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Abstract

Water markets even though not perfect and require a lot of effort to establish are considered as a robust tool to address water management issues around the world. However, the existing literature does not provide an optimal water resource management policy. To create a perfect water market, the government needs to identify the potential number of suppliers/producers and consumers of water against various extraction/supply/production rates of water, i.e., to identify a supply and a demand curve for number of suppliers/producers of water against each production rate in economy. This article presents a theory which is practically applicable for an optimal dynamical water resource management policy. (JEL H20, H23, H27)

Keywords: Water market, Production rate, Dynamic efficiency, Adjustment path, Equilibrium

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1 Introduction

Water is a scarce natural resource upon which the very existence of life depends. The water on earth is in abundance, however, only 0.3 percent of that is usable, and the rest, i.e., 99.7 percent is in the soil, icecaps, oceans, and floating in the atmosphere. A huge fraction of usable water has still not been made available for use. The prospects of water resource management are formidable including financial, regulatory, and institutional hindrances regarding policy formulation. A huge fraction of population still does not have access to safe drinking water. Without appropriate policy measures, water resource management in an optimal manner is not achievable.

Agricultural water management is the use of water for agricultural purposes in an optimal manner, i.e., to provide crops and animals the water they need to enhance productivity, and at the same time avoid wastage of water, and save it for other purposes including ecosystem balance. Around 70 percent of global freshwater is consumed for agricultural purposes, however, water use efficiency is less than 50 percent in majority of countries. Due to lack of proper management, changing environment, and wasteful utilization of water, fresh water supply has been increasingly getting scarce. Downward trends both in quality and quantity of water in various parts of the world are daunting challenges both for safe drinking water and sustainability of ecological balance. To address these issues, efficiency in water use is required.

For efficiency in water use, the government needs to create a perfect water market on the principle of economic efficiency. This is essential to address overuse of water on part of free riders. A price attached to the use can be instrumental in demand management. The other component of market is supply side, which needs to be augmented. It requires engineering and/or infrastructure solutions to enhance water supply, such as construction of dams, weirs, and desalination, etc. Formal water markets involve the transformation of water public property rights to one where some water use rights are divisible, transferable, privately managed that can be bought or sold (in whole or part). Creation of a water market can allocate resources efficiently. The first stage involves the establishment of enabling institutions, e.g., this includes having available information on current and sustainable (capped) water extractions, hydrology, regulations, legislation, and enforcement to govern water markets. The second stage involves trade facilitation including the assessment of trade benefits, monitoring supply of water, and reduction in transaction costs. The third stage involves revisiting and reform of existing water markets. Water markets even though not perfect and require a lot of effort to establish are considered as a robust tool to address water management issues around the world. However, the existing literature does not provide an optimal water resource management policy.

Weinberg, Kling and Wilen (1993) show that although water markets will not generally achieve a least-cost solution, they may be a practical alternative to economically efficient, but informationally intensive, environmental policies such as Pigouvian taxes. Bjornlund and McKay (1998) shows that more efficient irrigators are willing to pay a higher price for water, whereas the least efficient farmers are willing to sell at a lower price, showing that the buyers with high value of marginal product are willing to pay a price in excess of the value of the income generated by the sellers with low value of marginal product. Carey, Sunding and Zilberman (2002) considers the allocation of water by markets that are only imperfectly developed, in which prices are not publicly known and in which there is no centralized trading location. Romano and Leporati (2002) examines the distributive impacts on the relevant population (in particular on the poor and the most vulnerable groups. Bjornlund (2003) discusses the operational mechanism of a water exchange in Victoria, Australia, and analyses the outcome of the first five years of operation. Nieuwoudt and Armitage (2004) studies demand-side responses to water allocation in two irrigation districts in South Africa by investigating how water markets can lead to more efficient water allocation and use. Gómez-Limón and Martinez (2006) develops a multi-criteria methodology to simulate irrigation water markets at basin level. Brown (2006) shows that much more water changes hands via leases than via sales of water rights. Chong and Sunding (2006) advocates transferable water rights. In Brennan (2006), the nature of the seasonal water market is examined using a theoretical model and empirical evidence from the Victorian market. In van Heerden, Blignaut and Horridge (2008), a static computable general equilibrium model of South Africa is adapted to compare new taxes on water demand by two industries, namely forestry, and irrigated field crops. In Zaman, Malano and Davidson (2009), the integration of an economic trading model with a hydrologic water allocation model is discussed. Hanak and Stryjewski (2012) provides an overview of the policy context for water marketing and the related practice of groundwater banking and summarizes trends in both areas. Wheeler, Garrick, Loch and Bjornlund (2013) shows how Australia provides a leading example of a government buying back water for the environment. Bakker (2014) reviews the literature relevant to market environmentalism in the water sector, focusing on five themes: the privatization of resource ownership and management, the commercialization of resource management organizations, the environmental valuation and pricing of resources, the marketization of trading and exchange mechanisms, and the liberalization of governance. Wheeler, Loch, Crase, Young and Grafton (2017) attempts to fill the existing water market development gap and provide an initial framework (the water market readiness assessment (WMRA)) to describe the policy and administrative conditions/reforms necessary to enable governments/jurisdictions to develop water trading arrangements that are efficient, equitable and within sustainable limits.

In order to create a perfect water market, the government needs to identify the potential number of suppliers/producers and consumers of water against various extraction/supply/production rates of water, i.e., to identify a supply and a demand curve for number of suppliers/producers of water against each production rate in economy. Graph A in figure 1 shows the number of producers of water on x-axis, and the production rate on y-axis. The upward sloping curve is the supply curve for number of producers (both private and public) of water against each production rate, and the downward sloping curve is the demand curve. Graph B illustrates the supply and demand of quantity of water against various prices, and both supply and demand collectively determine the equilibrium price. Both graphs are connected, i.e., if perpendiculars from various points on demand and supply curves are drawn on x, and y-axes in graph A, the areas correspond to the abscissas/horizontal coordinates for demand and supply curves in graph B. As number of producers of water multiplied by the production rate in graph A determines the quantity of water supplied and demanded at various prices in graph B, both graphs and hence equilibria occur simultaneously. However, the government has a non-tax revenue constraint for production contracts/leasing out water extraction facilities/tradable licenses fee, which needs to be satisfied for an optimal water production/supply level, therefore, the design of policy involves an order for graph B, and A, i.e., to first derive an optimal level of contract/lease/license fee subject to the non-tax revenue constraint based on graph B, followed by a policy design based on graph A for an optimal number of producers of water, and the production rate subject to the constraint imposed by the optimal policy in graph B, i.e., the change in inventory/storage of water per unit time. For an optimal contract/lease/license fee, the efficiency loss in post-fee equilibrium as well as that during the adjustment of water market is minimized subject to the fee revenue constraint. When the government leases out the water extraction facility, the producers' cost becomes equal to the production cost plus the lease/contract/license fee, which pushes the water market out of equilibrium (assuming that before the government adopted the leasing out/contract/license fee, water demand was equal to water supply, and the equilibrium was inefficient due to an externality of over-use of water). Both supply and demand of water adjust over time and the market attains the final equilibrium. The mechansim for adjustment of water market is based on the presumption that when the market is out of equilibrium, the decisions of producers and consumers are not coordinated at current price. The efficiency losses during the water market adjustment must be taken into account to find an optimal policy. The optimal fee policy based on graph B decides the constraint for an optimal policy based on graph A, i.e., to find an optimal number of producers of water and the production rate by minimizing the social damage in terms of inadequate/excessive number of producers/suppliers of water in initial equilibrium as well as during the adjustment process to the final equilibrium subject to a change in water inventory/storage per unit time (a constraint determined by derivation of an optimal policy in graph B). As soon as the government adopts a policy to vary the number of water producers/suppliers and the production/extraction rate, an equilibrium does not result instantaneously, and rather the market of water producers undergo an adjustment mechanism to achieve the final equilibrium, i.e., where the supply and demand of water producers become equal.

While deriving an optimal policy for number of producers based on graph A, social damage both in initial equilibrium as well as during the adjustment process has been minimized subject to the change in water inventory/storage per unit time.

The remainder of this paper is organized as follows: Section 2 presents the water market model. Section 3 provides a solution of the water market model with a contract/lease/license fee. Section 4 derives a dynamically optimal water market policy. Section 5 presents the water producers and production rate model. Section 6 provides a solution of the water producers model with a production policy. Section 7 derives a dynamic optimal production policy for water producers model. Section 8 provides a summary of findings and conclusion. Appendix elaborates detailed mathematical steps in derivations in the text.

2 The Water Market Model

Suppose an imperfect water market exists and demand equals supply. There are four types of infinitely-lived market agents, i.e., a representative or a unit mass of producer who uses some engineering technique to make water useable and supplies that to the government who stores water and sells to the consumer of water. The government has a dual role, i.e., as a middleman between the producer and consumer, and also as a policy maker. Government sets the price equal to marginal cost (marginal cost of producer plus the marginal cost of storage by the government). The government as a policy maker increases the marginal cost of producer by imposing a contract/lease/license fee to control the overuse of water. When a shock happens to the water market, the market goes out of equilibrium, and the price is adjusted by government to bring the final equilibrium after shock. Although, government is more informed than other economic agents and can play a coordination role among agents, however, still the information of government is far from perfect regarding the new water supply and demand patterns after the shock, so the government adjusts the price based on the changing size of inventory/storage of water. Suppose the demand of water contracts in agriculture due to some innovative technology which improves the water efficiency of crops. As the supply and demand are not equal any longer, the market is out of equilibrium, and the excess supply will be reflected from a bigger inventory/storage volume of water with the government. Government will react to this bigger volume of storage by decreasing the price, which will lead the producer to produce less based on an altered profit maximizing condition after a price decrease. After some adjustment the water market will arrive at the final equilibrium with both a lower price and quantity than before. Equilibrium in water market is defined as follows:

(i) The producer maximizes profit and the consumer maximizes utility subject to their respective constraints.

(ii) Supply of water equals demand and the storage volume with government stays the same.

The equilibrium conditions, i.e., Routh–Hurwitz stability criterion, which provides a necessary and sufficient condition for the stability of a linear dynamical system have been elaborated in Section 3. Government as a middleman does not change the price during water market equilibrium, as it sets price equal to marginal cost and does not deviate from that. However, after a shock happens to the water market and equilibrium no longer holds, the government changes price only during the adjustment of the market to the final equilibrium and once the market attains the equilibrium again, the government stays put.

For mathematical purposes, the problem of each of the economic agents is considered and solved such that their objective is achieved, and the equations resulting from their individual solutions are solved simultaneously to arrive at the collective water market outcome. Linearity of supply and demand curves is assumed which is reasonable as far as the final equilbrium is not too off the initial equilibrium after shock.

2.1 Government-Water Storage

Government in the role of middleman buys water from producers at a price equal to the producers' marginal cost, stores it and sells to the consumers at a price equal to the marginal cost of production plus the marginal cost of storage. Storage is a phase between supply and demand of water. If the level of storage remains the same, it implies that the supply and demand rate of water is the same. A change in the level of storage implies a change in either of the rates, i.e., supply, demand or both (at different rates). If due to a supply shock, the supply curve shifts to right without a change in demand, the water storage goes up at the existing price (equal to marginal cost), and the price decreases to equalize the new marginal cost in final equilibrium. Similarly, if due to a demand shock, the demand curve shifts to right while there is no change in supply, the size of water storage reduces at current price, and the price increases to the new marginal cost in the final equilibrium. This implies that a price change is inversely related to water storage change, ceteris paribus. If both supply and demand curves shift but the water storage size stays put, the price of water will also remain the same. Water storage is central to both supply and demand shocks as each shock operates through a change in size of water storage.

The following mechanism is operative to bring about the price changes described above: The government maintains a water storage through buying water from producers and selling to consumers. It costs government more to maintain a bigger storage size. If supply and demand rates of water are the same, the size of water storage does not change, and the price of water also stays put. Suppose as a result of a technological advancement, the marginal cost of supply of water decreases, and the production/extraction rate of water goes up, whereas demand remains the same. The water market is no longer in equilibrium, and the water storage size increases at the current price. As the government's marginal cost of storage has gone up (while that of producer has gone down, and the total marginal cost has also decreased, i.e., the marginal cost of producer plus the government's marginal cost of storage), the government will try to reduce the water storage size by reducing the price to bring demand up along demand curve. After making price follow some adjustment, the government will finally set the price equal to the new marginal cost and the market will settle at final equilibrium Mathematically, the movement of price of water by government as middleman is captured as follows:

$$
\begin{aligned}
\textit{Price change} &\propto \textit{change in water storage.}\\
P &= \textit{price change.}\\
M_B &= m_B - m_{Bs} = \textit{change in water storage,}\\
m_B &= \textit{water storage at time } t,\\
m_{Bs} &= \textit{water storage in steady state equilibrium.}\\
\textit{Input} - \textit{output} &= \frac{dm_B}{dt} = \frac{d(m_B - m_{Bs})}{dt} = \frac{dM_B}{dt},\\
\text{or } M_B &= \int (\textit{input} - \textit{output})\, dt.\\
\textit{Price change} &\propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or}\\
P &= -K_{Bm} \int (\textit{supply rate} - \textit{demand rate})\, dt.
\end{aligned}
$$

K_{Bm} is proportionality constant; *supply* and *demand rates* are quantity of water per unit time. The negative sign reflects that when (*supply rate* $-$ *demand rate*) is positive, price goes down. After rearranging the above expression, we get:

$$\int (\textit{supply rate} - \textit{demand rate})\, dt = -\frac{P}{K_{Bm}}, \text{ or}$$

$$\int (w_{Bi} - w_{B0})\, dt = -\frac{P}{K_{Bm}}, \tag{1}$$

$$
\begin{aligned}
w_{Bi} &= \textit{supply rate},\\
w_{B0} &= \textit{demand rate},\\
K_{Bm} &= \textit{dimensional constant.}
\end{aligned}
$$

Suppose at $t = 0$, *supply rate* $=$ *demand rate*, i.e., market is in equilibrium, substituting which, eq. (1) becomes as follows:

$$\int (w_{Bis} - w_{B0s})\, dt = 0. \tag{2}$$

The subscript s stands for steady state equilibrium. $P = 0$ when market is in equilibrium. Subtracting eq. (2) from (1) gives:

$$\int (w_{Bi} - w_{Bis})\, dt - \int (w_{B0} - w_{B0s})\, dt = -\frac{P}{K_{Bm}}, \text{ or}$$

$$\int (W_{Bi} - W_{B0})\, dt = -\frac{P}{K_{Bm}}, \tag{3}$$

where $w_{Bi} - w_{Bis} = W_{Bi} = \textit{change in supply rate},$

$w_{B0} - w_{B0s} = W_{B0} = \textit{change in demand rate.}$

P, W_{Bi} and W_{B0} are deviation variables, i.e., they reflect a deviation from equilibrium value, which implies that their initial values are zero. Eq. (3) can also be expressed as:

$$P = -K_{Bm} \int W_B dt = -K_{Bm} M_B, \tag{4}$$

where $W_B = W_{Bi} - W_{B0}$. If price of water changes on account of an input other than a change in water storage volume, eq. (4) can be expressed as:

$$P = -K_{Bm} \int W_B dt + B_B = -K_{Bm} M_B + B_B. \tag{4a}$$

This is due to the fact that in a linear dynamical model, inputs can get added. The water storage volume can also get an exogenous shock which is not the same as the price feedback.

2.2 Water Extractor/Producer/Supplier

The water extractor/producer/supplier's problem is to maximize present discounted value of future stream of profits. The zero value, i.e., the present value for $t = 0$, is as follows:

$$V(0) = \int_0^\infty \left[\alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t)\right] e^{-\rho_{Bp} t} dt, \tag{5}$$

α denotes the market price fraction charged by water extractor to government. ρ_{Bp} reflects the discount rate; $l(t)$ (labor) and $i(t)$ (level of investment) are *control variables* and $k(t)$ is *state variable*. Water extractor's problem can be written as:

$$\underset{\{l(t),i(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t)\right] e^{-\rho_{Bp} t} dt,$$

subject to the constraints that

$\dot{k}(t) = i(t) - \delta k(t)$ (state equation, describing how state variable changes with time),

$k(0) = k_s$ (initial condition),

$k(t) \geq 0$ (non-negativity constraint on state variable),

$k(\infty)$ free (terminal condition).

The expression for current-value Hamiltonian is as follows:

$$\widetilde{H} = \alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t) + \mu_{Bp}(t)\left[i(t) - \delta k(t)\right]. \tag{6}$$

Maximizing conditions are given below:

(*i*) $l^*(t)$ and $i^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial l} = 0$ and $\frac{\partial \widetilde{H}}{\partial i} = 0$,

(*ii*) $\dot{\mu_{Bp}} - \rho_{Bp}\mu_{Bp} = -\frac{\partial \widetilde{H}}{\partial k}$,

(*iii*) $\dot{k}^* = \frac{\partial \widetilde{H}}{\partial \mu_{Bp}}$ (this just gives back the state equation),

(*iv*) $\lim_{t\to\infty} \mu_{Bp}(t) k(t) e^{-\rho_{Bp} t} = 0$ (the transversality condition).

First two conditions are:

$$\frac{\partial \widetilde{H}}{\partial l} = 0, \tag{7}$$

$$\frac{\partial \widetilde{H}}{\partial i} = 0, \tag{8}$$

and

$$\dot{\mu_{Bp}} - \rho_{Bp}\mu_{Bp} = -\frac{\partial \widetilde{H}}{\partial k}. \tag{9}$$

After a price increase, the water extractor's profit maximizing condition gets modified and prompts him to supply more water (details in appendix). Let $p =$ market price of water at which government supplies water to consumers, $c_B =$ a reference/feasible minimum price for water extractor to decide whether to operate or not.

$$W_{Bp} = \textit{Change in water extraction/production volume due to change in price.}$$

The condition $p - c_B \geq 0$ provides the water extractor/producer an incentive to supply more water, i.e.,

$$W_{Bp} \propto \alpha(p - c_B), \text{ or}$$

$$W_{Bp} = K_{Bp}(p - c_B). \tag{10}$$

When the water market is in equilibrium, $W_{Bp} = 0$, i.e.,

$$0 = K_{Bp}(p_s - c_{Bs}). \tag{11}$$

K_{Bp} is a proportionalityconstant. p_s and c_{Bs} reflect the equilibrium values. If we subtract eq. (11) from eq. (10), we get:

$$W_{Bp} = K_{Bp}\left[(p - p_s) - (c_B - c_{Bs})\right] = -K_{Bp}\left(C_B - P\right) = -K_{Bp}\varepsilon_B, \tag{12}$$

W_{Bp}, C_B and P reflect corresponding deviation values from those at the steady state.

2.3 Consumers of Water

There are two major types of consumers of water, i.e., the producers involved in production activities using water as an input, and the final consumer. The problems of both types of consumers are discussed below:

Producers Using Water as an Input:

The producer of a commodity using water as an input has a problem of maximizing present discounted value of future streams of profits. The zero value, i.e., the present value for $t = 0$, is as follows:

$$V(0) = \int_0^\infty \left[p_{Bc}(t)F\left(K\left(t\right), L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t)\right]e^{-rt}dt, \tag{13}$$

p_{Bc} is price of commodity being produced by the producer; r reflects discount rate. $L(t)$ (labor input), $I(t)$ (investment), and $w_{Bc}(t)$ (quantity of water as an input) are *control variables* and $K(t)$ is the *state variable*. The producer's (as consumer of water) problem can be written as

$$\underset{\{L(t), I(t), w_{Bc}(t)\}}{Max} V(0) = \int_0^\infty \left[p_{Bc}(t)F\left(K\left(t\right), L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t)\right]e^{-rt}dt,$$

subject to the constraints that

$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, describing how the state variable changes with time),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (non-negativity constraint on state variable),

$K(\infty)$ free (terminal condition).

The expression for current-value Hamiltonian is as follows:

$$\widetilde{H} = p_{Bc}(t)F\left(K\left(t\right), L\left(t\right)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t) + \mu(t)\left[I(t) - \delta K(t)\right]. \tag{14}$$

Maximizing conditions are given below:

(i) $L^*(t)$, $I^*(t)$ and $w^*_{Bc}(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L} = 0$, $\frac{\partial \widetilde{H}}{\partial I} = 0$ and $\frac{\partial \widetilde{H}}{\partial w_{Bc}} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty} \mu(t)K(t)e^{-rt} = 0$ (the transversality condition).
First two conditions are:

$$\frac{\partial \widetilde{H}}{\partial L} = 0, \tag{15}$$

$$\frac{\partial \widetilde{H}}{\partial I} = 0, \tag{16}$$

$$\frac{\partial \widetilde{H}}{\partial w_{Bc}} = 0, \tag{17}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}. \tag{18}$$

After a water price increase, the producer using water as an input will reduce water consumption to satisfy profit maximization condition (see detail in appendix). If demand change is proportional to price change (or otherwise if linearization of demand schedule around equilibrium is a reasonable assumption), we have:

$$\textit{Change in demand} \propto P, \text{ or}$$

$$W_{Bc} = -K_{Bc}P. \tag{19}$$

W_{Bc} is deviation in demand with respect to the equilibrium value after a price change, i.e., P. K_{Bc} is proportionality constant, and the negative sign is reflective of the fact that when price increases, the demand of water goes down.

Final Consumer:

The consumer's problem is to maximize present discounted value of future stream of utilities. The zero value, i.e., the present value for $t = 0$, is as follows:

$$V(0) = \int_0^\infty U_{Bc}(w_{Bc}(t))e^{-\rho_{Bc}t}dt, \tag{20}$$

ρ_{Bc} reflects the discount rate, and $w_{Bc}(t)$, i.e., the amount the consumer chooses for consumption is *control variable.* Consumer's problem can be written as:

$$\underset{\{w_{Bc}(t)\}}{Max}\, V(0) = \int_0^\infty U_{Bc}(w_{Bc}(t))e^{-\rho_{Bc}t}dt,$$

subject to the constraints that

$\dot{a}(t) = r(t)a(t) + w(t) - p(t)w_{Bc}(t)$ (state equation, describing how state variable changes with time). $a(t)$ is asset holdings (a *state variable*); and $w(t)$ and $r(t)$ are time path of wages and return on assets respectively.
$a(0) = a_s$ (initial condition),
$a(t) \geq 0$ (non-negativity constraint on state variable),
$a(\infty)$ free (terminal condition).
The expression for current-value Hamiltonian is as follows:

$$\widetilde{H} = U_{Bc}(w_{Bc}(t)) + \mu_{Bc}(t)\left[r(t)a(t) + w(t) - p(t)w_{Bc}(t)\right]. \tag{21}$$

Maximizing conditions are given below:
(i) $w^*_{Bc}(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial w_{Bc}} = 0$,
(ii) $\dot{\mu_{Bc}} - \rho_{Bc}\mu_{Bc} = -\frac{\partial \widetilde{H}}{\partial a}$,
(iii) $\dot{a}^* = \frac{\partial \widetilde{H}}{\partial \mu_{Bc}}$ (this just gives back the state equation),
(iv) $\lim_{t\to\infty} \mu_{Bc}(t)a(t)e^{-\rho_{Bc}t} = 0$ (the transversality condition).
First two conditions are:

$$\frac{\partial \widetilde{H}}{\partial w_{Bc}} = U'_{Bc}(w_{Bc}(t)) - \mu_{Bc}(t)p(t) = 0, \tag{22}$$

and

$$\dot{\mu_{Bc}} - \rho_{Bc}\mu_{Bc} = -\frac{\partial \widetilde{H}}{\partial a} = -\mu_{Bc}(t)r(t). \tag{23}$$

If price of water goes up, the consumer's utility maximizing condition at current consumption level modifies to the following inequality:

$$\frac{\partial \widetilde{H}}{\partial w_{Bc}} = U'_{Bc}(w_{Bc}(t)) - \mu_{Bc}(t)p(t) < 0,$$

which reflects that after price goes up, the consumer will reduce consumption of water for utility maximization condition to get satisfied. If demand change is proportional to price change (or otherwise if linearization of demand schedule around equilibrium is a reasonable assumption), we have:

$$\textit{Change in demand} \propto P, \text{ or}$$

$$W_{Bc} = -K_{Bc}P. \tag{24}$$

W_{Bc} is deviation in demand with respect to the equilibrium value after a price change, i.e., P. K_{Bc} is proportionality constant, and the negative sign is reflective of the fact that when price increases, the demand of water goes down. The demand going down implies that people economize on use of water, and reduce wastages as they face a higher cost for wasting water.

3 Solution of the Water Market Model with a Contract/Lease/License Fee

From (4a), (12) and (24):

$$\begin{aligned} \frac{dP(t)}{dt} &= -K_{Bm}W_B(t), \\ W_{Bp}(t) &= -K_{Bp}\varepsilon_B(t), \\ \varepsilon_B(t) &= C_B(t) - P(t), \\ W_{Bc} &= -K_{Bc}P. \end{aligned}$$

In the absence of a shock, we have

$$W_B(t) = W_{Bp}(t) - W_{Bc}(t).$$

The above expressions imply that

$$\begin{aligned} \frac{dP(t)}{dt} &= -K_{Bm}\left[W_{Bp}(t) - W_{Bc}(t)\right] \\ &= -K_{Bm}\left[-K_{Bp}\varepsilon_B(t) + K_{Bc}P(t)\right] \\ &= -K_{Bm}\left[-K_{Bp}C_B(t) + (K_{Bp} + K_{Bc})P(t)\right], \end{aligned}$$

or

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}C_B(t). \quad (25)$$

If a per unit water extraction fee is imposed on producer at $t = 0$, i.e., $C_B(t) = T$, the above expression becomes

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}T. \quad (26)$$

The Routh–Hurwitz stability criterion (a necessary and sufficient condition for stability of a linear dynamical system depicted by the above differential equation) is as follows: $K_{Bm}(K_{Bp}+K_{Bc}) > 0$. As K_{Bm}, K_{Bp} and K_{Bc} are defined as positive numbers, the stability condition holds, which ensures that after a shock the water market arrives at a new equilibrium through some adjustment mechanism. If the fee is charged from buyer instead of producer per unit of water consumption, the producer will take into account the price faced by him/her, i.e.,

$$\varepsilon_B(t) = T - P(t), \tag{27}$$

which leads to the following expression:

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}T.$$

The above expression is the same as eq. (26), however, the solution/dynamic adjustment path will depend on initial conditions. The solution of eq. (26) with initial conditions of a producer's fee is as follows:

$$P(t) = C_1 + C_2 e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}. \tag{28}$$

Putting values of C_1 and C_2 in the above expression, we get:

$$P(t) = \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}. \tag{29}$$

When $t = 0$, $P(0) = 0$ (initial condition). When $t = \infty$, $P(\infty) = \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}$ (final value). In final equilibrium, supply equals demand, which has been verified in appendix.

4 A Dynamically Optimal Water Market Policy

Pre-policy water market equilibrium is inefficient, and the imposition of producer fee leads to an efficient equilibrium. However, there are some efficiency losses on the adjustment path of the water market to the new efficient equilibrium. After a fee is imposed on water producer, the supply of water shrinks, the market forces come into play and the water market adjusts to the final equilibrium. The price and quantity of water in final equilibrium are dependent on supply and demand elasticities. The level of water storage rises if supply is higher than demand and goes down otherwise. When demand and supply again become equal, the water market is in final equilibrium. When demand and supply are not equal, either water supply and/or consumption is being lost at that point in time. The total production and/or consumption lost in terms of quantity is the efficiency loss and can be expressed as follows:

$$EL_B = -\left[\int_{-\infty}^{0} W_{Bp}(\infty)dt + M(t)\right]. \tag{30}$$

After imposition of water fee, the supply of water shrinks by $K_{Bp}T$. As the demand of water has not yet changed, the level of water storage also decreases by $K_{Bp}T$. The water market is out of equilibrium, and drifts toward the final equilibrium through market forces. The price of water is changed by government to bring the final equilibrium. The government earns the following amount as producer fee revenue (PFR):

$$PFR = T\left[w_{Bpi}(0) - K_{Bp}\left\{T - P(t)\right\}\right]. \tag{31}$$

The problem of minimizing efficiency loss with T as a control variable subject to constraint that revenue from imposition of producer fee must be greater than or equal to G_B in a given time, is as follows:

$$\min_{T} EL_B \quad \text{s.t.} \quad PFR \geq G_B.$$

The constraint is binding. Lagrangian for the problem of minimizing efficiency loss is as follows:

$$\begin{aligned}
\mathcal{L} &= -\int_{-\infty}^{0} W_{Bp}(\infty)dt - M(t) + \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - P(t)\right\}\right]\right] \\
&= \int_{-\infty}^{0} \left[K_{Bp}T - \frac{K_{Bp}^2 T}{K_{Bp} + K_{Bc}}\right] dt \\
&+ \frac{1}{K_{Bm}}\left[\frac{K_{Bp}T}{K_{Bp} + K_{Bc}} - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}T\right] \\
&+ \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} + \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0} \frac{K_{Bp}K_{Bc}T}{K_{Bp} + K_{Bc}} dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}T}{K_{Bp} + K_{Bc}} - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}T\right] \\
&+ \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} + \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]\right].
\end{aligned}$$

Derivative of Lagrangian with respect to T leads to the following expression:

$$T = \frac{\lambda w_{Bpi}(0) - \left[\int_{-\infty}^{0} \frac{K_{Bp}K_{Bc}}{K_{Bp}+K_{Bc}} dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}\right]\right]}{2\lambda K_{Bp}\left[1 - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}}{K_{Bp}+K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right]}. \tag{32}$$

Similarly derivative of Lagrangian with respect to λ gives:

$$G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} + \frac{K_{Bp}T}{K_{Bp} + K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]. \tag{33}$$

Putting eq. (32) into (33), we obtain:

$$
\begin{aligned}
G_B &= \frac{\lambda w_{Bpi}^2(0) - w_{Bpi}(0)J_B}{2\lambda Q_B} - \left(\frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B}\right)^2 Q_B, \\
4\lambda^2 Q_B G_B &= 2\lambda^2 w_{Bpi}^2(0) - 2\lambda w_{Bpi}(0)J_B - \lambda^2 w_{Bpi}^2(0) - J_B^2 + 2\lambda w_{Bpi}(0)J_B, \\
4\lambda^2 Q_B G_B &= 2\lambda^2 w_{Bpi}^2(0) - \lambda^2 w_{Bpi}^2(0) - J_B^2, \\
4\lambda^2 Q_B G_B &= \lambda^2 w_{Bpi}^2(0) - J_B^2, \\
\text{where } T &= \frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B}, \\
Q_B &= K_{Bp}\left[1 - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}}{K_{Bp}+K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right], \\
J_B &= \int_{-\infty}^{0} \frac{K_{Bp}K_{Bc}}{K_{Bp}+K_{Bc}} dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}\right].
\end{aligned}
$$

This implies that

$$\left\{w_{Bpi}^2(0) - 4Q_B G_B\right\}\lambda^2 = J_B^2.$$

$$\lambda = \frac{J_B}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}.$$

Eq. (32) can also be written as

$$T = \frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B}. \tag{34}$$

After putting value of λ in above expression, we obtain:

$$T = -\frac{\frac{J_B w_{Bpi}(0)}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}} - J_B}{\frac{2Q_B J_B}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}},$$

$$T = \frac{w_{Bpi}(0) - \sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}{2Q_B}. \tag{35}$$

The second order condition shows that efficiency loss has been minimized (see appendix). Suppose government has a revenue target of $1000 to be generated through imposition of producer fee. The initial equilibrium value is 100, and the value of each parameter, i.e., K_{Bm}, K_{Bp} and K_{Bc} is equal to one. Plugging these values in eq. (35) yields

$$T = \frac{100 - \sqrt{10000 - 4000}}{2} = 11.27,$$

where $Q_B = 1 - 0.5 + 0.5e^{-2t}$, and at $t = 0$, $Q = 1$. $PFR = T\left[w_{Bpi}(0) - K_{Bp}\left\{T - P(t)\right\}\right] = 1000$. The optimal producer fee is \$11.27 per unit of water.

5 The Water Producers and Production Rate Model

Please refer to graph A in figure 1, where the *number of water producers per unit time* are plotted along x-axis, and the *water production rate*, i.e., *quantity of water produced/extracted per producer* is plotted along y-axis. The upward sloping curve is the supply curve for number of producers, i.e., the number of water producers/extractors the society can have (both from public and private sector) against each production/extraction rate. The positive relationship between the water production rate and the number of producers is on account of the fact that a higher production/extraction by the existing producers due to some incentives also encourages new entrants in water industry. The downward sloping curve is the demand curve (including both public and private demand) for number of water producers/extractors against each production rate. The negative relationship indicates that for a higher water production/extraction rate, the demand for number of producers is lower. The point of intersection of both curves denotes the equilibrium. At a production rate where demand is higher than supply, the production rate will go up until the number of water producers/extractors becomes equal on both curves. If supply is higher than demand at a certain production rate, the production rate will decrease until the number of producers are in equilibrium. Suppose the number of water producers on supply curve equals that on demand curve with an equilibrium water production/extraction rate. The following infinitely-lived economic agents are there: private and public sector in the role of having a demand for certain number of water producers against a production rate; a representative –or a unit mass of– water producers who produce/extract water to supply to the government as middleman; and public and private sector as a whole which supplies a certain number of water producers at a certain production rate. The adjustment mechanism for water production rate is based on the fact that there is a lack of coordination among economic agents regarding new supply and demand patterns regarding number of water producers against each production rate after a supply or demand shift. Suppose the supply and demand of number of water producers are in equilibrium. A rightward shift in demand occurs due to which the demand of water producers becomes greater than the supply at production rate before demand shock. There is a higher demand than supply regarding number of water producers. The existing producers will increase production/extraction rate and new entrants will enter water industry. This will lead to higher water production rate, and a higher number of producers in final equilibrium. The equilibrium is defined as given below:

(i) Producers maximize profit, public and private sector maximizes utility in the role of having a certain demand for water, and the public sector maximizes net benefit of public service for society, subject to their respective constraints mentioned in Section 5.
(ii) The supply of water producers equals demand and the production rate stays put in equilibrium. The equilibrium conditions are based on Routh–Hurwitz stability criterion (a necessary and sufficient condition for a linear dynamical system to be stable), and are mentioned in Section 6. The production rate for the public and private sector is given. Producer does not change water production/extraction rate during equilibrium, however, he/she does so during the phase of disequilibrium. The government formulates a policy to enhance or reduce water production/extraction either by shifting supply and/or demand schedules depending on the objectives to be achieved. The final equilibrium after implementation of policy does not result instantaneously, and rather the production rate, and number of water producers/extractors adjust over time to lead to final equilibrium. The basis of adjustment is self-interest by economic agents. Some social damage occurs during the adjustment of number of producers and the water production rate, which is defined as the sum of too many or too few water producers/extractors before the new equilibrium arrives. The new equilibrium is (more) efficient as compared to the initial equilibrium.
For mathematical derivation of results, the objectives of various economic agents are maximized subject to constraints and then the resulting expressions are solved simultaneously to find expression regarding the collective outcome of their individual and independent decisions in self-interest. Linearity of demand and supply schedules is assumed which is a reasonable assumption if both initial and final equilibriums are not too far from each other.

5.1 Water Extractor/Producer/Supplier

The water producer extracts water and supplies to the government for storage for onward supply to consumer. When the number of producers and production rate are in equilibrium, demand of water producers equals supply. If the number of producers change, that must be on account of a supply or demand shock, or both at different rates. The cumulative number of water producers, their entry rate (supply) and demand are linked as follows: When demand of water producers shifts to the right while supply remains the same, the cumulative number of water producers is lower than the demand at existing water extraction rate, the water production/extraction rate goes up to equalize supply and demand in final equilibrium. If supply shifts to the right while demand stays put, the cumulative number of water producers increases at the existing water extraction rate, and the production rate reduces to bring final equilibrium. The above discussion concludes that there exists a negative relationship between number of water producers and production/extraction rate. The following mechanism is at work: Suppose the number of water producers and the extraction rate are in equilibrium. If supply of producers shifts to the right due to a reduced marginal cost of production as it will lead to new firms' entry into water industry, the production rate goes down, and the demand for producers increases as a feedback effect of reduced production rate. The adjustment of number of producers and the production rate depends on how water producers react to the shock, and the direction and magnitude of shock. For mathematical illustration of the producer's choice, let us take into consideration profit maximizing decision of the water producer as follows:

5.1.1 One Time Period Problem

We consider profit maximization by producer for one time period where the water producer does not take into account future time periods. The purpose is to provide a simple intuition to the reader. A more complex dynamic problem is discussed later. Water producer's objective is as follows:

$$\Theta = U_c(c) - \varsigma_A(m_A(c, e_A)), \tag{36}$$

where
Θ = net benefit of water producer,
$U_c(c)$ = benefit of the producer by producing,
c = quantity of water extracted per producer (production rate in a dynamic setting),
m_A = total number of water producers in economy,
e_A = other factors which affect the total number of producers,
$\varsigma_A(m_A(c, e_A))$ = cost as a function of total number of producers in economy (increasing in number of producers).
The derivative of Θ with respect to c is given below:

$$U_c'(c) - \varsigma_A'(m_A(c, e_A))m_{A1}'(c, e_A) = 0. \tag{37}$$

If supply of water producers shifts to right due to a reduced water extraction cost faced by producers, new entrants get an incentive to enter water industry, and number of producers get out of equilibrium. With more water producers in number, the term $\varsigma_A'(m_A(c, e_A))$ is higher at existing value of c. As the term, $m_{A1}'(c, e_A)$ is a function of c, which has not changed yet, therefore, $m_{A1}'(c, e_A)$ is the same as before, and the water producer faces the following inequality as a modified profit maximization condition:

$$\frac{\partial \Theta}{\partial c} = U_c'(c) - \varsigma_A'(m_A(c, e_A))m_{A1}'(c, e_A) < 0, \tag{38}$$

which suggests that after supply shock, the water producer reduces production/extraction rate to maximize net benefit. If profit maximizing values of number of water producers are plotted against respective production/extraction rate, a downward sloping curve is obtained with number of water producers/extractors on x-axis, and production/extraction rate on y-axis. This is defined as *cumulative number of producers curve.*

5.1.2 Dynamic Optimization

The water producer's long run problem (dynamic context) is to maximize present discounted value of future stream of net benefits. The producer's present value at $t = 0$ is given below:

$$V(0) = \int_0^\infty [U_c(c) - \varsigma_A(m_A(c, e_A))]\, e^{-\varpi t} dt, \tag{39}$$

where ϖ is discount rate. $c(t)$ is *control variable*, and $m_A(t)$ is *state variable.* Net benefit maximization problem of water producer/extractor is as follows:

$$\underset{\{c(t)\}}{Max} V(0) = \int_0^\infty [U_c(c) - \varsigma_A(m_A(c, e_A))]\, e^{-\varpi t} dt,$$

subject to the following constraints:
$\dot{m_A}(t) = m'_{A1}(c(t), e_A(c(t), z_A))\dot{c}(t) + m'_{A2}(c(t), e_A(c(t), z_A))\ e'_{A1}(c(t), z_A)\dot{c}(t)$ (state equation, depicting how state variable changes with time; z_A denote exogenous variables),
$m_A(0) = m_{As}$ (initial condition),
$m_A(t) \geq 0$ (non-negativity constraint on state variable),
$m_A(\infty)$ free (terminal condition).
Current-value Hamiltonian is given below:

$$\widetilde{H} = U_c(c(t)) - \varsigma_A(m_A(c(t), e_A(c(t), z_A))) + \mu_A(t)\dot{c}(t) \left[\begin{array}{c} m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A))* \\ e'_{A1}(c(t), z_A) \end{array} \right]. \tag{40}$$

The maximizing conditions are listed below:
(i) $c^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial c} = 0$,
(ii) $\dot{\mu_A} - \varpi \mu_A = -\frac{\partial \widetilde{H}}{\partial m_A}$,
(iii) $\dot{m_A}^* = \frac{\partial \widetilde{H}}{\partial \mu_A}$ (this just gives back the state equation),
(iv) $\lim\limits_{t \to \infty} \mu_A(t) m_A(t) e^{-\varpi t} = 0$ (the transversality condition).
(i) and (ii) conditions are as follows:

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial c} &= U_c'(c(t)) - \varsigma_A'(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A)) * \\ e_{A1}'(c(t), z_A) \end{array} \right\} \\ &+ \mu_A(t)\dot{c}(t) * \left[\begin{array}{c} m_{A11}''(c(t), e_A(c(t), z_A)) + m_{A12}''(c(t), e_A(c(t), z_A)) e_{A1}'(c(t), z_A) + \\ m_{A21}''(c(t), e_A(c(t), z_A)) e_{A1}'(c(t), z_A) + m_{A22}''(c(t), e_A(c(t), z_A)) e_{A1}'^2(c(t), z_A) + \\ m_{A2}'(c(t), e_A(c(t), z_A)) e_{11}''(c(t), z_A) \end{array} \right] \\ &= 0. \end{aligned} \tag{41}$$

and

$$\dot{\mu_A} - \varpi \mu_A = -\frac{\partial \widetilde{H}}{\partial m_A} = \varsigma_A'(m_A(c(t), e_A(c(t), z_A))). \tag{42}$$

In steady state, $\dot{c}(t) = 0$, substituting which in eq. (41), the following expression is obtained:

$$U_c'(c(t)) - \varsigma_A'(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A)) * \\ e_{A1}'(c(t), z_A) \end{array} \right\} = 0.$$

Suppose a positive shock shifts supply to the right, then at current water extraction rate, the number of water extractors is higher, and the same is the case with the term $\varsigma_A'(m_A(c(t), e_A(c(t), z_A)))$. The term multiplying $\varsigma_A'(m_A(c(t), e_A(c(t), z_A)))$, i.e., $m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A)) e_{A1}'(c(t), z_A)$ is a function of water extraction rate which is the same as before. Therefore, water producer/extractor faces the following expression after shock:

$$\frac{\partial \widetilde{H}}{\partial c} < 0.$$

The water producer/extractor will reduce the extraction rate for maximizing net benefits in the dynamic context after supply shock. Hence, a negative relationship exists between number of water producers/extractors and extraction rate. If supply equals demand regarding number of water producers in economy, there is an equilibrium. However, if due to a shock the supply rate is no longer equal to demand rate, and the economic agents do not respond to the water extraction rate on account of a difference in supply and demand rates, the water extraction rate will keep on changing until the saturation point arrives. This explanation can be depicted mathematically as follows:

$$Water\ production/extraction\ rate\ change \propto change\ in\ number\ of\ producers.$$

$$C = production\ rate\ change.$$
$$M_A = m_A - m_{As} = change\ in\ number\ of\ producers,$$
$$m_A = number\ of\ producers\ at\ time\ t,$$
$$m_{As} = number\ of\ producers\ in\ steady\ state\ equilibrium.$$
$$Input - output = \frac{dm_A}{dt} = \frac{d(m_A - m_{As})}{dt} = \frac{dM_A}{dt},$$
$$\text{or } M_A = \int (input - output)\, dt.$$
$$Production\ rate\ change \propto \int (inflow/supply\ rate - required/demand\ rate)\, dt, \text{ or}$$
$$C = -K_c \int (inflow/supply\ rate - required/demand\ rate)\, dt,$$

K_c is proportionality constant; *inflow/supply* and *required/demand rates* are inflow of new entrants and demand of number of producers in water industry respectively. When (*inflow/supply rate* $-$ *required/demand ra* is positive, C is negative, i.e., the water production/extraction rate reduces. The above expression can also be written as:

$$\int (inflow/supply\ rate - required/demand\ rate)\, dt = -\frac{C}{K_c}, \text{ or}$$

$$\int (w_{Ai} - wA_0)\, dt = -\frac{C}{K_c}, \tag{43}$$

$$w_{Ai} = inflow/supply\ rate,$$
$$w_{A0} = required/demand\ rate,$$
$$K_c = dimensional\ constant.$$

At time $t = 0$, *inflow/supply rate* $=$ *required/demand rate*, and eq. (43) becomes:

$$\int (w_{Ais} - w_{A0s})\, dt = 0. \tag{44}$$

The subscript s is for steady state equilibrium, where $C = 0$. Subtracting eq. (44) from (43) gives:

$$\int (w_{Ai} - w_{Ais})\, dt - \int (w_{A0} - wA_{0s})\, dt = -\frac{C}{K_c}, \text{ or}$$

$$\int (W_{Ai} - W_{A0})\, dt = -\frac{C}{K_c}, \tag{45}$$

where $w_{Ai} - w_{Ais} = W_{Ai} = \textit{change in inflow/supply rate},$

$w_{A0} - w_{A0s} = W_{A0} = \textit{change in required/demand rate}.$

C, W_{Ai} and W_{A0} are deviation variables with zero initial value, as they indicate deviation from equilibrium values. Eq. (45) is given by:

$$C = -K_c \int W_A dt = -K_c M_A, \tag{46}$$

where $W_A = W_{Ai} - W_{A0}$. If C gets changed due to some other input, that can be added to the above expression as follows (inputs can get added in a linear dynamical system):

$$C = -K_c \int W_A dt + E_A = -K_c M_A + E_A. \tag{46a}$$

M_A gets affected due to feedback of water production/extraction rate, however, it can also have an exogenous input just like C.

5.2 Private Sector as a Supplier of Water Producers/Extractors

The public and private sectors both supply and demand water producers/extractors, however, only one of their roles is presented here. Total supply and demand is a sum of that of both public and private sectors. In this section, the role the private sector plays as a supplier of water producers/extractors is presented. The private sector has a problem of maximizing present discounted value of future stream of net benefit for economy, and present value at $t = 0$, is given below:

$$V(0) = \int_0^\infty \left[U_{pr}\left(n_{pr}\right) - \varsigma_{pr}(c\left(n_{pr}\right))\right] e^{-r_{pr}t} dt. \tag{47}$$

$U_{pr}\left(n_{pr}\right)$ is increasing in number of water producers/extractors, i.e., the private sector draws a higher utility, the more the number of water producers. $\varsigma_{pr}(c\left(n_{pr}\right))$ is their cost which is a positive function of the water extraction rate. The cost curve as a plot of cost against water production rate is concave downward, i.e., decreasing in slope.

r_{pr} is discount rate. $n_{pr}(t)$ denotes *control variable*, and $c(t)$ has been defined as the *state variable*. The private sector's problem is given below:

$$\underset{\{n_{pr}(t)\}}{Max} V(0) = \int_0^\infty \left[U_{pr}\left(n_{pr}\right) - \varsigma_{pr}(c\left(n_{pr}\right))\right] e^{-r_{pr}t} dt,$$

subject to the following constraints:

$\dot{c}(t) = c'(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right)$ (state equation, describing how the state variable changes with time),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),
$c(\infty)$ free (terminal condition).
Current-value Hamiltonian is given below:

$$\widetilde{H} = U_{pr}\left(n_{pr}(t)\right) - \varsigma_{pr}(c\left(n_{pr}\left(t\right)\right)) + \mu(t)\ c'(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right). \tag{48}$$

Maximizing conditions can be expressed as follows:
(i) $n_{pr}{}^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial n_{pr}} = 0$,
(ii) $\dot{\mu_{pr}} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c}$,
(iii) $\dot{c}^* = \frac{\partial \widetilde{H}}{\partial \mu_{pr}}$ (this just gives back the state equation),
(iv) $\lim\limits_{t\to\infty} \mu_{pr}(t)c(t)e^{-r_{pr}t} = 0$ (the transversality condition).
(i) and (ii) are given below:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} = U'_{pr}\left(n_{pr}(t)\right) - \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))\ c'\left(n_{pr}\left(t\right)\right) + \mu_{pr}(t)\ c''(n_{pr}\left(t\right))\dot{n}_{pr}\left(t\right) = 0, \tag{49}$$

and

$$\dot{\mu_{pr}} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right)). \tag{50}$$

During equilibrium, $\dot{n}_{pr}\left(t\right) = 0$, and we can express $\frac{\partial \widetilde{H}}{\partial n_{pr}}$ as follows:

$$U'_{pr}\left(n_{pr}(t)\right) - \varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))\ c'\left(n_{pr}\left(t\right)\right) = 0.$$

If water production/extraction rate increases, the term $\varsigma'_{pr}(c\left(n_{pr}\left(t\right)\right))$ decreases, and the private sector's first order condition for dynamic optimization gets modified to:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} > 0.$$

The private sector will increase supply of water producers after production rate shock. If supply curve is linear (or linearization around steady state is a reasonable assumption), and change in number of water producers supplied is directly proportional to water extraction rate, we get the following expression:

$$W_{pr}(t) = -K_{pr}\left[\epsilon\left(t\right) - C(t)\right] = -K_{pr}\eta(t), \tag{51}$$

where $\epsilon\left(t\right) = e - e_s$; e is a reference water production rate parameter. The private sector takes it as a reference for decision making with which the variation in production rate is compared. $W_{pr}(t)$ is change in number of water producers/extractors by private sector as a supplier of producers from initial equilibrium value, with an initial value of zero. Due to a time delay between change in water production/extraction rate and change in the number of producers supplied, a time lag term has been introduced in the above equation leading to:

$$W_{pr}(t) = -K_{pr}\eta(t - \tau_{d1}). \tag{52}$$

5.3 Public Sector/Government as a Demander of Water Producers/Extractors

This section presents the role of public sector as a demander of water producers/extractors. The private sector also acts as a demander and the total demand is the sum of demand of both public and private sectors. In the role of a demander, the public sector has the problem of maximizing present discounted value of future stream of net benefits for economy, and present value at $t = 0$, is given below:

$$V(0) = \int_0^\infty \left[U_{pu}\left(n_{pu}\right) - \varsigma_{pu}(c\left(n_{pu}\right))\right] e^{-r_{pu}t} dt, \tag{53}$$

$U_{pu}\left(n_{pu}\right)$ is public service benefit for economy, increasing in number of water producers/extractors and concave downward. $\varsigma_{pu}(c\left(n_{pu}\right))$ is their cost to encourage production, and is a positive function of the water production rate. The cost curve, i.e., a plot of cost against water production rate is concave upward, i.e., increasing in slope.

r_{pu} is discount rate. $n_{pu}(t)$ denotes *control variable*,and $c(t)$ has been defined as the *state variable*. The public sector's problem is given below:

$$\underset{\{n_{pu}(t)\}}{Max}\, V(0) = \int_0^\infty \left[U_{pu}\left(n_{pu}\right) - \varsigma_{pu}(c\left(n_{pu}\right))\right] e^{-r_{pu}t} dt,$$

subject to the following constraints:

$\dot{c}(t) = c'(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right)$ (state equation, describing how the state variable changes with time),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),

$c(\infty)$ free (terminal condition).

The current-value Hamiltonian for this case is

$$\widetilde{H} = U_{pu}\left(n_{pu}(t)\right) - \varsigma_{pu}(c\left(n_{pu}\left(t\right)\right)) + \mu_{pu}(t)\ c'(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right). \tag{54}$$

Maximizing conditions can be expressed as follows:

(i) ${n_{pu}}^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial n_{pu}} = 0$,

(ii) $\dot{\mu_{pu}} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c}$,

(iii) $\dot{c}^* = \frac{\partial \widetilde{H}}{\partial \mu_{pu}}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty} \mu_{pu}(t)c(t)e^{-r_{pu}t} = 0$ (the transversality condition).

(i) and (ii) are given below:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} = U'_{pu}\left(n_{pu}(t)\right) - \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right))\ c'\left(n_{pu}\left(t\right)\right) + \mu_{pu}(t)\ c''\left(n_{pu}\left(t\right)\right)\dot{n}_{pu}\left(t\right) = 0. \tag{55}$$

and

$$\dot{\mu_{pu}} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right)). \tag{56}$$

During equilibrium, $\dot{n}_{pu}\left(t\right) = 0$, and we can express $\frac{\partial \widetilde{H}}{\partial n_{pu}}$ as follows:

$$U'_{pu}\left(n_{pu}(t)\right) - \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right))\ c'\left(n_{pu}\left(t\right)\right) = 0.$$

If water production/extraction rate increases, the term $\varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right))$ increases, and the public sector's first order condition for dynamic optimization gets modified to:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} < 0.$$

The public sector will decrease demand of water producers after production rate shock. If demand curve is linear (or linearization around steady state is a reasonable assumption), and change in number of water producers demanded is directly proportional to water extraction rate, we get the following expression:

$$W_{pu}(t) = K_{pu}\left[\epsilon\left(t\right) - C(t)\right] = -K_{pu}C(t), \tag{57}$$

$W_{pu}(t)$ is change in number of water producers/extractors by public sector as a demander of producers from initial equilibrium value, with an initial value of zero. Due to a time delay between change in water production/extraction rate and change in the number of producers demanded, a time lag term has been introduced in the above equation leading to:

$$W_{pu}(t) = -K_{pu}C(t - \tau_{d2}). \tag{58}$$

6 Solution of the Water Producers Model with a Production Policy

We solve the model for $\tau_{d1} = \tau_{d2} = 0$. Eq. $(46a)$, (51), and (57) are reproduced as follows:

$$
\begin{aligned}
\frac{dC}{dt} &= -K_c W_A(t), \\
W_{pr}(t) &= -K_{pr}\left[\epsilon\,(t) - C(t)\right], \\
W_{pu}(t) &= -K_{pu} C(t), \\
W_A(t) &= W_1(t) - W_{pu}(t), \\
&= D(t) + W_{pr}(t) - W_{pu}(t). \\
\text{where } D(t) &= W_{Ai}(t) - W_{A0}(t).
\end{aligned}
$$

When there is no exogenous shock regarding number of water producers or production/extraction rate, $D(t) = 0$. Suppose government shifts supply leftward by adopting a policy of size A, i.e.,

$$
W_{pr}(t) = -K_{pr}\left[A - C(t)\right],
$$

which implies that

$$
\begin{aligned}
\frac{dC(t)}{dt} &= -K_c\left[W_{pr}(s) - W_{pu}(t)\right] \\
&= -K_c\left[K_{pr}C\,(t) - K_{pr}A + K_{pu}C(t)\right] \\
&= -K_c\left[-K_{pr}A + \left(K_{pr} + K_{pu}\right)C(t)\right].
\end{aligned}
$$

After rearranging, we obtain the following expression:

$$
\frac{dC(t)}{dt} + K_c(K_{pr} + K_{pu})C(t) = K_c K_{pr} A. \tag{59}
$$

The Routh–Hurwitz stability criterion for the dynamical system represented by the above expression is $K_c(K_{pr}+K_{pu}) > 0$. As K_c, K_{pr} and K_{pu} have been defined as positive numbers, $K_c(K_{pr}+K_{pu}) > 0$, which ensures that after a shock, a new equilibrium is arrived at, after following an adjustment path. The above expression is solved as given below:
The differential equation's characteristic function is given below:

$$
x + K_c(K_{pr} + K_{pu}) = 0,
$$

which has a single root as given below:

$$
x = -K_c(K_{pr} + K_{pu}).
$$

The complementary solution can be written as

$$C_c(t) = C_2 e^{-[K_c(K_{pr}+K_{pu})]t}.$$

The particular solution can be expressed as follows:

$$C_p(t) = C_1.$$

The solution is as given below:

$$C(t) = C_1 + C_2 e^{-[K_c(K_{pr}+K_{pu})]t}. \tag{60}$$

By substituting the above expression into the differential equation, we obtain the following expression:

$$-K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t} + K_c(K_{pr}+K_{pu})C_1 + K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t} = K_c K_{pr} A,$$

$$C_1 = \frac{K_{pr}A}{K_{pr}+K_{pu}}.$$

After substituting the initial conditions and value of C_1 in eq. (60), the following expression results:

$$C(0) = \frac{K_{pr}A}{K_{pr}+K_{pu}} + C_2 = 0,$$
$$C_2 = -\frac{K_{pr}A}{K_{pr}+K_{pu}}.$$

Plugging in the values of C_1 and C_2 in eq. (60) yields:

$$C(t) = \frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}. \tag{61}$$

The initial conditions are $t = 0$, $C(0) = 0$, and in the final steady state equilibrium, $t = \infty$, $C(\infty) = \frac{K_{pr}A}{K_{pr}+K_{pu}}$.

7 A Dynamic Optimal Production Policy for Water Producers Model

The equilibrium before adoption of water production policy needed to be improved upon, this is why government wanted to adopt a water policy. Also, there are some efficiency losses on the dynamic adjustment path to the new equilibrium. Adding the equilibrium and adjustment path inefficiencies, we get the total social damage which needs to be minimized. The government either shifts the demand or the supply curve through a production policy. Suppose it shifts supply leftward. The number of water producers/extractors and the production rate adjust over time to bring final equilibrium, at which the production/extraction rate is higher and the number of water producers is lower as compared to those in the initial equilibrium. If supply of water producers is higher than demand, the number of producers are excessive and vice versa. The excessive number or a shortage of water producers is the social damage at a certain point in time. By summing up the social damage in equilibrium and that on the dynamic adjustment path, the total damage in terms of number of water producers/extractors is obtained as follows:

$$SD = -\left[\int_{-\infty}^{0} W_{pr}(\infty)dt + M_A(t)\right]. \tag{62}$$

From eq. (51), a change in number of water producers/extractors on account of implementation of production policy is given below:

$$W_{pr}(t) = -K_{pr}\left[A - C(t)\right],$$
$$\text{or } w_{prf}(t) - w_{pri}(0) = -K_{pr}\left[A - C(t)\right],$$

where $w_{pri}(0)$ is number of water producers/extractors supplied in initial equilibrium and $w_{prf}(t)$ is new value after water production policy is adopted, as $W_{pr}(t)$ is a deviation variable, i.e., deviation from initial steady state value. A change in quantity of production/extraction per unit time is given below:

$$\Delta QP = A\left[w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right]. \tag{63}$$

The problem of minimizing the social damage subject to a change in quantity of production being greater than or equal to G_A $\left(\text{change in quantity of production per unit time} = \frac{dM_B}{dt}\right)$ can be expressed as follows:

$$\min_A SD \quad \text{s.t.} \quad \Delta QP \geq G_A\left(= \frac{dM_B}{dt}\right).$$

A is choice variable, i.e., the size of production policy. The constraint is binding, and Lagrangian can be expressed as follows:

$$\mathcal{L} = -M_A(t) - \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left\{w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right\}\right].$$

Expression from eq. (46a) is given below:

$$C(t) = -K_c M_A + E_A.$$

The value of E_A is found by imposing initial conditions as shown below:

$$\begin{aligned} C(0) &= -K_c M_A(0) + E_A, \\ 0 &= K_c K_{pr} A + E_A, \\ E_A &= -K_c K_{pr} A \end{aligned}$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) + K_c K_{pr} A\right].$$

Therefore, the Lagrangian can now be written as:

$$\begin{aligned} \mathcal{L} &= -\int_{-\infty}^{0} W_{pr}(\infty)dt - M_A(t) + \lambda\left[G_A - A\left\{w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right\}\right] \\ &= \int_{-\infty}^{0}\left[K_{pr}A - \frac{K_{pr}^2 A}{K_{pr}+K_{pu}}\right]dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right] \\ &+ \lambda\left[G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right] \\ &= \int_{-\infty}^{0}\frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}}dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right] \\ &+ \lambda\left[G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right]. \end{aligned}$$

Derivative of Lagrangian with respect to A leads to the following expression:

$$A = \frac{\lambda w_{pri}(0) - \left[\int_{-\infty}^{0}\frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}}dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right]\right]}{2\lambda K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \quad (64)$$

Similarly derivative of Lagrangian with respect to λ gives:

$$G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] = 0. \quad (65)$$

Putting eq. (64) into (65), we obtain:

$$
\begin{aligned}
G_A &= \frac{\lambda w_{pri}^2(0) - w_{pri}(0)J_A}{2\lambda Q_A} - \left(\frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A}\right)^2 Q_A,\\
4\lambda^2 Q_A G_A &= 2\lambda^2 w_{pri}^2(0) - 2\lambda w_{pri}(0)J_A - \lambda^2 w_{pri}^2(0) - J_A^2 + 2\lambda w_{pri}(0)J_A,\\
4\lambda^2 Q_A G_A &= 2\lambda^2 w_{pri}^2(0) - \lambda^2 w_{pri}^2(0) - J_A^2,\\
4\lambda^2 Q_A G_A &= \lambda^2 w_{pri}^2(0) - J_A^2,\\
\text{where } T &= \frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A},\\
Q_A &= K_{pr}\left[1 - \frac{K_{pr}}{K_{pr}+K_{pu}} + \frac{K_{pr}}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}\right],\\
J_A &= \int_{-\infty}^{0} \frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}} dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right].
\end{aligned}
$$

This implies that

$$
\left\{w_{pri}^2(0) - 4Q_A G_A\right\}\lambda^2 = J_A^2.
$$

$$
\lambda = \frac{J_A}{\sqrt{w_{pri}^2(0) - 4Q_A G_A}}.
$$

Eq. (64) can also be written as

$$
A = \frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A}. \tag{66}
$$

After putting value of λ in above expression, we obtain:

$$
A = \frac{\frac{J_A w_{pri}(0)}{\sqrt{w_{pri}^2(0) - 4Q_A G_A}} - J_A}{\frac{2J_A Q_A}{\sqrt{w_{pri}^2(0) - 4Q_A G_A}}},
$$

$$
A = \frac{w_{pri}(0) - \sqrt{w_{pri}^2(0) - 4Q_A G_A}}{2Q_A}. \tag{67}
$$

A is a policy for an optimal number of water producers/extractors in a dynamical setting. The second order condition shows that efficiency loss has been minimized (see appendix).

8 Conclusion

When government adopts a dynamically optimal water market policy, e.g., imposes a production fee, and shifts the water production/extraction curve to the left, the water market goes out of equilibrium, with a presemption that it was in equilibrium before the implementation of the policy. The water production/extraction and the market price of water adjust over time and the water market eventually attains the final equilibrium. The final equilibrium is (more) efficient as compared to the initial equilibrium, however, some efficiency is lost during the adjustment of the market. Eq (35) presents a dynamically optimal water market policy after minimizing the efficiency losses during adjustment of the market. The expression involves production, demand, and government storage curves' slopes and initial pre-policy water equilibrium quantity.

For a dynamically optimal production policy for water producers model, we develop a water production model involving number of water producers and production/extraction rate. The model can predict the adjustment path and the final equilibrium after a supply/demand or production rate shock. A dynamic optimal water production policy has been derived by minimizing social damage in terms of excessive number of water producers/extractors on the adjustment path to final equilibrium after the government adopts the policy subject to a certain change in water quantity per unit time. The area under the demand curve is social benefit in terms of water quantity per unit time when water production is in equilibrium. Eq. (67) presents the expression for the optimal production policy depending on parameters $w_{pri}(0)$, G_A, K_c, K_{pr}, K_{pu}, τ_{d1} and τ_{d2}.

9 Appendix:

9.1 Water Extractor/Producer/Supplier

The water extractor/producer/supplier's problem is to maximize present discounted value of future stream of profits. The zero value, i.e., the present value for $t = 0$, is as follows:

$$V(0) = \int_0^\infty \left[\alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t)\right] e^{-\rho_{Bp} t} dt, \tag{68}$$

α denotes the market price fraction charged by water extractor to government. ρ_{Bp} reflects the discount rate; $l(t)$ (labor) and $i(t)$ (level of investment) are *control variables* and $k(t)$ is *state variable.* Water extractor's problem can be written as:

$$\underset{\{l(t),i(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t)\right] e^{-\rho_{Bp} t} dt,$$

subject to the constraints that

$\dot{k}(t) = i(t) - \delta k(t)$ (state equation, describing how state variable changes with time),

$k(0) = k_s$ (initial condition),

$k(t) \geq 0$ (non-negativity constraint on state variable),

$k(\infty)$ free (terminal condition).

Current-value Hamiltonian can be written as follows:

$$\widetilde{H} = \alpha p(t) f\left(k\left(t\right), l\left(t\right)\right) - w(t) l\left(t\right) - r(t) i(t) + \mu_{Bp}(t)\left[i(t) - \delta k(t)\right]. \tag{69}$$

Maximizing conditions are given below:
(*i*) $l^*(t)$ and $i^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial l} = 0$ and $\frac{\partial \widetilde{H}}{\partial i} = 0$,
(*ii*) $\dot{\mu_{Bp}} - \rho_{Bp}\mu_{Bp} = -\frac{\partial \widetilde{H}}{\partial k}$,
(*iii*) $\dot{k}^* = \frac{\partial \widetilde{H}}{\partial \mu_{Bp}}$ (this just gives back the state equation),
(*iv*) $\lim\limits_{t\to\infty} \mu_{Bp}(t) k(t) e^{-\rho_{Bp} t} = 0$ (the transversality condition).
First two conditions are:

$$\frac{\partial \widetilde{H}}{\partial l} = \alpha p(t) f_l'\left(k\left(t\right), l\left(t\right)\right) - w(t) = 0, \tag{70}$$

$$\frac{\partial \widetilde{H}}{\partial i} = -r(t) + \mu_{Bp}(t) = 0, \tag{71}$$

and

$$\dot{\mu_{Bp}} - \rho_{Bp}\mu_{Bp} = -\frac{\partial \widetilde{H}}{\partial k} = -\left[\alpha p(t) f_k'\left(k\left(t\right), l\left(t\right)\right) - \delta \mu_{Bp}(t)\right]. \tag{72}$$

Substituting values of $\dot{\mu_{Bp}}$ and μ_{Bp} from eq. (71) into eq. (72) yields

$$\alpha p(t) f_k'\left(k\left(t\right), l\left(t\right)\right) - (\rho_{Bp} + \delta) r(t) + \dot{r}(t) = 0.$$

If $p(t)$ increases, producer faces following inequalities at existing levels of labor and investment:

$$\alpha p(t) f_l'\left(k\left(t\right), l\left(t\right)\right) - w(t) > 0,$$
$$\alpha p(t) f_k'\left(k\left(t\right), l\left(t\right)\right) - (\rho_{Bp} + \delta) r(t) + \dot{r}(t) > 0.$$

After a price increase, the water extractor's profit maximizing condition gets modified and prompts him to supply more water (details in appendix). Let p = market price of water at which government supplies water to consumers, c_B = a reference/feasible minimum price for water extractor to decide whether to operate or not.

W_{Bp} = *Change in water extraction/production volume due to change in price.*

The condition $p - c_B \geq 0$ provides the water extractor/producer an incentive to supply more water, i.e.,

$$W_{Bp} \propto \alpha(p - c_B), \text{ or}$$

$$W_{Bp} = K_{Bp}(p - c_B). \tag{73}$$

When the water market is in equilibrium, $W_{Bp} = 0$, i.e.,

$$0 = K_{Bp}(p_s - c_{Bs}). \tag{74}$$

K_{Bp} is a proportionalityconstant. p_s and c_{Bs} reflect the equilibrium values. If we subtract eq. (11) from eq. (10), we get:

$$W_{Bp} = K_{Bp}\left[(p - p_s) - (c_B - c_{Bs})\right] = -K_{Bp}\left(C_B - P\right) = -K_{Bp}\varepsilon_B, \tag{75}$$

W_{Bp}, C_B and P reflect corresponding deviation values from those at the steady state.

9.2 Consumers of Water

There are two major types of consumers of water, i.e., the producers involved in production activities using water as an input, and the final consumer. The problems of both types of consumers are discussed below:

Producers Using Water as an Input:

The producer of a commodity using water as an input has a problem of maximizing present discounted value of future streams of profits. The zero value, i.e., the present value for $t = 0$, is as follows:

$$V(0) = \int_0^\infty \left[p_{Bc}(t)F\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t)\right]e^{-rt}dt, \tag{76}$$

p_{Bc} is price of commodity being produced by the producer; r reflects discount rate. $L(t)$ (labor input), $I(t)$ (investment), and $w_{Bc}(t)$ (quantity of water as an input) are *control variables* and $K(t)$ is the *state variable*. The producer's (as consumer of water) problem can be written as

$$\underset{\{L(t),I(t),w_{Bc}(t)\}}{Max} V(0) = \int_0^\infty \left[p_{Bc}(t)F\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t)\right]e^{-rt}dt,$$

subject to the constraints that

$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, describing how the state variable changes with time),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (non-negativity constraint on state variable),

$K(\infty)$ free (terminal condition).

The expression for current-value Hamiltonian is as follows:

$$\widetilde{H} = p_{Bc}(t)F\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - w(t)L\left(t\right) - \Re(t)I(t) - p(t)w_{Bc}(t) + \mu(t)\left[I(t) - \delta K(t)\right]. \tag{77}$$

Maximizing conditions are given below:

(i) $L^*(t)$, $I^*(t)$ and $w^*_{Bc}(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial\widetilde{H}}{\partial L} = 0$, $\frac{\partial\widetilde{H}}{\partial I} = 0$ and $\frac{\partial\widetilde{H}}{\partial w_{Bc}} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial\widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial\widetilde{H}}{\partial\mu}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty}\mu(t)K(t)e^{-rt} = 0$ (the transversality condition).

First two conditions are:

$$\frac{\partial \widetilde{H}}{\partial L} = p_{Bc}(t)\dot{F}_2\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - w(t) = 0, \tag{78}$$

$$\frac{\partial \widetilde{H}}{\partial I} = -\Re(t) + \mu(t) = 0, \tag{79}$$

$$\frac{\partial \widetilde{H}}{\partial w_{Bc}} = p_{Bc}(t)\dot{F}_3\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - p(t) = 0, \tag{80}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K} = -\left[p(t)\dot{F}_1\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - \delta\mu(t)\right]. \tag{81}$$

If price of water goes up, the producer as consumer of water faces a modified condition, i.e.,

$$p_{Bc}(t)\dot{F}_3\left(K\left(t\right), L\left(t\right), w_{Bc}(t)\right) - p(t) < 0.$$

After a water price increase, the producer using water as an input will reduce water consumption to satisfy profit maximization condition (see detail in appendix). If demand change is proportional to price change (or otherwise if linearization of demand schedule around equilibrium is a reasonable assumption), we have:

$$\textit{Change in demand} \propto P, \text{ or}$$

$$W_{Bc} = -K_{Bc}P. \tag{82}$$

W_{Bc} is deviation in demand with respect to the equilibrium value after a price change, i.e., P. K_{Bc} is proportionality constant, and the negative sign is re‡ective of the fact that when price increases, the demand of water goes down.

9.3 Solution of the Water Market Model with a Contract/Lease/License Fee

From $(4a)$, (12) and (24):

$$\begin{aligned} \frac{dP(t)}{dt} &= -K_{Bm}W_B(t), \\ W_{Bp}(t) &= -K_{Bp}\varepsilon_B(t), \\ \varepsilon_B(t) &= C_B(t) - P(t), \\ W_{Bc} &= -K_{Bc}P. \end{aligned}$$

In the absence of a shock, we have

$$W_B(t) = W_{Bp}(t) - W_{Bc}(t),$$

The above expressions imply that

$$\begin{aligned}\frac{dP(t)}{dt} &= -K_{Bm}\left[W_{Bp}(t) - W_{Bc}(t)\right] \\ &= -K_{Bm}\left[-K_{Bp}\varepsilon_B(t) + K_{Bc}P(t)\right] \\ &= -K_{Bm}\left[-K_{Bp}C_B(t) + (K_{Bp} + K_{Bc})P(t)\right],\end{aligned}$$

or

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}C_B(t). \tag{83}$$

If a per unit water extraction fee is imposed on producer at $t = 0$, i.e., $C_B(t) = T$, the above expression becomes

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}T. \tag{84}$$

The Routh–Hurwitz stability criterion (a necessary and sufficient condition for stability of a linear dynamical system depicted by the above differential equation) is as follows: $K_{Bm}(K_{Bp}+K_{Bc}) > 0$. As K_{Bm}, K_{Bp} and K_{Bc} are defined as positive numbers, the stability condition holds, which ensures that after a shock the water market arrives at a new equilibrium through some adjustment mechanism. If the fee is charged from buyer instead of producer per unit of water consumption, the producer will take into account the price faced by him/her, i.e.,

$$\varepsilon_B(t) = T - P(t). \tag{85}$$

which leads to the following expression:

$$\frac{dP(t)}{dt} + K_{Bm}(K_{Bp} + K_{Bc})P(t) = K_{Bm}K_{Bp}T.$$

The above expression is the same as eq. (26), however, the solution/dynamic adjustment path will depend on initial conditions. The solution of eq. (26) with initial conditions of a producer's fee is as follows:
The above differential equation's characteristic function is as follows:

$$x + K_{Bm}(K_{Bp} + K_{Bc}) = 0,$$

which has a single root, i.e.,

$$x = -K_{Bm}(K_{Bp} + K_{Bc}),$$

giving the complementary solution as

$$P_c(t) = C_2 e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}.$$

The particular solution has the form

$$P_p(t) = C_1,$$

and the solution has the following form:

$$P(t) = C_1 + C_2 e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}. \tag{86}$$

The constant C_1 is determined by substitution into the differential equation as follows:

$$-K_{Bm}(K_{Bp}+K_{Bc})C_2e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}+K_{Bm}(K_{Bp}+K_{Bc})C_1+K_{Bm}(K_{Bp}+K_{Bc})C_2e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} = K_{Bm}K_B$$

$$C_1 = \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}.$$

C_2 is determined by the initial condition as follows:

$$P(0) = \frac{K_{Bp}T}{K_{Bp}+K_{Bc}} + C_2 = 0,$$
$$C_2 = -\frac{K_{Bp}T}{K_{Bp}+K_{Bc}}.$$

Substituting the values of C_1 and C_2 in eq. (86), we get:

$$P(t) = \frac{K_{Bp}T}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}. \tag{87}$$

When $t = 0$, $P(0) = 0$ (initial condition). When $t = \infty$, $P(\infty) = \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}$ (final value). In final equilibrium, supply equals demand, and in order to verify that we proceed as follows:

$$\begin{aligned}
w_{Bp}(\infty) &= w_{Bc}(\infty), \text{ or} \\
w_{Bp}(0) - K_{Bp}\left[T - P(\infty)\right] &= w_{Bc}(0) - K_{Bc}P(\infty), \\
-K_{Bp}\left[T - P(\infty)\right] &= -K_{Bc}P(\infty), \\
-K_{Bp}\left[T - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}\right] &= -K_{Bc}\frac{K_{Bp}T}{K_{Bp}+K_{Bc}}, \\
-\frac{K_{Bc}K_{Bp}T}{K_{Bp}+K_{Bc}} &= -\frac{K_{Bc}K_{Bp}T}{K_{Bp}+K_{Bc}},
\end{aligned}$$

which is true as

$$w_{Bp}(0) = w_{Bc}(0).$$

9.4 A Dynamically Optimal Water Market Policy

Pre-policy water market equilibrium is inefficient, and the imposition of producer fee leads to an efficient equilibrium. However, there are some efficiency losses on the adjustment path of the water market to the new efficient equilibrium. After a fee is imposed on water producer, the supply of water shrinks, the market forces come into play and the water market adjusts to the final equilibrium. The price and quantity of water in final equilibrium are dependent on supply and demand elasticities. The level of water storage rises if supply is higher than demand and goes down otherwise. When demand and supply again become equal, the water market is in final equilibrium. When demand and supply are not equal, either water supply and/or consumption is being lost at that point in time. The total production and/or consumption lost in terms of quantity is the efficiency loss and can be expressed as follows:

$$EL_B = -\left[\int_{-\infty}^{0} W_{Bp}(\infty)dt + M(t)\right]. \tag{88}$$

After imposition of water fee, the supply of water shrinks by $K_{Bp}T$. As the demand of water has not yet changed, the level of water storage also decreases by $K_{Bp}T$. The water market is out of equilibrium, and drifts toward the final equilibrium through market forces. The price of water is changed by government to bring the final equilibrium. The government earns the following amount as producer fee revenue (PFR):

$$PFR = T\left[w_{Bpi}(0) - K_{Bp}\left\{T - P(t)\right\}\right]. \tag{89}$$

The problem of minimizing efficiency loss with T as a control variable subject to constraint that revenue from imposition of producer fee must be greater than or equal to G_B in a given time, is as follows:

$$\min_T EL_B \quad \text{s.t.} \quad PFR \geq G_B.$$

The constraint is binding. Lagrangian for the problem of minimizing efficiency loss is as follows:

$$\begin{aligned}
\mathcal{L} &= -\int_{-\infty}^{0} W_{Bp}(\infty)dt - M(t) + \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - P(t)\right\}\right]\right] \\
&= \int_{-\infty}^{0}\left[K_{Bp}T - \frac{K_{Bp}^2 T}{K_{Bp}+K_{Bc}}\right]dt \\
&+ \frac{1}{K_{Bm}}\left[\frac{K_{Bp}T}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}T\right] \\
&+ \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0}\frac{K_{Bp}K_{Bc}T}{K_{Bp}+K_{Bc}}dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}T}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}T\right] \\
&+ \lambda\left[G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]\right].
\end{aligned}$$

Derivative of Lagrangian with respect to T leads to the following expression:

$$T = \frac{\lambda w_{Bpi}(0) - \left[\int\limits_{-\infty}^{0} \frac{K_{Bp}K_{Bc}}{K_{Bp}+K_{Bc}}dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}\right]\right]}{2\lambda K_{Bp}\left[1 - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right]}. \tag{90}$$

Similarly derivative of Lagrangian with respect to λ gives:

$$G_B - T\left[w_{Bpi}(0) - K_{Bp}\left\{T - \frac{K_{Bp}T}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}T}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right\}\right]. \tag{91}$$

Putting eq. (32) into (33), we obtain:

$$\begin{aligned}
G_B &= \frac{\lambda w_{Bpi}^2(0) - w_{Bpi}(0)J_B}{2\lambda Q_B} - \left(\frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B}\right)^2 Q_B,\\
4\lambda^2 Q_B G_B &= 2\lambda^2 w_{Bpi}^2(0) - 2\lambda w_{Bpi}(0)J_B - \lambda^2 w_{Bpi}^2(0) - J_B^2 + 2\lambda w_{Bpi}(0)J_B,\\
4\lambda^2 Q_B G_B &= 2\lambda^2 w_{Bpi}^2(0) - \lambda^2 w_{Bpi}^2(0) - J_B^2,\\
4\lambda^2 Q_B G_B &= \lambda^2 w_{Bpi}^2(0) - J_B^2,\\
\text{where } T &= \frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B},\\
Q_B &= K_{Bp}\left[1 - \frac{K_{Bp}}{K_{Bp}+K_{Bc}} + \frac{K_{Bp}}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t}\right],\\
J_B &= \int\limits_{-\infty}^{0} \frac{K_{Bp}K_{Bc}}{K_{Bp}+K_{Bc}}dt + \frac{1}{K_{Bm}}\left[\frac{K_{Bp}}{K_{Bp}+K_{Bc}} - \frac{K_{Bp}}{K_{Bp}+K_{Bc}}e^{-\left[K_{Bm}(K_{Bp}+K_{Bc})\right]t} + K_{Bm}K_{Bp}\right].
\end{aligned}$$

This implies that

$$\left\{w_{Bpi}^2(0) - 4Q_B G_B\right\}\lambda^2 = J_B^2.$$

$$\lambda = \frac{J_B}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}.$$

After putting value of λ in above expression, we obtain:

$$T = -\frac{\frac{J_B w_{Bpi}(0)}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}} - J_B}{\frac{2Q_B J_B}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}}.$$

Eq. (90) can also be written as

$$T = \frac{\lambda w_{Bpi}(0) - J_B}{2\lambda Q_B}. \tag{92}$$

After substituting value of λ in eq. (92), we get:

$$T = \frac{w_{Bpi}(0) - \sqrt{w_{Bpi}^2(0) - 4Q_B G_B}}{2Q_B}. \tag{93}$$

The second order condition for minimization of efficiency loss is as follows:
Lagrangian can be written as

$$\mathcal{L} = J_B T + \lambda \left[G_B - T \left(w_{Bpi}(0) - Q_B T \right) \right].$$

The Bordered Hessian matrix of Lagrange function is as follows:

$$BH = \begin{bmatrix} 0 & w_{Bpi}(0) - 2Q_B T \\ w_{Bpi}(0) - 2Q_B T & \frac{2Q_B J_B}{\sqrt{w_{Bpi}^2(0) - 4Q_B G_B}} \end{bmatrix}.$$

As the determinant of the above matrix is negative, i.e., $-\left(w_{Bpi}(0) - 2Q_B T\right)^2 < 0$, it implies that the efficiency loss is minimized.

9.5 A Dynamic Optimal Production Policy for Water Producers Model

The equilibrium before adoption of water production policy needed to be improved upon, this is why government wanted to adopt a water policy. Also, there are some efficiency losses on the dynamic adjustment path to the new equilibrium. Adding the equilibrium and adjustment path inefficiencies, we get the total social damage which needs to be minimized. The government either shifts the demand or the supply curve through a production policy. Suppose it shifts supply leftward. The number of water producers/extractors and the production rate adjust over time to bring final equilibrium, at which the production/extraction rate is higher and the number of water producers is lower as compared to those in the initial equilibrium. If supply of water producers is higher than demand, the number of producers are excessive and vice versa. The excessive number or a shortage of water producers is the social damage at a certain point in time. By summing up the social damage in equilibrium and that on the dynamic adjustment path, the total damage in terms of number of water producers/extractors is obtained as follows:

$$SD = -\left[\int_{-\infty}^{0} W_{pr}(\infty) dt + M_A(t) \right]. \tag{94}$$

From eq. (51), a change in number of water producers/extractors on account of implementation of production policy is given below:

$$W_{pr}(t) = -K_{pr} \left[A - C(t) \right],$$
$$\text{or } w_{prf}(t) - w_{pri}(0) = -K_{pr} \left[A - C(t) \right],$$

where $w_{pri}(0)$ is number of water producers/extractors supplied in initial equilibrium and $w_{prf}(t)$ is new value after water production policy is adopted, as $W_{pr}(t)$ is a deviation variable, i.e., deviation from initial steady state value. A change in quantity of production/extraction per unit time is given below:

$$\Delta QP = A\left[w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right]. \tag{95}$$

The problem of minimizing the social damage subject to a change in quantity of production being greater than or equal to G_A $\left(\text{change in quantity of production per unit time} = \frac{dM_B}{dt}\right)$ can be expressed as follows:

$$\min_A SD \quad \text{s.t.} \quad \Delta QP \geq G_A\left(= \frac{dM_B}{dt}\right).$$

A is choice variable, i.e., the size of production policy. The constraint is binding, and Lagrangian can be expressed as follows:

$$\mathcal{L} = -M_A(t) - \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left\{w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right\}\right].$$

Expression from eq. (46a) is given below:

$$C(t) = -K_c M_A + E_A.$$

The value of E_A is found by imposing initial conditions as shown below:

$$\begin{aligned} C(0) &= -K_c M_A(0) + E_A, \\ 0 &= K_c K_{pr} A + E_A, \\ E_A &= -K_c K_{pr} A \end{aligned}$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) + K_c K_{pr} A\right].$$

Therefore, the Lagrangian can now be written as:

$$
\begin{aligned}
\mathcal{L} &= -\int_{-\infty}^{0} W_{pr}(\infty)dt - M_A(t) + \lambda\left[G_A - A\left\{w_{pri}(0) - K_{pr}\left\{A - C(t)\right\}\right\}\right] \\
&= \int_{-\infty}^{0}\left[K_{pr}A - \frac{K_{pr}^2 A}{K_{pr}+K_{pu}}\right]dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right] \\
&+ \lambda\left[G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0}\frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}}dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr} A\right] \\
&+ \lambda\left[G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right].
\end{aligned}
$$

Derivative of Lagrangian with respect to A leads to the following expression:

$$
A = \frac{\lambda w_{pri}(0) - \left[\int_{-\infty}^{0}\frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}}dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr}A\right]\right]}{2\lambda K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \quad (96)
$$

Similarly derivative of Lagrangian with respect to λ gives:

$$
G_A - A\left[w_{pri}(0) - K_{pr}\left\{A - \frac{K_{pr}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] = 0. \quad (97)
$$

Putting eq. (96) into (97), we obtain:

$$
\begin{aligned}
G_A &= \frac{\lambda w_{pri}^2(0) - w_{pri}(0)J_A}{2\lambda Q_A} - \left(\frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A}\right)^2 Q_A, \\
4\lambda^2 Q_A G_A &= 2\lambda^2 w_{pri}^2(0) - 2\lambda w_{pri}(0)J_A - \lambda^2 w_{pri}^2(0) - J_A^2 + 2\lambda w_{pri}(0)J_A, \\
4\lambda^2 Q_A G_A &= 2\lambda^2 w_{pri}^2(0) - \lambda^2 w_{pri}^2(0) - J_A^2, \\
4\lambda^2 Q_A G_A &= \lambda^2 w_{pri}^2(0) - J_A^2, \\
\text{where } T &= \frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A}, \\
Q_A &= K_{pr}\left[1 - \frac{K_{pr}}{K_{pr}+K_{pu}} + \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right], \\
J_A &= \int_{-\infty}^{0}\frac{K_{pr}K_{pu}A}{K_{pr}+K_{pu}}dt + \frac{1}{K_c}\left[\frac{K_{pr}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t} + K_c K_{pr}A\right].
\end{aligned}
$$

This implies that

$$\left\{w_{pri}^2(0) - 4Q_AG_A\right\}\lambda^2 = J_A^2.$$

$$\lambda = \frac{J_A}{\sqrt{w_{pri}^2(0) - 4Q_AG_A}}.$$

λ must be positive as the social damage increases with an increase in G_A. Eq. (96) can also be written as

$$A = \frac{\lambda w_{pri}(0) - J_A}{2\lambda Q_A}. \tag{98}$$

After putting value of λ in above expression, we obtain:

$$A = \frac{\frac{J_A w_{pri}(0)}{\sqrt{w_{pri}^2(0) - 4Q_AG_A}} - J_A}{\frac{2J_AQ_A}{\sqrt{w_{pri}^2(0) - 4Q_AG_A}}},$$

$$A = \frac{w_{pri}(0) - \sqrt{w_{pri}^2(0) - 4Q_AG_A}}{2Q_A}. \tag{99}$$

The second order condition for minimization of efficiency loss is as follows:
Lagrangian can be written as

$$\mathcal{L} = J_AA + \lambda\left[G_A - A\left(w_{pri}(0) - Q_AA\right)\right].$$

The Bordered Hessian matrix of Lagrange function is as follows:

$$BH = \begin{bmatrix} 0 & w_{pri}(0) - 2Q_AA \\ w_{pri}(0) - 2Q_AA & \frac{2Q_AJ_A}{\sqrt{w_{pri}^2(0) - 4Q_AG_A}} \end{bmatrix}.$$

As the determinant of the above matrix is negative, i.e., $-\left(w_{pri}(0) - 2Q_AA\right)^2 < 0$, it implies that the efficiency loss is minimized.

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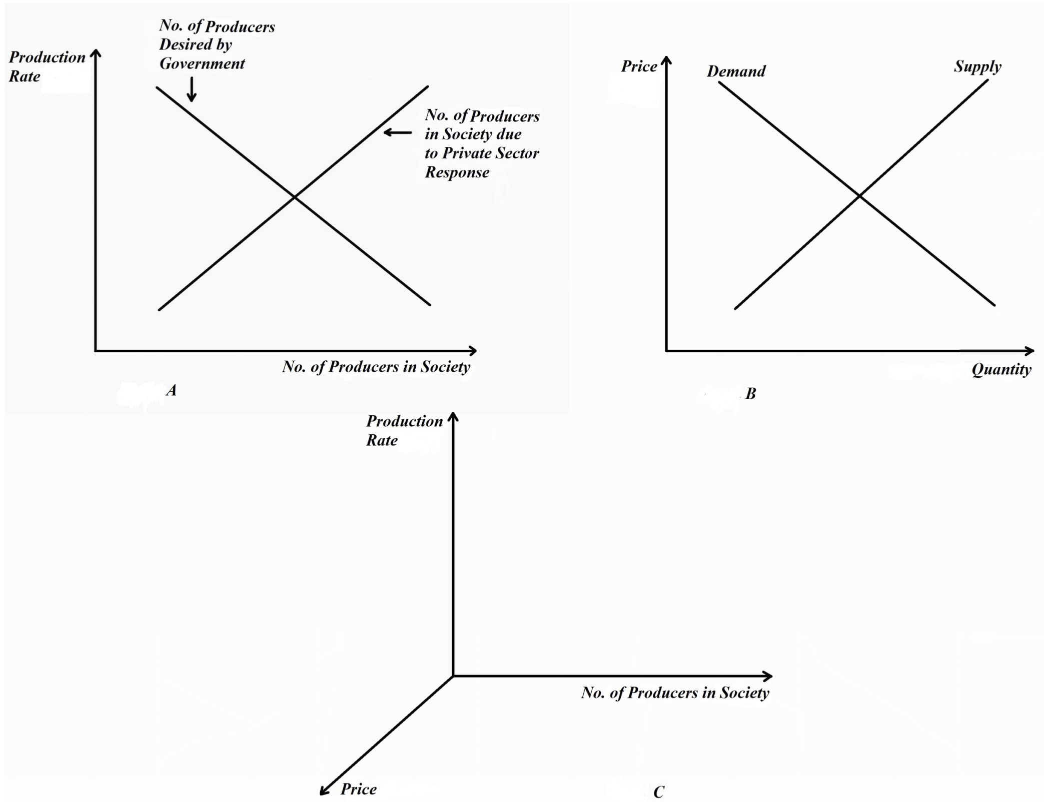

Figure 1: Theory of an optimal dynamical water resource management policy.