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MEMORY AND ASSET PRICING MODELS WITH HETEROGENEOUS BELIEFS

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Abstract

The paper discusses the role of memory in asset pricing models with heterogeneous beliefs. In particular, we were interested in how memory in the fitness measure affects stability of evolutionary adaptive systems and survival of technical trading. In order to obtain an insight into this matter two cases were analyzed; a two-type case of fundamentalists versus contrarians and a three-type case of fundamentalists versus opposite biases. It has been established that increasing memory strength has a stabilizing effect on dynamics, though it is not able to eliminate speculative traders' short-run profit seeking behaviour from the market. Furthermore, opposite biases do not seem to lead to chaotic dynamics, even when there are no costs for fundamentalists. Apparently some (strong) trend extrapolator beliefs are needed in order to trigger chaotic asset price fluctuations.

Key Words: asset pricing, biased beliefs, contrarians, fitness measure, fundamentalists, heterogeneous beliefs, memory strength, stability

JEL Classification: C60, D83, D84, E32

Contents

1 Introduction

There is an important paradigm shift taking place in economics and finance in the last decade; from the representative, rational agent approach towards a behavioural, agent-based approach with heterogeneous boundedly rational agents (*cf.* Sargent, 1993; Thaler, 1994; Hommes, 2006). While the traditional approach makes use of simple, analytically tractable models with a representative, perfectly rational agent, the new behavioural approach utilizes computational and numerical methods on agent-based simulation models. By now, there is a rather extensive literature available on computationally oriented agent-based simulation models of artificial markets (*cf.* LeBaron, 2006). However, there is also an important stream developing in the literature, which endeavours to maintain at least to same extent analytically tractable heterogeneous agent models, for which theoretical results are obtained supporting numerical simulation results (*cf.* Hommes, 2006). Such an approach uses a mixture of analytical and computational tools of nonlinear economic dynamics.

Heterogeneous agent models are present in various fields of economic analysis, such as market maker models, exchange rate models, monetary policy models, overlapping generations models and models of socio-economic behaviour. Yet the field with the most systematic and perhaps most promising nonlinear dynamic approach seems to be asset price modelling. Contributions of Brock and Hommes (1998), LeBaron (2000), Hommes *et al*. (2002), Chiarella and He (2002), Chiarella *et al*. (2003), Gaunersdorfer *et al*. (2003), Brock *et al*. (2005), Hommes *et al*. (2005), and Brock and Hommes (2006) thoroughly demonstrate how a simple standard pricing model is able to lead to complex dynamics that makes it extremely hard to predict the evolution of prices in asset markets. The main framework of analysis of such asset pricing models constitutes a financial market application of the evolutionary selection of expectation rules, introduced by Brock and Hommes (1997a) and called the adaptive belief system.

As a model in which different agents have the ability to switch beliefs, the adaptive belief system in a standard discounted value asset pricing set-up is derived from mean-variance maximization and extended to the case of heterogeneous beliefs (Hommes, 2006, p. 47). It can be formulated in terms of deviations from a benchmark fundamental and therefore used in experimental and empirical testing of deviations from the rational expectations benchmark. Agents are boundedly rational, act independently of each other and select a forecasting or investment strategy based upon its recent relative performance. The key feature of such systems, which often incorporate active learning and adaptation, is endogenous heterogeneity (*cf.* LeBaron, 2002), which means that markets can move through periods that support diverse population of beliefs, and others where these beliefs and strategies might collapse down to a very small set.

As such, these models are highly nonlinear systems, generating a wide range of dynamical behaviours, ranging from simple convergence to a stable steady state to very irregular and unpredictable fluctuations, which are highly sensitive to noise. Sophisticated traders, such as fundamentalists or rational arbitrageurs typically act as a stabilizing force, pushing prices in the direction of the rational expectations fundamental value. Technical traders, such as feedback traders, trend extrapolators and contrarians typically act as a destabilizing force, pushing prices away from the fundamental. When the proportion of chartists believing in a trend exceeds some critical value, the price trend becomes reinforced and the belief becomes self-fulfilling, causing prices to deviate from fundamentals. Nonlinear interaction between fundamental traders and chartists can lead to deviations from the fundamental price in the short run, when price trends are reinforced due to technical trading, and mean reversion in the long run, when more agents switch back to fundamental strategies when the deviation from fundamental price becomes too large.

The mixture of different trader types therefore leads to diverse dynamics exhibiting some stylized, qualitative features observed in practice on financial markets (*cf.* Beja and Goldman, 1980; Campbell *et al.*, 1997; Johnson *et al.*, 2003), e.g. persistence in asset prices, unpredictability of returns at daily horizon, mean reversion at long horizons, excess volatility, clustered volatility, and leptokurtosis of asset returns. The important finding so far was that irregular and chaotic behaviour is caused by rational choice of prediction strategies in the bounded-rationality framework, and that this also exhibits quantitative features of asset price fluctuations, observed in financial markets. Namely, due to differences in beliefs these models generate a high and persistent trading volume, which is in sharp contrast to no trade theorems in rational expectations models. Fractions of different trading strategies fluctuate over time and simple technical trading rules can survive evolutionary competition. On average, technical analysts may even earn profits comparable to the profits earned by fundamentalists or value traders.

While recent literature on asset price modelling focuses mainly on impacts of heterogeneity of beliefs in the standard adaptive belief system as set up by Brock and

Hommes (1997a) on the market dynamics and stability on one hand, and the possibility of survival of such 'irrational', speculative traders in the market on the other, several crucial issues regarding the foundations of asset price modelling and its underlying theoretical findings remain open and indeterminate. According to LeBaron (2002), one of those issues is related to heterogeneity in investors' time horizon; both their planning and their evaluation perspective. Namely, it has been scarcely addressed so far how memory in the fitness measure, i.e. the share of past information that boundedly rational economic agents take into account as decision makers, affects stability of evolutionary adaptive systems and survival of technical trading. The motivation behind this paper is therefore to lay foundations for a competent and critical theoretical analysis of setting this modelling assumption in a simple, analytically tractable asset pricing model.

Memory strength represents the share of past fitness in the performance measure of an asset pricing model, which determines fractions of respective belief types and consequently affects the asset price. Memory strength is thus one of the parameters of the asset pricing model that could decisively influence our inferences on stability of evolutionary adaptive systems and survival of technical trading (*cf.* Verbič, 2006). The cause of much of the above described dynamics can be related to the interaction between traders with differing views of the past. Agents with a short-term perspective are expected to both influence the market in terms of increasing volatility and create an evolutionary space where they are able to prosper. Changing the population to more long-memory types should lead to a reliable convergence in strategies, which would be a useful benchmark test. Memory or perhaps better said – lack thereof – is therefore an important aspect of the market that is likely to keep it from converging and prevent the elimination of 'irrational', speculative strategies from the market.

The main research hypothesis to be analyzed in the paper thus states that additional memory in the fitness measure of an asset pricing model with heterogeneous beliefs has stabilizing effects on evolutionary adaptive systems and unfavourable consequences for survival of technical trading. In so doing, one needs to have in mind that both short and long term perspective are equally important in economics and finance. In order to be able to adequately examine our research hypothesis, both analytical and numerical analysis will have to be employed and complemented. Therefore, we shall first expand the asset pricing model to include more memory, and then solve it both analytically and numerically. Two cases are going to be analyzed, hopefully sufficiently general to cover some main aspects of financial markets; (1) a two-type case of fundamentalists versus contrarians and (2) a three-type case of fundamentalists versus opposite biased beliefs. Complementing the stability analysis with local bifurcation theory, we will also be able to analyze numerically the effects of adding different amounts of additional memory to fitness measure on stability of the standard asset pricing model and survival of technical trading. Thus the analysis of both local and global stability can be performed for different combinations of trader types in the market.

The outline of the paper is as follows. In Chapter 2, after a brief description of typical results in such heterogeneous agent models, a short overview is given of the scarce contributions on memory analysis to date. In Chapter 3 an asset pricing model with heterogeneous beliefs with endogenous evolutionary switching of strategies is presented, forming the groundwork for analysis of economic fluctuations and the underlying rules relating to the formation of expectations. In so doing, the role of memory in the fitness measure and possible consequences for the outcomes of such models are stressed. In Chapters 4 and 5 the two aforementioned cases are being examined both analytically and numerically; the main results regarding effects of the different types of market traders on market stability are presented, together with effects of changing memory on market movements of different economic categories. In the final chapter the essential findings of the paper are summarized.

2 Memory and Performance of Heterogeneous Agents

In order to demonstrate how different types of market traders can affect market stability, we shall briefly employ some results from asset pricing models with heterogeneous, adaptive beliefs for different simple, linear predictors, except with no additional memory being included or analyzed. Particularly, two cases with two trader types are going to be taken into consideration; fundamentalists versus trend chasers and fundamentalists versus contrarians. Both were already quite extensively analyzed in Brock and Hommes (1998, pp. 1248-1258), but will serve us henceforth in analyzing and explaining our analytical and numerical results.

In the first case the asset pricing model consists of fundamentalists with some information gathering costs that are necessary in general to obtain understanding of how markets work and to be able to price according to the efficient market hypothesis fundamental value (*cf.* Fama, 1991), and trend chasers that follow previous asset price movements with given intensity of trend extrapolation. When trend chasers extrapolate weakly, we have a unique, globally stable steady state, which is called the fundamental steady state. If costs for predictor of fundamentalists are equal to zero, both types have equal weight, while when these costs are nonzero the mass of fundamentalists decreases to zero with costs of predictor or intensity of choice approaching infinity. This is understandable, since at the fundamental price additional cost carries no extra profit, and therefore the mass of the most profitable strategy increases. When trend chasers extrapolate strongly and intensity of choice increases, we get a primary bifurcation, i.e. a pitchfork bifurcation of the fundamental steady state, in which two additional non-fundamental (stable) steady states arise; one above and the other below the fundamental. As intensity of choice further increases, a secondary bifurcation occurs, *viz*. a Hopf bifurcation, where the non-fundamental steady states also become unstable and two attracting invariant circles are created around them.

The question was whether invariant circles break into strange attractors (*cf.* Eckmann and Ruelle, 1985; Hommes, 1991; Palis and Takens, 1993) when intensity of choice further increases. In fact, it can be shown that the system has a homoclinic point that indicates chaotic dynamics when intensity of choice increases. It can be demonstrated with the use of bifurcation diagrams and largest Lyapunov exponent plots (*cf.* Brock and Hommes, 1998, pp. 1250-1253) that the quasi-periodic dynamics evolve to chaotic dynamics. It can also be illustrated with the use of time series plots that prices are characterized by switching between an unstable phase of (depending on initial state) upward or downward trend and stable phase with prices close to the fundamental value. Without noise, this is quite regular, but with small dynamic noise added to the IID dividend process, switching becomes highly irregular and unpredictable. Trend chasers in the presence of fundamentalists therefore trigger irregular switching between phases of optimism and pessimism.

In the second case the asset pricing model consists of fundamentalists (again with some information gathering costs) and contrarians, where the latter quote the asset price contra previous asset price movements with given intensity of trend extrapolation. In case of weak contrarians we again have a unique, globally stable fundamental steady state. But in case of strong contrarians and increasing intensity of choice we get a period-doubling bifurcation of the fundamental steady state in which a (stable) two-cycle is created, with one point above and the other one below the fundamental. We therefore get oscillations of prices around the fundamental value. As intensity of choice further increases, a secondary bifurcation occurs, again a Hopf bifurcation, where the stable period two-cycle becomes unstable and two attracting invariant circles are created around each of two unstable period two-points; one lying above and the other below the fundamental price. The dynamics at this stage is either periodic or quasi-periodic, jumping back and forth between the two circles.

We can again ask ourselves, whether invariant circles break into strange attractors, when intensity of choice further increases. Using the phase plot one can demonstrate (*cf.* Brock and Hommes, 1998, pp. 1255-1257) that large intensity of choice leads to a system that is close to having a homoclinic intersection between stable and unstable manifolds of the fundamental steady state. In fact, it can be shown that the system has homoclinic orbits, which indicates chaotic dynamics. It can be shown by the use of bifurcation diagrams and largest Lyapunov exponent plots that the periodic and quasi-periodic dynamics evolve to chaotic dynamics after the secondary bifurcation. Similarly, it can be demonstrated with the use of time series plots that prices are characterized by irregular switching between a stable phase with value close to the fundamental and an unstable phase of up and down price oscillations with increasing amplitude. Contrarians therefore trigger irregular fluctuations around the fundamental.

Now we can introduce some of the scarce contributions of authors that were analyzing memory in such heterogeneous agent models to date. LeBaron (2002) was using simulated agent-based financial markets of individuals following relatively simple behavioural rules that are updated over time. Actually, time was an essential and critical feature of the model. It has been argued that someone believing that the world is stationary should use all available information in forming his or her beliefs, while if one views the world as constantly in a state of change, then it will be better to use time series reaching a shorter length into the past. The dilemma is thus seen as an evolutionary challenge where long-memory agents, using lots of past data, are pitted against short-memory agents to see who takes over the market. Essentially, results of two market simulations were presented. The first one was consisting of traders with many different memory lengths, drawn from a uniform distribution between six months and twenty years, while the second simulation restricted the agents' memory lengths to be between 16-20 years. Four variables were examined; logarithms of the price series, trading volume in units of shares subjected to trade, returns, and dividend-price ratio.

In the first simulation with horizons between six months and twenty years the price series indeed exhibited the expected linear trend driven by the constant dividend growth, but the prices seemed to take large deviations around this trend. Furthermore, volume was not a large fraction of the shares outstanding, but it was not going to zero as it should if the agents' beliefs had been converging to each other. Returns also demonstrated some features of actual markets, since there were large spikes corresponding to large up and down movements in the market. These movements corresponded to the well documented nongausseineity of financial return series. Also, the volatility in the market seemed to be clumped with periods of relative calm and periods of large activity. The dividend-price ratio, which compared movements of the equity price series with its underlying fundamental, should have been a constant if there were no changes in the underlying riskiness of the equity security. Yet from the simulation results it was clear that large and persistent deviations had occurred.

With the use of the second simulation, where the population of agents was long memory (with horizons between sixteen and twenty years), the author examined whether much of the variability and instability in the market was really coming from the presence of short-memory traders. The conjecture was generally being confirmed. Namely, the price series was much more stable, while the trading volume was near zero, except for a few brief jumps. Moreover, the returns were also generally stable with the exception of a few jumps, and the dividend-price ratio was very close to being constant. Numerical results were claimed to be checked by theoretical calculations, where the market was apparently approaching the theoretical benchmark of the well-defined homogeneous agent equilibrium.

Honkapohja and Mitra (2003) provided basic analytical results for dynamics of adaptive learning when the learning rule had finite memory and the presence of random shocks precluded exact convergence to the rational expectations equilibrium. Authors focused on the case of learning a stochastic steady state. Even though their work is not done in the heterogeneous agent setting, the results they obtained are interesting for our analysis. Their fundamental outcome was that the expectational stability principle, which plays a central role in situations of complete learning, as discussed e.g. in Evans and Honkapohja (2001), retains its importance in the analysis of incomplete learning, though it takes a new form. In the models that were analyzed expectational stability guaranteed stationarity of the dynamics of the learning economy and unbiasedness of the forecasts.

The authors also noted that their approach to incomplete learning was quite different from that of Hommes and Sorger (1998), who introduced the notion of consistent expectations equilibrium in a similar nonlinear setup. In the consistent expectations equilibrium the perceived law of motion was linear and thus misspecified, but it was required that the sample mean and autocorrelations coincide with their theoretical counterparts. There could be different types of consistent expectations equilibria, such as steady states, period cycles or even chaotic solutions. The relationship between bounded memory learning and consistent expectations equilibria were not clear-cut, but it was implied that such processes were approximately consistent expectations equilibria when the memory length was sufficiently large. This follows since the sample mean is unbiased and the covariances are small for memory length large enough.

Besides memory in the fitness measure we can also have memory in the expectation rules of the model. As we shall observe in the next chapter, the latter is somewhat less analytically tractable phenomenon, since including more preceding price deviations immediately increases the dimension of the system. This is indeed also the case with memory in the fitness measure, but there the performance measure can be written as weighted sum of contemporaneous realized profits and past fitness, thus keeping the increase in dimension under control. Usually only one memory lag is taken into account in the expectation rule to attain analytical tractability, but an appropriate numerical analysis could always be employed. Due to the lack of existing analyses we can currently mainly speculate about the effects of changing memory in expectation rules on stability of evolutionary adaptive systems and survival of technical trading; especially in interaction with changing memory in the fitness measure. We could expect that incorporating much memory in the expectation rule with more or less equal weights given to past prices produces an average price forecast close to the fundamental value. On the other hand, with most weight given to the last observations, the expectation rule is likely to be more of the trend following kind with all the accompanying results.

In this manner Chiarella *et al*. (2006) proposed a dynamic financial market model in which demand for traded assets had both a fundamentalist and a chartist component in the boundedly rational framework. The chartist demand was governed by the difference between current price and a (long-run) moving average. By examining the price dynamics of the moving average rule they found out that an increase of the window length of the moving average rule can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour. The analysis of the corresponding stochastic model was able to explain various market price phenomena, including temporary bubbles, sudden market crashes, price resistance and price switching between different levels. However, in this paper we will focus on the memory in the fitness measure.

In the end of our overview let us mention the effects of changing memory in a cobweb model, since this could be beneficial for analysis of properties of heterogeneous agent models in general. Chiarella *et al*. (2003) studied the dynamics of the traditional cobweb model with risk averse heterogeneous producers who seek to learn the distribution of asset prices using a geometric decay processes, with both finite and infinite fading memory. With constant absolute risk aversion the dynamics of the model has been characterized with respect to the length of memory window and the memory decay rate of the process. It was found that an increase of the memory decay rate played a stabilizing role in the local stability of the steady state price when memory was infinite, but this role became less clear when memory was finite. It has been shown that (quasi-)periodic solutions and strange or even chaotic attractors could be created through a Hopf bifurcation when memory was infinite, but also through a flip bifurcation in case of finite memory.

3 The Heterogeneous Agents Model

The adaptive belief system employs a mechanism, which deals with interaction between fractions of market traders of different types, and distance between the fundamental and the actual price. Financial markets are thus viewed as an evolutionary system, where price fluctuations are driven by an evolutionary dynamics between different expectation schemes. Pioneering work in this field has been done by Brock and Hommes (1997a), who attempted to conciliate the two main perspectives concerning economic fluctuations, i.e. the new classical and the Keynesian view (*cf.* Hommes, 2006, pp. 1-5), and the underlying rules relating to the formation of expectations. In order to get some insight into possible ways of theoretical analysis to follow, we shall describe a simple, analytically tractable version of the asset pricing model as constructed by Brock and Hommes (1998). The model can be viewed as composed of two simultaneous parts; present value asset pricing and evolutionary selection of strategies, resulting in equilibrium pricing equation and fractions of belief types equation. We shall also make an indication of where memory in the fitness measure (and in expectation rules) enters the model and how it might affect the analysis.

3.1 Present Value Asset Pricing

The model incorporates one risky asset and one risk free asset. The latter is perfectly elastically supplied at given gross return R, where $R = 1 + r$. Investors of different types h have different beliefs about the conditional expectation and the conditional variance of modelling variables based on a publicly available information set consisting of past prices and dividends. The present value asset pricing part of the adaptive demand system is used to model each investor type as a myopic mean variance maximizer of expected wealth demand, $W_{h,t}$, for the risky asset:

$$W_{h,t+1} = RW_{h,t} + (p_{t+1} + y_{t+1} - Rp_t)z_{h,t}\,, \tag{1}$$

where p_t is the price (ex dividend) at time t per share of risky asset, y_t is an IID dividend process at time t of the risky asset, $z_{h,t}$ is number of shares purchased at date t by agent of type h, and $R_{t+1} = p_{t+1} + y_{t+1} - Rp_t$ is the excess return.

In order to perform myopic mean variance maximization of expected wealth demand for risky asset of type h, we seek for $z_{h,t}$ that solves:

$$\max_{z_{h,t}} \left\{ E_{h,t}W_{t+1} - \frac{1}{2} aV_{h,t}W_{t+1} \right\} \tag{2}$$

and thus:

$$z_{h,t} = \frac{E_{h,t}\left[p_{t+1} + y_{t+1} - Rp_t\right]}{aV_{h,t}\left[p_{t+1} + y_{t+1} - Rp_t\right]} = \frac{1}{a\sigma^2} E_{h,t}\left[p_{t+1} + y_{t+1} - Rp_t\right], \tag{3}$$

where the belief about expected value of wealth at time $t + 1$, conditional on all publicly available information at time t, for a trader of type h is $E_{h,t}W_{t+1}$, the belief about conditional variance is $V_{h,t}W_{t+1}$, and there is a risk factor $k = \frac{1}{a\sigma^2}$ present. Beliefs about the conditional variance of excess return are assumed constant and the same for all types of investors, i.e. $V_{h,t} = \sigma^2$. All traders are assumed to be equally risk averse with a given risk aversion parameter a, which is constant over time. Gaunersdorfer (2000) investigated the case of time varying variance and supported the assumption of a constant and homogeneous variance term.

Solving this optimization problem produces quantities of shares purchased by agents of different types, which enables us to seek for the equilibrium between the constant supply of the risky asset per trader z^s and the sum of demands:

$$\sum_{h=1}^{H} n_{h,t} k E_{h,t}\left[p_{t+1} + y_{t+1} - Rp_t\right] = z^s , \tag{4}$$

where the fraction of traders of type h out of altogether H types at time t is denoted by $n_{h,t}$, where $\sum_{h=1}^{H} n_{h,t} = 1$. The price of the risky asset is determined by market clearing, which can be seen by rewriting expression (4) in the form:

$$Rp_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left[p_{t+1} + y_{t+1}\right] - a\sigma^2 z^s , \tag{5}$$

where $a\sigma^2 z^s$ is the risk premium. The latter is an extra amount of money that traders get for holding the risky asset. Traders will only purchase the risky asset if its expected value is equal or higher than the expected value of the risky asset. Since the outcome of the risky asset is uncertain, a risk premium is associated with it.

In the simplest case of IID dividends with mean $\overline{y}$ and with traders having correct beliefs about dividends, i.e. $E_{h,t}\left[y_{t+1}\right]=\overline{y}$, the market price of the risky asset p_t at time t is determined by:

$$Rp_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left[p_{t+1}\right] + \overline{y} - a\sigma^2 z^s + \varepsilon_t \, , \tag{6}$$

where a noise term ε_t is included, which represents random fluctuations in the supply of risky shares. Considering a special case with constant zero supply of outside shares, i.e. $z^s = 0$, we obtain:

$$Rp_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left[p_{t+1}\right] + \overline{y} + \varepsilon_t \, .$$

If we instead consider for a moment the case of homogeneous beliefs with no noise and all traders being rational, the pricing equation simplifies to:

$$Rp_t = E_t\left[p_{t+1}\right] + \overline{y} - a\sigma^2 z^s \, . \tag{7}$$

Since the effects of dividend beliefs on realized dividends have shown to be less notable than the effects of price beliefs on realized prices, rational expectations are imposed on the former.

In equilibrium the expectations of the price will be the same and equal to the fundamental price. The constant fundamental value of the price of the risky asset p^* in the case of homogeneous beliefs is derived from the expression:

$$Rp^* = p^* + \overline{y} - a\sigma^2 z^s \, . \tag{8}$$

By imposing a transversality condition to expression (7) with infinitely many solutions we exclude bubble solutions (*cf.* Cuthbertson, 1996) and expression (8) now has only one solution. We are thus able to derive the fundamental price as the discounted sum of expected future dividends:

$$p^* = \frac{1}{R-1}\left[\overline{y} - a\sigma^2 z^s\right]. \tag{9}$$

By simplification of the fundamental price equation for the case of the IID dividend process with constant conditional expectation we thus obtain the standard benchmark notion of the 'fundamental', i.e. $p_t^* = \frac{\overline{y}}{r}$, to be used in the model hereinafter.

Taking into account the appropriate form of heterogeneous beliefs of future prices, i.e. including some deterministic function $f_{h,t}$, which can differ across trader types:

$$E_{h,t}\left[p_{t+1}\right] = E_t\left[p_{t+1}^*\right] + E_{h,t}\left[x_{t+1}\right] = p_{t+1}^* + f_h(x_{t-1},...,x_{t-L}),$$

we restrict beliefs about the next deviation of the actual from the fundamental price, x_t, to deterministic functions of past deviations from the fundamental:

$$E_{h,t}\left[p_{t+1}\right] = p^* + f_h(x_{t-1},...,x_{t-L}), \tag{10}$$

where L is the number of lags of past information, taken into account. Since the deterministic function in the expectation rule depends on preceding price deviations, it can also be seen as including memory. However, due to rapidly increasing analytical complexity, *viz*. including more preceding price deviations rapidly increases the dimension of the system, this issue has so far mainly been neglected. In this paper we are focusing on the memory in the fitness measure and will thus include only one lag in the memory in the expectation rule, i.e. $f_h(x_{t-1})$.

The equilibrium pricing equation itself can thus finally be rewritten in terms of deviations from the fundamental price, $x_t = p_t - p^*$:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left[x_{t+1}\right] = \sum_{h=1}^{H} n_{h,t} f_{h,t} \ . \tag{11}$$

The particular form of deterministic function in the forecasting or expectation rule is thus what determines different types of heterogeneous agents in an adaptive belief system. In general, we distinguish between two typical investor types; fundamentalists and 'noise traders' or technical analysts. Fundamentalists believe that the price of an asset is defined solely by its efficient market hypothesis fundamental value (Fama, 1991), i.e. the present value of the stream of future dividends. Since they have no knowledge about other beliefs and fractions, $f_{h,t} \equiv 0$. Actual financial data show that fundamentalists have a stabilizing effect on prices (De Grauwe and Grimaldi, 2006).

Technical analysts or chartists, on the other hand, believe that asset prices are not completely determined by fundamentals, but may be predicted by inferences on past prices. Depending on the purpose of analysis, it is possible to distinguish between (pure) trend chasers with expectation rule $f_{h,t} = g_h x_{t-1}; g_h > 0$, (pure) contrarians with expectation rule $f_{h,t} = g_h x_{t-1}; g_h < 0$, and (pure) biased beliefs with expectation rule $f_{h,t} = b_h$, where g_h is the trend and b_h is the bias (difference between p^* and trader's belief of p^*) of the trader of type h.

3.2 Evolutionary Selection of Strategies

In order to be able to understand the dynamics of fractions of different trader types, we consider the appropriate formulations of realized excess return R_t from expression (1), and demand of different types of market traders, $z_{h,t-1}$, defined by expression (3). Taking again into account the nature of the dividend process $y_t = \overline{y} + \delta_t$ with constant conditional expectation, $\overline{y} = E\left[y_{t+1}\right]$, and assumed distribution $y_t \sim \text{IID}\,\mathrm{N}(0, \vartheta^2)$, we are thus able to formulate profits for a particular type of traders in each period as the product of realized excess return and number of shares purchased by traders of that type:

$$\pi_{h,t} = R_t z_{h,t-1} - C_h = (p_t + y_t - Rp_{t-1}) k E_{h,t-1}\left[p_t + y_t - Rp_{t-1}\right] - C_h , \tag{12}$$

where C_h represents the costs traders have to pay to use strategy h. Albeit introducing additional analytical complexity, we usually take into account the costs for predictor of particular trader type, since more information-intense predictors are evidently more costly. It is of course convenient to rewrite profits of different types of traders in terms of deviations from the benchmark fundamental:

$$\pi_{h,t} = (x_t - Rx_{t-1} + \delta_t) k E_{h,t-1}\left[x_t - Rx_{t-1}\right] - C_h . \tag{13}$$

The fitness function or performance measure of each trader type can now be defined in terms of its realized profits. In fact, it can be expressed as the weighted sum of realized profits, i.e. as the sum of current realized profits and a share of past fitness, which is in turn defined as past realized profits:

$$U_{h,t} = wU_{h,t-1} + (1-w)\pi_{h,t} , \tag{14}$$

where current realized profits are defined in the following final form:

$$\pi_{h,t} = k(x_t - Rx_{t-1})(f_{h,t-1} - Rx_{t-1}) - C_h . \tag{15}$$

The fitness function can also be rewritten in the following expanded form with exponentially declining weights:

$$U_{h,t} = w^{t-1}(1-w)\pi_{h,1} + w^{t-2}(1-w)\pi_{h,2} + \ldots + w(1-w)\pi_{h,t-1} + (1-w)\pi_{h,t} .$$

In case of the equilibrium pricing equation, herein formulated as the sum over trader types of products of a fraction of particular trader type and its deterministic function, the fitnesses enter the adaptive belief system before the equilibrium price is observed. This is suitable for analyzing the asset pricing model as an explicit nonlinear difference equation. Even though nonlinear asset pricing dynamics can be modelled either as a deterministic or a stochastic process, only the latter enables investigating the effects of noise upon the asset pricing dynamics.

The share of past fitness in the performance measure is expressed by the parameter w; $0 \leq w \leq 1$, called memory strength. When the value of this parameter is zero ($w = 0$), the fitness is given by most recent net realized profit. Due to analytical tractability this is presently for the most part the case in the existing literature on asset pricing models with heterogeneous agents, though not in this paper. Namely, the main contribution of this paper is to analyze the case of nonzero memory in the fitness measure. When the memory strength parameter takes a positive value, some share of current realized profits in any given period is taken into account when calculating the performance measure in the next time period. In case that the value of memory strength parameter amounts to one of course the entire accumulated wealth is taken into account.

The expression (14) for the fitness function is somewhat different that the one used in Brock and Hommes (1998), where the coefficient of the current realized profits was fixed to 1. Namely, if we rewrite the memory strength parameter as $w = 1 - \frac{1}{T}$, where T is considered to be a specific number of time periods, we obtain the following expression for the fitness function:

$$U_{h,t} = \left(1 - \frac{1}{T}\right) U_{h,t-1} + \frac{1}{T} \pi_{h,t}, \tag{16}$$

which is equivalent to taking the last T observations into account with equal weight (as benchmark). When T approaches infinity, memory parameter approaches 1 and the entire accumulated wealth is taken into account. We thus believe the expression (14) to be a more suitable formulation of the fitness measure than the one used in Brock and Hommes (1998), and in several other contributions.

Finally, we can express fractions of belief types, $n_{h,t}$, which are updated in each period, as a discrete choice probability by a multinomial logit model:

$$n_{h,t} = \frac{\exp\left[\beta U_{h,t-1}\right]}{\sum_{i=1}^{H} \exp\left[\beta U_{i,t-1}\right]}, \tag{17}$$

by using parameter β, determining the intensity of choice. The latter measures how fast economic agents switch between different prediction strategies; if the value of intensity of choice is zero then all trader types have equal weight and the mass of traders distributes itself evenly across the set of available strategies, while on the other hand the entire mass of traders tends to use the best predictor, i.e. the strategy with the highest fitness, when the intensity of choice approaches infinity (the neoclassical limit).

Trader fractions are therefore determined by fitness and intensity of choice. Rationality in the asset pricing model is evidently bounded, since fractions are ranked according to fitness, but not all agents choose the best predictor. To ensure that fractions of belief types depend only upon observable deviations from the fundamental at any given time period, fitness function in the fractions of belief types equation may only depend on past fitness and past return. This indeed ensures that past realized profits are observable quantities that can be used in predictor selection.

This completes the overview of the simple, analytically tractable asset pricing model in the adaptive belief system framework. Since it primarily represents the share of past fitness in the performance measure, memory can also be thought of as a share of past information that boundedly rational economic agents take into account as decision makers. When the value of the memory strength parameter is nonzero, past realized profits take an active role in determining the asset price in the simultaneous adaptive belief system. Though memory strength indeed appears in the fitness measure equation, memory in fact spreads through the model using appropriate deterministic functions representing different belief types. Namely, with memory affected fitness measure of each trader type enters discrete choice probability equation, which determines fractions of respective belief types, consequently affecting the asset price, which is modelled as the sum over trader types of products of a fraction of particular trader type and its deterministic function. Memory strength is thus one of the parameters of the asset pricing model that could decisively influence our inferences on stability of evolutionary adaptive systems and survival of technical trading.

4 Fundamentalists versus Contrarians

The first case we are going to examine is a two-type heterogeneous agents model with fundamentalists and contrarians as market participants. Fundamentalist exhibit deterministic function of the form:

$$f_{1,t} \equiv 0 \tag{18}$$

and have some positive information gathering costs C, i.e. $C > 0$. Contrarians exhibit deterministic function:

$$f_{2,t} = g x_{t-1}; \quad g < 0 \tag{19}$$

and zero information gathering costs. It is thus the case of fundamentalists versus pure contrarians. We have the following fractions of belief types equation:

$$n_{h,t} = \frac{\exp\left[\beta U_{h,t-1}\right]}{\exp\left[\beta U_{1,t-1}\right] + \exp\left[\beta U_{2,t-1}\right]}; \quad h = 1,2. \tag{20}$$

For convenience we shall also introduce difference in fractions m_t:

$$m_t = n_{1,t} - n_{2,t} = \frac{\exp\left[\beta U_{1,t-1}\right] - \exp\left[\beta U_{2,t-1}\right]}{\exp\left[\beta U_{1,t-1}\right] + \exp\left[\beta U_{2,t-1}\right]} = \tanh\left[\frac{\beta}{2}\left(U_{1,t-1} - U_{2,t-1}\right)\right]. \tag{21}$$

Finally, we have the fitness measure equation of each type:

$$U_{1,t} = wU_{1,t-1} + (1-w)\left[-kRx_{t-1}\left(x_t - Rx_{t-1}\right) - C\right], \tag{22}$$

$$U_{2,t} = wU_{2,t-1} + (1-w)\left[k\left(x_t - Rx_{t-1}\right)\left(gx_{t-2} - Rx_{t-1}\right)\right]. \tag{23}$$

In order to be able to analyze memory in our heterogeneous asset pricing model, we shall first determine the position and stability of the steady state and the period two-cycle in relation to the memory strength parameter. We will also examine the possible qualitative changes in dynamics. Then we will perform some numerical simulations to combine global stability analysis with local stability analysis.

4.1 Position of the Steady State

In our two-type heterogeneous agents model of fundamentalists versus contrarians the equilibrium pricing equation has the following form:

$$Rx_t = n_{2,t} g x_{t-1} = \frac{1-m_t}{2} g x_{t-1}, \tag{24}$$

where $n_{1,t} - n_{2,t} = m_t$ and $n_{1,t} + n_{2,t} = 1$. The difference in fractions of belief types equation, on the other hand, has the following form:

$$m_t = \tanh\left[\frac{\beta}{2}\left(w\left(U_{1,t-2} - U_{2,t-2}\right) - (1-w)\left(kgx_{t-3}\left(x_{t-1} - Rx_{t-2}\right) + C\right)\right)\right]. \tag{25}$$

All equations are of course rewritten in terms of the deviations from the fundamental price, $x_t = p_t - p^*$, as in Brock and Hommes (1998) and Brock *et al*. (2005), because the computation is then more convenient.

A steady state price deviation x is a fixed point of the system, if it satisfies $x = f(x)$ for mapping $f(x)$. In our two-type heterogeneous agents model of fundamentalists versus contrarians this implies:

$$Rx = \frac{1-m}{2} gx, \tag{26}$$

where either $x^{eq} = 0$, or $R = \frac{1-m^*}{2} g$ and thus $m^* = 1 - \frac{2R}{g}$. In the former case we get the fundamental steady state, where the price is equal to its fundamental value and the difference in fractions is:

$$m^{eq} = \tanh\left[\frac{\beta}{2}\left(w\left(U_1^{eq} - U_2^{eq}\right) - (1-w)C\right)\right].$$

Since it follows from expressions (22) and (23) that $U_1^{eq} = -C$ and $U_2^{eq} = 0$, the steady state difference in fractions simplifies:

$$m^{eq} = \tanh\left[\frac{\beta}{2}\left(-wC-(1-w)C\right)\right] = \tanh\left[-\frac{\beta C}{2}\right]. \tag{27}$$

Possible other (non-fundamental) steady states should satisfy:

$$m^{*} = \tanh\left[\frac{\beta}{2}\left(w(U_1^* - U_2^*) - (1-w)\left(kgx^*\left(x^* - Rx^*\right) + C\right)\right)\right]. \tag{28}$$

Since it can be derived that $U_1^* = -kRx^{*2}\left(1-R\right) - C$ and $U_2^* = kx^{*2}\left(1-R\right)\left(g-R\right)$, we obtain the following expression:

$$m^{*} = \tanh\left[\frac{\beta}{2}\left(w\left(kRx^{*2}\left(R-1\right)-C\right) - wkx^{*2}\left(1-R\right)\left(g-R\right) - \left(1-w\right)\left(kgx^{*2}\left(1-R\right)+C\right)\right)\right],$$

which further simplifies:

$$m^{*} = \tanh\left[-\frac{\beta}{2}\left(kgx^{*2}\left(1-R\right)+C\right)\right]. \tag{29}$$

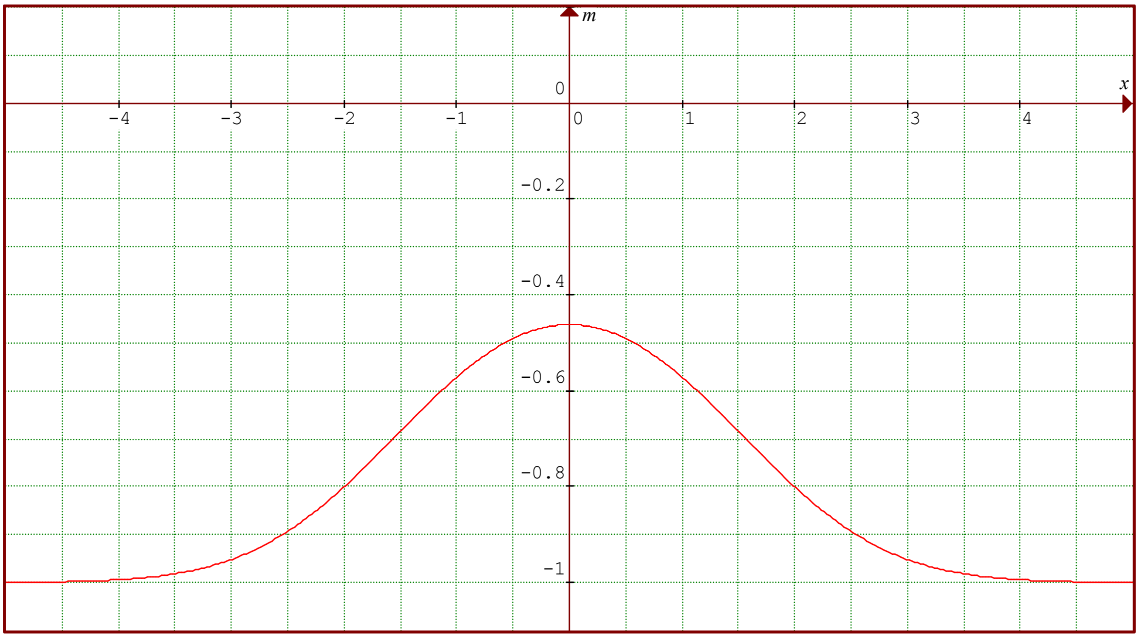

Figure 1: Difference in the fractions of belief types for values of parameters $\beta = 1$, $k = 1.0$, $g = -3.0$, $R = 1.1$ and $C = 1.0$

Therefore we can write the following lemma.

Lemma 1: The fundamental steady state in case of fundamentalists versus contrarians is a unique steady state of the system. Memory does not affect the position of this steady state.

Proof of Lemma 1: Since $g < 0$, $\frac{2R}{g} > 0$ holds and expression $m^* = 1 - \frac{2R}{g}$ is always greater than 1. On the other hand, the value of the hyperbolic tangent function is by definition between –1 and 1. In fact, since $k > 0$, $g < 0$, $R > 1$, $C > 0$ and the variable x is squared, the right-hand side of expression (29) is always between –1 and 0 (see Figure 1). Expression (29) thus never gives a solution and the fundamental steady state $(0,\ m^{eq})$ is a unique steady state of the system. Since there is no memory strength parameter in expression (27) and thus also in expression (26), memory does not affect the position of this steady state. ***QED***

4.2 Stability of the Steady State

In order to be able to analyze stability of the steady state we shall rewrite our system as a difference equation:

$$X_t = F_1\left(X_{t-1}\right), \tag{30}$$

where $X_{t-1} = \left(x_{1,t-1}, x_{2,t-1}, x_{3,t-1}, u_{1,t-1}, u_{2,t-1}\right)$ is a vector of new variables, which are defined as:

$x_{1,t-1} := x_{t-1}$;

$x_{2,t-1} := x_{t-2}$;

$x_{3,t-1} := x_{t-3}$;

$u_{1,t-1} := U_{1,t-2}$;

$u_{2,t-1} := U_{2,t-2}$.

We therefore obtain the following 5-dimensional first-order difference equation:

$$x_{1,t} = x_t = \frac{1}{R} n_{2,t} g x_{1,t-1} =$$

$$= \frac{1}{R} g x_{1,t-1} \frac{\exp\left[\beta U_{2,t-1}\right]}{\exp\left[\beta U_{1,t-1}\right] + \exp\left[\beta U_{2,t-1}\right]} = \frac{1}{R} g x_{1,t-1} \frac{\exp\left[\beta u_{2,t}\right]}{\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]}, \quad (31)$$

$$x_{2,t} = x_{t-1} = x_{1,t-1}, \quad (32)$$

$$x_{3,t} = x_{t-2} = x_{2,t-1}, \quad (33)$$

$$u_{1,t} = U_{1,t-1} = w u_{1,t-1} + (1-w)\left[-kR x_{2,t-1}\left(x_{1,t-1} - R x_{2,t-1}\right) - C\right], \quad (34)$$

$$u_{2,t} = U_{2,t-1} = w u_{2,t-1} + (1-w)\left[k\left(x_{1,t-1} - R x_{2,t-1}\right)\left(g x_{3,t-1} - R x_{2,t-1}\right)\right]. \quad (35)$$

The local stability of a steady state is determined by the eigenvalues of the Jacobian matrix. Thus we shall first compute the Jacobian matrix JF_1 of the 5-dimensional map, given by expression (30):

$$JF_1 = \begin{bmatrix} \frac{\partial x_{1,t}}{\partial x_{1,t-1}} & \frac{\partial x_{1,t}}{\partial x_{2,t-1}} & \frac{\partial x_{1,t}}{\partial x_{3,t-1}} & \frac{\partial x_{1,t}}{\partial u_{1,t-1}} & \frac{\partial x_{1,t}}{\partial u_{2,t-1}} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{\partial u_{1,t}}{\partial x_{1,t-1}} & \frac{\partial u_{1,t}}{\partial x_{2,t-1}} & 0 & w & 0 \\ \frac{\partial u_{2,t}}{\partial x_{1,t-1}} & \frac{\partial u_{2,t}}{\partial x_{2,t-1}} & \frac{\partial u_{2,t}}{\partial x_{3,t-1}} & 0 & w \end{bmatrix}, \quad (36)$$

with different derivatives given by the expressions hereinafter.

$$\frac{\partial x_{1,t}}{\partial x_{1,t-1}} = \frac{1}{R} g \left[\frac{\exp\left[\beta u_{2,t}\right]}{\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]} + \left(1-w\right) k g \beta x_{1,t-1} x_{3,t-1} \frac{\exp\left[\beta u_{1,t}\right] \exp\left[\beta u_{2,t}\right]}{\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)^2} \right]$$

$$\frac{\partial x_{1,t}}{\partial x_{2,t-1}} = -g^2 k \beta (1-w) x_{1,t-1} x_{3,t-1} \frac{\exp\left[\beta u_{1,t}\right] \exp\left[\beta u_{2,t}\right]}{\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)^2}$$

$$\frac{\partial x_{1,t}}{\partial x_{3,t-1}} = \frac{1}{R} g^2 k\beta(1-w) x_{1,t-1}\left(x_{1,t-1} - R x_{2,t-1}\right) \frac{\exp\left[\beta u_{1,t}\right]\exp\left[\beta u_{2,t}\right]}{\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)^2}$$

$$\frac{\partial x_{1,t}}{\partial u_{1,t-1}} = -\frac{1}{R} g\beta w x_{1,t-1} \frac{\exp\left[\beta u_{1,t}\right]\exp\left[\beta u_{2,t}\right]}{\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)^2}$$

$$\frac{\partial x_{1,t}}{\partial u_{2,t-1}} = \frac{1}{R} g\beta w x_{1,t-1} \frac{\exp\left[\beta u_{1,t}\right]\exp\left[\beta u_{2,t}\right]}{\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)^2}$$

$$\frac{\partial u_{1,t}}{\partial x_{1,t-1}} = \left(w-1\right) kR x_{2,t-1}$$

$$\frac{\partial u_{1,t}}{\partial x_{2,t-1}} = \left(w-1\right) kR\left(x_{1,t-1} - 2R x_{2,t-1}\right)$$

$$\frac{\partial u_{2,t}}{\partial x_{1,t-1}} = \left(1-w\right) k\left(g x_{3,t-1} - R x_{2,t-1}\right)$$

$$\frac{\partial u_{2,t}}{\partial x_{2,t-1}} = \left(1-w\right) kR\left(-x_{1,t-1} - g x_{3,t-1} + 2R x_{2,t-1}\right)$$

$$\frac{\partial u_{2,t}}{\partial x_{3,t-1}} = \left(1-w\right) kg\left(x_{1,t-1} - R x_{2,t-1}\right)$$

At the fundamental steady state $X^{eq} = (0, 0, 0, -C, 0)$ the Jacobian matrix becomes:

$$JF_1(X^{eq}) = \begin{bmatrix} \frac{1}{R} n_2^{eq} g & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & w \end{bmatrix}, \tag{37}$$

where $n_2^{eq} = \dfrac{1}{\exp\left[-\beta C\right]+1}$. A straightforward computation shows that the characteristic equation, $\det\left[JF_1\left(X^{eq}\right) - \lambda \mathrm{I}\right] = 0$, is in our case given by:

$$g(\lambda) = \left(\frac{1}{R} n_2^{eq} g - \lambda\right)\lambda^2\left(w-\lambda\right)^2 = 0\,, \tag{38}$$

with solutions (eigenvalues): $\lambda_1 = \frac{1}{R} n_2^{eq} g$, $\lambda_{2,3} = 0$ and $\lambda_{4,5} = w$. The steady state X^{eq} is stable for $|\lambda| < 1$; therefore in cases $-R < g n_2^{eq} < R$ and $w < 1$.

For the case of infinite memory, w = 1, the fitness remains fixed and therefore the fractions remain fixed. Given these fixed fractions, the price dynamics is generated by a linear system. This can be stable, unstable or have eigenvalues on the unit circle, depending on the initial fractions. In fact, our Jacobian matrix (37) reduces to the form:

$$JF_1(X^{eq} | w = 1) = \begin{bmatrix} \frac{1}{R} n_2^{eq} g & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{39}$$

which has the characteristic equation given by $\left(\frac{1}{R} n_2^{eq} g - \lambda \right) \lambda^2 = 0$, with eigenvalues $\lambda_1 = \frac{1}{R} n_2^{eq} g$ and $\lambda_{2,3} = 0$, and with stability condition $-R < g n_2^{eq} < R$.

Thus we can write the following lemma.

Lemma 2: The fundamental steady state in case of fundamentalists versus contrarians is globally stable for $-R < g < 0$. Memory does not affect the stability of this steady state.

Proof of Lemma 2: From the characteristic equation (38) we can observe three eigenvalues, where two of them are in fact double eigenvalues. The first eigenvalue assures stability when $-R < g n_2^{eq} < R$, while the second and third (double) eigenvalue always assure stability. The fundamental steady state is stable for $-\frac{R}{n_2^{eq}} < g < \frac{R}{n_2^{eq}}$, but since n_2^{eq} depends on other parameters of the system and g < 0, stability is (more conveniently) guaranteed at least for $-R < g < 0$. Since the memory strength parameter is represented (only) by the third (double) eigenvalue, memory does not affect the stability of the steady state, as has been shown by the reduced system. ***QED***

4.3 Bifurcations and the Period Two-cycle

A bifurcation is a qualitative change of the dynamical behaviour that occurs when parameters are varied (Brock and Hommes, 1998). A specific type of bifurcation that occurs when one parameter is varied is called a co-dimension one bifurcation. There are several types of such bifurcations, *viz*. period doubling, saddle-node and Hopf bifurcations. The first type has eigenvalue –1 of the Jacobian matrix, the second type has eigenvalue 1 and the third type has complex eigenvalues on the unit circle.

If we take a look at the eigenvalue λ_1, which we are in our case interested in, we can observe that a saddle-node bifurcation can never occur. Namely, the expression:

$$1 = \frac{1}{R} n_2^{eq} g \tag{40}$$

can never hold, since the left-hand side is a positive constant and the right-hand side is always negative for $g < 0$, $R > 0$ and $n_2^{eq} > 0$. On the other hand, the expression:

$$-1 = \frac{1}{R} n_2^{eq} g \tag{41}$$

may be satisfied for $n_2^{eq} \neq 0$, since both sides of the expression are then negative. Thus a (primary) period doubling bifurcation may occur in our model for the following β-value:

$$\beta^* = \frac{1}{C} \ln\left[-\frac{R}{R+g} \right], \tag{42}$$

which has been computed by plugging in $n_2^{eq} = \dfrac{1}{\exp\left[-\beta C\right]+1}$ into expression (41) and solving for the memory strength parameter β.

Now we can check the existence of a period two-cycle $\left\{(x^*, m^*), (-x^*, m^*)\right\}$. Taking into account that $U_1^* = kRx^{*2}\left(1+R\right) - C$ and $U_2^* = kx^{*2}\left(1+R\right)\left(g+R\right)$, a period two-cycle occurs when $-R = \dfrac{1-m^*}{2} g$ and thus $m^* = 1 + \dfrac{2R}{g}$ satisfies:

$$m^* = \tanh\left[\frac{\beta}{2}\Big(w\big(kRx^{*2}(1+R)-C\big)-wkx^{*2}(1+R)(g+R)-(1-w)\big(kgx^{*2}(1+R)+C\big)\Big)\right],$$

which further simplifies to the form:

$$m^* = \tanh\left[-\frac{\beta}{2}\big(kgx^{*2}(1+R)+C\big)\right]. \tag{43}$$

Therefore we can write the following lemma.

Lemma 3: In case of fundamentalists versus contrarians the fundamental steady state (0, m^{eq}) is unstable for $g < -2R$ and there exists a period two-cycle $\{(x^*, m^*), (-x^*, m^*)\}$. For $-2R < g < -R$ there are two possibilities: (1) if $m^* = 1 + \frac{2R}{g} < m^{eq}$ then (0, m^{eq}) is the unique, globally stable steady state, while (2) if $m^* = 1 + \frac{2R}{g} > m^{eq}$ then the steady state (0, m^{eq}) is unstable and there exists a period two-cycle $\{(x^*, m^*), (-x^*, m^*)\}$. Memory does not affect the position of the period two-cycle.

Proof of Lemma 3: For $g < -2R$ it is clear from the expression for eigenvalue λ_1 of the characteristic equation (38) that the fundamental steady state is unstable. Furthermore, since $0 < m^* < 1$, the expression (43) has two solutions, x^* and $-x^*$. If expression (41) is satisfied, it then follows from expressions $m^* = 1 + \frac{2R}{g}$ and (43) that $\{(x^*, m^*), (-x^*, m^*)\}$ is a period two-cycle. Finally, for $-2R < g < -R$, the fundamental steady state is unstable and expression (43) has solutions $\pm x^*$ if and only if $m^* > m^{eq} = \tanh\left[-\frac{\beta C}{2}\right]$ (see Figure 2). Since the memory strength parameter does not affect the difference in fractions of belief types, memory does not affect the position of the period two-cycle. ***QED***

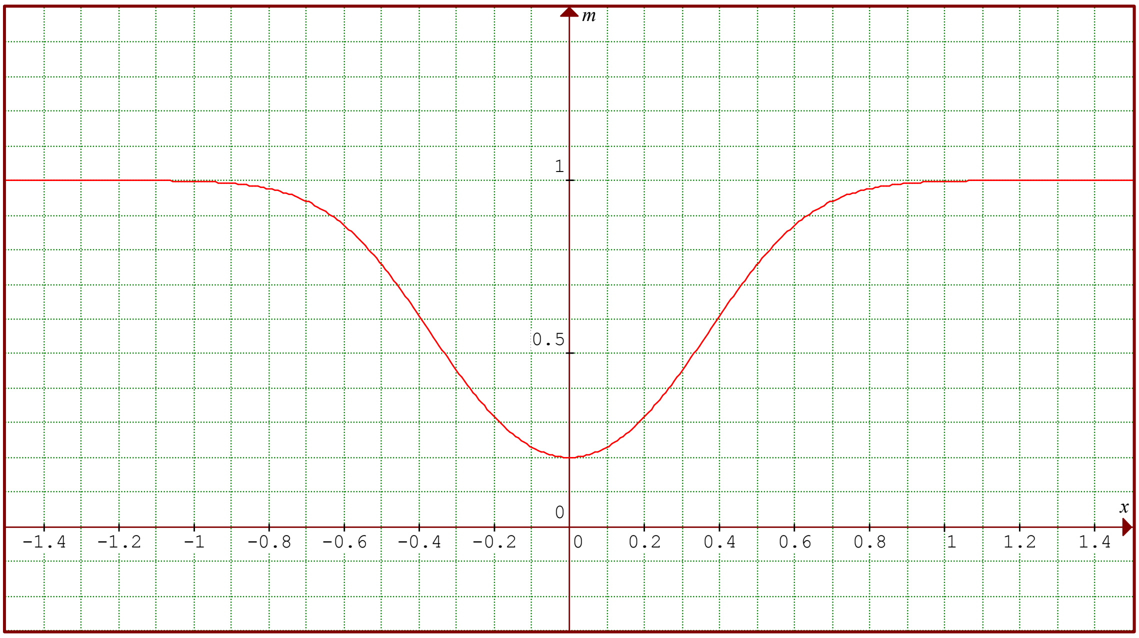

Figure 2: Difference in the fractions of belief types for values of parameters
$\beta = 1$, $k = 1.0$, $g = -3.0$, $R = 1.1$ and $C = 0.2$

As in the paper of Brock and Hommes (1998), very strong contrarians with $g < -2R$ may lead to the existence of a period two-cycle, even when there are no costs for fundamentalists ($C = 0$). When the fundamentalists' costs are positive ($C > 0$), strong contrarians with $-2R < g < -R$ may lead to a period two-cycle. As the intensity of choice increases to $\beta = \beta^*$, a period doubling bifurcation occurs in which the fundamental steady state becomes unstable and a (stable) period two-cycle is created, with one point above and the other one below the fundamental.

When the intensity of choice further increases, we are likely to find a value $\beta = \beta^{**}$, for which the period two-cycle becomes unstable and a Hopf bifurcation of this period two-cycle occurs, as in Brock in Hommes (1998). The model would then get an attractor consisting of two invariant circles around each of the two (unstable) period two-points, one lying above and the other one below the fundamental. Immediately after such a Hopf bifurcation, the price dynamics is either periodic or quasi-periodic, jumping back and forth between the two circles. The proof of this phenomenon is not straightforward due to the non-zero period points, although the 5-dimensional system (31) – (35) is still symmetric with respect to the origin. We shall thus demonstrate the occurrence of the Hopf bifurcation and the emergence of the attractor numerically in the next section.

4.4 Numerical Analysis

Our numerical analysis in case of fundamentalists and contrarians is going to be conducted for fixed values of parameters R = 1.1, k = 1.0, C = 1.0 and g = –1.5. We shall thus vary the intensity of choice parameter $\beta = \{4, 9, 25\}$ and of course the memory strength parameter $w = \{0, 0.3, 0.6, 0.9\}$. Four analytical tools will be used from the ones that are available in nonlinear economic dynamics; bifurcation diagrams, largest Lyapunov characteristic exponent (LCE) plots, phase plots, and time series plots. However, we will not discuss them here in more detail, since they are fairly well-known; instead we will direct the interested reader to more detailed discussions in Arrowsmith and Place (1990), Shone (1997), Brock and Hommes (1998), and Hommes (2004).

Bifurcation Diagrams

Dynamical behaviour of the system can first and foremost be determined by investigating bifurcation diagrams. In Figure 3 the bifurcation diagrams for four different values of the memory strength parameter are presented. We can observe that for low values of β we have a stable steady state, i.e. the fundamental steady state. As has been proven in Lemma 1, the position of this steady state, i.e. $x^{eq} = 0$, is independent of the memory, which is clearly demonstrated by the simulations. For increasing β a (primary) period doubling bifurcation occurs at $\beta = \beta^*$; the steady state becomes unstable and a stable period two-cycle appears, as proven in Lemma 3. As can be seen from the simulations, this bifurcation value is also independent of the memory and by the use of expression (42) it amounts to $\beta^* = 1.01$. Stability of the steady state is thus unaffected by the memory, as proven in Lemma 2.

For further increasing β indeed a (secondary) Hopf bifurcation occurs at $\beta = \beta^{**}$, as has been claimed in Section 4.3; the period two-cycle becomes unstable and an attractor appears consisting of two invariant circles around each of the two (unstable) period two-points, one lying above and the other one below the fundamental. It is a supercritical Hopf bifurcation, where the steady state gradually changes either into an unstable equilibrium or into an attractor (*cf.* Guckenheimer and Holmes, 1983; Frøyland, 1992; Brock, 1993; Kuznetsov, 1995). The position of the period two-cycle is independent of the memory, but it is not independent of the intensity of choice, as can be seen from expression (43). In our case of fundamentalists versus contrarians, the position of the period two-cycle is $x^* = \pm 0.126\sqrt{20 - 20.232\beta^{-1}}$, which approximately amounts to $x^* \approx \pm 0.5$ in all three cases with regard to the intensity of choice.

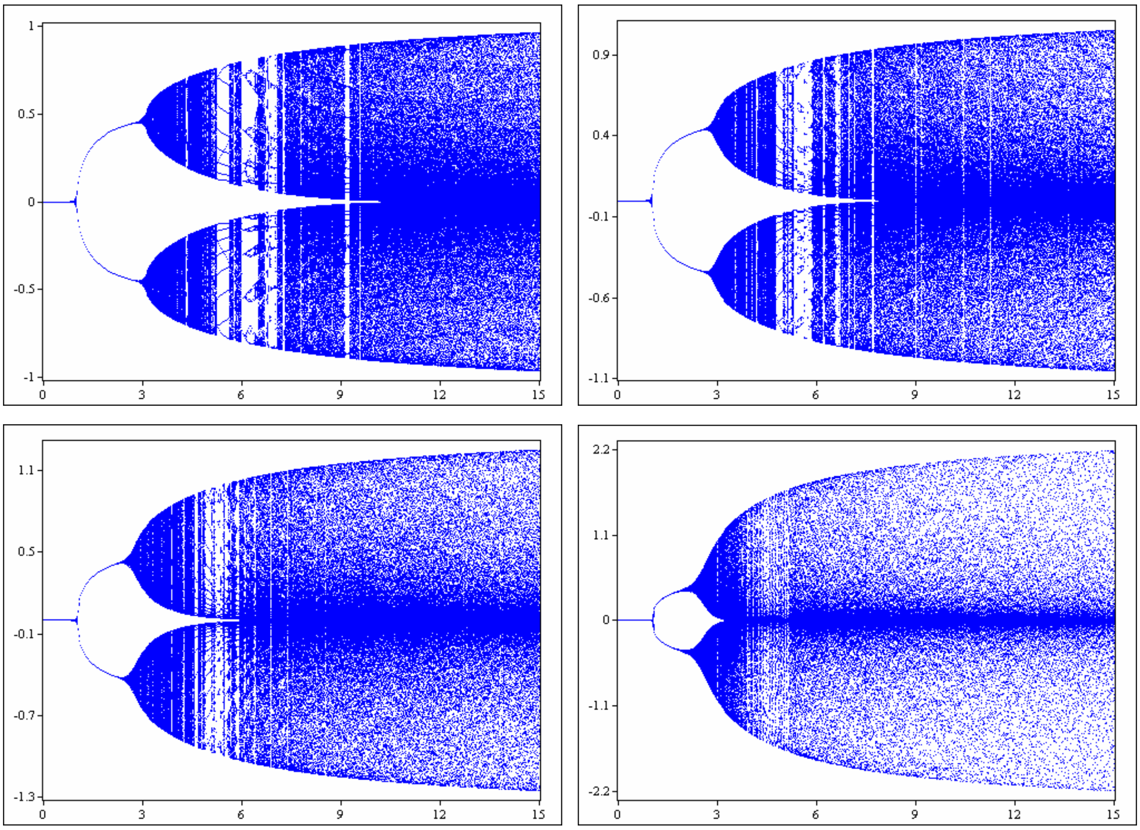

Figure 3: Bifurcation diagrams in case of fundamentalists versus contrarians

Notes: Horizontal axis represents the intensity of choice (β). Vertical axis represents deviations of the price from the fundamental value (x). The diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

Numerical simulations suggest that the secondary bifurcation value also does not vary with changing memory strength parameter and approximately amounts to $\beta^{**} \approx 3$. For $\beta > \beta^{**}$ chaotic dynamical behaviour appears, which is interspersed with many (mostly higher order) stable cycles. Such a bifurcation route to chaos was also called the rational route to randomness (Brock and Hommes, 1997a), while the last part of it has been referred to as the breaking of an invariant circle (Hommes, 2004, pp. 40-43).

Largest LCE Plots

By examining largest Lyapunov characteristic exponent (LCE) plots of β we arrive to the same conclusions about the dynamical behaviour of the system. Namely, it can be seen from Figure 4 that the largest LCE is smaller than 0 and the system is thus stable until the primary bifurcation, which happens at β^* = 1.01 and is independent of memory. At the bifurcation value, a qualitative change in dynamics occurs, i.e. a period doubling bifurcation and we obtain a stable period two-cycle. Largest LCE is again smaller than 0 and the system is thus stable until the secondary bifurcation, which occurs at $\beta^{**} \approx 3$. At this bifurcation value, again a qualitative change in dynamics occurs, i.e. a Hopf bifurcation, but the dynamics hereon is more complicated.

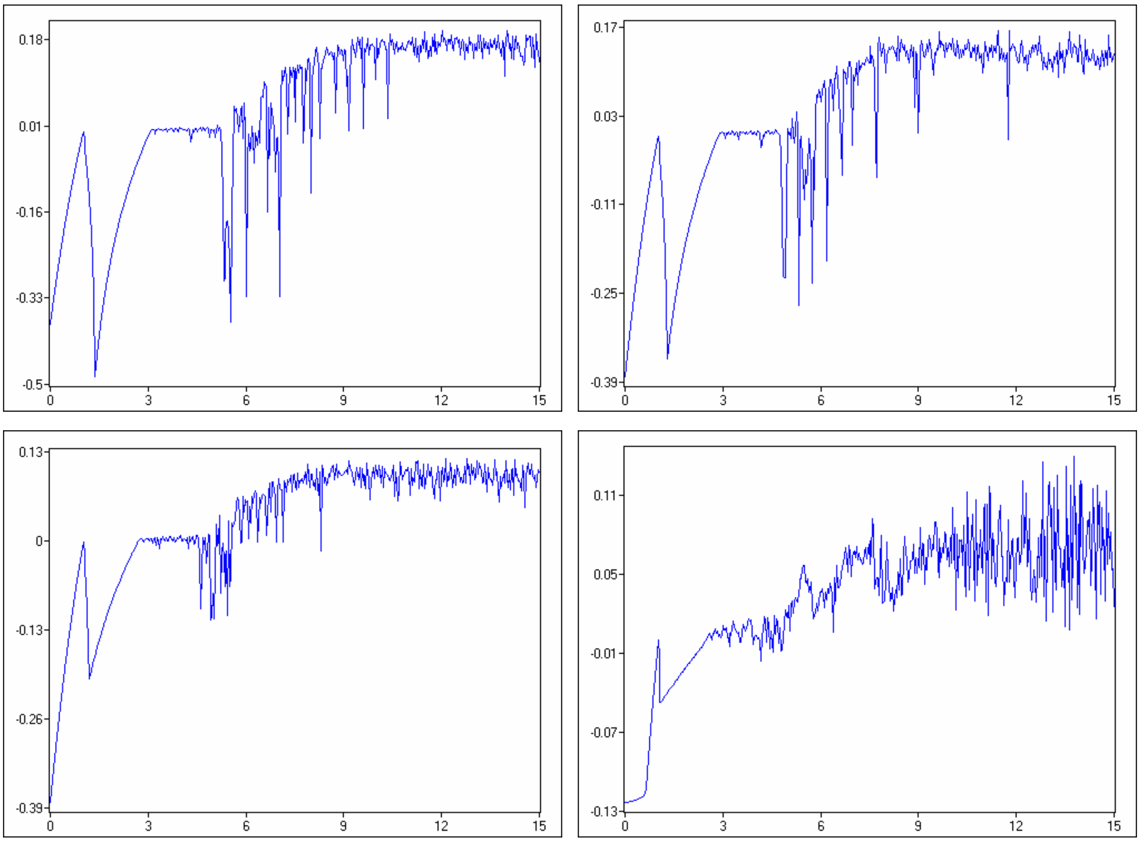

Figure 4: Largest LCE plots of β in case of fundamentalists versus contrarians

Notes: Horizontal axis represents the intensity of choice (β). Vertical axis represents the value of the largest LCE. The plots differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

We can observe that for $w = \{0, 0.3, 0.6\}$ the largest LCE after β^{**} is non-positive, but close to 0, which implies quasi-periodic dynamics. After some transient period the largest LCE becomes mainly positive with exceptions, which implies chaotic dynamics, interspersed with stable cycles. In fact, the largest LCE plot has a fractal structure (*cf.* Brock and Hommes, 1998, p. 1258). In case of w = 0.9 the global dynamics after β^{**} immediately becomes chaotic. Memory thus certainly affects the dynamics after the secondary bifurcation. Since the latter is a period doubling bifurcation, we are talking about period doubling routes to chaos.

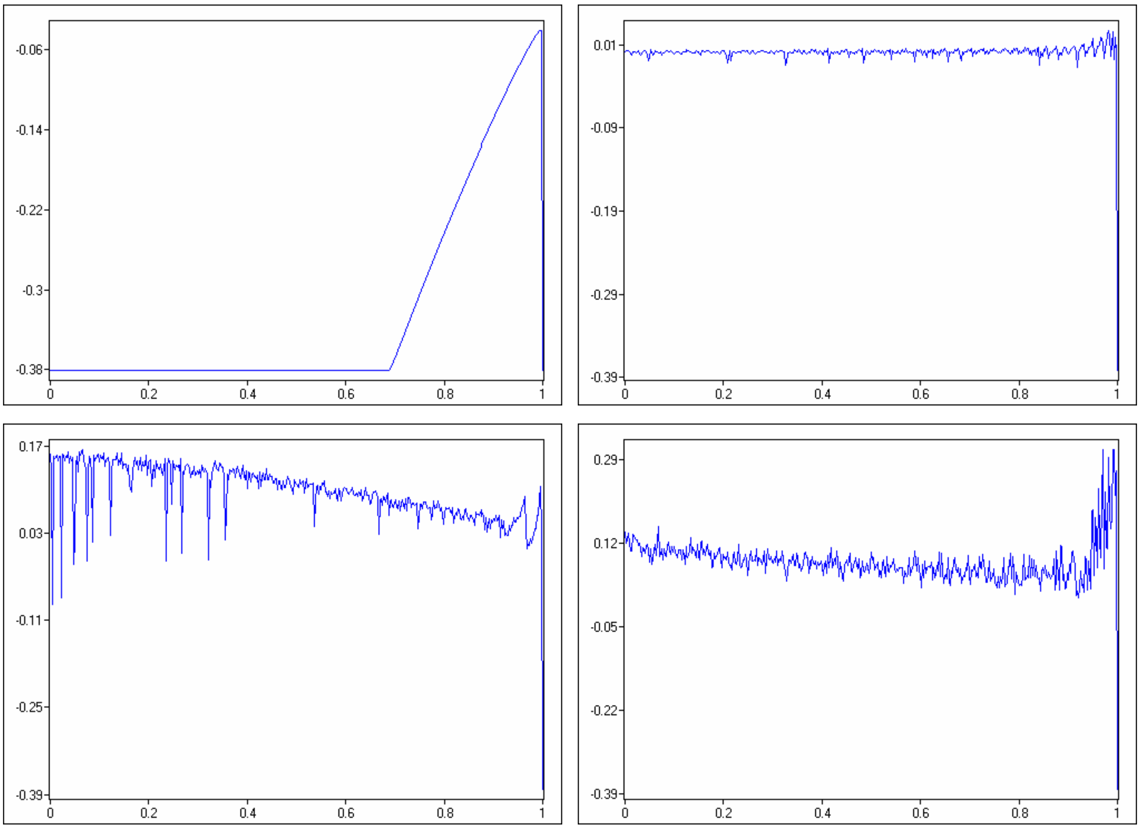

Figure 5: Largest LCE plots of w in case of fundamentalists versus contrarians

Notes: Horizontal axis represents the memory strength (w). Vertical axis represents the value of the largest LCE. The plots differ with respect to the intensity of choice parameter β; upper left corresponds to $\beta = 0$, upper right to $\beta = 4$, lower left to $\beta = 9$ and lower right to $\beta = 25$.

Besides the already observed distinction with respect to w we should also point out that the largest LCE indeed reaches lower maximum value for higher memory strength, but also has higher volatility. Therefore, at first sight, at higher intensity of choice there seems to be less divergence of nearby initial states in case of less memory in the model. This has been examined by largest Lyapunov characteristic exponent (LCE) plots of w. As can be seen from Figure 5, the statement is only partially correct. Namely, the interval of increasingly chaotic behaviour at high values of β, i.e. w = (0.9, 1), is quite narrow, though it broadens with increasing β. The rest of the interval, i.e. w = (0, 0.9), exhibits decreasingly chaotic dynamics at high values of β.

Phase Plots

Next, we shall examine plots of the attractors in the (x_t, x_{t-1}) plane and in the (x_t, $n_{1,t}$) plane without noise and with IID noise added to the supply of risky shares. Attractors in the (x_t, $n_{2,t}$) plane are just flipped (rotated by 180 degrees) images of attractors in the (x_t, $n_{1,t}$) plane and will thus not be separately examined. In the upper left plot of each of the four parts of Figures 6 and 7 we can first observe the appearance of an attractor for the intensity of choice beyond the secondary bifurcation value. The orbits converge to such an attractor consisting of two invariant 'circles' around each of the two (unstable) period two-points, one lying above and the other one below the fundamental value. Though we are topologically speaking about circles, the actual shape of such an attractor can be quite diverse, as seen from the figures.

As the intensity of choice increases, the circles 'move' closer to each other. In the upper right and lower left plot of each of the four parts of Figures 6 and 7 we can observe that the system seems to be already close to having a homoclinic orbit. The stable manifold of the fundamental steady state, $W^s(0,m^{eq})$, contains the vertical segment, x^{eq} = 0, whereas the unstable manifold, $W^u(0,m^{eq})$, has two branches, one moving to the right and one to the left. Both of them are then 'folding back' close to the stable manifold.

Namely, as Brock and Hommes (1998, p. 1254) have proven for the asset pricing model without additional memory, at infinite intensity of choice and strong contrarians, $g < -R$, that unstable manifold $W^u(0,-1)$ is bounded and all orbits converge to the saddle point (0, –1). In particular, all points of the unstable manifold converge to (0, –1) and are thus also on the stable manifold. Consequently, the system has homoclinic orbits for infinite intensity of choice. In case of strong contrarians and high intensity of choice it is therefore reasonable to expect that we obtain a system that is close to having a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state. This is indeed what can be observed from the lower left plot of each of the four parts of Figures 6 and 7 and it suggests the occurrence of chaos for high intensity of choice. As can be seen from the lower right plot of each of the four parts of Figures 6 and 7, adding small dynamic noise to the system does not alter our findings.

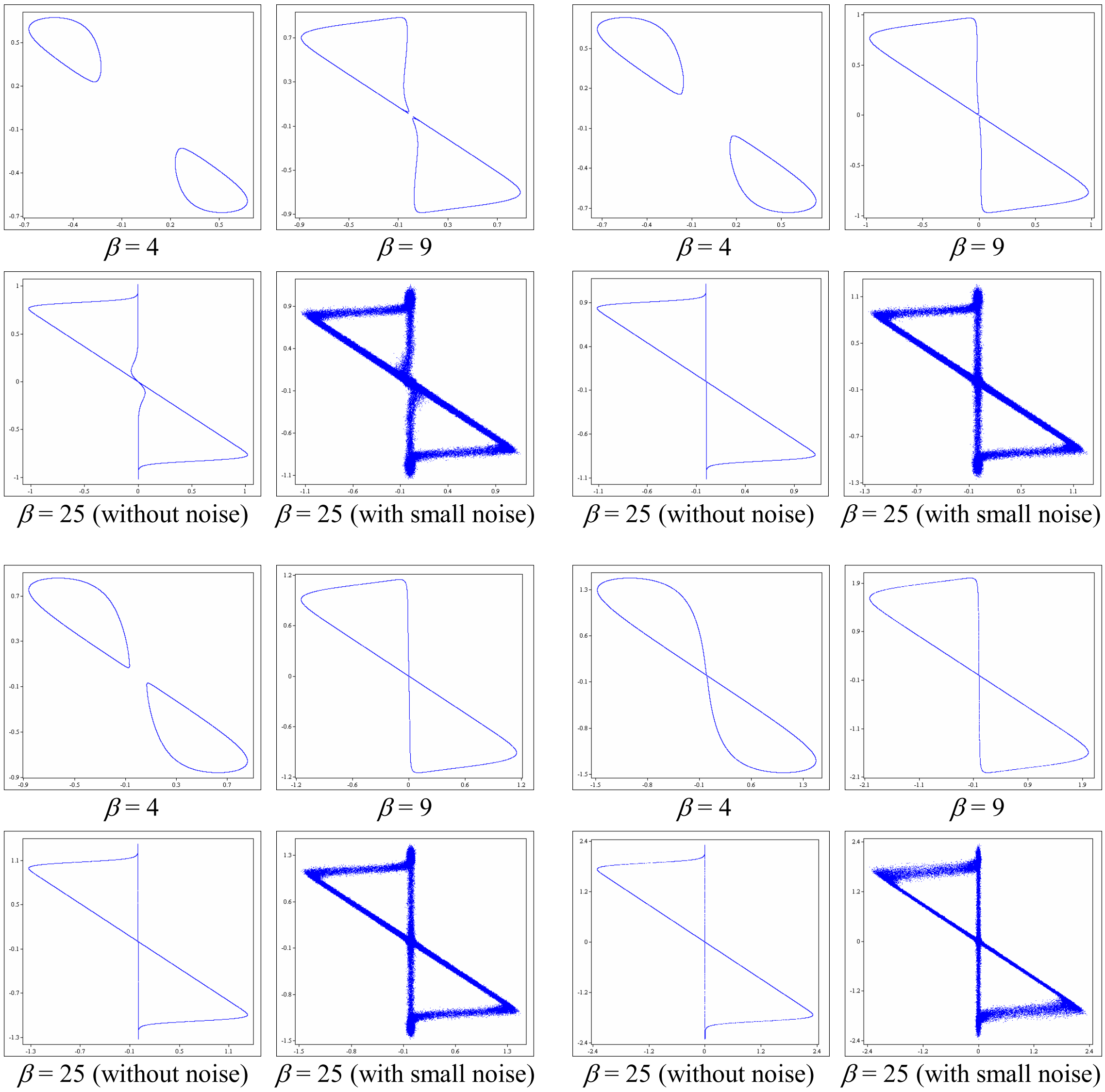

Figure 6: Phase plots of (x_t, x_{t-1}) in case of fundamentalists versus contrarians

Notes: Horizontal axis represents deviations of the price from the fundamental value (x_t). Vertical axis represents lagged deviations of the price from the fundamental value (x_{t-1}). The groups of four diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

The observed dynamic behaviour is quite similar to the chaotic price fluctuations in the cobweb model with costly rational versus free naive expectations in Brock and Hommes (1997a). In particular, the geometric shape of the (strange) attractors of the 5-dimenional asset pricing model with costly fundamentalism versus contrarians is very similar to the geometric shape of the strange attractors in the 2-dimensional cobweb demand-supply model with costly rational versus naive expectations. Similar finding was already established by Brock and Hommes (1998) for their 3-dimenional asset pricing model with costly fundamentalism versus contrarians.

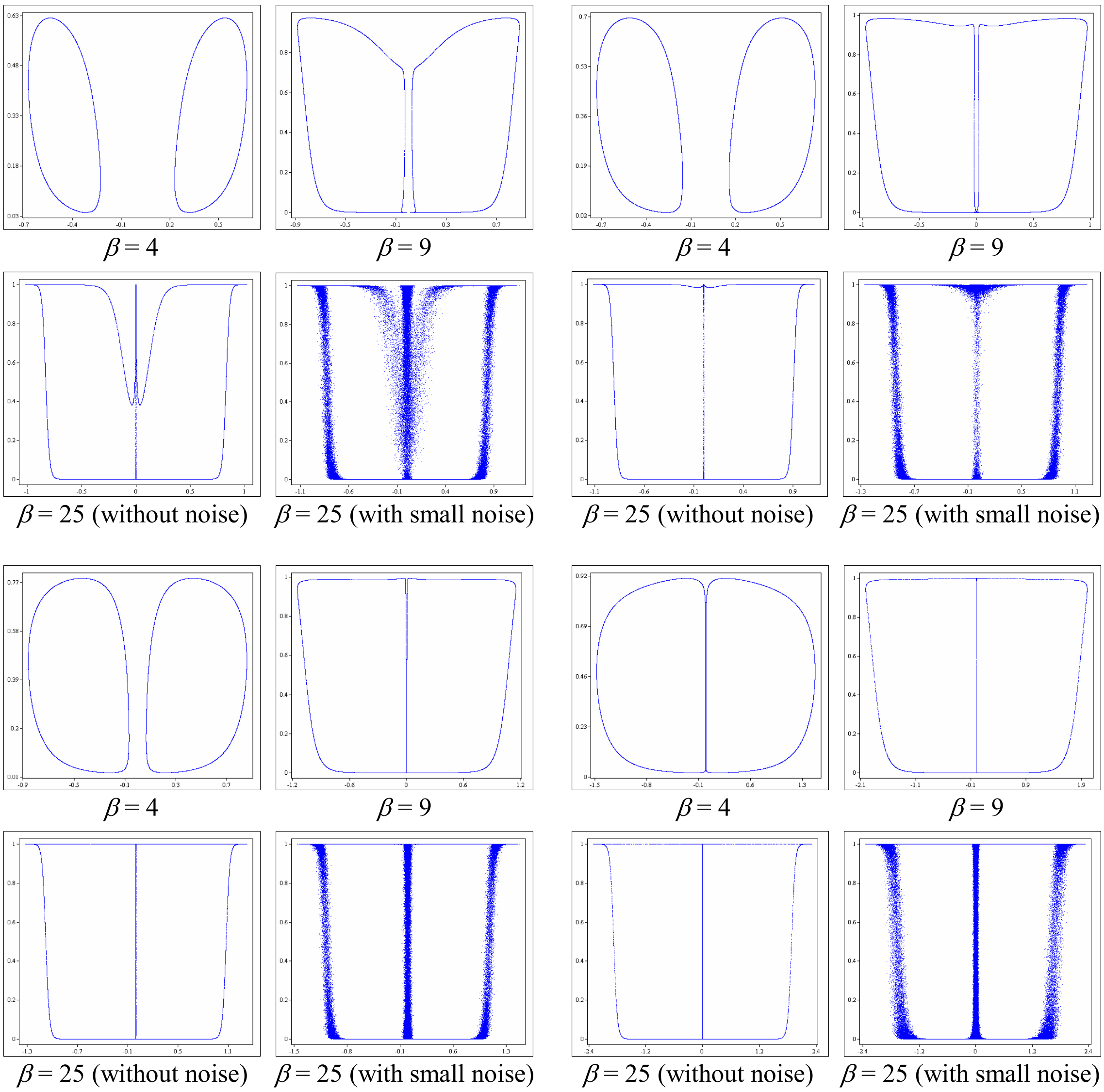

Figure 7: Phase plots of $(x_t, n_{1,t})$ in case of fundamentalists versus contrarians

Notes: Horizontal axis represents deviations of the price from the fundamental value (x_t). Vertical axis represents the fraction of fundamentalists $(n_{1,t})$. The groups of four diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

Again, we can observe from Figures 6 and 7 that memory has an impact on the global dynamics of the system. Namely, both the convergence of the system to an attractor consisting of two invariant 'circles' around each of the two unstable period two-points and the 'moving' of the circles closer to each other seem to be happening faster (at lower intensity of choice) when more memory is present in the model. Moreover, at the same intensity of choice we seem to be closer to obtaining a system that has a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state when the memory strength is higher. When there is no additional memory (w = 0), this homoclinic intersection is just being indicated, even for β = 25, but when the memory strength increases considerably, the homoclinic intersection between the stable and unstable manifolds of the fundamental steady state becomes distinctive.

Time Series Plots

Finally, we shall examine time series plots of deviations of the price from the fundamental value and of the fraction of fundamentalists. Since the fraction of contrarians is just the unity complement of the fraction of fundamentalists, i.e. $n_{1,t} + n_{2,t} = 1$, the former will thus not be separately graphically examined. Figures 8a and 8b show some time series corresponding to the attractors in Figures 6 and 7, with and without noise added to the supply of risky shares. Similarly to the findings of Brock and Hommes (1998), we can observe that the asset prices are characterized by an irregular switching between a stable phase with prices close to their (unstable) fundamental value and an unstable phase of up and down price fluctuations with increasing amplitude.

This irregular switching is of course reflected in the fractions of fundamentalists and contrarians in the market. Namely, when the oscillations of the price around the unstable steady state gain sufficient momentum, it becomes profitable for the trader to follow efficient market hypothesis fundamental value despite the costs that are involved in this strategy. The fraction of fundamentalists approaches unity and the asset price stabilizes. But then the nonzero costs of fundamentalists bring them into position where they are unable to compete in the market; the fraction of fundamentalists rapidly decrease to zero, while the fraction of contrarians with no costs approaches unity with equal speed. The higher the intensity of choice, *ceteris paribus*, the faster this transition is complete; when β approaches the neoclassical limit, the entire mass of traders tends to use the best predictor with respect to costs, i.e. the strategy with the highest fitness.

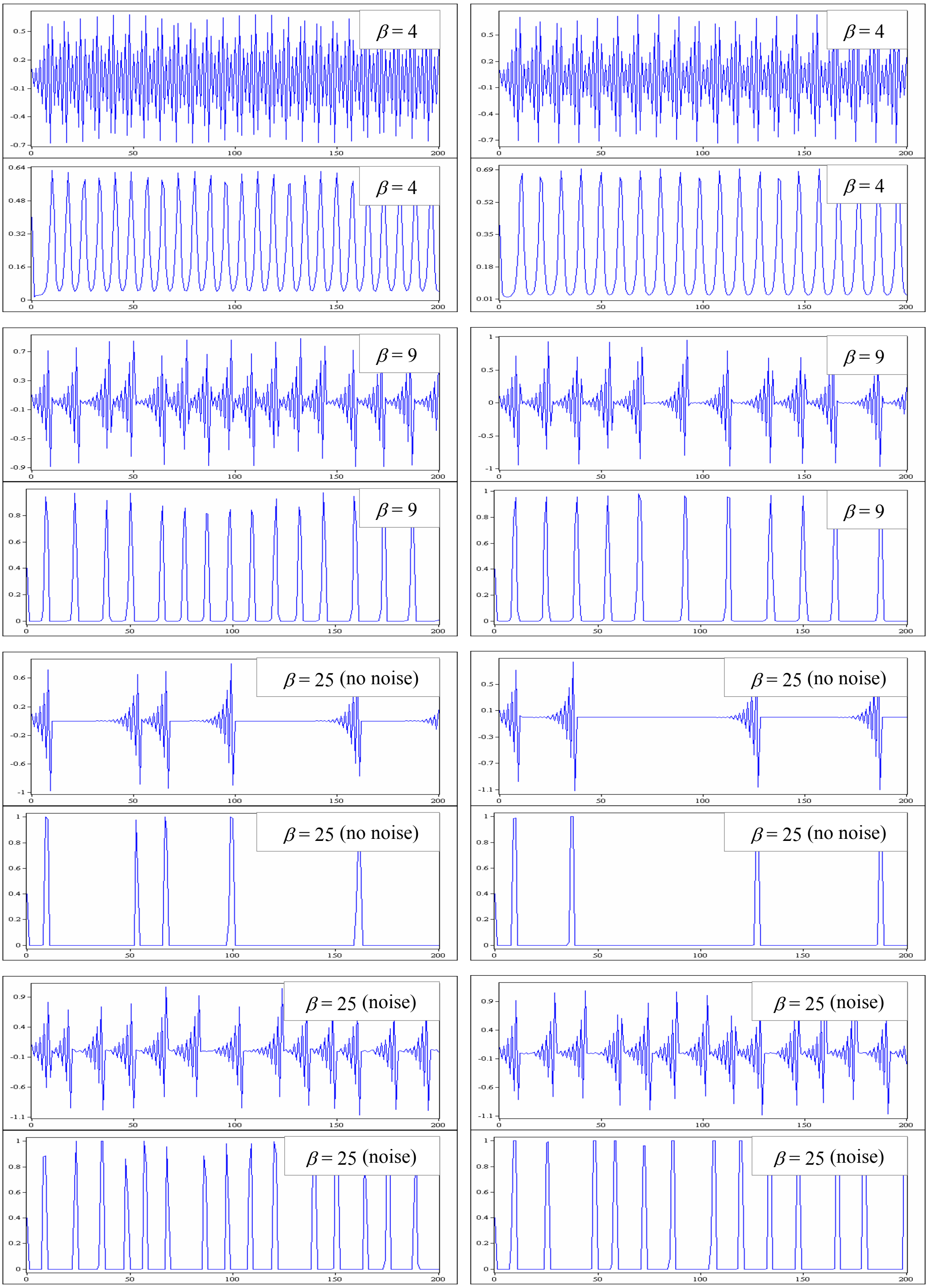

Figure 8a: Time series of prices and fractions in case of fundamentalists versus contrarians

Notes: Horizontal axis represents the time (t). Vertical axis in each pair of time series plots first represents deviations of the price from the fundamental value (x_t), and then the fraction of fundamentalists ($n_{1,t}$). The plots on the left-hand side and the right-hand side of the figure differ with respect to the memory strength parameter w; the ones on the left correspond to $w = 0$, while the ones on the right to $w = 0.3$.

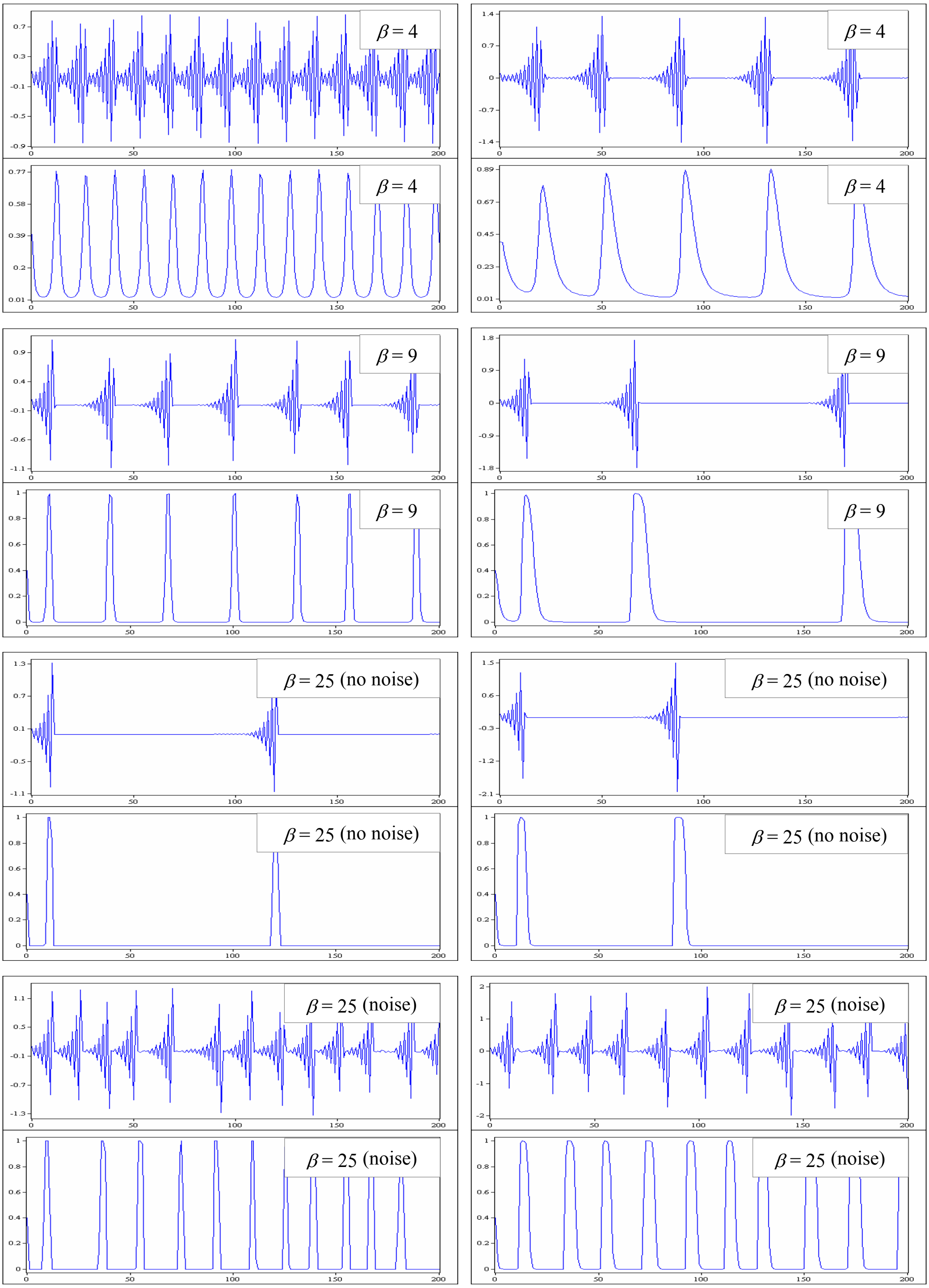

Figure 8b: Time series of prices and fractions in case of fundamentalists versus contrarians

Notes: Horizontal axis represents the time (t). Vertical axis in each pair of time series plots first represents deviations of the price from the fundamental value (x_t), and then the fraction of fundamentalists ($n_{1,t}$). The plots on the left-hand side and the right-hand side of the figure differ with respect to the memory strength parameter w; the ones on the left correspond to $w = 0.6$, while the ones on the right to $w = 0.9$.

Additional memory does not change the pattern of asset prices *per se*, but it does affect its period. Namely, at the same intensity of choice and higher memory strength the period of this irregular cycle appears to be elongated on average, in a way that the stable phase with prices close to their fundamental value lasts longer, while the duration of the unstable phase of up and down price fluctuations does not change significantly. The effect of including more memory thus mainly appears to be stabilizing with regard to asset prices. With regard to fractions of different trader types we could say that including additional (though still finite amount of) memory affects the transition from the short period of fundamentalists' dominance to the longer period of contrarians' dominance in the market. This transition takes more time to complete at the same intensity of choice. More but finite memory thus causes the traders to stick longer to the strategy that has been profitable in the past, but might not be so profitable in the recent periods. Addition of small dynamic noise to the supply of risky shares makes the effects of additional memory in the model less distinct, though they can still be observed.

5 Fundamentalists versus Opposite Biased Beliefs

The second case we are going to examine is a three-type heterogeneous agents model with fundamentalists and opposite biased beliefs as market participants. Fundamentalist again exhibit deterministic function of the form:

$$f_{1,t} \equiv 0\,, \tag{44}$$

though this time with no information gathering costs, i.e. $C = 0$. Biased beliefs exhibit deterministic functions:

$$f_{2,t} = b_2; \quad b_2 > 0\,, \tag{45}$$

$$f_{3,t} = b_3; \quad b_3 < 0\,, \tag{46}$$

for optimist and pessimist biases, respectively. In this paper we will mainly focus on the symmetric case. Biases also exhibit zero information gathering costs. We have the following fractions of belief types equation:

$$n_{h,t} = \frac{\exp\left[\beta U_{h,t-1}\right]}{\sum_{i=1}^{3}\exp\left[\beta U_{i,t-1}\right]}; \quad h = 1,2,3\,. \tag{47}$$

Finally, we have the fitness measures of each type:

$$U_{1,t} = wU_{1,t-1} + (1-w)\left[-kRx_{t-1}\left(x_t - Rx_{t-1}\right)\right], \tag{48}$$

$$U_{2,t} = wU_{2,t-1} + (1-w)\left[k\left(x_t - Rx_{t-1}\right)\left(b_2 - Rx_{t-1}\right)\right], \tag{49}$$

$$U_{3,t} = wU_{3,t-1} + (1-w)\left[k\left(x_t - Rx_{t-1}\right)\left(b_3 - Rx_{t-1}\right)\right]. \tag{50}$$

In order to be again able to analyze memory in our heterogeneous asset pricing model, we shall first determine the position and stability of the steady state, and then examine the possible qualitative changes in dynamics; all in relation to the memory strength parameter. Then we shall perform some numerical simulations to combine global stability analysis with local stability analysis.

5.1 Position of the Steady State

In our three-type heterogeneous agents model of fundamentalists versus biased beliefs, we shall again start by rewriting our system as a difference equation:

$$X_t = F_2\left(X_{t-1}\right), \tag{51}$$

where $X_{t-1} = \left(x_{1,t-1}, x_{2,t-1}, u_{1,t-1}, u_{2,t-1}, u_{3,t-1}\right)$ is a vector of new variables, defined as:

$x_{1,t-1} := x_{t-1}$;

$x_{2,t-1} := x_{t-2}$;

$u_{1,t-1} := U_{1,t-2}$;

$u_{2,t-1} := U_{2,t-2}$;

$u_{3,t-1} := U_{3,t-2}$.

We therefore obtain the following 5-dimensional first-order difference equation:

$$\begin{aligned} x_{1,t} = x_t &= \frac{1}{R}\left(n_{2,t}b_2 + n_{3,t}b_3\right) = \frac{1}{R}\left(\frac{\exp\left[\beta U_{2,t-1}\right]}{\sum_{i=1}^{3}\exp\left[\beta U_{i,t-1}\right]}b_2 + \frac{\exp\left[\beta U_{3,t-1}\right]}{\sum_{i=1}^{3}\exp\left[\beta U_{i,t-1}\right]}b_3\right) = \\ &= \frac{1}{R}\left(\frac{\exp\left[\beta u_{2,t}\right]}{\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]}b_2 + \frac{\exp\left[\beta u_{3,t}\right]}{\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]}b_3\right), \end{aligned} \tag{52}$$

$$x_{2,t} = x_{t-1} = x_{1,t-1}, \tag{53}$$

$$u_{1,t} = U_{1,t-1} = wu_{1,t-1} + (1-w)\left[-kRx_{2,t-1}\left(x_{1,t-1} - Rx_{2,t-1}\right)\right], \tag{54}$$

$$u_{2,t} = U_{2,t-1} = wu_{2,t-1} + (1-w)\left[k\left(x_{1,t-1} - Rx_{2,t-1}\right)\left(b_2 - Rx_{2,t-1}\right)\right], \tag{55}$$

$$u_{3,t} = U_{3,t-1} = wu_{3,t-1} + (1-w)\left[k\left(x_{1,t-1} - Rx_{2,t-1}\right)\left(b_3 - Rx_{2,t-1}\right)\right]. \tag{56}$$

Our three-type heterogeneous agents model of fundamentalists versus biased beliefs in general can have the following steady state price deviations:

$$x = \frac{1}{R}\left(n_2 b_2 + n_3 b_3\right). \tag{57}$$

We obtain the fundamental steady state for $b_2 = -b_3 = b > 0$ (opposite biased beliefs), where $x^{eq} = 0$. This is implied by $u_1^{eq} = u_2^{eq} = u_3^{eq} = 0$ and consequently by $n_1^{eq} = n_2^{eq} = n_3^{eq} = \frac{1}{3}$, originating from the rewritten expression (47).

By performing a generalization we can write the following lemma.

Lemma 4: The fundamental steady state in case of fundamentalists versus opposite biased beliefs is a unique steady state of the system. Memory does not affect the position of this steady state.

Proof of Lemma 4: We will prove a more general result for the case with h = 1, …, H purely biased types b_h (including fundamentalists with $b_1 = 0$). Proceeding from the non-transformed variables the system is:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} b_h , \tag{58}$$

$$n_{h,t} = \frac{\exp\left[\beta\left(wU_{h,t-2} + (1-w)\left[k\left(x_{t-1} - Rx_{t-2}\right)\left(b_h - Rx_{t-2}\right)\right]\right)\right]}{\sum_{i=1}^{3}\exp\left[\beta\left(wU_{i,t-2} + (1-w)\left[k\left(x_{t-1} - Rx_{t-2}\right)\left(b_i - Rx_{t-2}\right)\right]\right)\right]}; \quad 1 \le h \le H. \tag{59}$$

After subtracting off identical terms from the exponents of both numerator and denominator in expression (59) we obtain a new expression for the fractions:

$$n_{h,t} = \frac{\exp\left[\beta\left(wU^{\circ}_{h,t-2} + (1-w)k\left(x_{t-1} - Rx_{t-2}\right)b_h\right)\right]}{\sum_{i=1}^{H}\exp\left[\beta\left(wU^{\circ}_{i,t-2} + (1-w)k\left(x_{t-1} - Rx_{t-2}\right)b_i\right)\right]}; \quad 1 \le h \le H, \tag{60}$$

where $U^{\circ}_{h,t}$ is the fitness of trader type h, adjusted by subtracting off identical terms as above. The dynamic system defined by (58) and (60) is thus of the form:

$$Rx_t = V_{\beta k}(x_{t-1} - Rx_{t-2}), \tag{61}$$

where the right-hand side function is defined as:

$$V_{\beta k}(y_t)=\frac{\exp\left[\beta\left(wU^{\circ}_{h,t-2}(y_{t-1})+(1-w)kb_h y_t\right)\right]}{\sum_{i=1}^{H}\exp\left[\beta\left(wU^{\circ}_{i,t-2}(y_{t-1})+(1-w)kb_i y_t\right)\right]}=\sum_{h=1}^{H}b_h n_h=\langle b_h\rangle. \qquad (62)$$

Since it follows from (55) and (56) that $U_h^*=kx^*\left(1-R\right)\left(b_h-Rx^*\right)$, steady states of expressions (58) and (60) or expression (61) are determined by:

$$Rx^*=V_{\beta k}(x^*-Rx^*)=V_{\beta k}(-rx^*), \qquad (63)$$

where $r = R - 1$. Since a steady state has to satisfy expression (63), following Brock and Hommes (1998, p. 1271), a straightforward computation shows that:

$$\frac{\mathrm{d}}{\mathrm{d}y}V_{\beta k}(y)=\sum_{h=1}^{H}\left(\frac{\beta kb_h\exp[\beta kb_h y]}{\sum_{i=1}^{H}\exp[\beta kb_i y]}-\frac{\exp[\beta kb_h y]}{\left(\sum_{i=1}^{H}\exp[\beta kb_i y]\right)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}y}\left(\sum_{i=1}^{H}\exp[\beta kb_i y]\right)\right)b_h=$$

$$=\sum_{h=1}^{H}\left(\beta kn_h b_h^2-\beta kn_h b_h\sum_{h=1}^{H}n_h b_h\right)=\sum_{h=1}^{H}\left(\beta kn_h b_h^2-\beta kn_h b_h\langle b_h\rangle\right)=$$

$$=\beta k\left[\langle b_h^2\rangle-\langle b_h\rangle^2\right]>0, \qquad (64)$$

where the inequality follows from the fact that the term between square brackets can be interpreted as the variance of the stochastic process, where each b_h is drawn with probability n_h. Therefore, $V_{\beta k}(y)$ is increasing and $V_{\beta k}(-rx^*)$ decreasing in x^*. It then follows from expression (63) that the steady state x^* has to be unique. From expression (62) we obtain $V_{\beta k}(0)=\sum_{h=1}^{H}\frac{b_h}{H}=\bar{b}$, so that x^* equals the fundamental steady state if and only if $\bar{b}=0$, i.e. when all biases are exactly balanced. Since there is no memory strength parameter left in expressions (63) and $V_{\beta k}(0)$, memory does not affect the position of this steady state. It has to be mentioned though, that our derivation holds for finite intensity of choice, since fractions are only then all positive. ***QED***

5.2 Stability of the Steady State and Bifurcations

The local stability of a steady state is again determined by the eigenvalues of the Jacobian matrix. Thus we shall first compute the Jacobian matrix JF_2 of the 5-dimensional map, given by expression (51):

$$JF_2 = \begin{bmatrix} \frac{\partial x_{1,t}}{\partial x_{1,t-1}} & \frac{\partial x_{1,t}}{\partial x_{2,t-1}} & \frac{\partial x_{1,t}}{\partial u_{1,t-1}} & \frac{\partial x_{1,t}}{\partial u_{2,t-1}} & \frac{\partial x_{1,t}}{\partial u_{3,t-1}} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{\partial u_{1,t}}{\partial x_{1,t-1}} & \frac{\partial u_{1,t}}{\partial x_{2,t-1}} & w & 0 & 0 \\ \frac{\partial u_{2,t}}{\partial x_{1,t-1}} & \frac{\partial u_{2,t}}{\partial x_{2,t-1}} & 0 & w & 0 \\ \frac{\partial u_{3,t}}{\partial x_{1,t-1}} & \frac{\partial u_{3,t}}{\partial x_{2,t-1}} & 0 & 0 & w \end{bmatrix}, \qquad (65)$$

with different derivatives given by the expressions hereinafter.

$$\begin{aligned} \frac{\partial x_{1,t}}{\partial x_{1,t-1}} = \frac{1}{R}\beta k(1-w)\frac{1}{\left(\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]\right)^2}\cdot\Big[\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]\cdot\Big(\exp\left[\beta u_{2,t}\right]b_2\left(b_2 - Rx_{2,t-1}\right)+ \\ +\exp\left[\beta u_{3,t}\right]b_3\left(b_3 - Rx_{2,t-1}\right)\Big)+\left(\exp\left[\beta u_{2,t}\right]b_2+\exp\left[\beta u_{3,t}\right]b_3\right)\cdot \\ \cdot\left[\exp\left[\beta u_{1,t}\right]Rx_{2,t-1}-\exp\left[\beta u_{2,t}\right]\left(b_2 - Rx_{2,t-1}\right)-\exp\left[\beta u_{3,t}\right]\left(b_3 - Rx_{2,t-1}\right)\right]\Big] \end{aligned}$$

$$\begin{aligned} \frac{\partial x_{1,t}}{\partial x_{2,t-1}} = \beta k(1-w)\frac{1}{\left(\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]\right)^2}\cdot\Big[\sum_{i=1}^{3}\exp\left[\beta u_{i,t-1}\right]\cdot\Big(\exp\left[\beta u_{2,t}\right]b_2\left(-x_{1,t-1}-b_2+2Rx_{2,t-1}\right)+ \\ +\exp\left[\beta u_{3,t}\right]b_3\left(-x_{1,t-1}-b_3+2Rx_{2,t-1}\right)\Big)+\left(\exp\left[\beta u_{2,t}\right]b_2+\exp\left[\beta u_{3,t}\right]b_3\right)\cdot \\ \cdot\left[\sum_{i=1}^{3}\exp\left[\beta u_{i,t-1}\right]\cdot\left(x_{1,t-1}-2Rx_{2,t-1}\right)+\exp\left[\beta u_{2,t}\right]b_2+\exp\left[\beta u_{3,t}\right]b_3\right]\Big] \end{aligned}$$

$$\frac{\partial x_{1,t}}{\partial u_{1,t-1}} = -\frac{1}{R}w\beta\frac{\exp\left[\beta u_{1,t}\right]}{\left(\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]\right)^2}\cdot\left(\exp\left[\beta u_{2,t}\right]b_2+\exp\left[\beta u_{3,t}\right]b_3\right)$$

$$\frac{\partial x_{1,t}}{\partial u_{2,t-1}} = \frac{1}{R}w\beta\frac{\exp\left[\beta u_{2,t}\right]}{\left(\sum_{i=1}^{3}\exp\left[\beta u_{i,t}\right]\right)^2}\cdot\left(b_2\left(\exp\left[\beta u_{1,t}\right]+\exp\left[\beta u_{3,t}\right]\right)-b_3\exp\left[\beta u_{3,t}\right]\right)$$

$$\frac{\partial x_{1,t}}{\partial u_{3,t-1}} = -\frac{1}{R} w\beta \frac{\exp\left[\beta u_{3,t}\right]}{\left(\sum_{i=1}^{3} \exp\left[\beta u_{i,t}\right]\right)^2} \cdot \left(b_2 \exp\left[\beta u_{2,t}\right] - b_3\left(\exp\left[\beta u_{1,t}\right] + \exp\left[\beta u_{2,t}\right]\right)\right)$$

$$\frac{\partial u_{1,t}}{\partial x_{1,t-1}} = (w-1)kRx_{2,t-1}$$

$$\frac{\partial u_{1,t}}{\partial x_{2,t-1}} = (w-1)kR\left(x_{1,t-1} - 2Rx_{2,t-1}\right)$$

$$\frac{\partial u_{2,t}}{\partial x_{1,t-1}} = (1-w)k\left(b_2 - Rx_{2,t-1}\right)$$

$$\frac{\partial u_{2,t}}{\partial x_{2,t-1}} = (1-w)kR\left(-x_{1,t-1} - b_2 + 2Rx_{2,t-1}\right)$$

$$\frac{\partial u_{3,t}}{\partial x_{1,t-1}} = (1-w)k\left(b_3 - Rx_{2,t-1}\right)$$

$$\frac{\partial u_{3,t}}{\partial x_{2,t-1}} = (1-w)kR\left(-x_{1,t-1} - b_3 + 2Rx_{2,t-1}\right)$$

At the fundamental steady state X^{eq} = (0, 0, 0, 0, 0) the Jacobian matrix becomes:

$$JF_2(X^{eq}) = \begin{bmatrix} \frac{2}{3R}(1-w)k\beta b^2 & -\frac{2}{3}(1-w)k\beta b^2 & 0 & \frac{1}{3R}w\beta b & -\frac{1}{3R}w\beta b \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ (1-w)kb & (w-1)kRb & 0 & w & 0 \\ (w-1)kb & (1-w)kRb & 0 & 0 & w \end{bmatrix}. \quad (66)$$

The characteristic equation, $\det\left[JF_2\left(X^{eq}\right) - \lambda \mathrm{I}\right] = 0$, is in this case given by:

$$g(\lambda) = -\left(\lambda^2 - \left(w - \frac{2}{3R}k\beta b^2(w-1)\right)\lambda - \frac{2}{3}k\beta b^2(w-1)\right)\lambda\left(w-\lambda\right)^2 = 0, \quad (67)$$

which has the following five solutions, with one of them being double: $\lambda_1 = 0$, $\lambda_{2,3} = w$ and $\lambda_{4,5} = \frac{1}{6R}\left(2b^2\beta k(1-w) + 3Rw \pm \sqrt{\left(2b^2\beta k(w-1) - 3Rw\right)^2 - 24b^2\beta k(1-w)R^2}\right)$.

The fundamental steady state is stable for $|\lambda|<1$, which in our case limited to the product of eigenvalues $\lambda_{4,5}$ being smaller than one, i.e. $-\frac{2}{3}k\beta b^2(w-1)<1$. In terms of the intensity of choice this happens for $\beta<-\frac{3}{2kb^2(w-1)}$, while in terms of the memory strength this is guaranteed for $w<1-\frac{3}{2k\beta b^2}$.

For the case of infinite memory, w = 1, our Jacobian matrix (66) reduces to the form:

$$JF_2(X^{eq}|w=1)=\begin{bmatrix}\frac{2}{3R}(1-w)k\beta b^2 & -\frac{2}{3}(1-w)k\beta b^2\\ 1 & 0\end{bmatrix}, \qquad (68)$$

which has the characteristic equation given by $\lambda^2-\frac{2}{3}(1-w)k\beta b^2\left(\frac{1}{R}\lambda-1\right)=0$, with eigenvalues $\lambda_{1,2}=\frac{1}{6R}\left(2b^2\beta k(1-w)\pm b\sqrt{2\beta k(w-1)\left(2b^2\beta k(w-1)+12R^2\right)}\right)$.

Thus we can write the following lemma.

Lemma 5: The fundamental steady state in case fundamentalists versus opposite biased beliefs is globally stable for $\beta<-\frac{3}{2kb^2(w-1)}$. Memory affects the stability of this steady state by restricting it to the given interval of the parameter value.

Proof of Lemma 5: From the characteristic equation (67) we can observe five eigenvalues. The first three eigenvalues always assure stability, while the last two eigenvalues limit stability. Given $k>0$, $b>0$, $\beta\geq 0$, $R>1$ and $0\leq w\leq 1$, the condition for stability in terms of β implies $\beta<-\frac{3}{2kb^2(w-1)}$. Similarly, the condition for stability in terms of w indicates $w<1-\frac{3}{2k\beta b^2}$. Memory therefore affects the stability of the steady state as shown. ***QED***

If we now take a look at the eigenvalues $\lambda_{4,5}$ of the characteristic equation (67), which are of interest in our case, we can observe that a saddle-node bifurcation would occur for:

$$\beta = \frac{3R}{2b^2k(1-R)}. \tag{69}$$

This can never hold, since $\beta \geq 0$ and the left-hand side is always non-negative, while $R > 1$ and the right-hand side is always negative. On the other hand, a period doubling bifurcation would occur for:

$$\beta = \frac{3R(w+1)}{2b^2k(R+1)(w-1)}. \tag{70}$$

This can never hold either, since $\beta \geq 0$ and the left-hand side is again always non-negative, while $0 \leq w \leq 1$ and the right-hand side is either negative or not defined.

The only qualitative change left of the three discussed in Section 4.3 is the Hopf bifurcation. For this to occur, a complex conjugate pair of eigenvalues has to cross the unit circle. Eigenvalues $\lambda_{4,5}$ are complex for $\left(2b^2\beta k(w-1)-3Rw\right)^2 - 24b^2\beta k(1-w)R^2 < 0$, which produces the following interval of values:

$$\frac{R\left(3w-6R-2\sqrt{R(R-w)}\right)}{2b^2k(w-1)} < \beta < \frac{R\left(3w-6R+2\sqrt{R(R-w)}\right)}{2b^2k(w-1)}. \tag{71}$$

We therefore have the following lemma.

Lemma 6: There exists an intensity of choice value β^* such that the fundamental steady state, which is stable for $0 \leq \beta < \beta^*$, becomes unstable and remains such for $\beta > \beta^*$. For $\beta^* = -\dfrac{3}{2kb^2(w-1)}$ the system exhibits a Hopf bifurcation. Memory affects the emergence of this bifurcation, *viz*. with more memory the bifurcation occurs later.

Proof of Lemma 6: When β increases, terms with β in the expressions for the eigenvalues $\lambda_{4,5}$ increase as well, and one of the eigenvalues has to cross the unit circle at some critical $\beta = \beta^*$. The fundamental steady state thus becomes unstable. Since it is obvious from the characteristic equation (67) that for all $\beta \geq 0$ we have $g(1) > 0$ and $g(-1) < 0$, a bifurcation has to occur. At the moment of the bifurcation the product of eigenvalues $\lambda_{4,5}$ has to be equal one, i.e. $-\frac{2}{3}k\beta b^2(w-1) = 1$. This happens either when we have two real eigenvalues with product equal to one or a complex conjugate pair of eigenvalues. Since $\beta^* = -\frac{3}{2kb^2(w-1)}$ falls into the interval (71) for any given finite memory strength, we can conclude that for $\beta = \beta^*$ the eigenvalues have to be complex and thus a Hopf bifurcation occurs. Since the memory strength parameter is present in the expression for β^*, memory affects the emergence of this bifurcation; the higher the value of this parameter, the higher the bifurcation value. ***QED***

As we have just established, in case of fundamentalists versus opposite biased beliefs increasing intensity of choice to switch predictors destabilizes the fundamental steady state. This happens through a Hopf bifurcation. We can thus conclude, as did Brock and Hommes (1998) for the simpler version of the model, that in the presence of biased agents the first step towards complicated price fluctuations is different from that in the presence of contrarians. This fact does not change when we take memory into account.

5.3 Numerical Analysis

Our numerical analysis in case of fundamentalists and opposite biased beliefs is going to be conducted for fixed values of parameters $R = 1.1$, $k = 1.0$, $b_2 = 0.2$ and $b_3 = -0.2$. We shall thus vary the memory strength parameter $w = \{0, 0.3, 0.6, 0.9\}$ and the intensity of choice parameter β. The latter is going to be varied with regard to the memory strength parameter in the interval $\beta = [50, ..., 35000]$, because memory in this case substantially affects the bifurcation value (see Figure 9). The same four analytical tools will be used from the ones that are available in nonlinear economic dynamics than in the case of fundamentalists versus contrarians; bifurcation diagrams, largest Lyapunov characteristic exponent (LCE) plots, phase plots, and time series plots.

Bifurcation Diagrams

Dynamical behaviour of the system can again first and foremost be determined by investigating bifurcation diagrams. In Figure 9 the bifurcation diagrams for four different values of the memory strength parameter are presented. We can observe that for low values of β we have a stable steady state, i.e. the fundamental steady state. As has been proven in Lemma 4, the position of this steady state, i.e. $x^{eq} = 0$, is independent of the memory, which is clearly demonstrated by the simulations. For increasing β a (primary and only) bifurcation occurs at $\beta = \beta^*$, i.e. a Hopf bifurcation; the steady state becomes unstable and an attractor appears consisting of an invariant circle around the (unstable) steady state. It is again a supercritical Hopf bifurcation, where the steady state gradually changes either into an unstable equilibrium or into an attractor.

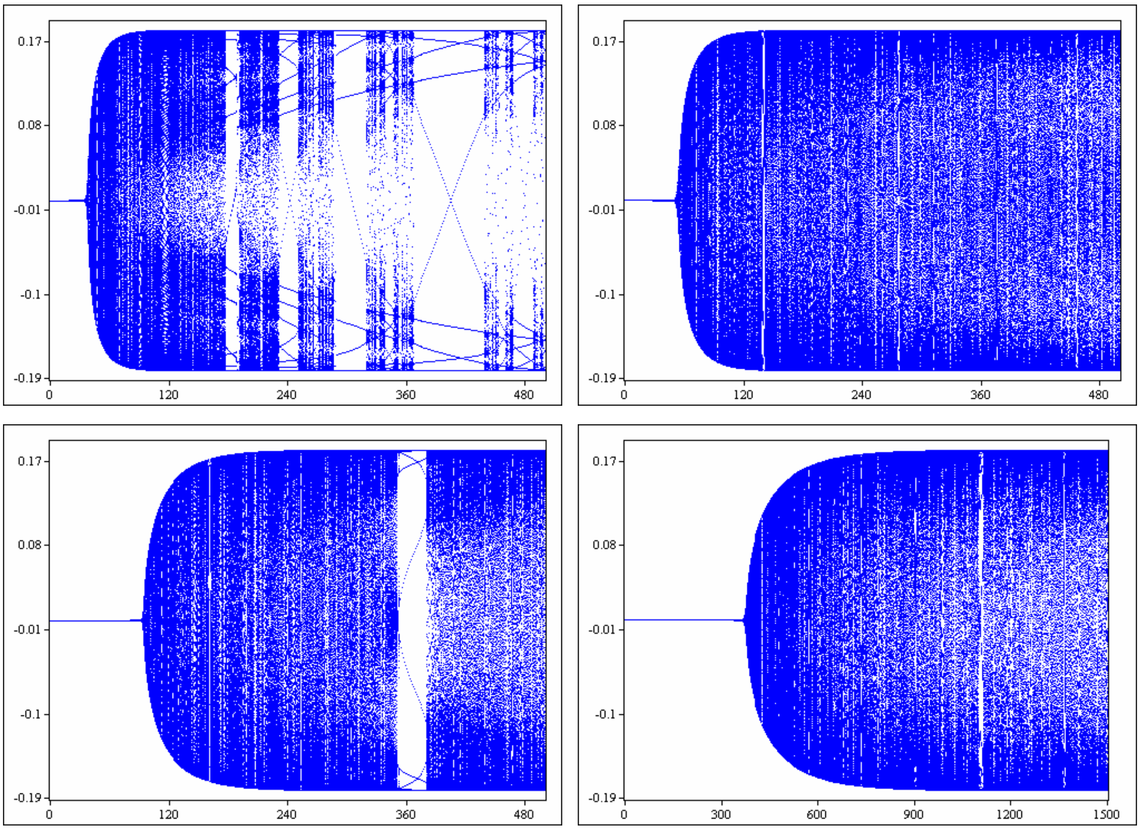

Figure 9: Bifurcation diagrams in case of fundamentalists versus opposite biased beliefs

Notes: Horizontal axis represents the intensity of choice (β). Vertical axis represents deviations of the price from the fundamental value (x). The diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

The bifurcation value varies with changing memory strength parameter, as given by expression in Lemma 6; for $w = 0$ it amounts to $\beta^* = 37.5$, for $w = 0.3$ it is equal to $\beta^* =$

53.57, for w = 0.6 it amounts to β^* = 93.75 and for w = 0.9 it is equal to β^* = 375. As can also be seen from Figure 9, at higher memory strength the bifurcation occurs later. For $\beta > \beta^*$ complex dynamical behaviour appears, which is interspersed with stable cycles. As we have already discovered in Section 5.2, irrespective of the amount of additional memory that is taken into account such a (bifurcation) route to complicated dynamics is different from that in the presence of contrarians, where we observed period doubling route to chaos (rational route to randomness). We shall now examine this route more thoroughly with other analytical tools of nonlinear economic dynamics.

Largest LCE Plots

By examining largest Lyapunov characteristic exponent (LCE) plots of β we arrive to more precise conclusions about the dynamical behaviour of the system. Namely, it can be seen from Figure 10 that the largest LCE is smaller than 0 and the system is thus stable until the primary (and only) bifurcation, which happens at $\beta = \beta^*$ and is dependent on memory. At the bifurcation value, a qualitative change in dynamics occurs, i.e. a Hopf bifurcation. The dynamics hereon is somewhat more complicated. Namely, we can observe that the largest LCE after $\beta = \beta^*$ is non-positive, but mainly close to 0, which implies periodic and quasi-periodic dynamics, i.e. for high values of the intensity of choice only regular (quasi-)periodic fluctuations around the unstable fundamental steady state occur. An important finding is that the predominating quasi-periodic dynamics does not seem to evolve to chaotic dynamics and the route to complex dynamics is indeed different from the routes examined so far.

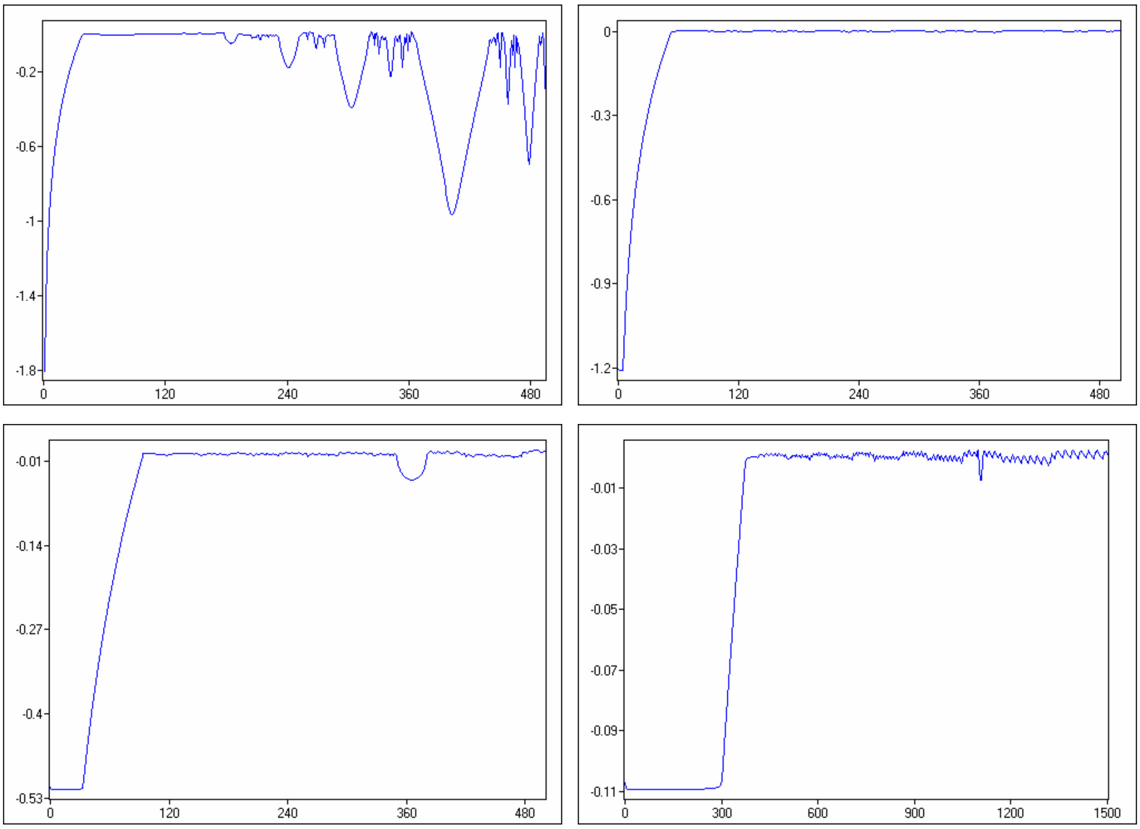

Figure 10: Largest LCE plots of β in case of fundamentalists versus opposite biased beliefs

Notes: Horizontal axis represents the intensity of choice (β). Vertical axis represents the value of the largest LCE. The plots differ with respect to the memory strength parameter *w*; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

The amount of memory affects the dynamics of the system primarily by determining its bifurcation value, while the dynamics of the system after the Hopf bifurcation is qualitatively not changed much by additional memory. The bifurcation value can be clearly observed not only on Figure 10, where the steady state is stable until the value of the largest LCE approaches 0, but for $\beta = \{100, 500, 1000\}$ also on Figure 11, where the steady state is stable after the value of the largest LCE digresses from 0 for the last time (at high memory strength value). As can be seen from Figures 10 and 11, after the incidence of the bifurcation higher value of the memory strength parameter causes the dynamics to be less periodic and more quasi-periodic; consequently there tends to be less interspersion of the quasi-periodic dynamics with stable cycles. The dynamics therefore converges to the purely quasi-periodic behaviour with increasing memory strength.

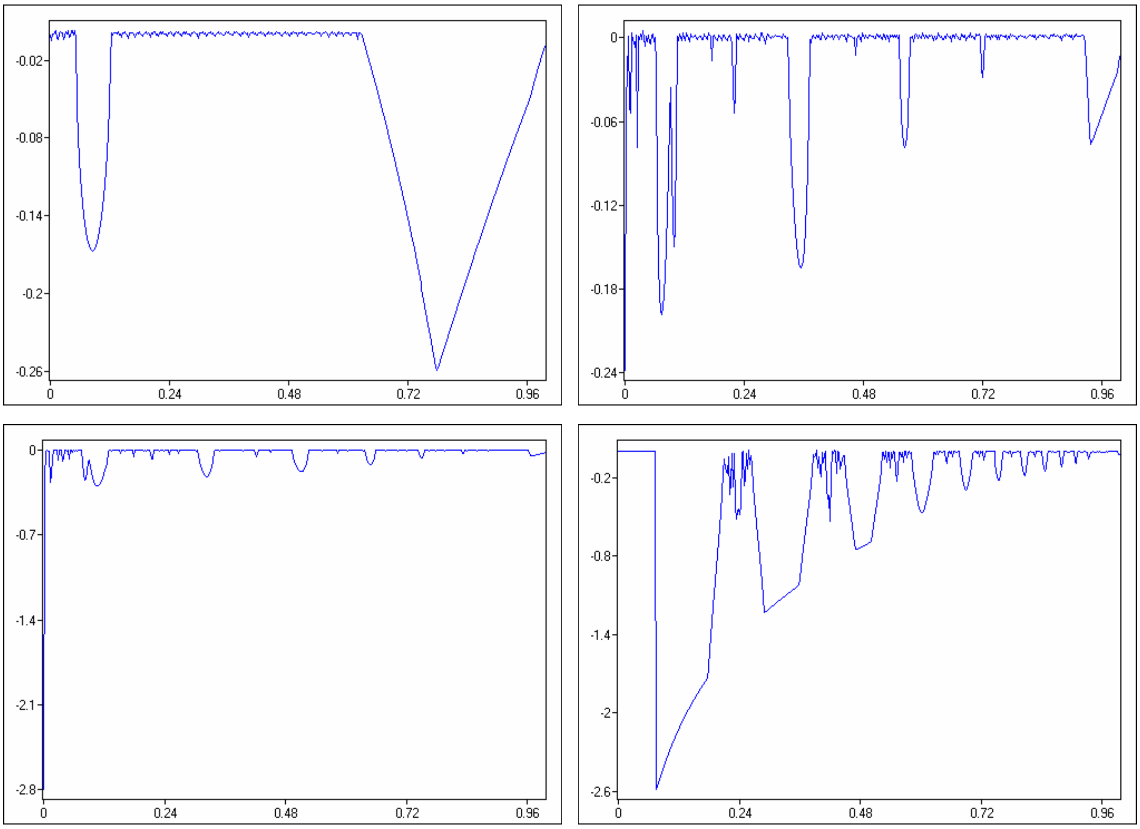

Figure 11: Largest LCE plots of w in case of fundamentalists versus opposite biased beliefs

Notes: Horizontal axis represents the memory strength (w). Vertical axis represents the value of the largest LCE. The plots differ with respect to the intensity of choice parameter β; upper left corresponds to $\beta = 100$, upper right to $\beta = 500$, lower left to $\beta = 1000$ and lower right to $\beta = 5000$. The largest LCE is calculated for $0 \leq w \leq 0.995$, since no numerical convergence could be achieved for $w = 1$.

Phase Plots

Next, we shall examine plots of the attractors in the planes, determined by (x_t, x_{t-1}), (x_t, $n_{1,t}$) and (x_t, $n_{2,t}$). Attractors in the (x_t, $n_{3,t}$) plane are just mirror images of attractors in the (x_t, $n_{2,t}$) plane and will thus not be separately examined. In the upper left plot of each of the four parts of Figures 12 and 14 we can first observe the appearance of an attractor for the intensity of choice beyond the bifurcation value. The orbits converge to such an attractor consisting of an invariant 'circle' around the (unstable) fundamental steady state. Though we are again topologically speaking about circles, the actual shape of such an attractor can be quite diverse, as seen from the figures. The attractor obtained in the (x_t, $n_{1,t}$) plane is somewhat different. Namely, the unstable steady state dissipates into numerous points and evolves into a 'loop' shape, as shown in Figure 13.

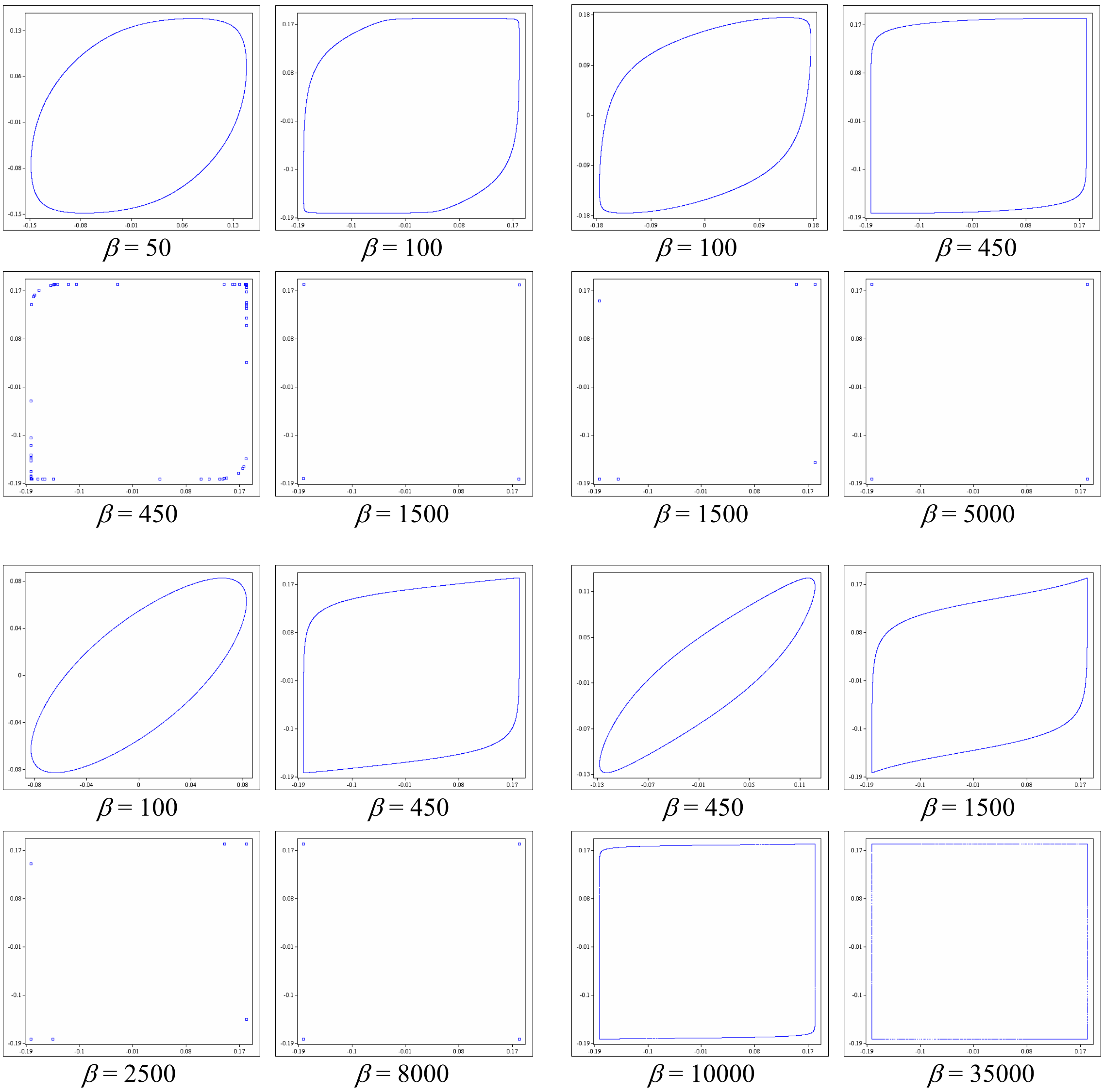

Figure 12: Phase plots of (x_t, x_{t-1}) in case of fundamentalists versus opposite biases

Notes: Horizontal axis represents deviations of the price from the fundamental value (x_t). Vertical axis represents lagged deviations of the price from the fundamental value (x_{t-1}). The groups of four diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

As the intensity of choice increases, the dynamics remains periodic or quasi-periodic; in case of past deviations of prices from the fundamental value and fractions of biased beliefs the invariant circle slowly changes its shape into a ‘(full) square’ (see Figures 12 and 14), while in case of fractions of fundamentalists the loop slowly changes into a ‘three-sided square’ (see Figure 13). For high values of intensity of choice we seem to obtain (stable) higher period cycles; in case of past deviations of prices from the fundamental value and fractions of biased beliefs we seem to attain a stable period four-cycle, while in the case of fractions of fundamentalists it is difficult to obtain any solid indications based solely on numerical simulations due to convergence problems for very high values of intensity of choice. In the latter case we can observe stable period four- and six-cycles, though (see lower right plot of each of the four parts of Figure 13).

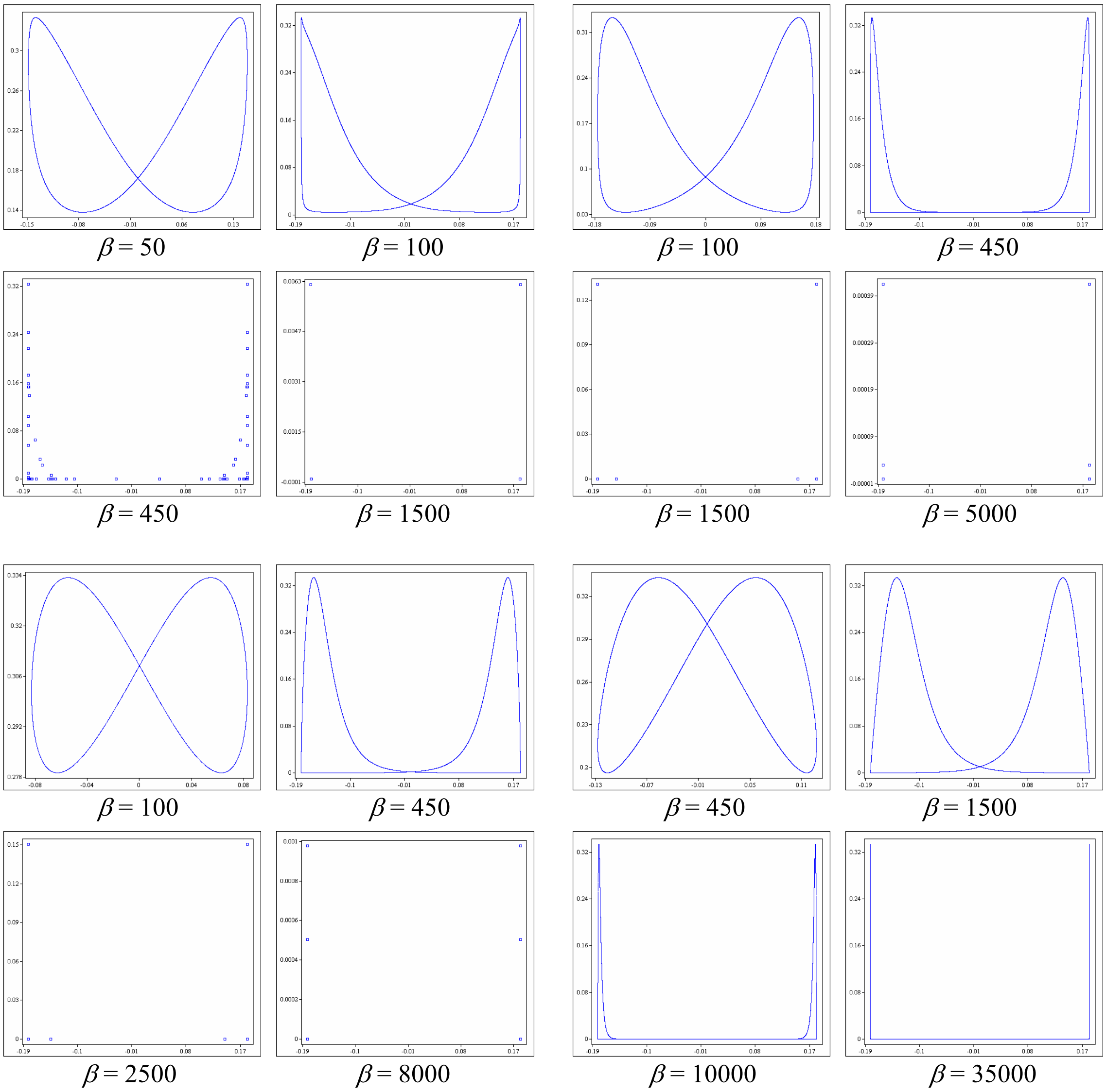

Figure 13: Phase plots of (x_t, $n_{1,t}$) in case of fundamentalists versus opposite biases

Notes: Horizontal axis represents deviations of the price from the fundamental value (x_t). Vertical axis represents the fraction of fundamentalists ($n_{1,t}$). The groups of four diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

Indeed, Brock and Hommes (1998) proved for the case of exactly opposite biased beliefs and infinite intensity of choice in their simpler version of the model without additional memory that the system has a stable four-cycle attracting all orbits, except for hairline cases converging to the unstable fundamental steady state. Additionally, they discovered that for all three trader types average profits along the four-cycle equal b^2.

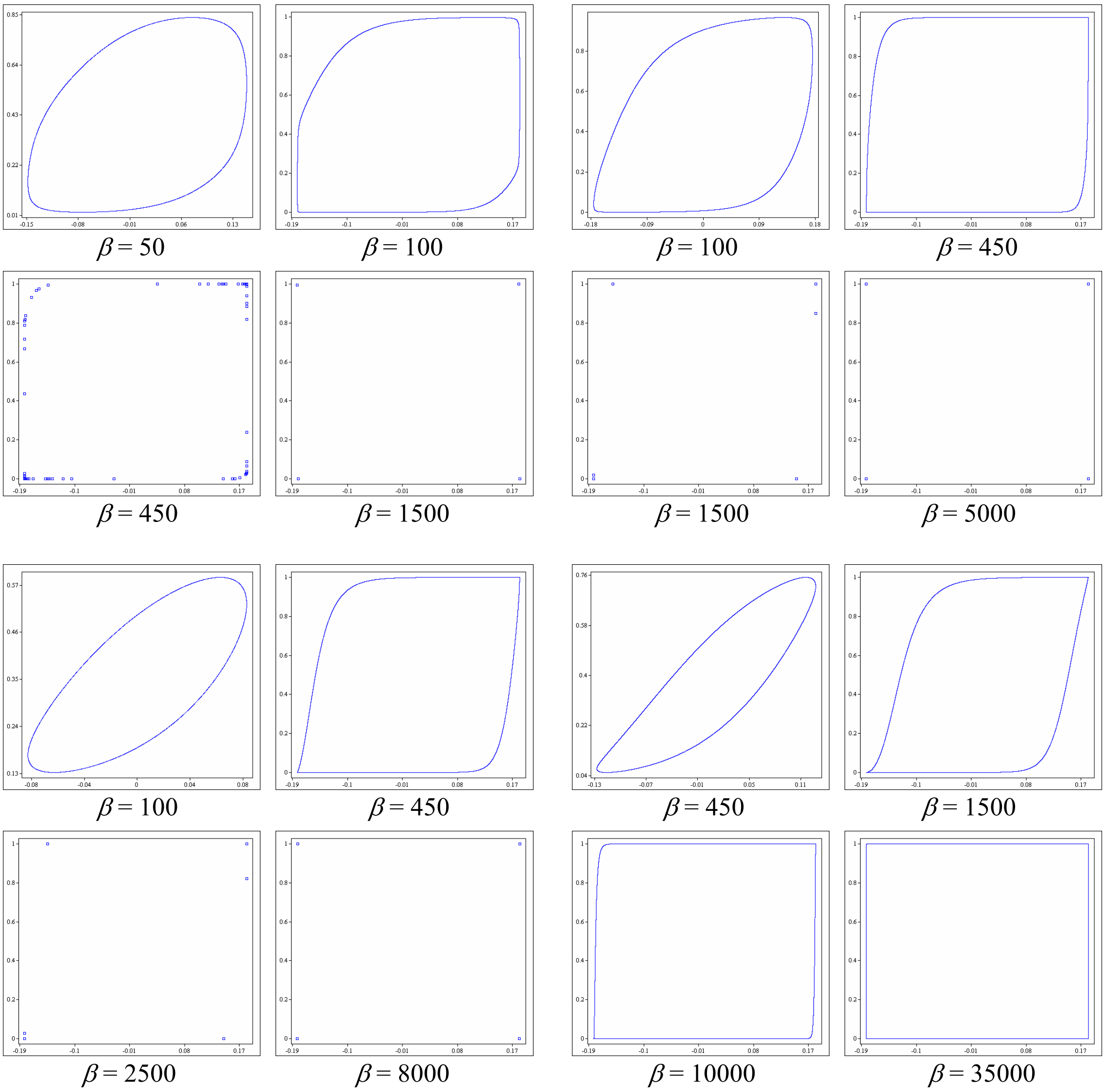

Figure 14: Phase plots of $(x_t, n_{2,t})$ in case of fundamentalists versus opposite biases

Notes: Horizontal axis represents deviations of the price from the fundamental value (x_t). Vertical axis represents the fraction of optimistic biased beliefs ($n_{2,t}$). The groups of four diagrams differ with respect to the memory strength parameter w; upper left corresponds to $w = 0$, upper right to $w = 0.3$, lower left to $w = 0.6$ and lower right to $w = 0.9$.

Again, we can observe from Figures 12–14 that the memory has an impact on the dynamics of the system. Namely, both the convergence of the system to an attractor and the further development of such an attractor seem to be dependent on the value of the memory strength parameter. The precise impact of memory is somewhat more difficult to establish due to the dependence of the bifurcation value on memory strength and subsequent need to choose higher intensities of choice with higher memory strength in order to demonstrate different nature of attractors of the system. However, we can still establish that at the same intensity of choice (after the bifurcation value) the system apparently needs less additional memory in order to develop a specific stage of an attractor or even a (stable) higher period cycle. This is in accordance with Figure 11, where we can observe that at higher intensity of choice (after the bifurcation value) we are more likely to obtain purely periodic dynamics with less memory and that we need more memory to obtain purely quasi-periodic dynamics. Nonetheless, it has to be emphasized that the model generates no strange (chaotic) attractors, as indicated before.

Time Series Plots

Finally, we shall examine time series plots of deviations of the price from the fundamental value and of the fractions of all three types of traders. Figures 15a and 15b show some time series corresponding to the attractors in Figures 12-14. Similarly to the findings of Brock and Hommes (1998), we can observe that opposite biases may cause perpetual oscillations around the fundamental, even when there are no costs for fundamentalists, but can not lead to chaotic movements. Furthermore, as has already been indicated by appearance of stable higher period cycles for high intensities of choice, in a three type world, even when there are no costs and memory is infinite, fundamentalist beliefs can not drive out opposite purely biased beliefs, when the intensity of choice to switch strategies is high.

Hence, following the argumentation of Brock and Hommes (1998, p. 1260), the market can protect a biased trader from his own folly if he is part of a group of traders whose biases are ‘balanced’ in the sense that they average out to zero over the set of types. Centralized market institutions can make it difficult for unbiased traders to prey on a set of biased traders provided they remain ‘balanced’ at zero. On the other hand, in a pit trading situation unbiased traders could learn which types are balanced and simply take the opposite side of the trade. In such situations biased traders would be eliminated, whereas a centralized trading institution could ‘protect’ them.

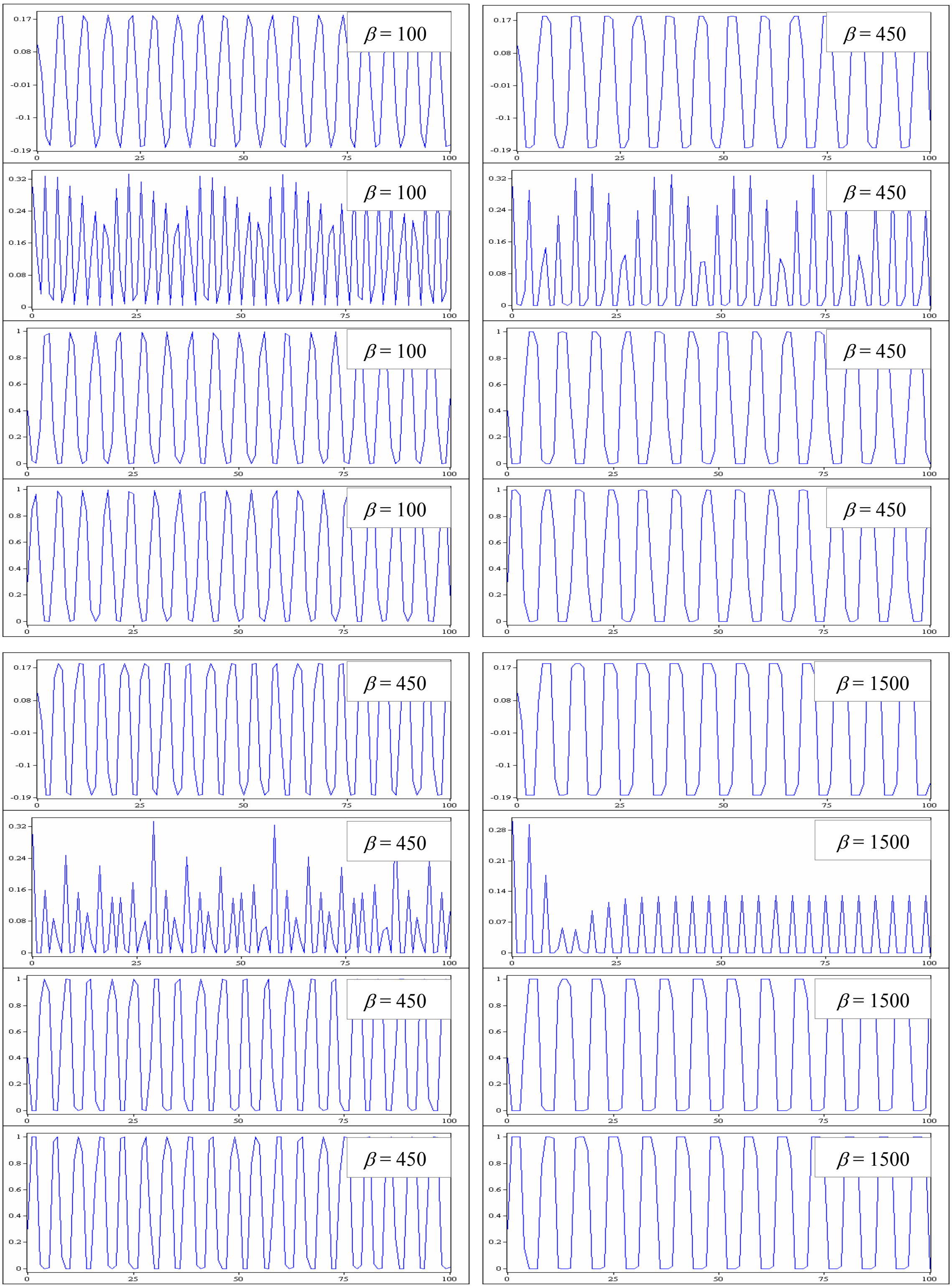

Figure 15a: Time series of prices and fractions in case of fundamentalists versus opposite biases

Notes: Horizontal axis represents the time (t). Vertical axis in each set of time series plots represents deviations of the price from the fundamental value (x_t), and the fractions of fundamentalists ($n_{1,t}$), optimistic biased beliefs ($n_{2,t}$) and pessimistic biased beliefs ($n_{3,t}$). The plots on the left-hand side and the right-hand side of the figure differ with respect to the memory strength parameter w; the ones on the left correspond to $w = 0$, while the ones on the right to $w = 0.3$.

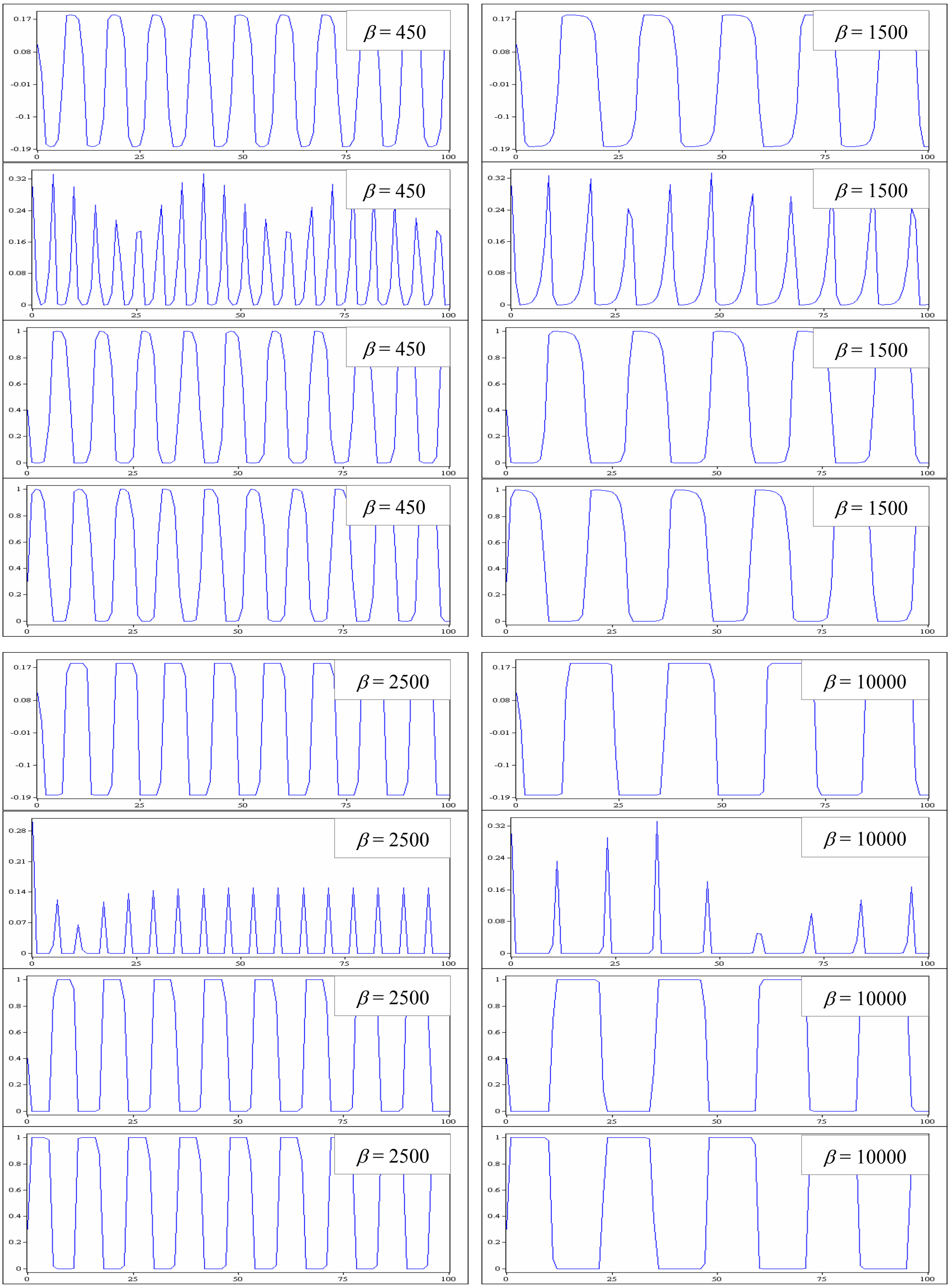

Figure 15b: Time series of prices and fractions in case of fundamentalists versus opposite biases

Notes: Horizontal axis represents the time (t). Vertical axis in each set of time series plots represents deviations of the price from the fundamental value (x_t), and the fractions of fundamentalists ($n_{1,t}$), optimistic biased beliefs ($n_{2,t}$) and pessimistic biased beliefs ($n_{3,t}$). The plots on the left-hand side and the right-hand side of the figure differ with respect to the memory strength parameter w; the ones on the left correspond to $w = 0.6$, while the ones on the right to $w = 0.9$.

Additional memory does not change the pattern of asset prices and trader fractions *per se*, but it does affect its period. Namely, at the same intensity of choice and higher memory strength the period of these cycles appears to be elongated on average, in a way that both the negative and the positive deviation of the price from the fundamental value last longer. The same is valid for fractions ob biased traders, while in the case of fractions of fundamentalists the prolongation of the period of the irregular cycle appears in the form of less frequent 'spikes', which is understandable, since more persistent deviations of prices from the fundamental imply more space for biased traders and less chance for appearance of the fundamentalists. More memory causes the traders to stick longer to the strategy that has been profitable in the past, but might not be so profitable in the recent periods; therefore the system approaches purely quasi-periodic dynamics when the memory strength increases at given intensity of choice.

6 Conclusion

Computational models are becoming increasingly important in economics, since they allow many aspects at the micro level and details of the interaction among agents to be modelled and simulated. Heterogeneity is likely to play a key role in this approach, and agent-based computational asset pricing models thus deserve high priority in research. We have investigated an asset pricing model with heterogeneous beliefs where agents select a predictor from a finite set based upon past performance as measured by realized profits. If all traders had been identical and rational, the model would have essentially reduced to Lucas (1978) asset pricing model and under the additional assumption of the IID dividend process the asset price dynamics would have been extremely simple, *viz*. there would have been one constant price equal to the efficient market hypothesis fundamental value at each and every point in time.

On the other hand, in a heterogeneous agent financial market evolutionary dynamics may lead to persistent deviations from the fundamental price and highly irregular, even chaotic asset price fluctuations, when the intensity of choice to switch prediction strategies becomes high. However, a problem becomes increasingly apparent with such simulation models, i.e. there tend to be too many degrees of freedom and too many parameters. One of such issues that we pursued in this paper to make less indeterminate and unresolved relates to memory strength. Memory strength represents the share of past fitness in the performance measure of an asset pricing model and can also be thought of as the share of past information that boundedly rational economic agents take into account as decisions makers. We were interested in how this parameter affects stability of evolutionary adaptive systems and survival of technical trading. In order to obtain an insight into this matter two cases were analyzed; a two-type case of fundamentalists versus contrarians and a three-type case of fundamentalists versus opposite biases.

In a market with fundamentalists and contrarians the fundamental steady state is the unique steady state of the system, which arises for low values of intensity of choice. Memory affects neither the position of this steady state nor its stability. For increasing intensity of choice a primary bifurcation, i.e. a period doubling bifurcation occurs; the steady state becomes unstable and a stable period two-cycle appears. Both the primary bifurcation value and the position of the period two-cycle are independent of the memory. For further increasing intensity of choice a secondary bifurcation, i.e. a supercritical Hopf bifurcation occurs; the period two-cycle becomes unstable and an attractor appears consisting of two invariant circles around each of the two (unstable) period two-points, one lying above and the other one below the fundamental. For high intensity of choice chaotic asset price dynamics occurs, interspersed with many stable period cycles. Such a bifurcation route to chaos is often called the rational route to randomness.

In case of strong contrarians and high intensity of choice it is reasonable to expect that we obtain a system that is close to having a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state, which indicates the occurrence of chaos. There exists a certain limited interval of memory strength values, for which at a given intensity of choice we are more likely to obtain such a system with more additional memory in the model. A rational choice between fundamentalists' and contrarians' beliefs triggers situations that do not reach fruition due to practical considerations and are thus unattainable; the so-called 'castles in the air', as Brock and Hommes (1998, p. 1258) would put it. As a consequence we obtain market instability, characterized by irregular up and down oscillations around the unstable efficient market hypothesis fundamental price. Additional memory lengthens on average the period of this irregular cycle and mainly appears to be stabilizing with regard to asset prices.

In a market with fundamentalists and opposite biases the fundamental steady state is again the unique steady state of the system, arising for low values of intensity of choice. Memory does not affect the position of this steady state, but it affects its stability. For increasing intensity of choice a supercritical Hopf bifurcation occurs; the steady state becomes unstable and an attractor appears. Memory affects the emergence of this bifurcation; the higher the memory strength, the higher the bifurcation value. More memory thus has a stabilizing effect on dynamics. For high intensity of choice the dynamical behaviour is more complex. However, irrespective of the amount of additional memory such a route to complicated dynamics is different from that in the presence of contrarians. Namely, after the bifurcation value only regular (quasi-)periodic fluctuations around the unstable fundamental steady state occur. Consequently, an important finding is that the predominating quasi-periodic dynamics does not seem to evolve to chaotic dynamics.

After the incidence of the bifurcation higher value of the memory strength parameter causes the dynamics to be less periodic and more quasi-periodic; the dynamics therefore converges to the purely quasi-periodic behaviour with increasing memory strength.

Opposite biases may cause perpetual oscillations around the fundamental, even without costs for fundamentalists, but can not lead to chaotic movements. Furthermore, in a three type world, even when there are no costs and memory is infinite, fundamentalist beliefs can not drive out opposite purely biased beliefs, when the intensity of choice to switch strategies is high. Hence, following the argumentation of Brock and Hommes (1998, p. 1260), the market can protect a biased trader from his own folly if he is part of a group of traders whose biases are balanced.

In conclusion, both our analytical work and our numerical simulations suggest that biases alone do not trigger chaotic asset price fluctuations. Sensitivity to initial states and irregular switching between different phases seem to be triggered by trend extrapolators; in our case by contrarians. Apparently, some (strong) trend extrapolator beliefs are needed, such as strong trend followers or strong contrarians, in order to trigger chaotic asset price fluctuations. A key feature of our heterogeneous beliefs model is that the irregular fluctuations in asset prices are triggered by a rational choice in prediction strategies, based upon realized profits, *viz*. the observed deviations from the fundamentals are driven by short-run profit seeking. We can also talk about rational animal spirits that, according to Brock and Hommes (1997b), exhibit some qualitative features of asset price fluctuations in the actual financial markets, such as the autocorrelation structure of prices and returns.

The analyzed model is quite simple and stylized. One may thus question the validity and generality of the results. Do similar results also hold for asset pricing model with more than two assets or even in a general equilibrium framework, with a higher-dimensional equilibrium pricing equation? Brock and Hommes (1998, p. 1266) suggested an affirmative answer. How would a non-constant conditional variance of excess return change the results? Would the dynamics change dramatically with an increase in the number of trader types? These and some other issues exceed the purpose of this paper and are left for further research. When the consequences of changing memory on stability of evolutionary adaptive systems and survival of technical trading are well understood, we will be able to formulate asset pricing models in a more consistent and efficient manner. Optimistically, expanded and improved these models may yield further (seemingly) counter-intuitive results in terms of methods to stabilize, predict, or improve on current market institutions.

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