of the components of $\boldsymbol{F}_\alpha$ as follows:

$$a_\alpha = \sum_{j=1}^{n_\alpha} \lambda_\alpha^j, \qquad A_\alpha = \sum_{j=1}^{n_\alpha} F_\alpha^j, \tag{8}$$

and assemble a_α and A_α, respectively, to define country-level vectors:

$$\boldsymbol{a} = (a_\alpha \mid \alpha = 1, \dots, m), \qquad \boldsymbol{A} = (A_\alpha \mid \alpha = 1, \dots, m).$$

A component A_α of the country-level governing equation has economic significance, as explained in Remark 1.

Remark 1. For the replicator dynamics, the original governing equation (2) satisfies the conservation law of population: $\sum_{i=1}^{n} \lambda_i = 1$ in (5). The reduced equation inherits this conservation law as $\sum_{\alpha=1}^{m} a_\alpha = 1$. The component A_α of the country-wise equation is evaluated to

$$\begin{aligned} A_\alpha &= \sum_{j=1}^{n_\alpha} (v_\alpha^j - \bar{v})\lambda_\alpha^j = \sum_{j=1}^{n_\alpha} v_\alpha^j \lambda_\alpha^j - \bar{v} \sum_{j=1}^{n_\alpha} \lambda_\alpha^j = (\bar{v}_\alpha - \bar{v})\, a_\alpha \\ &= \{(\text{weighted average utility of country } \alpha) \\ &\quad - (\text{weighted average utility of the whole world})\} \times (\text{country } \alpha\text{'s population}), \end{aligned}$$

where $\bar{v}_\alpha = (\sum_{j=1}^{n_\alpha} v_\alpha^j \lambda_\alpha^j)/a_\alpha$. Thus, the reduced equation inherits the form of replicator dynamics; a country's population increases when its average utility exceeds the global average and decreases when it falls short. This inheritance establishes a correspondence between place-level original properties and country-level reduced ones:

Original place-level equation F_i		Reduced country-level equation A_α
Place's population λ_α^j	$\Rightarrow$	Country's population a_α
Place's utility v_α^j		Country's utility $\bar{v}_\alpha$
Place's replicator dynamics $(v_\alpha^j - \bar{v})\lambda_\alpha^j$		Country's counterpart $(\bar{v}_\alpha - \bar{v})\, a_\alpha$

□

4.2 Simple Example of Reduced Equation

A global–local system with two identical countries serves as a hierarchical analogue of the standard two-location economy, widely used in economic geography (e.g., Fujita et al., 1999). A simple example of this system is depicted in Fig. 3; each country contains $n_1 = n_2 = n_*$ local places.

We define country-level variables as $a_\alpha = \sum_{j=1}^{n_*} \lambda_\alpha^j$ and $A_\alpha = \sum_{j=1}^{n_*} F_\alpha^j$ $(\alpha = 1, 2)$ by (8). Then, using the general procedure presented in Section 4.3, we can derive the reduced governing equation for a single economic parameter $\boldsymbol{\tau} = \tau$ as

$$\begin{pmatrix} \mathrm{d}A_1 \\ \mathrm{d}A_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial A_1}{\partial \tau} \\ \frac{\partial A_2}{\partial \tau} \end{pmatrix} \mathrm{d}\tau + \text{h.o.t.} = \mathbf{0}. \tag{9}$$

Here, $J_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are components of the reduced Jacobian matrix, and "h.o.t." denotes higher-order terms. This equation also applies to two distinct countries and is useful for analyzing trade competition between them.

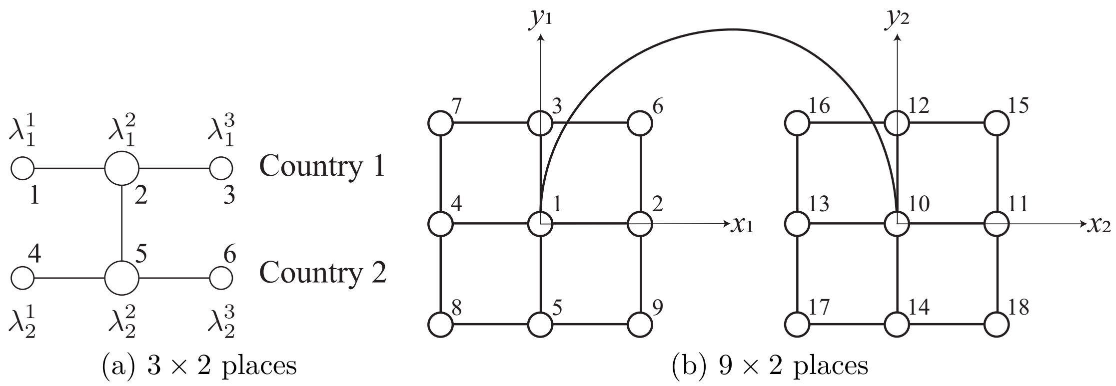


Figure 3: Global–local system comprising two identical countries

When $J_{11}J_{22} - J_{12}J_{21} \neq 0$, the reduced equation (9) can be solved as

$$\begin{pmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \end{pmatrix} = T\mathrm{d}\tau + \text{h.o.t.}$$

with the population gradient matrix:

$$T = \begin{pmatrix} \frac{\partial a_1}{\partial \tau} \\ \frac{\partial a_2}{\partial \tau} \end{pmatrix} = -\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial A_1}{\partial \tau} \\ \frac{\partial A_2}{\partial \tau} \end{pmatrix}.$$

The signs of the components $\frac{\partial a_\alpha}{\partial \tau}$ ($\alpha = 1, 2$) indicate whether the populations of the two countries increase or decrease in response to a change $\mathrm{d}\tau$ in the economic parameter. This matrix is helpful in the analysis of the international trade competition in Sections 7–9.

If the two countries are in an identical state, we have $J_{11} = J_{22}$, $J_{12} = J_{21}$, and $\frac{\partial A_1}{\partial \tau} = \frac{\partial A_2}{\partial \tau}$. When $J_{11} \neq J_{12}$, the reduced equation (9) has a unique solution of the identical state. When $J_{11} = J_{12}$, the reduced equation undergoes a break bifurcation.

4.3 Reduction to Country-level Equation

We perform a coordinate transformation from the original n-dimensional place-level vectors $\boldsymbol{\lambda}$ and $\boldsymbol{F}$ to the m-dimensional country-level vectors $\boldsymbol{a}$ and $\boldsymbol{A}$, as follows:

$$\boldsymbol{\lambda} = H \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}, \quad \tilde{H}^\top \boldsymbol{F} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix}. \tag{10}$$

Here, $\boldsymbol{b}$ and B are auxiliary vectors of dimension $(n-m)$, and H and $\tilde{H}$ are transformation matrices given in Appendix B.1.[7]

An incremental equation in terms of $\mathrm{d}\boldsymbol{a}$ in Proposition 1 can be derived by the standard reduction analysis method via the elimination of $\mathrm{d}\boldsymbol{b}$ from the incremental governing equation (cf. Ikeda and Murota, 2019, p.51). While the details of this reduction is worked out in Appendix B.2, notations are prepared in Lemma 1.

Lemma 1. *The partial derivatives $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ and $G = \partial \boldsymbol{F}/\partial \boldsymbol{\tau}$ of the original governing equation $\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ in (2) are transformed as*

$$\tilde{H}^\top J H = \begin{pmatrix} J_a & J_{ab} \\ J_{ba} & J_b \end{pmatrix}, \quad \tilde{H}^\top G = \begin{pmatrix} G_a \\ G_b \end{pmatrix}. \tag{11}$$

Then, a reduced Jacobian matrix and a reduced economic parameter influence matrix are defined respectively as follows: $\hat{J} = J_a - J_{ab}J_b^{-1}J_{ba}$ and $\hat{G} = G_a - J_{ab}J_b^{-1}G_b$.

Proposition 1. *If the matrix J_b is nonsingular, the reduced governing equation in terms of $\mathrm{d}\boldsymbol{a}$ is obtained as*

$$\mathrm{d}\boldsymbol{A} = \hat{J}\mathrm{d}\boldsymbol{a} + \hat{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \boldsymbol{0}. \tag{12}$$

Proof. See Appendix B.2 for the proof. □

If the matrix $\hat{J}$ is nonsingular, the reduced equation (12) can be solved for $\mathrm{d}\boldsymbol{a}$, yielding the *population equation*:

$$\mathrm{d}\boldsymbol{a} = T\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}$$

with the *population gradient matrix*:

$$T = -\hat{J}^{-1}\hat{G}. \tag{13}$$

This matrix captures changes in the population distribution across countries and plays a central role in analyzing the effects of international trade freeness, as discussed in Sections 7–9.

[7]Since $\tilde{H}$ is invertible, $\boldsymbol{F} = \boldsymbol{0}$ is equivalent to both $\boldsymbol{A} = (\underbrace{0, \dots, 0}_{m \text{ times}})^\top$ and $\boldsymbol{B} = (\underbrace{0, \dots, 0}_{n-m \text{ times}})^\top$.

Original Place-Level Governing Equation:
n equations and population variables

$\Downarrow$ Place-to-country reduction (Section 4.3)

Reduced Country-Level Governing Equation:
m equations and population variables

$\downarrow$ Reduction to $(m-1)$-dimensional simplex (Section 5.2)

Further Reduced Country-Level Governing Equation:
$(m-1)$ equations and population variables

Figure 4: Flowchart of the recursive reduction process

5 Further Reduction and Inverse Analysis

Section 4 introduced the place-to-country reduction from n local places to m countries. Certain dynamics, such as replicator and logit dynamics, enables a further reduction to an $(m-1)$-dimensional simplex. Figure 4 illustrates the reduction process, which is well suited for large-scale matrix analysis.

Lemma 2. *For dynamics that satisfy* $\sum_{i=1}^{n} F_i(\boldsymbol{\lambda}, \boldsymbol{\tau}) = 0$ *in* (4) *and* $\sum_{i=1}^{n} \lambda_i = 1$ *in* (5)*, the following country-level identities hold:*

$$\sum_{\alpha=1}^{m} \mathrm{d}a_\alpha = 0, \quad \sum_{\alpha=1}^{m} \mathrm{d}A_\alpha = 0, \quad \sum_{\alpha=1}^{m} \frac{\partial A_\alpha}{\partial \tau_k} = 0 \quad (k = 1, \dots, p). \tag{14}$$

Proof. See Appendix B.3 for the proof. □

5.1 Simple Example: Two Countries

We consider two countries ($m = 2$) with a single economic parameter $\boldsymbol{\tau} = \tau$ under replicator dynamics. The two-dimensional reduced equation given in (9):

$$\begin{pmatrix} \mathrm{d}A_1 \\ \mathrm{d}A_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial A_1}{\partial \tau} \\ \frac{\partial A_2}{\partial \tau} \end{pmatrix} \mathrm{d}\tau + \text{h.o.t.} = \mathbf{0} \tag{15}$$

can be further reduced to a one-dimensional equation, as shown below.

Using $\mathrm{d}a_2 = -\mathrm{d}a_1$ and $\frac{\partial A_1}{\partial \tau} = -\frac{\partial A_2}{\partial \tau} \equiv g$, which follow from (14) with $m = 2$, and introducing a projection matrix $P = (1, -1)^\top$ ($m = 2$ in the general case in (50)), we obtain

$$\mathrm{d}A_1 - \mathrm{d}A_2 = P^\top \begin{pmatrix} \mathrm{d}A_1 \\ \mathrm{d}A_2 \end{pmatrix}, \quad \begin{pmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \end{pmatrix} = P\mathrm{d}a_1, \quad \begin{pmatrix} \frac{\partial A_1}{\partial \tau} \\ \frac{\partial A_2}{\partial \tau} \end{pmatrix} = Pg.$$

We then derive a one-dimensional governing equation in terms of the variable $\mathrm{d}a_1$ from (15) as

$$\begin{aligned} \mathrm{d}A_1 - \mathrm{d}A_2 &= P^\top \left\{ \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} P \mathrm{d}a_1 + Pg\,\mathrm{d}\tau + \text{h.o.t.} \right\} \\ &= (1,-1) \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{d}a_1 + (1,-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} g\,\mathrm{d}\tau + \text{h.o.t.} \\ &= (J_{11} - J_{12} - J_{21} + J_{22})\,\mathrm{d}a_1 + 2g\,\mathrm{d}\tau + \text{h.o.t.} = 0. \end{aligned} \tag{16}$$

This reduced equation enables both forward and inverse analyses:

- In the forward analysis, when $\mathit{\Delta} = J_{11} - J_{12} - J_{21} + J_{22} \neq 0$, (16) can be solved as $\mathrm{d}a_1 \approx -\frac{2g}{\mathit{\Delta}}\mathrm{d}\tau$.
- In the inverse analysis, when $g \neq 0$, (16) can be solved as $\mathrm{d}\tau \approx -\frac{\mathit{\Delta}}{2g}\mathrm{d}a_1$.

5.2 Further Reduction and Inverse Analysis

The m-dimensional reduced equation in (12):

$$\mathrm{d}\boldsymbol{A} = \hat{J}\mathrm{d}\boldsymbol{a} + \hat{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \mathbf{0}$$

can be further reduced to an $(m-1)$-dimensional simplex. We generalize the case of $m = 2$ in Section 5.1 to arrive at Proposition 2.

Proposition 2. *The further reduced equation in $(m-1)$-dimensional simplex is given by:*

$$\tilde{J}\mathrm{d}\tilde{\boldsymbol{a}} + \tilde{G}\mathrm{d}\boldsymbol{\tau} + h.o.t. = \mathbf{0} \tag{17}$$

with an $(m-1)$-dimensional vector $\mathrm{d}\tilde{\boldsymbol{a}} = (\mathrm{d}a_1, \dots, \mathrm{d}a_{m-1})^\top$, an $(m-1) \times (m-1)$ matrix $\tilde{J}$, and an $(m-1) \times p$ matrix $\tilde{G}$.

Proof. See Appendix B.4 for the proof and the definition of $\tilde{J}$ and $\tilde{G}$. □

In economic geography models, population distributions are typically derived for fixed values of economic parameters. In contrast, this paper presents a multi-parameter inverse analysis that identifies a specific economic parameter vector $\boldsymbol{\tau} = \boldsymbol{\tau}^*$ that realizes the target country-level population distribution $\boldsymbol{a} = \boldsymbol{a}^*$. Section 8.1 presents the application of this analysis.

In the case of $p = m - 1$ (see Remark 2 for other cases), the matrix $\tilde{G}$ in (17) becomes an $(m-1) \times (m-1)$ square matrix and is generically invertible. Then, (17) can be approximately solved as

$$\mathrm{d}\boldsymbol{\tau} \approx -\tilde{G}^{-1}\tilde{J}\mathrm{d}\tilde{\boldsymbol{a}}. \tag{18}$$

The inverse problem is then formulated as follows:

$$\text{Inverse problem: find } \boldsymbol{\tau} = \boldsymbol{\tau}^* \text{ such that } \boldsymbol{a} = \boldsymbol{a}^*, \text{ using (18).} \tag{19}$$

In solving the reduced governing equation (12), we employ two analytical approaches: (i) comparative statics and (ii) inverse analysis. In comparative statics, commonly used in economic geography, the country-level reduction introduced in the previous section is sufficient. In contrast, inverse analysis requires the further reduction presented in this section.

Remark 2. If $1 \leq p < m - 1$, the inverse problem in (19) cannot be solved exactly. If $p > m - 1$, the number of economic parameters exceeds the number of equations, resulting in a non-unique solution set for $\boldsymbol{\tau}^*$. □

6 Analysis by Population Gradient Matrix

Changes in a country's tariff rate often raise serious concerns in other countries, as evidenced by ongoing international trade tensions. We highlight the population gradient matrix T in (13) as a systematic tool for analyzing the effects of tariffs, as discussed in Sections 7–9. We examine the influence of trade freeness parameters $\tau_k = \phi_k$ and define the *population gradient component* of this matrix as $t_{\alpha k} = \frac{\partial a_\alpha}{\partial \phi_k}$, i.e.,

$$T = (t_{\alpha k} \mid \alpha = 1, \dots, m;\ k = 1, \dots, p)\,. \tag{20}$$

A parameter ϕ_k is considered influential for country α's population a_α if the absolute value of $t_{\alpha k}$ is large. For example, if $t_{\alpha k} > 0$, then the population a_α increases with ϕ_k, as explained in the following definition:

Definition 1. Country α is said to be in a *strong position* if its population a_α increases with ϕ_k, and in a *weak position* if it decreases. □

We can formalize this definition as follows: When ϕ_k increases, country α is in a

$$\begin{cases} \text{strong position} & \text{if } t_{\alpha k} > 0, \\ \text{weak position} & \text{if } t_{\alpha k} < 0. \end{cases} \tag{21}$$

Country α is in an *absolutely strong* (or *absolutely weak*) position if $t_{\alpha k} > 0$ (or $t_{\alpha k} < 0$) for any $\phi_k \in (0, 1)$.

For dynamics that satisfy the conservation law $\sum_{\alpha=1}^{m} a_\alpha = 1$, differentiating this law with respect to ϕ_k yields

$$\sum_{\alpha=1}^{m} t_{\alpha k} = 0 \quad (k = 1, \dots, p). \tag{22}$$

Since $t_{\alpha k}$ typically takes both positive and negative values, at least one country must satisfy $t_{\alpha k} > 0$ and another $t_{\alpha k} < 0$. The following holds for multiple countries:

> (1) When ϕ_k increases, at least one country is in a strong position ($t_{\alpha k} > 0$) and at least one is in a weak position ($t_{\alpha k} < 0$). (2) When ϕ_k decreases, countries previously in strong positions become weak, and vice versa.

In the two-country case, an increase in ϕ_k places one country in a strong position and the other in a weak one. When ϕ_k decreases, these positions are interchanged.

7 Analysis of the UK, France, and Germany

We apply the proposed theoretical framework to analyze trade competition among the UK, France, and Germany. We construct a global–local system representing these three countries, based on their 2020 populations during the pre-Brexit period. We employ the Helpman model (1998) presented in Section 3.3 and specify its parameters as $\sigma = 5.0$, $\mu = 0.75$, and $A = 1$.[8]

7.1 Spatial Modeling and Trade Framework

According to the Population Pyramids of the World (1950–2100),[9] the ratio of the populations of the UK, France, and Germany in 2020 is given as

$$67,351,861 \ : \ 65,905,277 \ : \ 83,628,708 \ \approx \ 31.1\% \ : \ 30.4\% \ : \ 38.6\%. \qquad (23)$$

Based on this ratio, the number of places representing the UK, France, and Germany is set to 34, 33, and 42, respectively. Figure 2 depicts a spatial model[10] of the three countries, where discrete local places are located at grid points of square lattices, and goods are transported along the lattice. The number of local places assigned to each country reflects the relative size of its economy. One place in the UK (London) is directly connected to places in France (Paris) and Germany (Frankfurt), as indicated by the red curves. France and Germany are continuously linked by the red grid lines.

We define a national trade freeness parameter for each country as

$$\phi_\alpha \quad (\alpha = \mathrm{UK}, \mathrm{Fra}, \mathrm{Ger}, \mathrm{EU})$$

for each country, where "EU" denotes the combination of France and Germany. We introduce the international trade freeness parameter for the import of country α from another country β as

$$\phi_{\beta\to\alpha} \quad (\alpha, \beta = \mathrm{UK}, \mathrm{Fra}, \mathrm{Ger}, \mathrm{EU}; \ \alpha \neq \beta).$$

[8]The values of σ and μ used in this paper are the same as those used by Redding and Rossi-Hansberg (§3.9, 2017).

[9]See https:// www.populationpyramid.net.

[10]We employ a simple and uniform grid to focus on the study of the competition of the three countries, while it is possible to employ a finer and nonuniform grid to express local properties more accurately.

We employ two types of tariffs:

$$\begin{cases} \text{reciprocal tariff :} & \phi_{\beta\to\alpha} = \phi_{\alpha\to\beta}, \\ \text{asymmetric tariff :} & \phi_{\beta\to\alpha} \neq \phi_{\alpha\to\beta}. \end{cases}$$

We adopt the following assumption regarding the control of trade freeness:

Assumption 1. A country α controls its national trade freeness ϕ_α and the import trade freeness $\phi_{\beta\to\alpha}$, while another country β controls ϕ_β and $\phi_{\alpha\to\beta}$.

France and Germany are assumed to form a seamless economy represented by a single trade freeness parameter ϕ_{EU}, being defined as

$$\phi_{\text{EU}} \equiv \phi_{\text{Fra}} = \phi_{\text{Ger}} = \phi_{\text{Fra}\to\text{Ger}} = \phi_{\text{Ger}\to\text{Fra}} \tag{24}$$

and have the same level of tariffs for the trade between the UK, that is,

$$\phi_{\to\text{UK}} \equiv \phi_{\text{Fra}\to\text{UK}} = \phi_{\text{Ger}\to\text{UK}}, \quad \phi_{\to\text{EU}} \equiv \phi_{\text{UK}\to\text{Fra}} = \phi_{\text{UK}\to\text{Ger}}. \tag{25}$$

Based on these trade freeness parameters, we explore several scenarios: **pre-Brexit EU single market** of the three countries (Section 7.2), the **UK's post-Brexit trade strategy** on its trade position against the EU (comprising France and Germany) in Section 8, and the **EU's post-Brexit trade strategy** against the UK (Section 9).

7.2 The EU Single Market

We consider an idealized model of the pre-Brexit EU single market, characterized by the single national trade freeness parameter:

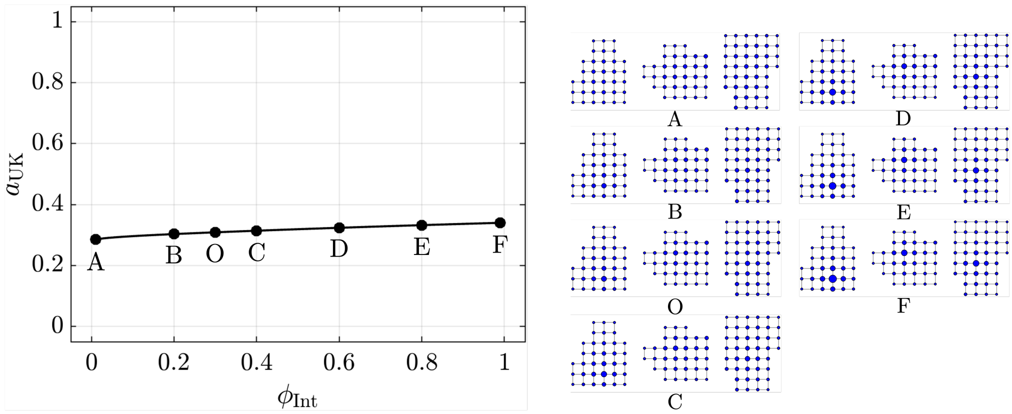
$$\phi \equiv \phi_{\text{UK}} = \phi_{\text{EU}}$$

and a reciprocal tariff:

$$\phi_{\text{Int}} \equiv \phi_{\to\text{UK}} = \phi_{\to\text{EU}}$$

for the trade between the UK and the EU (comprising France and Germany).

First, we examine how trade liberalization affects the UK's population. Figure 5(a) plots ϕ_{Int}–a_{UK} curve for $\phi = 0.3$. The country-level population distribution at point O, where $\phi_{\text{Int}} = 0.3$, is $\boldsymbol{a} = (a_{\text{UK}}, a_{\text{Fra}}, a_{\text{Ger}}) \approx (0.309, 0.299, 0.392)$ and is close to the ratio 31.1%: 30.4%: 38.6% in (23) of the real data from 2020 in the



(a) ϕ_{Int}–a_{UK} curve for $\phi = 0.3$ (b) Population distribution

Figure 5: a_{UK} plotted against ϕ_{Int} for $\phi = 0.3$ obtained by comparative statics

pre-Brexit era. Accordingly, we designate the point O, with $(\phi, \phi_{\mathrm{Int}}) = (0.3, 0.3)$, as the representative state of the EU single market for the three countries, and call it *origin point*. In the place-level population distribution at point O in Panel (b), agglomeration around the hubs of direct international trade is observed (e.g., London, UK). Agglomeration intensifies as trade liberalization advances (from points C to F as ϕ_{Int} increases), benefiting these trading hubs. In contrast, the population distribution becomes more uniform with increasing protectionism (from points B to A as ϕ_{Int} decreases).

7.2.1 Local Analysis Using Population Gradient

We conduct a local analysis at the origin point O, using the population gradient matrix T in (20). This matrix for $\boldsymbol{a} = (a_{\mathrm{UK}}, a_{\mathrm{EU}})$ and $\boldsymbol{\tau} = (\phi_{\mathrm{Int}}, \phi, \sigma, \mu)$ at this point is evaluated to

$$T = \begin{pmatrix} t_{\mathrm{UK},\phi_{\mathrm{Int}}} & t_{\mathrm{UK},\phi} & t_{\mathrm{UK},\sigma} & t_{\mathrm{UK},\mu} \\ t_{\mathrm{EU},\phi_{\mathrm{Int}}} & t_{\mathrm{EU},\phi} & t_{\mathrm{EU},\sigma} & t_{\mathrm{EU},\mu} \end{pmatrix} = \begin{pmatrix} +0.0546 & -0.0209 & -0.0000 & +0.0055 \\ -0.0546 & +0.0209 & +0.0000 & -0.0055 \end{pmatrix}.$$

We observe that $t_{\mathrm{UK},\phi_{\mathrm{Int}}} = 0.0546 > 0$, which corresponds to the positive slope of the curve at the origin point O in Fig. 5(a). Since a_{UK} increases with ϕ_{Int}, the UK is in a strong position as trade liberalization progresses (cf. (21)). This result is ironic, as events such as Brexit (a decrease in ϕ_{Int}) would not benefit but rather undermine the UK.

In the three countries' cooperative domestic development (an increase in ϕ), the UK with $t_{\mathrm{UK},\phi} = -0.0209 < 0$ is in a weak position, while the EU with $t_{\mathrm{EU},\phi} = 0.0209 > 0$ is in a strong position reciprocally (see the end of Section 6 for the reciprocity of two countries).

The economic parameters σ and μ have population gradient components of much smaller absolute values than ϕ_{Int} and ϕ, and are therefore less influential.[11]

We investigate the competition between France and Germany within the EU, using

$$T = \begin{pmatrix} t_{\text{UK},\phi_{\text{Int}}} & t_{\text{UK},\phi} \\ t_{\text{Fra},\phi_{\text{Int}}} & t_{\text{Fra},\phi} \\ t_{\text{Ger},\phi_{\text{Int}}} & t_{\text{Ger},\phi} \end{pmatrix} = \begin{pmatrix} +0.0546 & -0.0209 \\ -0.0097 & -0.0031 \\ -0.0449 & +0.0240 \end{pmatrix}$$

for $\boldsymbol{a} = (a_{\text{UK}}, a_{\text{Fra}}, a_{\text{Ger}})$, evaluated at the origin point O. France and Germany are in weak positions under trade liberalization (an increase in ϕ_{Int}), like the EU. When ϕ increases, France is in a weak position but Germany is in a strong position. These findings suggest that France and Germany have conflicting interests, which must be carefully managed within the EU single market. It is understandable that the UK and the other EU countries had a stake in each other's economic activities during the pre-Brexit period. As we have seen, the population gradient matrix T is useful in analyzing such a stake.

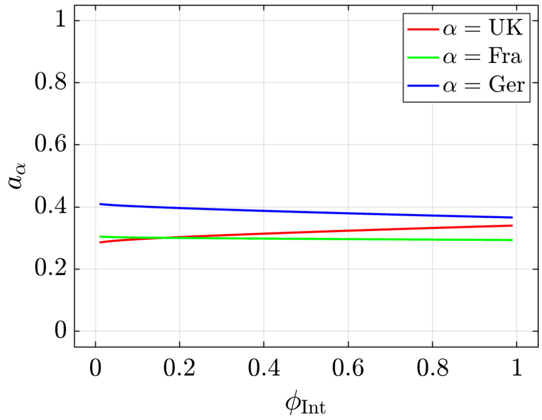
7.2.2 Global Analysis

Having observed local trends at the origin point O, we now turn to the analysis of global trends beyond this point. Figure 6 plots the populations of the three countries against ϕ_{Int} for $\phi = 0.3$ in Panel (a) and against ϕ for $\phi_{\text{Int}} = 0.3$ in Panel (b). As ϕ_{Int} increases and trade liberalization progresses in Panel (a), the UK's population share a_{UK} increases monotonically, while the populations in France and Germany decrease. Thus, in this trade liberalization, the UK is in a strong position globally, and the other two countries are in weak positions globally. When ϕ increases and the infrastructure development progresses in Panel (b), the UK is in a weak position globally and Germany is in a strong position globally, while France is initially in a weak position but transitions to a strong position as ϕ exceeds 0.5. An increase in national trade freeness ϕ benefits Germany but not the UK, due to its relatively smaller domestic market. As we have seen, the UK and the two EU countries have conflicting interests in the EU single market.

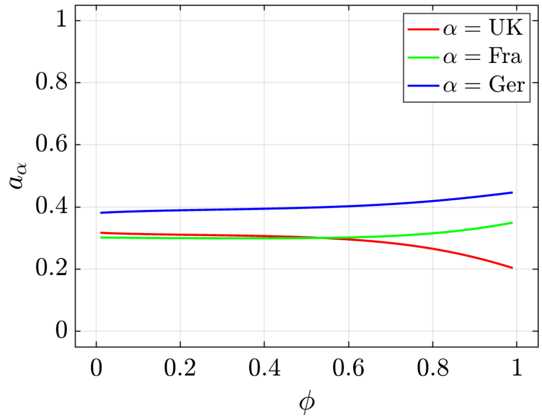
Figure 7(a) shows the weighted price index[12] against ϕ_{Int} for $\phi = 0.3$. When ϕ_{Int} increases from the original value 0.3, promoting trade liberalization, the price index decreases, especially for the UK. Conversely, as ϕ_{Int} decreases from 0.3, the

[11]This is especially the case for $\sigma > 4$ and $\mu < 0.8$, which do not lead to excessive agglomeration (cf. Appendix C).

[12]See (41) in Appendix B for the definition of this index.

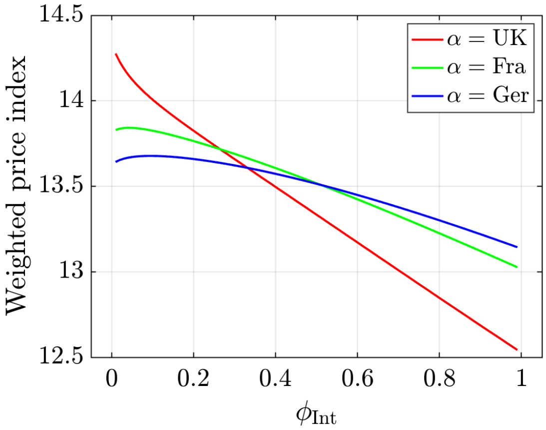


(a) Plot against ϕ_{Int} for $\phi = 0.3$

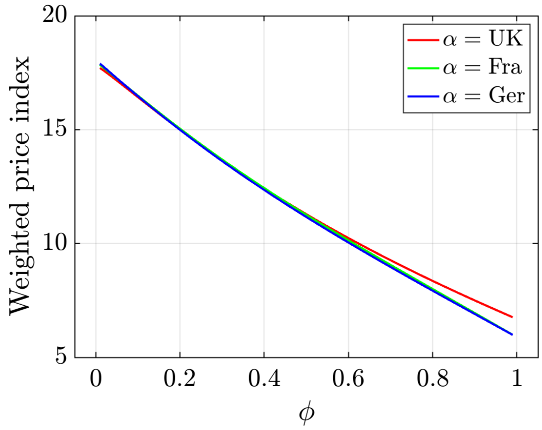


(b) Plot against ϕ for $\phi_{\text{Int}} = 0.3$

Figure 6: Populations' dependence on trade freeness parameters

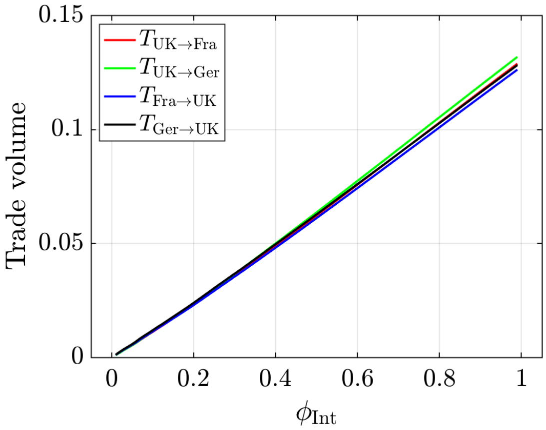


(a) Plot against ϕ_{Int} for $\phi = 0.3$

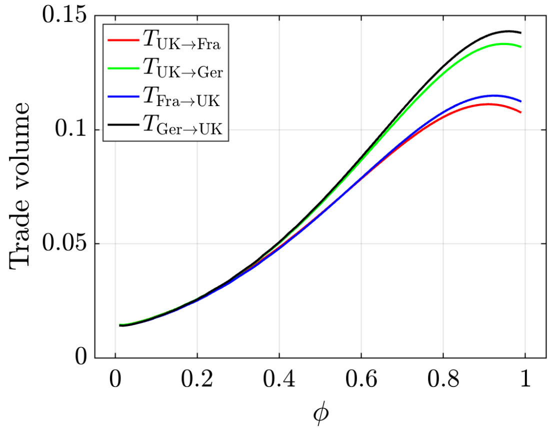


(b) Plot against ϕ for $\phi_{\text{Int}} = 0.3$

Figure 7: Weighted price index's dependence on trade freeness parameters



(a) Plot against ϕ_{Int} for $\phi = 0.3$



(b) Plot against ϕ for $\phi_{\text{Int}} = 0.3$

Figure 8: Trade volume's dependence on trade freeness parameters

price index increases sharply, reflecting economic disintegration, especially for the UK. As shown in Panel (b), the price index decreases in all three countries with ϕ, as a benefit of infrastructure development.

Figure 8 shows trade volume (cf. (42) for its definition). As shown in Panel (a), the trade volume between the UK and the two EU countries increases constantly with ϕ_{Int}. As shown in Panel (b), the trade volume initially increases with ϕ but decreases over $\phi = 0.9$.

We conclude with a few remarks on the influence of Brexit. As ϕ_{Int} decreases from the original value of 0.3, modeling Brexit, (i) the population a_{UK} decreases (cf. Fig. 5) and (ii) the trade volume decreases sharply (Fig. 8(a)). The former aligns with the study by Di Iasio and Wahba (2022), which states: "the UK has become less attractive to EU potential and current immigrants." The latter agrees with De Lucio et al. (2024), who reports: "We find that Spanish exports and imports to the UK decreased by 24% and 27%, respectively, compared to the period before the Brexit referendum."

8 The UK's Post-Brexit Trade Strategy

The UK left the EU in 2021, marking the end of its economic integration with the EU. We analyze how changes in tariffs affect population distributions across countries to elucidate how the UK should implement trade policy to attract mobile workers in the post-Brexit period.

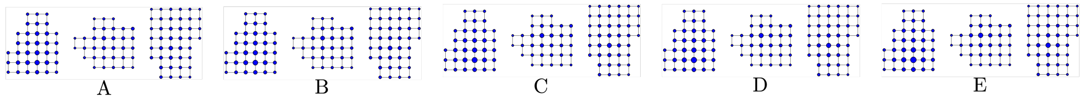
The UK increases its national trade freeness, ϕ_{UK}, from an initial value of 0.3, and selects an appropriate value for its import trade freeness, $\phi_{\to\mathrm{UK}}$ ($= \phi_{\mathrm{EU}\to\mathrm{UK}}$), defined in (25). The EU (comprising France and Germany) maintains a single market with $\phi_{\mathrm{EU}} = 0.3$ (cf. (24)). For imports from the UK, denoted by $\phi_{\to\mathrm{EU}}$, the EU has the option to choose between two types of tariffs

$$\phi_{\to\mathrm{EU}} = \begin{cases} \phi_{\to\mathrm{UK}} & \text{for reciprocal tariff,} \\ 0.3 & \text{for asymmetric tariff.} \end{cases} \tag{26}$$

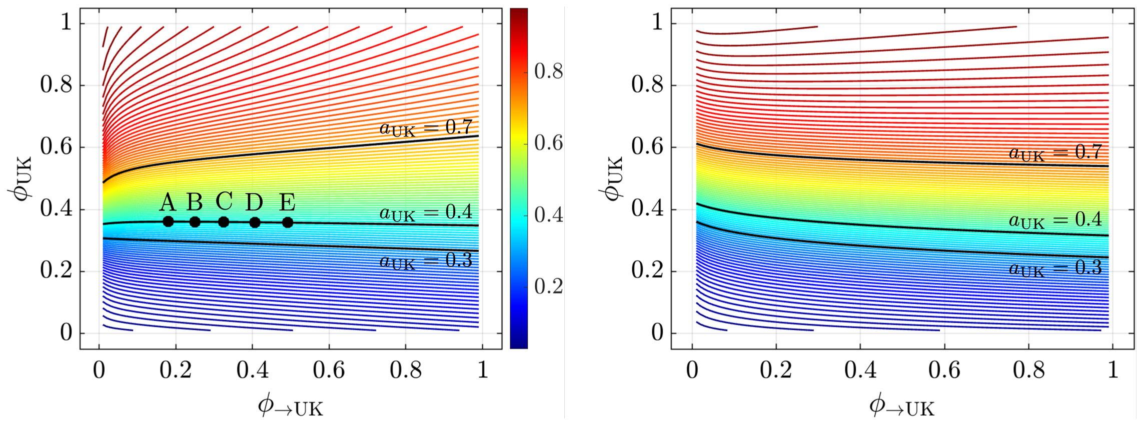
8.1 Contour Maps for the UK's Population

By the inverse analysis presented in Section 5.2, we can find the values of trade freeness parameters ϕ_{UK} and $\phi_{\to\mathrm{UK}}$ that realize a specific country-level population distribution $\boldsymbol{a} = \boldsymbol{a}^*$ ($\boldsymbol{a} = (a_{\mathrm{UK}}, a_{\mathrm{Fra}}, a_{\mathrm{Ger}})$). Under the reciprocal tariff, we obtained sets of parameter values $(\phi_{\to\mathrm{UK}}, \phi_{\mathrm{UK}})$ for a chain of population distributions $\boldsymbol{a}^* = (0.40, 0.26 + \epsilon, 0.34 - \epsilon)$ for several values of ϵ, yielding the same $a_{\mathrm{UK}} = 0.4$, but different values for a_{Fra} and a_{Ger}. Figure 9(a) shows population distributions A–F within the three countries obtained in this manner, and the associated locations in the space of $(\phi_{\to\mathrm{UK}}, \phi_{\mathrm{UK}})$ are plotted as points A–F in Panel (b). Such points, at finer intervals, form a black contour line. Thus, the trade option $(\phi_{\to\mathrm{UK}}, \phi_{\mathrm{UK}})$ that achieves a_{UK} ($= 0.4$) is not unique and should be chosen based on other factors, such as the promotion of industries, the reduction of inflation, and changes in tariffs.

Figures 9(b) and (c) show contour maps of a_{UK} in the space of $(\phi_{\to\mathrm{UK}}, \phi_{\mathrm{UK}})$ for the two types of tariffs in (26). We can see from the contour line for $a_{\mathrm{UK}} = 0.3$ with a negative slope that the required level of ϕ_{UK} tends to decrease as $\phi_{\to\mathrm{UK}}$ enlarges under both types of tariffs. Accordingly, the liberalization favors the UK for $a_{\mathrm{UK}} = 0.3$. If the UK promotes its infrastructure to realize a very high target of $a_{\mathrm{UK}} = 0.7$, a reversed trend emerges under the reciprocal tariff: Its contour line has a positive slope and the required level of ϕ_{UK} tends to increase with $\phi_{\to\mathrm{UK}}$. Thus, trade liberalization becomes unfavorable. The mechanism behind this reversed trend is analyzed below.



(a) Population distributions with $a_{\text{UK}} = 0.4$ (reciprocal tariff)



(b) Reciprocal tariff

(c) Asymmetric tariff

Figure 9: An inverse analysis for pinpointing $a_{\text{UK}} = 0.4$ (points A–E) and contour maps of a_{UK} in the space of $(\phi_{\to\text{UK}}, \phi_{\text{UK}})$

8.2 Local Analysis Using Population Gradient

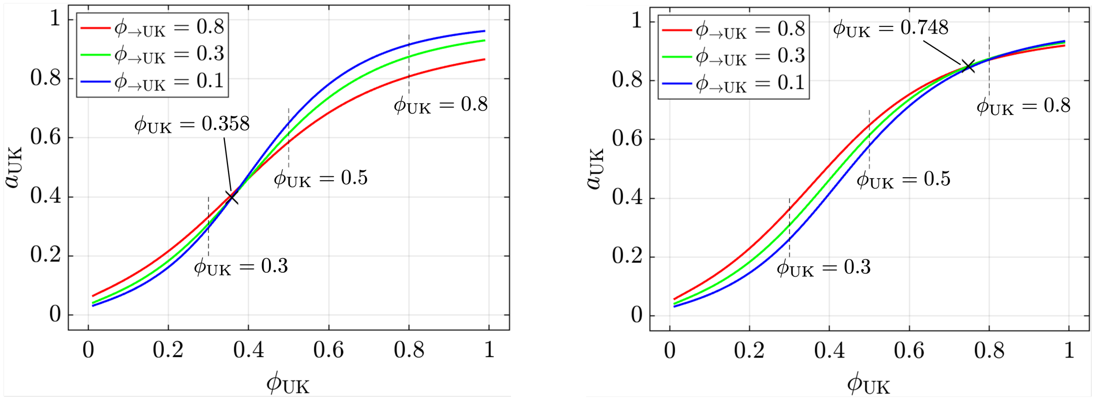
We conduct a local analysis at the origin point O (cf. Fig. 5), referring to the population gradient matrix in (20):

$$T = \begin{pmatrix} t_{\text{UK},\phi_{\text{UK}}} & t_{\text{UK},\phi_{\to\text{UK}}} \\ t_{\text{EU},\phi_{\text{UK}}} & t_{\text{EU},\phi_{\to\text{UK}}} \end{pmatrix} = \begin{cases} \begin{pmatrix} +1.437 & +0.055 \\ -1.437 & -0.055 \end{pmatrix} & \text{for reciprocal tariff,} \\ \begin{pmatrix} +1.437 & +0.167 \\ -1.437 & -0.167 \end{pmatrix} & \text{for asymmetric tariff} \end{cases} \tag{27}$$

for $\boldsymbol{a} = (a_{\text{UK}}, a_{\text{EU}})$ and $\boldsymbol{\tau} = (\phi_{\to\text{UK}}, \phi_{\text{UK}})$, evaluated at the origin point. Since both $t_{\text{UK},\phi_{\text{UK}}}$ and $t_{\text{UK},\phi_{\to\text{UK}}}$ are positive under both types of tariffs, the UK is in a strong position with respect to increases in both ϕ_{UK} and $\phi_{\to\text{UK}}$. The EU (comprising France and Germany) is in a weak position under both types of tariffs, since the strong and weak positions are reciprocal between the UK and the EU (cf. Section 6). The gradient $t_{\text{UK},\phi_{\text{UK}}} = 1.437$ has a much larger value compared to $t_{\text{UK},\phi_{\to\text{UK}}} = 0.055,\ 0.167$; accordingly, ϕ_{UK} is predicted to be more influential on a_{UK} than $\phi_{\to\text{UK}}$.

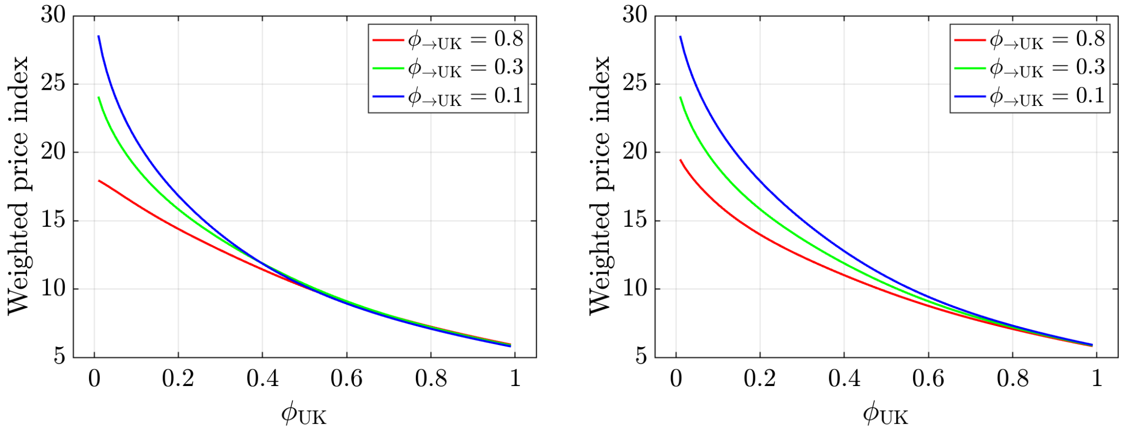
8.3 Global Analysis of the UK's Trade Position

Following the local analysis, we now proceed to a global analysis. Figure 10(a) plots a_{UK} against ϕ_{UK} for $\phi_{\to\text{UK}} = 0.1$, 0.3, and 0.8, under the reciprocal tariff at the left



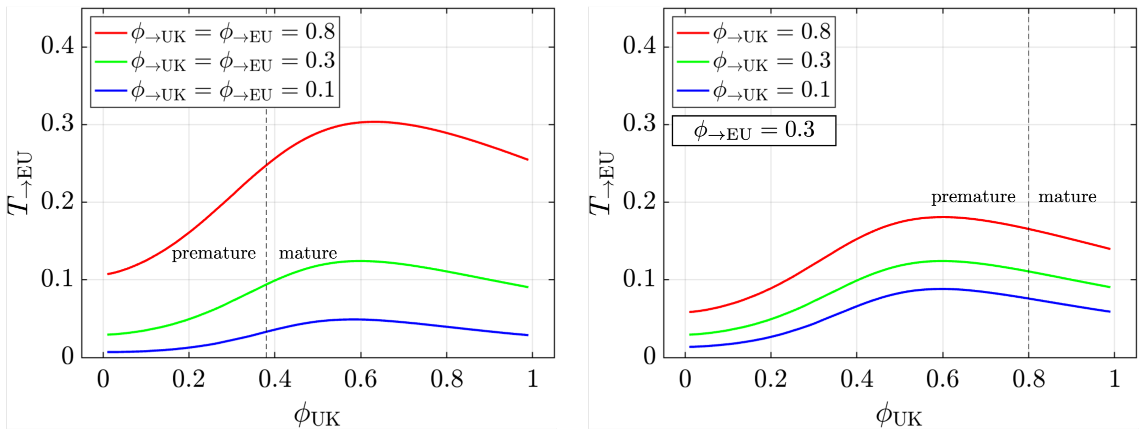
Reciprocal tariff ($\phi_{\to EU} = \phi_{\to UK}$) Asymmetric tariff ($\phi_{\to EU} = 0.3$)

(a) Population share of the UK



Reciprocal tariff ($\phi_{\to EU} = \phi_{\to UK}$) Asymmetric tariff ($\phi_{\to EU} = 0.3$)

(b) Weighted price index of the UK



Reciprocal tariff ($\phi_{\to EU} = \phi_{\to UK}$) Asymmetric tariff ($\phi_{\to EU} = 0.3$)

(c) Volume $T_{\to EU}$ of the export from the UK to the EU

Figure 10: The influence of the domestic economic development on the population share, price index, and export volume of the UK

and the asymmetric one at the right. When ϕ_{UK} increases from 0 to 1, developing its domestic transportation infrastructure, a_{UK} monotonically increases from below 0.1 to above 0.9. Panel (b) shows a monotonic decrease in the UK's price index as ϕ_{UK} increases, aligning with the rise in the UK population a_{UK} in Panel (a). As shown in Panel (c), the volume $T_{\to\text{EU}}$ of UK exports[13] to the EU increases with ϕ_{UK}, reaches a peak near $\phi_{\text{UK}} \approx 0.6$, and then declines.

Figure 11(a) plots a_{UK} for $\phi_{\to\text{UK}}$ for $\phi_{\text{UK}} = 0.3$, 0.35, and 0.4 under both types of tariffs. Note that $\phi_{\to\text{UK}}$ has a smaller influence on a_{UK} than ϕ_{UK}, as predicted with reference to the population gradient in (27). Under the asymmetric tariff shown at the right, as $\phi_{\to\text{UK}}$ increases, the population a_{EU} increases on both the curve A'OB' for $\phi_{\text{UK}} = 0.3$ and the curve C'DE' for $\phi_{\text{UK}} = 0.4$. Such an increasing trend becomes conditional under the reciprocal tariff. The population a_{EU} increases along the curve AOB for $\phi_{\text{UK}} = 0.3$ but descends on the curve CDE for $\phi_{\text{UK}} = 0.4$.

We search for the general mechanism behind this reversed trend. Based on the sign of the gradient component $t_{\alpha,\phi_{\to\alpha}}$ of the $\phi_{\to\alpha}$ versus a_α curve in some country α at a given national trade freeness ϕ_α, we introduce the definition:

Definition 2. Country α's trade position is

$$\begin{cases} \text{premature} & \text{if } t_{\alpha,\phi_{\to\alpha}} > 0, \\ \text{mature} & \text{if } t_{\alpha,\phi_{\to\alpha}} < 0. \end{cases} \tag{28}$$

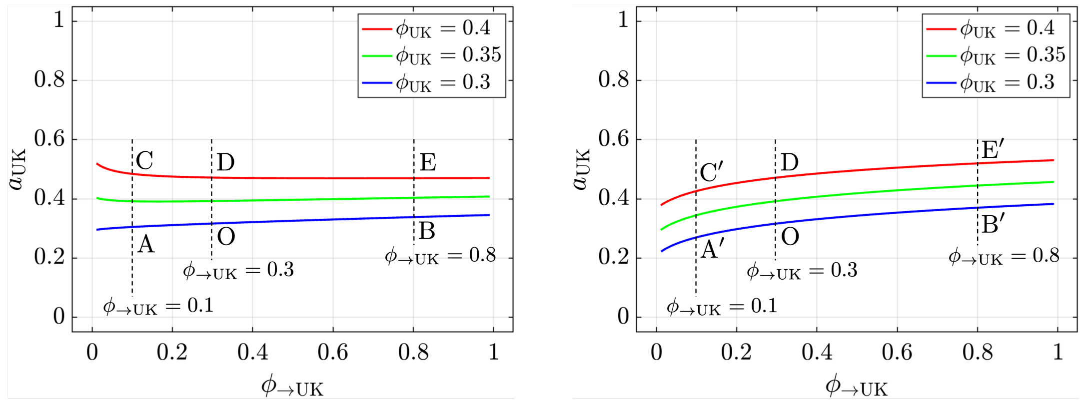
The origin point O, with $\phi_{\text{UK}} = 0.3$, in Fig. 11(a) corresponds to a premature trade position for both types of tariffs. When ϕ_{UK} increases to 0.4, the UK reaches a mature position under the reciprocal tariff, but remains in a premature position under the asymmetric tariff. Thus, the type of tariff matters. As shown in Table 1, according to the trend in the increase or decrease of a_{UK}, the direction of change in a_{UK} reverses depending on the UK's trade position.

Table 1: The trend of increase ($\nearrow$) or decline ($\searrow$) of a_{UK} and a_{EU} as $\phi_{\to\text{UK}}$ increases or declines

UK's Position	$\phi_{\to\text{UK}}$	a_{UK}	a_{EU}
Premature	$\nearrow$	$\nearrow$	$\searrow$
	$\searrow$	$\searrow$	$\nearrow$
Mature	$\nearrow$	$\searrow$	$\nearrow$
	$\searrow$	$\nearrow$	$\searrow$

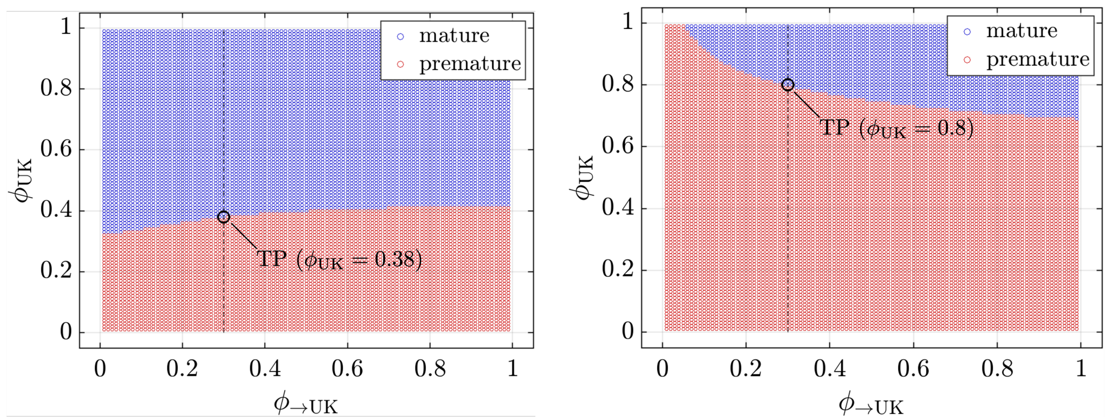
Figure 11(b) shows parameter zones of the UK's positions in the space $(\phi_{\to\text{UK}}, \phi_{\text{UK}})$

[13] The import of the UK displayed the same tendency as the export and, accordingly, is suppressed herein.



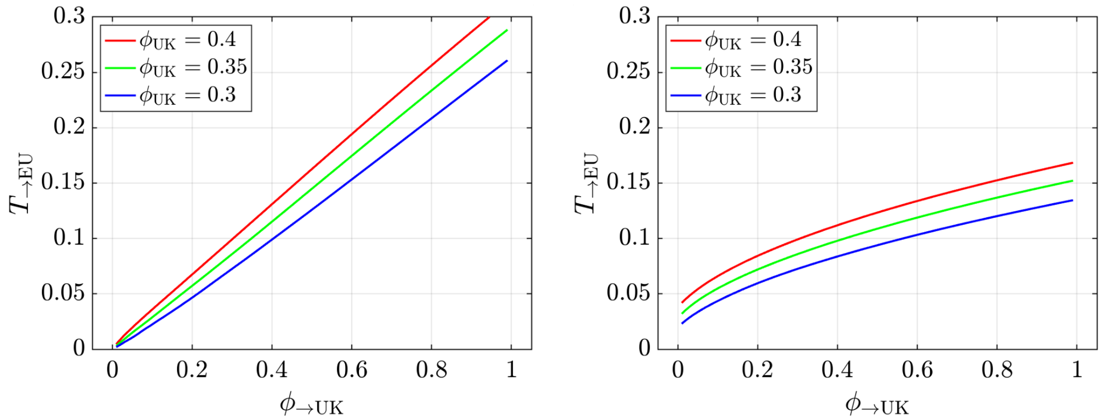
Reciprocal tariff ($\phi_{\rightarrow\mathrm{EU}} = \phi_{\rightarrow\mathrm{UK}}$) Asymmetric tariff ($\phi_{\rightarrow\mathrm{EU}} = 0.3$)

(a) Population share of the UK



Reciprocal tariff ($\phi_{\rightarrow\mathrm{EU}}$) Asymmetric tariff ($\phi_{\rightarrow\mathrm{EU}} = 0.3$)

(b) Parameter zones of trade positions for the UK



Reciprocal tariff ($\phi_{\rightarrow\mathrm{EU}} = \phi_{\rightarrow\mathrm{UK}}$) Asymmetric tariff ($\phi_{\rightarrow\mathrm{EU}} = 0.3$)

(c) Trade volume of the export $T_{\rightarrow\mathrm{EU}}$ of the UK to the EU

Figure 11: The influence of trade liberalization on the population, trade position, and trade volume of the UK

for both types of tariffs. The zone of a premature position is colored red and the zone of a mature one is colored blue (cf. (28)). Under the reciprocal tariff, as ϕ_{UK} increases from 0 along the vertical dashed line at $\phi_{\to\mathrm{EU}} = 0.3$, we encounter a point TP (turning point) at $\phi_{\mathrm{UK}} \approx 0.38$, transiting from a premature position to a mature position. Increasing $\phi_{\to\mathrm{UK}}$ ($= \phi_{\to\mathrm{EU}}$) under the reciprocal tariff, leading to trade liberalization, favors the UK during $0 < \phi_{\mathrm{UK}} < 0.38$ but is unfavorable during $\phi_{\mathrm{UK}} > 0.38$. Under the asymmetric tariff, TP resides at $\phi_{\mathrm{UK}} \approx 0.8$, and the premature zone widens, thereby favoring the UK.

Understanding the transition in a country's trade position is crucial. In a premature position, the trade liberalization is an attractive strategy. Once a country's trade position shifts from premature to mature, protectionism may become a more attractive strategy. We propose the following conjectures:

Conjecture 3. *(i) Liberalization benefits a country in a premature position, whereas protectionism favors it in a mature position. (ii) A premature position transitions into a mature one, as the country's domestic trade freeness grows to a certain level.*

As shown in Panel (c), the volume of exports from the UK increases with trade liberalization (an increase in $\phi_{\to\mathrm{UK}}$), especially under the reciprocal tariff.[14] As $\phi_{\to\mathrm{UK}}$ decreases from its original value of 0.3—such as occurred during Brexit—the trade volume of UK exports to the EU declines sharply, consistent with Freeman et al. (2025): "Our estimates imply that, in the short term, leaving the EU reduced worldwide UK exports by 6.4% and worldwide imports by 3.1%."

Remark 3. The transition from a premature to a mature position can be observed from the ϕ_{UK}–a_{UK} curves for different values of $\phi_{\to\mathrm{UK}}$ in Fig. 10(a), as indicated by the presence of a crossing point[15] ($\times$) among the curves. □

8.4 The UK's Trade Strategy

We investigate the UK's trade strategy to promote its population share a_{UK} by selecting appropriate values of two trade freeness parameters, $\phi_{\to\mathrm{UK}}$ and ϕ_{UK}, at the origin point O, where $a_{\mathrm{UK}} = 0.309$ and $(\phi_{\to\mathrm{UK}}, \phi_{\mathrm{UK}}) = (0.3, 0.3)$.

First, we deal with the choice of ϕ_{UK}. As we have seen in the plot of a_{UK} against ϕ_{UK} in Fig. 10(a), the UK's population a_{UK} increases with ϕ_{UK}. The UK consistently benefits from infrastructure development that raises ϕ_{UK} beyond its initial value of 0.3.

[14] The volume of imports from the EU has the same tendency and is suppressed herein.

[15] At this point, the three ϕ_{UK}–a_{UK} curves come very close together.

Next, we investigate the choice of $\phi_{\to\text{EU}}$. As worked out in Remark 4, for both types of tariffs, (i) a_{UK} increases with $\phi_{\to\text{UK}}$ and (ii) decreases as $\phi_{\to\text{UK}}$ decreases. Thus, the success or failure of the UK's trade strategy is not contingent on the EU's tariff choice.

The UK's recommended post-Brexit trade strategy is as follows: Domestically, invest in its infrastructure to increase national trade freeness. Internationally, pursue trade liberalization to raise import trade freeness and reduce the domestic price index. The present analysis demonstrates that trade liberalization benefits the UK in several aspects: an increase in population, a reduction in the price index, and an increase in trade volume. This analysis aligns with the statement by Brakman et al. (2018), based on gravity equation results: "Paradoxically, only a trade agreement with the EU can compensate for Brexit's trade losses."

Remark 4. We formulate the UK's trade strategy, referring to the relationship between $\phi_{\to\text{UK}}$ and a_{UK} in Fig. 11(a). (i) The UK upgrades the import trade freeness $\phi_{\to\text{UK}}$ by reducing the tariffs. If the EU adopts the reciprocal tariff ($\phi_{\to\text{EU}} = \phi_{\to\text{UK}}$), matching the UK's tariff rate, a_{UK} increases along OB. If the EU instead maintains $\phi_{\to\text{EU}} = 0.3$ (asymmetric tariff), a_{UK} increases along OB', yielding a greater increase than that along OB. Accordingly, the UK can successfully increase a_{UK} regardless of the EU's trade policy. (ii) The UK reduces the import trade freeness $\phi_{\to\text{UK}}$ by increasing the tariff. The population a_{UK} decreases along either OA or OA', regardless of the tariff chosen by the EU. Accordingly, the UK's trade strategy does not work out. □

9 The EU's Post-Brexit Trade Strategy

We investigate the trade strategy of the EU (comprising France and Germany), aimed at increasing its population share, $a_{\rm EU}$ $(= a_{\rm Fra} + a_{\rm Ger})$. The EU develops its infrastructure to enhance its national trade freeness $\phi_{\rm EU}$ from an initial value of 0.3 and controls the import trade freeness $\phi_{\to\rm EU}$ through the implementation of an appropriate import tariff. For imports from the EU, the UK is assumed to counteract by choosing between two types of tariffs:

$$\phi_{\to\rm UK} = \begin{cases} \phi_{\to\rm EU} & \text{for reciprocal tariff,} \\ 0.3 & \text{for asymmetric tariff.} \end{cases}$$

9.1 Local Analysis Using Population Gradient

At the origin point O (cf. Fig. 5), the population gradient matrix is evaluated as

$$T = \begin{pmatrix} t_{\rm EU,\phi_{EU}} & t_{\rm EU,\phi_{\to EU}} \\ t_{\rm UK,\phi_{EU}} & t_{\rm UK,\phi_{\to EU}} \end{pmatrix} = \begin{cases} \begin{pmatrix} +1.458 & -0.055 \\ -1.458 & +0.055 \end{pmatrix} & \text{for reciprocal tariff,} \\ \begin{pmatrix} +1.458 & +0.113 \\ -1.458 & -0.113 \end{pmatrix} & \text{for asymmetric tariff.} \end{cases} \tag{29}$$

Since the population gradient component $t_{\rm EU,\phi_{EU}} = 1.458$ is positive, the EU is in a strong position when $\phi_{\rm EU}$ increases, while the UK is reciprocally in a weak position. As the import trade freeness $\phi_{\to\rm EU}$ increases, according to the classification in (28), the EU is in a premature position under the asymmetric tariff ($t_{\rm EU,\phi_{\to EU}} = 0.113 > 0$) but in a mature position under the reciprocal tariff ($t_{\rm EU,\phi_{\to EU}} = -0.055 < 0$). Thus, the type of tariff matters for the EU, unlike the UK, which remains in a premature position under both types of tariffs at the origin point O (cf. Section 8.2).

9.2 Global Analysis of the EU's Trade Position

Figure 12(a) plots $a_{\rm EU}$ against $\phi_{\rm EU}$ for $\phi_{\to\rm EU} = 0.1$, 0.3, and 0.5. When $\phi_{\rm EU}$ increases from 0 to 1, $a_{\rm EU}$ increases from below 0.2 to near 1.0 under both types of tariffs. Accordingly, infrastructure development benefits the EU by increasing $\phi_{\rm EU}$ and, in turn, $a_{\rm EU}$.

Figure 12(b) plots $a_{\rm EU}$ against the EU's import trade freeness $\phi_{\to\rm EU}$ for $\phi_{\rm EU} =$ 0.3, 0.4 and 0.6. As $\phi_{\to\rm EU}$ increases, all the curves for the reciprocal tariff (shown on the left) descend but the curves for the asymmetric tariff (shown on the right) ascend or remain flat. A reduction in the tariff (i.e., an increase in $\phi_{\to\rm EU}$) is detrimental

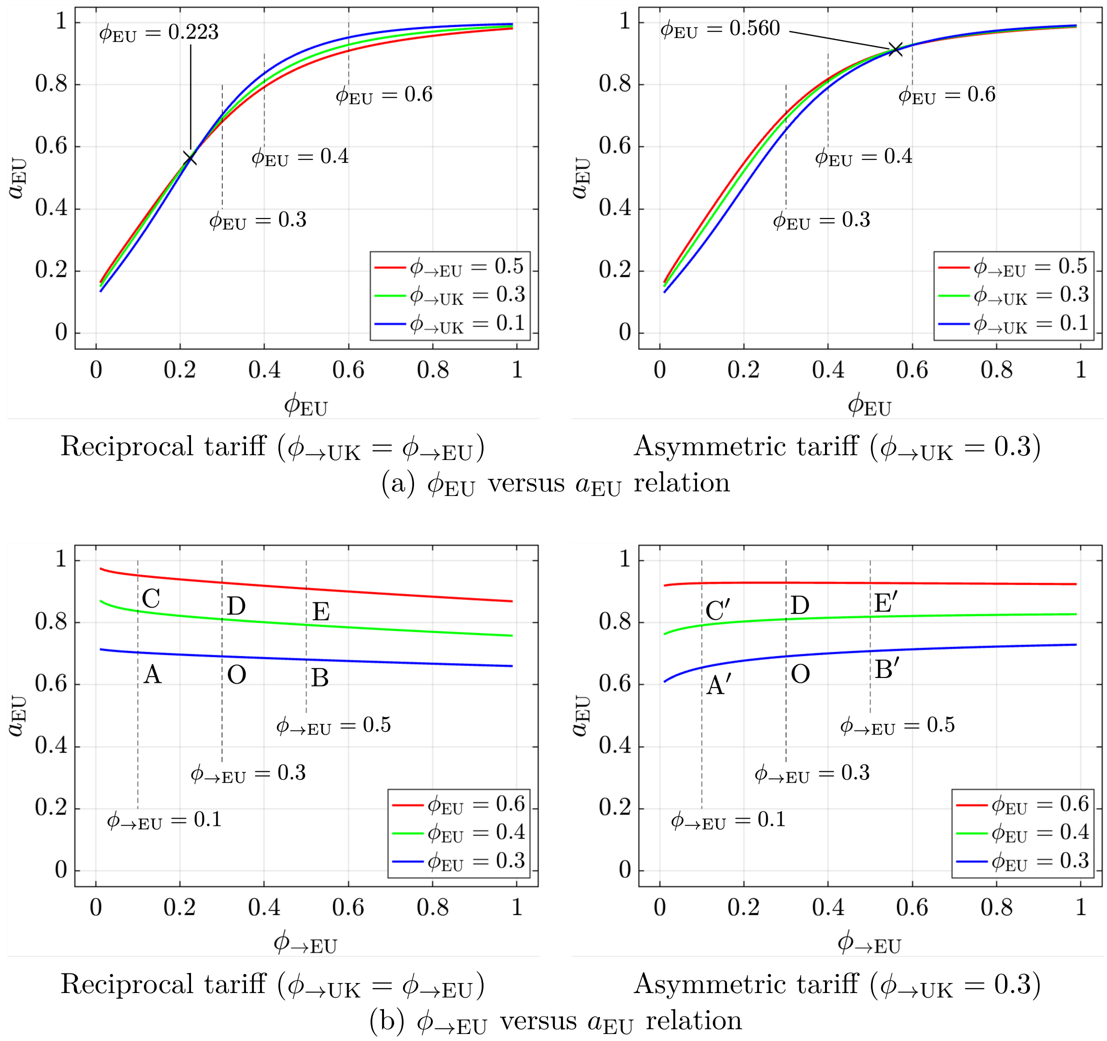


Figure 12: The influence of trade freeness parameters on the population share of the EU

under the reciprocal tariff but benefits the EU under the asymmetric tariff.

9.3 The EU's Trade Strategy

We investigate the EU's trade strategy to promote the EU's population share a_{EU} by setting appropriate values for trade freeness parameters $\phi_{\to EU}$ and ϕ_{EU} at the origin point O with $(\phi_{\to UK}, \phi_{UK}) = (0.3, 0.3)$.

First, we deal with the choice of ϕ_{EU}. Since a_{EU} increases with ϕ_{EU} (Fig. 12(a)), the EU consistently benefits from infrastructure development that increases ϕ_{EU}. Next, we deal with the choice of $\phi_{\to EU}$. As worked out in Remark 5, the success and failure of both (i) reducing the tariff and (ii) increasing it depend on the UK's choice of tariff rate. Thus, changes in the international trade environment present a double-edged sword, akin to a gamble. This strategic gamble is summarized in

Table 2. The recommended trade strategy for the EU is to maintain tariff levels—avoiding risky moves—and increase national trade freeness through its infrastructure development.

Remark 5. We formulate the EU's trade strategy, referring to Fig. 12(b). (i) The EU reduces the tariff and increases the import trade freeness $\phi_{\to \mathrm{EU}}$. If the UK remains in the state of $\phi_{\to \mathrm{UK}} = 0.3$ (asymmetric tariff), the EU's population a_{EU} increases along OB' (cf. Fig. 12(b)). This scenario benefits the EU. However, if the UK adopts a reciprocal tariff, a_{EU} decreases along OB, making the outcome unfavorable to the EU. (ii) The EU increases the tariff and reduces the import trade freeness $\phi_{\to \mathrm{EU}}$. If the UK imposes the retaliatory tariff of the same level (reciprocal tariff), a_{EU} increases along OA, and the EU's trade strategy is successful. If the UK keeps the level of trade freeness as $\phi_{\to \mathrm{UK}} = 0.3$ (asymmetric tariff), a_{EU} decreases along OA', indicating a failure of the EU's trade strategy. □

Table 2: Trade strategies of the EU and the UK and the resulting population winner

Country's Choice of Trade Tariff		Winner
EU	UK	(Population Gainer)
Reduce Tariff	Hold Tariff	EU
	Reduce Tariff	UK
Raise Tariff	Raise Tariff	EU
	Hold Tariff	UK

10 Concluding Remarks

This paper presented a place-to-country reduction method for analyzing international trade competition using a general equilibrium model of economic geography. The original place-level governing equation can be reduced to a country-level equation with significantly fewer degrees of freedom. The reduced equation yields informative country-level insights. Although the trade competition analysis around Brexit in this paper is based on the Helpman model (1998), the proposed reduction method is applicable to a broader class of models in economic geography and quantitative spatial economics.

We examined trade competition among the UK, France, and Germany around Brexit by incorporating internationally mobile workers. We successfully demonstrated the importance of infrastructure development. Moreover, we analyzed how changes in tariffs affect population distributions across countries to elucidate how the countries should implement trade policy to attract mobile workers.

Within the EU single market, trade liberalization reduces the price index across all three countries. This liberalization benefits the UK the most, resulting in a more rapid decrease in the price index and an increase in population. Joint infrastructure development across the three countries benefits the EU but not the UK.

The UK's post-Brexit trade environment is analyzed. The UK's population increases as a result of infrastructure development. Increasing import trade freeness—possibly through tariff reductions—benefits the UK, regardless of the EU's tariff policy. The recommended UK's strategy is to upgrade both national and import trade freeness.

In the EU's post-Brexit trade strategy, investing in the EU's domestic infrastructure always favors the EU. However, changes in import trade freeness have mixed effects depending on the tariff type, unlike the UK's trade strategy. When this trade freeness is increased by reducing the tariff, it favors the UK to opt for the reciprocal tariff by also reducing its tariff, but favors the EU if the UK retains the tariff level (asymmetric tariff). When the import trade freeness is decreased by raising the tariff, the EU gains population under the reciprocal tariff, but loses population if the UK retains the tariff level. Thus, altering the tariff level is a double-edged sword, as the EU's outcome depends critically on the UK's response.

When a country raises a tariff against another, how the affected country responds is critical. There may be a temptation to introduce a retaliatory tariff. However, our results show that such retaliatory tariffs are detrimental to both the UK and the EU. While tariffs can be imposed quickly and may seem attractive, this paper emphasizes that the importance of infrastructure development should not be overlooked. Such development takes time, possibly over the years.

We conclude this paper with a remark on the US's trade tactics. The US raised the tariffs on steel and aluminum imports in the President Trump's first term, and "metal production picked up, but higher costs slowed other industries" (Reuters, March 11, 2025). The analysis presented in this paper provides an insight into the outcome of the introduction of high tariffs by the US and retaliatory tariffs by other countries in 2025. The US has the privilege to raise the tariff first and then invest in domestic infrastructure, possibly spending some of the tariff income. Such tactics may strain other countries but also undermine the US economy by raising its domestic price index. These tactics are fragile, as their success depends on the responses of other countries' choice of tariff rates. A critical future task is to further unveil the mechanisms of trade competition involving the US.

Acknowledgements

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A Details of the Helpman Model

A.1 General Issues

To produce $x_j(\varphi)$ units of the φth differentiated good, a firm requires $f + cx_j(\varphi)$ units of labor. The total production cost for a firm located in place j is thus given by $w_j(f + cx_j(\varphi))$. Each firm located in place j maximizes its profit, which is given by

$$\pi_j(\varphi) = \sum_{i=1}^{n} p_{ji}(\varphi) q_{ji}(\varphi) \lambda_i - w_j \left\{ f + cx_j(\varphi) \right\}, \tag{30}$$

where $q_{ji}(\varphi)\lambda_i$ is the total demand in place i for the φth differentiated good produced in place j.

Under iceberg transportation costs (cf. Section 3.2), only a fraction $1/\tau_{ji}$ of each unit transported from place j to place i arrives. Consequently, the total supply is $x_j(\varphi) = \tau_{ji} q_{ji}(\varphi)\lambda_i$. Therefore, the first-order condition for profit maximization yields

$$p_{ji}(\varphi) = \frac{\sigma}{\sigma - 1} c \tau_{ji} w_j, \tag{31}$$

which is identical across the differentiated goods. Hereafter, we omit the argument φ from $p_{ji}(\varphi)$, $q_{ji}(\varphi)$, and $\pi_j(\varphi)$ since these functions are independent of φ.

In the model, market clearing holds given the spatial distribution of workers $\boldsymbol{\lambda}$. By the land market-clearing condition, the housing stock at each place is given by $S = \lambda_i h_i$. Combining this with $h_i = \frac{(1-\mu)Y_i}{r_i}$ from (6) yields the equilibrium housing price:

$$r_i = \frac{(1 - \mu) Y_i \lambda_i}{S}. \tag{32}$$

Using this equation, along with the assumption of public ownership, yields the expenditure of a worker residing in place i:

$$Y_i = w_i + (1 - \mu) \sum_{j=1}^{n} \lambda_j Y_j. \tag{33}$$

Using this equation, we obtain the equilibrium expenditure as a function of the wage w_i and the number of workers λ_i:

$$\boldsymbol{Y} = \left[I - (1 - \mu) \mathbf{1}\mathbf{1}^{\top} \mathrm{diag}[\boldsymbol{\lambda}] \right]^{-1} \boldsymbol{w}, \tag{34}$$

where $\boldsymbol{Y} = (Y_i) \in \mathbb{R}^n$, $\boldsymbol{w} = (w_i) \in \mathbb{R}^n$, $\mathbf{1} = (\underbrace{1, \ldots, 1}_{n})^\top$, diag[·] denotes the diagonal matrix with entries given by the vector inside the parentheses, and I is the $n \times n$ identity matrix. The labor market-clearing and zero-profit conditions are given by

$$\lambda_i = m_i \left[\left(\sum_{j=1}^{n} c\lambda_j q_{ij} \tau_{ij} \right) + f \right], \tag{35}$$

$$\sum_{j=1}^{n} \lambda_j q_{ij} (p_{ij} - c\tau_{ij} w_i) - w_i f = 0. \tag{36}$$

Substituting the price from (31) into (36) yields

$$\frac{1}{\sigma - 1} \sum_{j=1}^{n} c\lambda_j q_{ij} \tau_{ij} = f. \tag{37}$$

Using condition (35) and the above equation yields $m_i = \frac{\lambda_i}{\sigma f}$. Substituting this relation and (31) into the price index (7) yields

$$P_i = \left[\sum_{j=1}^{n} \frac{\lambda_j}{f\sigma} \left(\frac{\sigma}{\sigma - 1} c\tau_{ji} w_j \right)^{1-\sigma} \right]^{1/(1-\sigma)}. \tag{38}$$

Substituting q_{ij} in (6) into (37) and using (38) yield the following fixed point problem for the wage:

$$\mu \sum_{j=1}^{n} \left[\frac{\lambda_j Y_j \tau_{ij}^{1-\sigma} w_i^{1-\sigma}}{\sum_{k=1}^{n} \lambda_k \tau_{kj}^{1-\sigma} w_k^{1-\sigma}} \right] = w_i. \tag{39}$$

The wage w_i is implicitly determined by the equation above. Substituting the price index (38) and the housing price (32) into the indirect utility function v_i in (6) yields

$$v_i = \zeta \lambda_i^{\mu-1} Y_i^{\mu} \left(\sum_{j=1}^{n} \lambda_j (\tau_{ji} w_j)^{1-\sigma} \right)^{\mu/(\sigma-1)}, \tag{40}$$

where ζ is a constant depending on exogenous variables:

$$\zeta = \left(\frac{1}{f\sigma} \right)^{-\mu/(1-\sigma)} \left(\frac{\sigma c}{\sigma - 1} \right)^{-\mu} \left(\frac{1-\mu}{S} \right)^{-(1-\mu)}.$$

A.2 Key Indexes for Global–Local System

We highlight the price index and trade volume as key indicators of international trade competitiveness in Sections 7–9. Let $\mathcal{C}_\alpha$ denote the set of places in country α with

n_α places (cf. Section 4.1). The set of all places can be expressed as $\{1, 2, \ldots, n\} = \bigcup_{\alpha=1}^{m} \mathcal{C}_\alpha$. Denote by P_i and λ_i the price index and the population in place $i \in \mathcal{C}_\alpha$ in country α, respectively. The weighted price index in country α is defined as

$$\bar{P}_\alpha = \frac{\sum_{i \in \mathcal{C}_\alpha} \lambda_i P_i}{\sum_{i \in \mathcal{C}_\alpha} \lambda_i}. \tag{41}$$

We define the trade volume for each country based on the set $\mathcal{C}_\alpha$, where $\alpha \in \{1, 2, \ldots, m\}$. The value of exports from place j to place i is defined as $E_{ji} = \int_0^{m_j} p_{ji} q_{ji} \lambda_i \mathrm{d}\varphi$. For $i \in \mathcal{C}_\alpha$ and $j \in \mathcal{C}_\beta$, T_{ji} denotes the value of exports from place j in country β to place i in country α. Using this definition, the trade volume from country β to α is given by

$$T_{\beta \to \alpha} = \sum_{i \in \mathcal{C}_\alpha} \sum_{j \in \mathcal{C}_\beta} T_{ji}. \tag{42}$$

B Theoretical Details

B.1 Transformation Matrices in (10)

The coordinate transformation from the original variables and equations to the country-level counterparts is expressed as

$$\boldsymbol{\lambda} = H \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = (H_a, H_b) \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = H_a \boldsymbol{a} + H_b \boldsymbol{b}, \qquad \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix} = \tilde{H}^\top \boldsymbol{F} = \begin{pmatrix} \tilde{H}_a^\top \boldsymbol{F} \\ H_b^\top \boldsymbol{F} \end{pmatrix} \tag{43}$$

with $H = (H_a, H_b)$ and $\tilde{H} = (\tilde{H}_a, H_b)$.

The submatrices H_a and $\tilde{H}_a$ in (10) are defined to satisfy (8) as follows:

$$H_a = \begin{pmatrix} \frac{1}{n_1}\mathbf{1}_{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_m}\mathbf{1}_{n_m} \end{pmatrix}, \qquad \tilde{H}_a = \begin{pmatrix} \mathbf{1}_{n_1} & & \\ & \ddots & \\ & & \mathbf{1}_{n_m} \end{pmatrix}$$

with $\mathbf{1}_{n_\alpha} = \underbrace{(1, \ldots, 1)}_{n_\alpha \text{ times}}{}^\top$. The column vectors of H_b are constructed to be orthogonal to those of both H_a and $\tilde{H}_a$, as follows:

$$H_b = \begin{pmatrix} W_1 & & \\ & \ddots & \\ & & W_m \end{pmatrix} \quad \text{with} \quad W_\alpha \equiv \begin{pmatrix} n_i - 1 & & & \\ -1 & n_i - 2 & & \\ \vdots & \ddots & \ddots & \\ \vdots & \ddots & -1 & 1 \\ -1 & \cdots & \cdots & -1 \end{pmatrix} \quad (\alpha = 1, \ldots, m).$$

B.2 Proof of Proposition 1

We derive an incremental governing equation by considering two solutions of (2): $(\boldsymbol{\lambda}, \boldsymbol{\tau})$ and $(\boldsymbol{\lambda} + \mathrm{d}\boldsymbol{\lambda}, \boldsymbol{\tau} + \mathrm{d}\boldsymbol{\tau})$, satisfying $\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \boldsymbol{F}(\boldsymbol{\lambda} + \mathrm{d}\boldsymbol{\lambda}, \boldsymbol{\tau} + \mathrm{d}\boldsymbol{\tau}) = \mathbf{0}$. This results in the following incremental governing equation:

$$\begin{aligned} \mathrm{d}\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau}) &\equiv \boldsymbol{F}(\boldsymbol{\lambda} + \mathrm{d}\boldsymbol{\lambda}, \boldsymbol{\tau} + \mathrm{d}\boldsymbol{\tau}) - \boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\tau}) \\ &= J(\boldsymbol{\lambda}, \boldsymbol{\tau})\mathrm{d}\boldsymbol{\lambda} + G(\boldsymbol{\lambda}, \boldsymbol{\tau})\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \mathbf{0} \end{aligned} \tag{44}$$

for infinitesimally small increments $\mathrm{d}\boldsymbol{\lambda}$ and $\mathrm{d}\boldsymbol{\tau}$. Here, $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ and $G = \partial \boldsymbol{F}/\partial \boldsymbol{\tau}$.

Using the coordinate transformation (43), the incremental equation (44) becomes

$$\begin{pmatrix} \mathrm{d}\boldsymbol{A} \\ \mathrm{d}\boldsymbol{B} \end{pmatrix} = \tilde{H}^\top \mathrm{d}\boldsymbol{F} = \tilde{H}^\top JH \begin{pmatrix} \mathrm{d}\boldsymbol{a} \\ \mathrm{d}\boldsymbol{b} \end{pmatrix} + \tilde{H}^\top G \mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \boldsymbol{0}. \tag{45}$$

Using (11):

$$\tilde{H}^\top JH = \begin{pmatrix} J_a & J_{ab} \\ J_{ba} & J_b \end{pmatrix}, \quad \tilde{H}^\top G = \begin{pmatrix} G_a \\ G_b \end{pmatrix},$$

we decompose (45) into two equations:

$$\mathrm{d}\boldsymbol{A} = J_a \,\mathrm{d}\boldsymbol{a} + J_{ab} \,\mathrm{d}\boldsymbol{b} + G_a \mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \boldsymbol{0}, \tag{46}$$

$$\mathrm{d}\boldsymbol{B} = J_{ba} \,\mathrm{d}\boldsymbol{a} + J_b \,\mathrm{d}\boldsymbol{b} + G_b \mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \boldsymbol{0}. \tag{47}$$

From (47), we have $\mathrm{d}\boldsymbol{b} = -J_b^{-1} J_{ba} \mathrm{d}\boldsymbol{a} - J_b^{-1} G_b \mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}$ The substitution of this equation into (46) leads to (12) with $\hat{J} = J_a - J_{ab} J_b^{-1} J_{ba}$ and $\hat{G} = G_a - J_{ab} J_b^{-1} G_b$.

B.3 Proof of Lemma 2

For the global–local system, using (8), we can rewrite the conservation law in (5) into

$$\sum_{\alpha=1}^{m} \sum_{j=1}^{n_\alpha} \lambda_\alpha^j = \sum_{\alpha=1}^{m} a_\alpha = 1 \tag{48}$$

and the relation in (4) for the governing equation into

$$\sum_{\alpha=1}^{m} \sum_{j=1}^{n_\alpha} F_\alpha^j = \sum_{\alpha=1}^{m} A_\alpha = 0. \tag{49}$$

From (48) and (49), we have $\sum_{\alpha=1}^{m} \mathrm{d}a_\alpha = 0$ and $\sum_{\alpha=1}^{m} \mathrm{d}A_\alpha = 0$. From this equation, we have the third relation: $\sum_{\alpha=1}^{m} \frac{\partial A_\alpha}{\partial \tau_k} = 0$.

B.4 Proof of Proposition 2

In the m-dimensional reduced equation (12), $\mathrm{d}\boldsymbol{A} = \hat{J}\mathrm{d}\boldsymbol{a} + \hat{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}$, we can simultaneously reduce $\mathrm{d}\boldsymbol{a}$, $\mathrm{d}\boldsymbol{A}$, and $\hat{G}$ by using an $m \times (m-1)$ matrix:

$$P = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix}. \tag{50}$$

In preparation for this reduction, we introduce the following notations

$$\begin{aligned} \mathrm{d}\tilde{\boldsymbol{a}} &= (\mathrm{d}a_1, \ldots, \mathrm{d}a_{m-1})^\top, \\ \mathrm{d}\tilde{\boldsymbol{A}} &= (\mathrm{d}A_1 - \mathrm{d}A_m, \ldots, \mathrm{d}A_{m-1} - \mathrm{d}A_m)^\top, \\ \bar{G} &= \{g_{\alpha k} \mid \alpha = 1, \ldots, m-1;\ k = 1, \ldots, p\}. \end{aligned}$$

The reduction of $\boldsymbol{a}$, $\boldsymbol{A}$, and $\hat{G}$ is formalized in Lemma 3.

Lemma 3.

$$\mathrm{d}\boldsymbol{a} = P\mathrm{d}\tilde{\boldsymbol{a}}, \quad \mathrm{d}\tilde{\boldsymbol{A}} = P^\top \mathrm{d}\boldsymbol{A}, \quad \hat{G} = P\bar{G}. \tag{51}$$

Proof. First, from the first relation of (14), we have $\mathrm{d}a_m = -\sum_{\alpha=1}^{m-1} \mathrm{d}a_\alpha$. Then, using the expression of P in (50), we obtain

$$\begin{aligned} \mathrm{d}\boldsymbol{a} &= (\mathrm{d}a_1, \ldots, \mathrm{d}a_m)^\top \\ &= (\mathrm{d}a_1, \ldots, \mathrm{d}a_{m-1}, -\mathrm{d}a_1 - \cdots - \mathrm{d}a_{m-1})^\top \\ &= P(\mathrm{d}a_1, \ldots, \mathrm{d}a_{m-1})^\top = P\mathrm{d}\tilde{\boldsymbol{a}}. \end{aligned}$$

Next, using the second relation of (14), we can suppress one equation and introduce an $(m-1)$-dimensional vector:

$$\mathrm{d}\tilde{\boldsymbol{A}} = (\mathrm{d}A_1 - \mathrm{d}A_m, \ldots, \mathrm{d}A_{m-1} - \mathrm{d}A_m)^\top = P^\top \mathrm{d}\boldsymbol{A}.$$

Finally, from Lemma 2 and $g_{\alpha k} = \frac{\partial A_\alpha}{\partial \tau_k}$, we have $g_{mk} = -\sum_{\alpha=1}^{m-1} g_{\alpha k} \quad (k = 1, \ldots, p)$. It then follows that

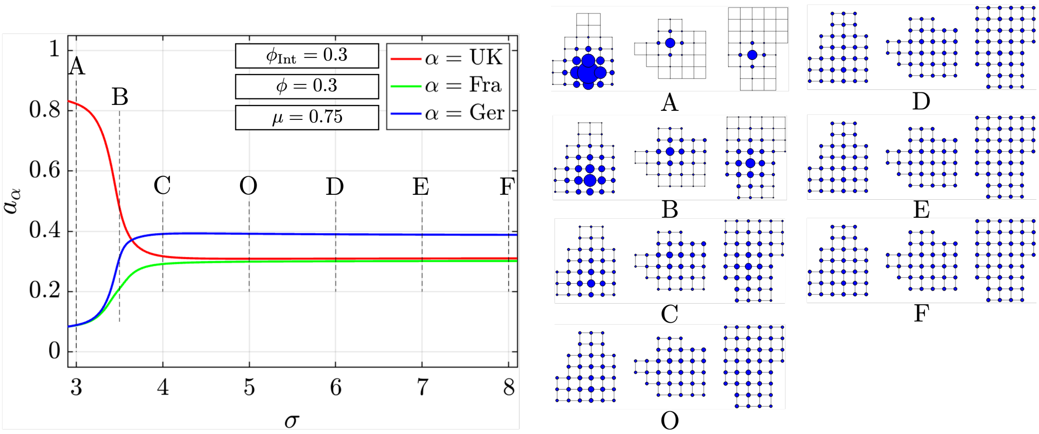
$$
\begin{aligned}
P\bar{G} &= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix} \{g_{\alpha k} \mid \alpha = 1, \dots, m-1;\ k = 1, \dots, p\} \\
&= \begin{pmatrix} & \bar{G} & \\ -\sum_{\alpha=1}^{m-1} g_{\alpha 1} & \cdots & -\sum_{\alpha=1}^{m-1} g_{\alpha p} \end{pmatrix} = \begin{pmatrix} & \bar{G} & \\ g_{m1} & \cdots & g_{mp} \end{pmatrix} \\
&= \{g_{\alpha j} \mid \alpha = 1, \dots, m;\ k = 1, \dots, p\} = \hat{G}.
\end{aligned}
$$

Therefore, we obtain $\hat{G} = P\bar{G}$. □

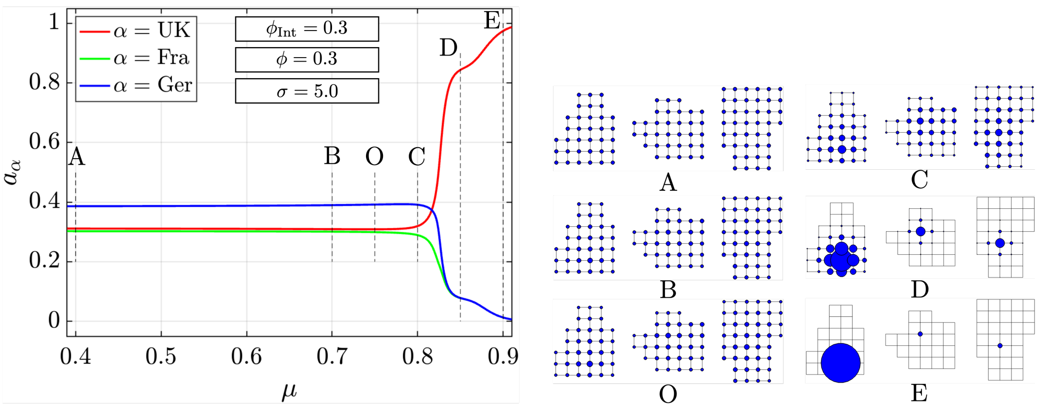
Using $\mathrm{d}\boldsymbol{A} = \hat{J}\mathrm{d}\boldsymbol{a} + \hat{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}$ in (12) and (51) in Lemma 3, we obtain

$$
\begin{aligned}
\mathrm{d}\tilde{\boldsymbol{A}} = P^\top \mathrm{d}\boldsymbol{A} &= P^\top(\hat{J}\mathrm{d}\boldsymbol{a} + \hat{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}) \\
&= P^\top(\hat{J}P\mathrm{d}\tilde{\boldsymbol{a}} + P\bar{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.}) \\
&= P^\top\hat{J}P\mathrm{d}\tilde{\boldsymbol{a}} + P^\top P\bar{G}\mathrm{d}\boldsymbol{\tau} + \text{h.o.t.} = \mathbf{0}.
\end{aligned}
$$

By setting $\tilde{J} = P^\top\hat{J}P$ and $\tilde{G} = P^\top P\bar{G}$, we obtain the relation (17).



(a) σ

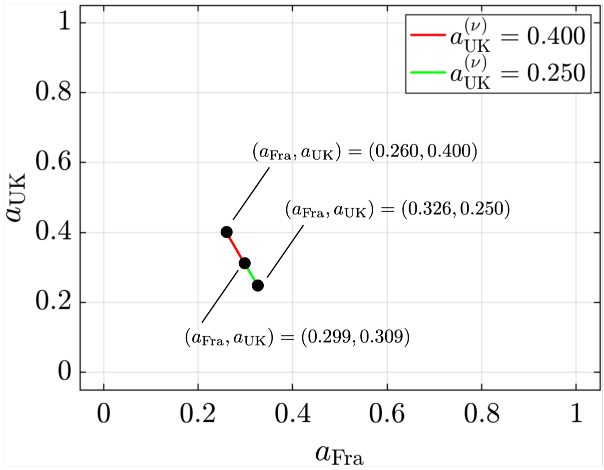


(b) μ

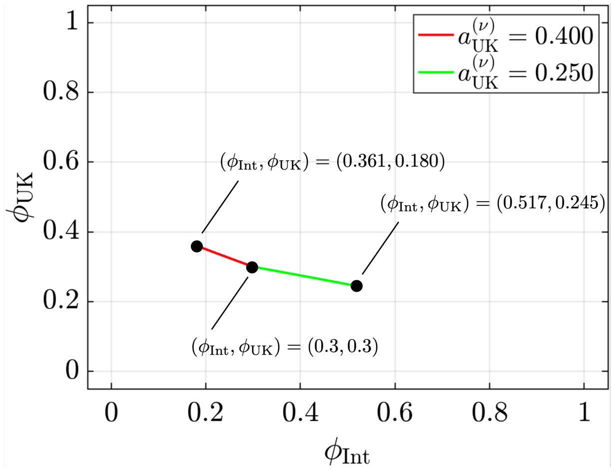
Figure 13: Influence of economic parameters σ and μ

C Influence of Economic Parameters

We investigate the influence of σ and μ under the setting $\phi_{\mathrm{Int}} = \phi = 0.3$. In the σ–a_{UK} curve in Fig. 13(a), a_{UK} remains nearly flat for $\sigma > 4$ but increases sharply as σ falls below 4. Figure 13(b) shows that a_{UK} remains nearly constant for $\mu < 0.8$ but rises sharply when $\mu > 0.8$. For $\sigma < 4$ or $\mu > 0.8$, excessive agglomeration occurs at the international trade hub of each country, particularly in London, UK. Accordingly, these parameter settings are considered unsuitable. In contrast, no excessive agglomeration occurs when $\sigma > 4$ and $\mu < 0.8$. Thus, the present setting of $(\sigma, \mu) = (5.0, 0.75)$ was chosen to qualitatively capture the population distribution within this parameter range.



(a) $(a_{\mathrm{UK}}, a_{\mathrm{Fra}})$



(b) $(\phi_{\mathrm{Int}}, \phi_{\mathrm{UK}})$

Figure 14: Convergence process toward the target population distribution $\boldsymbol{a} = (0.40, 0.26, 0.34)$ and $(0.25, 0.326, 0.424)$

D Details of the Numerical Inverse Analysis

We present an iterative procedure for inverse analysis. Given the initial values $(\boldsymbol{a}, \boldsymbol{\tau}) = (\boldsymbol{a}^{(0)}, \boldsymbol{\tau}^{(0)})$, and defining $\mathrm{d}\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{a}}^* - \tilde{\boldsymbol{a}}^{(0)}$ and $\mathrm{d}\boldsymbol{\tau} = \boldsymbol{\tau}^* - \boldsymbol{\tau}^{(0)}$, we rewrite (18) to approximate $\boldsymbol{\tau}^*$ as follows:

$$\boldsymbol{\tau}^* \approx \boldsymbol{\tau}^{(0)} - \tilde{G}^{-1}(\boldsymbol{a}^{(0)}, \boldsymbol{\tau}^{(0)})\tilde{J}(\boldsymbol{a}^{(0)}, \boldsymbol{\tau}^{(0)})(\tilde{\boldsymbol{a}}^* - \tilde{\boldsymbol{a}}^{(0)}).$$

The accuracy of this approximation can be improved by using the following iterative procedure:

$$\boldsymbol{\tau}^{(\nu+1)} \approx \boldsymbol{\tau}^{(\nu)} - \tilde{G}^{-1}(\boldsymbol{a}^{(\nu)}, \boldsymbol{\tau}^{(\nu)})\tilde{J}(\boldsymbol{a}^{(\nu)}, \boldsymbol{\tau}^{(\nu)})(\tilde{\boldsymbol{a}}^* - \tilde{\boldsymbol{a}}^{(\nu)}) \quad (\nu = 0, 1, 2, \ldots). \tag{52}$$

We conduct an inverse analysis to identify the combination of ϕ_{UK} and ϕ_{Int} that yields the target country-level population distribution $\boldsymbol{a} = \boldsymbol{a}^*$ ($\boldsymbol{a} = (a_{\mathrm{UK}}, a_{\mathrm{Fra}}, a_{\mathrm{Ger}})$). For example, by targeting the point $\boldsymbol{a}^* = (0.40, 0.26, 0.34)$, we identify the parameter set $(\phi_{\mathrm{UK}}, \phi_{\mathrm{Int}})$ that yields this distribution. We employ the iteration in (52), starting from an initial parameter set $(\phi_{\mathrm{UK}}^{(0)}, \phi_{\mathrm{Int}}) = (0.3, 0.3)$, which corresponds to the initial population distribution $\boldsymbol{a}^{(0)} = (0.309, 0.299, 0.392)$, referred to as the origin O (cf. Fig. 5). The iteration successfully converged to the target point A with parameter values $(\phi_{\mathrm{Int}}, \phi_{\mathrm{UK}}) = (0.188, 0.355)$ as shown in Fig. 14. A second target point, $\boldsymbol{a}^* = (0.25, 0.326, 0.424)$, was also successfully pinpointed.