A two-sector model of optimal growth in which labour is employed only in the industry of investment goods: A complete characterization of equilibrium paths

Guerrazzi, Marco

Department of Economics - DIEC, University of Genoa

28 September 2025

A Two-Sector Model of Optimal Growth in which Labour is Employed only in the Industry of Investment Goods: A Complete Characterization of Equilibrium Paths*

Marco Guerrazzi†

October 18, 2025

Abstract

In this paper, I develop a two-sector growth model with endogenous labour supply in which labour is employed only in the production of investment goods. From a theoretical point of view, I show that no matter the shape of the technology to produce consumption goods, a convex technology in the sector of investment goods is necessary and sufficient for a meaningful stationary solution and the determinacy of equilibrium trajectories. In addition, from a numerical perspective, I show that a calibrated version of the model relaxes the complementarity between the propensity to consume and to save and it is also able to provide a rationale for the procyclical patterns of the relative price of capital goods and the real wage.

Keywords: Capital accumulation; Investment; Consumption; Two-sector growth model, Relative price of capital goods, Real wage.

JEL Classification: C61; E21; E22; O11.

*Preliminary draft. Comments are welcome.

†Author contacts: Department of Economics - DIEC, University of Genoa, via F. Vivaldi n. 5, 16126 Genoa (Italy), Phone (+39) 010 2095225, Fax (+39) 010 2095269, E-mail guerrazzi@economia.unige.it.

1 Introduction

In two-sector models of optimal growth like the ones pioneered by Meade (1961) and Uzawa (1961) – in which there is an explicit distinction between the production technology of consumption and investment goods – labour and capital have to be optimally allocated over time in both industries. However, on account of its non-reproducibility, the way in which labour is directed in the two sectors reveals distinct trade-offs for the optimizing agent called in to choose the optimal amount of labour to supply in each industry. Specifically, in a given period, offering labour to employ in the sector of consumption goods implies to trade-off the relative forgone leisure – or the disutility from offering labour – against the possibility to produce and enjoy additional consumption goods. On the other side, offering labour to the sector of investment goods leads to trade-off the already mentioned forgone leisure against the value of a larger stock of capital that – in later periods – will have to be necessarily directed towards the production and the enjoyment of consumption goods.

While the former trade-off is typical of labour supply choices, the latter is closer to saving choices and it can be clearly singled out only in theoretical frameworks which allow for the production of final goods that do not directly contribute to the utility of the optimizing agent. For this reason, in this paper, I develop a frictionless two-sector model of optimal growth with endogenous labour supply and infinitely lived agents in which the non-reproducible factor is characterized by imperfect mobility, and it can employed only in the production of investment goods (cf. Cardi et al. 2020).[1] Specifically, relying on a centralized setting in which a unique optimizing agent takes all the relevant decisions in an efficient manner, I analyse how forgone leisure in the current period is optimally counterbalanced by the possibility to produce commodities that in the future will allow to produce welfare-enhancing consumption goods through the accumulation of capital made possible by producing investment goods. While Hashimoto and Sakuragawa (1999) analyse an OLG two-sector model in which labour enters only the production of consumption goods whereas Guerrazzi (2025a) does the same in the context of a framework with infinitely lived agents, to the best of my knowledge the reversed option has never been explored before.

From a theoretical perspective, relying on Cobb-Douglas technologies, logarithmic utility and separate CRRA disutility with respect to labour provision, I show that – no matter the shape of the technology prevailing in the sector that produces consumption goods – a convex technology in the sector of investment goods is necessary and sufficient for an economically meaningful stationary solution as well as for the determinacy of the equilibrium trajectories followed by the stock of capital, the level of consumption which are produced and enjoyed, and the labour input (cf. Guerrazzi, 2025a-b). In addition, parametrizing and calibrating the model by taking the US economy as reference and relying on some numerical simulations, I show that the present framework relaxes the complementarity between the propensities to consume and save of the optimizing agent, and it is also able to provide a rationale for the procyclical patterns of the relative price of capital goods and the real wage rate (cf. Sallivan, 1997; Beaudry et al. 2015).

[1] A two-sector economy in which capital is non-shiftable across industries is instead analysed by Ryder (1969).

The paper is arranged as follows. Section 2 illustrates the model economy. Section 3 explores its quantitative implications. Finally, Section 4 concludes.

2 Theoretical framework

I consider a model economy in which time (t) is continuous – so that $t \in \mathbb{R}_+$ – and there are two distinct productive sectors, one that produces consumption goods by contributing to the instantaneous utility of the optimizing agent, and another one in which are produced investment goods that instead boost the overall capital accumulation in the society (cf. Srinivasan, 1964). Consequently, in each instant, the total stock of capital available in the economy – say $K(t)$ – is broken down between the stock of capital allocated in the sector of consumption goods and the stock of capital allocated in the sector of investment goods, respectively denoted by $K_C(t)$ and $K_I(t)$. Therefore, full employment of the available capital implies that it will always hold that

$$K(t) = K_C(t) + K_I(t) \tag{1}$$

Reversing the setting developed by Hashimoto and Sakuragawa (1999) and Guerrazzi (2025a), I consider a situation in which labour is employed only in the industry that produces investment goods whereas capital is employed in both industries. Consequently, the productive technologies available in the two industries are assumed to be described by the following Cobb-Douglas expressions:

$$Y_I(t) = S_I (K_I(t))^{\alpha_I} (L_I(t))^{1-\alpha_I} \tag{2}$$

$$Y_C(t) = S_C (K_C(t))^{\alpha_C} \tag{3}$$

where $Y_I(t)$ $(Y_C(t))$ is the real output of the sector that produces investment (consumption) goods i, $S_I > 0$ $(S_C > 0)$ is an industry-specific measure of productivity, $\alpha_I \in (0, 1)$ $(\alpha_C > 0)$ is the value of the elasticity of output of the industry of investment (consumption) goods with respect to the stock of the employed capital in that sector, whereas $L_I(t)$ is the amount of labour employed in the sector of investment goods that in the case under scrutiny corresponds to total employment.

Since labour is completely omitted from the expression in eq. (3), the model economy described in this paper considers the situation in which there is full automation of production in the industry that supplies consumption goods and – at the same time – the immobility of the non-reproducible factor from the sector of investment goods (cf. Growiec, 2022; Cardi et al. 2020). Aiming at analysing the role of non-convexities in the production technologies, for the moment, I do not fix any upper bound for the elasticity of output with respect to the stock of capital employed in the sector of consumption goods, whereas I make the hypothesis of constant returns to scale with respect to capital and labour – or, alternatively, linearity in the employed capital – in the industry of investment goods. It is worth noticing, however, that the case in which the technology prevailing in the sector of investment goods becomes linear in the capital input, i.e., the situation in which α_I is equal to 1, corresponds to the case of full automation in both industries (cf. Guerrazzi, 2025a-b).

Moreover, assuming that the employed capital equally depreciates in both sectors, the capital accumulation law can be written as

$$\dot{K}(t) = Y_I(t) - \delta K(t) \tag{4}$$

where $\delta > 0$ is the common value of the depreciation rate of capital.

According to the expression in eq. (4), $Y_I(t)$ is the output of capital goods available for net investment to add to the stock of malleable machineries available in the economy (cf. Meade, 1961).

2.1 The social planner problem

In the model economy described above, from a centralized perspective, the decision to consume a certain amount of resources straightforwardly implies to allocate a certain share of the available capital in the sector of consumption goods by leaving the remaining fraction for the sector of investment goods which also may require the employment of some labour services (cf. Cai, 2006; Guerrazzi, 2025a-b).

On the one hand, the expression in eq. (3) implies that in each instant the quantity of capital to employ in the industry of consumption goods – for each unit of Y_C to produce and consume – is given by the following expression:

$$K_C(t) = \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}} \tag{5}$$

On the other hand, eq. (5) together with the hypothesis of full employment of the reproducible factor conveyed by eq. (1) imply that the residual quantity of capital to employ in the sector of investment goods can be written as

$$K_I(t) = K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}} \tag{6}$$

Consequently, given the expressions in (2), (4) and (6), a benevolent and well informed social planner endowed with logarithmic preferences over consumption and separate CRRA disutility with respect to labour provision will allocate labour and the available capital in the two sectors by solving the following intertemporal problem:

$$\begin{gathered}
\max_{\mathcal{C}_S \in \mathcal{A}_0(\overline{K})} \int_{t=0}^{\infty} \exp(-\rho t)\left(\ln Y_C(t) - \tfrac{1}{1+\gamma}(L_I(t))^{1+\gamma}\right) dt \\
\text{s.to} \\
\dot{K}(t) = S_I\left(K(t) - \left(\tfrac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)^{\alpha_I}(L_I(t))^{1-\alpha_I} - \delta K(t) \\
K(0) = \overline{K}
\end{gathered} \tag{7}$$

where $\mathcal{C}_S \equiv \begin{pmatrix} Y_C(\cdot) & L_I(\cdot) \end{pmatrix}$ is the set of control functions, $\mathcal{A}_0\left(\overline{K}\right)$ is the set of admissible control strategies, $\rho > 0$ is the discount rate, $\gamma > 0$ is the inverse of the elasticity of labour supply, whereas $\overline{K} > 0$ is the initial value of the total capital stock (cf. Benhabib and Farmer, 1996).

In order to have economically meaningful trajectories, the set of admissible control strategies $\mathcal{C}_S$ starting from the initial couple $\left\{0, \overline{K}\right\}$ is defined as

$$\mathcal{A}_0\left(\overline{K}\right) := \left\{\mathcal{C}_S \in \mathbb{L}^1_{\mathbf{loc}}\left(\mathbb{R}_+; \mathbb{R}^2_+\right) : K(t) \in \mathbb{R}_+ \quad \forall t \in \mathbb{R}_+\right\} \tag{8}$$

According to the definition in (8), $\mathcal{C}_S$ has to belong to the set of locally integrable (or summable) functions such that the level of consumption goods which are produced and consumed, the amount of labour employed in the sector of investment goods and the stock of capital are positive all over the relevant time horizon.

The first-order conditions (FOCs) for the social planner problem in (7) are given by

$$1 - \frac{\alpha_I S_I q(t) \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{\alpha_C \left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)}\right)^{1-\alpha_I}} = 0 \tag{9}$$

$$-\left(L_I(t)\right)^{\gamma} + \left(1-\alpha_I\right) S_I q(t) \left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)}\right)^{\alpha_I} = 0 \tag{10}$$

$$\dot{q}(t) = q(t) \left(\rho + \delta - \frac{\alpha_I S_I}{\left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)}\right)^{1-\alpha_I}}\right) \tag{11}$$

$$\lim_{t\to\infty} \exp\left(-\rho t\right) q(t) K(t) = 0 \tag{12}$$

where whereas $q(t)$ is the costate variable associated to the capital accumulation constraint.

Eq.s (9) and (10) are, respectively, the FOCs with respect to $Y_C(t)$ and $L_I(t)$ that have to hold in each instant. Moreover, the intertemporal relationship in eq. (11) conveys the optimal trajectory of the costate variable. Furthermore, the endpoint limit on the shadow value of capital in (12) is the required transversality condition.

Solving eq. (9) with respect to $S_I q(t)$ and plugging the result in eq. (10) would allow to pin down $L_I(t)$ as a function of $K(t)$ and $Y_C(t)$.[2] In this way, as it is usually done in the literature of two-sectors models, the motion of the economy could be framed within a bidimensional dynamical system that involves the production and the enjoyment of consumption goods as the control variable and the total

[2]Specifically, in each instant $L_I(t)$ should be equal to $\left(\left(\left(\left(1-\alpha_I\right)\alpha_C\right)/\alpha_I\right)\left(K(t)\left(Y_C(t)/S_C\right)^{-1/\alpha_C} - 1\right)\right)^{1/(1+\gamma)}$.

stock of capital as state variable (cf. Drabiki and Takayama, 1975; Galor, 1992). In what follows, aiming at preserving a certain analytical tractability, I will derive distinct Euler equation both for the production and the enjoyment of consumption goods as well as for the provision of labour to the sector of investment goods. In this direction, differentiating eq.s (9) and (10) with respect to time and using their implied expressions for $q(t)$ to simplify leads to the following expressions:

$$\frac{1-\alpha_I\left(1-\Phi\left(t\right)\right)}{\alpha_C\Phi\left(t\right)}\frac{\dot{Y}_C}{Y_C\left(t\right)}+\left(1-\alpha_I\right)\frac{\dot{L}_I\left(t\right)}{L_I\left(t\right)}=\frac{1-\alpha_I}{\Phi\left(t\right)}\frac{\dot{K}\left(t\right)}{K\left(t\right)}-\frac{\dot{q}\left(t\right)}{q\left(t\right)} \tag{13}$$

$$\frac{\alpha_I\left(1-\Phi\left(t\right)\right)}{\alpha_C\Phi\left(t\right)}\frac{\dot{Y}_C\left(t\right)}{Y_C\left(t\right)}+\left(\alpha_I+\gamma\right)\frac{\dot{L}_I\left(t\right)}{L_I\left(t\right)}=\frac{\alpha_I}{\Phi\left(t\right)}\frac{\dot{K}\left(t\right)}{K\left(t\right)}+\frac{\dot{q}\left(t\right)}{q\left(t\right)} \tag{14}$$

where $\Phi\left(t\right)\equiv\left(K\left(t\right)-\left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}\right)/K\left(t\right)$ is the share of capital allocated in the sector of investment goods.

Eq.s (13) and (14) allow us to straightforwardly retrieve the Euler equations for the production and the enjoyment of consumption goods and the level of employment in the sector of investment goods, namely

$$\frac{\dot{Y}_C\left(t\right)}{Y_C\left(t\right)}=\frac{\alpha_C\left(\left(1-\alpha_I\right)\gamma\frac{\dot{K}(t)}{K(t)}-\left(1+\gamma\right)\Phi\left(t\right)\frac{\dot{q}(t)}{q(t)}\right)}{\alpha_I\left(1+\gamma\right)\Phi\left(t\right)+\left(1-\alpha_I\right)\gamma} \tag{15}$$

$$\frac{\dot{L}_I\left(t\right)}{L_I\left(t\right)}=\frac{\alpha_I\frac{\dot{K}(t)}{K(t)}+\frac{\dot{q}(t)}{q(t)}}{\alpha_I\left(1+\gamma\right)\Phi\left(t\right)+\left(1-\alpha_I\right)\gamma} \tag{16}$$

The expressions in eq.s (15) and (16) show that the optimal growth rate of the output of consumption goods and the growth rate of the labour input employed in the sector of investment goods are given by a non-linear combinations between the growth rate of capital – which can be easily retrieved from eq.s (2), (4) and (6) – and the optimal growth rate of the costate variable implied by eq. (11). Obviously, eq.s (15) and (16) together with eq. (4) supplemented by the expressions in eq.s (2), (4) and (6) provide the 3×3 dynamical system that describes the whole motion of the model economy.

2.2 First-best pricing

The theoretical setting developed above describes the outcomes of a centralized economy in which all the relevant decisions are taken by a unique optimizing agent by considering the prevailing quantities but with no regard to any price signal. However, it is well known that in a decentralized setting in which there is a market for each good and for each factor, and where the two existing goods are then produced by atomistic profit-maximizing firms that take market prices as given, the value of the marginal product of capital must be continuously equalized across industries in order to verify the condition for the allocative efficiency of capital (cf. Huzawa, 1961; Senouci, 2011).[3] In a similar manner, continuous profit maximization requires that the value of the marginal productivity of labour employed in the sector of investment goods must be always equal to the ratio between the nominal wage rate and price of investment goods.

Consequently, in a competitive framework with continuous market clearing, the ratio between the marginal productivities of capital employed in the two industries must return the implicit path of the relative price of the two goods and – at the same time – the marginal productivity of labour in the sector of investment goods has to be equal the relative price of labour that implement the first-best allocation described by the solution of the problem in (7). Formally speaking, according to the expressions eq.s (2) and (3), the relative price of investment goods and the relative price of labour are, respectively, equal to

$$\frac{p_I(t)}{p_C(t)} = \frac{\alpha_C S_C \left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)} \right)^{1-\alpha_I}}{\alpha_I S_I \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1-\alpha_C}{\alpha_C}}} \tag{17}$$

$$\frac{w(t)}{p_I(t)} = (1-\alpha_I) S_I \left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)} \right)^{\alpha_I} \tag{18}$$

where $p_I(t)$ ($p_C(t)$) is the price of investment (consumption) goods that clears the relative market, whereas $w(t)$ is the nominal wage rate.

Obviously, eq.s (17) and (18) imply that the real wage, i.e., the measure of the purchasing power of workers employed in the sector of investment goods that, however, in a decentralized environment buy consumption goods will be given by

$$\frac{w(t)}{p_C(t)} = \frac{\alpha_C (1-\alpha_I) S_C \left(K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}} \right)}{\alpha_I \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1-\alpha_C}{\alpha_C}} L_I(t)} \tag{19}$$

On the one hand, eq. (17) shows that the relative price of investment goods is a decreasing function of the total factor productivity (TFP) in that sector relative to the TFP in the consumption good sector whereas its path depends on the allocation of labour and capital in the two industries over time (cf. Ferreira et al. 2014). On the other hand, recalling eq.s (5) and (6), eq. (19) reveals that the path followed by the real wage rate is an increasing (decreasing) function of the capital-labour ratio (stock of capital) prevailing (allocated) in the sector of investment (consumption) goods.

[3]Obviously, the consistency of this kind of behaviour at the decentralized level requires decreasing returns in the sector of consumption goods, i.e., it must hold that $\alpha_C \in (0,1)$.

2.3 Steady state and golden rule

In the model economy described above, steady-state allocations are defined as the set of triplets $\mathcal{S} := \{Y_C^*, L_I^*, K^*\} \in \mathbb{R}^3_{++}$ such that $\dot{Y}_C(Y_C^*, L_I^*, K^*) = \dot{L}_I(Y_C^*, L_I^*, K^*) = \dot{K}(Y_C^*, L_I^*, K^*) = 0$ whose point values will be also associated to the equilibrium level of the production of investment goods ($Y_I^* > 0$), the corresponding value of the amounts of capital allocated, respectively, in the sector of consumption goods ($K_C^* > 0$) and in the sector of investment goods ($K_I^* > 0$) as well as to the implied first-best relative prices of equilibrium (p_I^*/p_C^* and w^*/p_C^*). In case of asymptotic stability, some elements of $\mathcal{S}$ will be also characterized by the fact that $\lim_{t\to\infty} Y_C(t) = Y_C^* \wedge \lim_{t\to\infty} L_I(t) = L_I^* \wedge \lim_{t\to\infty} K(t) = K^*$, a feature that may lead to the convergence of all the remaining endogenous variables.

The long-run value of each endogenous variable as a function of the model's parameters can be easily retrieved as follows. First, setting $\dot{q}(t) = 0$ in eq. (11) by taking into consideration the result in eq. (6) allows us to find the steady-state level of the stock of capital per unit of labour allocated in the sector of investment goods, namely

$$\frac{K_I^*}{L_I^*} = \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{1}{1-\alpha_I}} \tag{20}$$

Second, setting $\dot{K}(t) = 0$ in eq. (4) and exploiting the results in eq.s (1), (2) and (20) pins down the ratio between the equilibrium stock of capital allocated in the sector of consumption goods and the level of employment in the sector of investment goods as

$$\frac{K_C^*}{L_I^*} = \frac{S_I (\rho + \delta (1 - \alpha_I))}{\delta (\rho + \delta)} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{21}$$

Third, the steady-state version of eq.s (9) and (10) together with eq.s (6), (20) and (21) allows us to retrieve the equilibrium level of employment, namely

$$L_I^* = \left(\frac{\alpha_C (1 - \alpha_I) \delta}{\rho + \delta (1 - \alpha_I)}\right)^{\frac{1}{1+\gamma}} \tag{22}$$

Eq. (22) shows that the long-run equilibrium amount of labour employed in the sector of investment goods is a decreasing function of the discount rate of the optimizing agent. This finding is perfectly consistent with the fact that the production of investment goods does not have an immediate impact its instantaneous utility. Moreover, it is worth noticing that when the technology prevailing in the sector of investment goods is linear in the employed capital, i.e., whenever $\alpha_I = 1$, the equilibrium amount of labour implied by eq. (22) becomes equal to zero by confirming that in this case there is actually full automation of production in both industries (cf. Guerrazzi, 2025a-b)

Forth, plugging eq. (22) into eq. (20) ((21)) pins down the equilibrium level of the stock of capital allocated in the sector of investment (consumption) goods so that

$$K_I^* = \left(\frac{\alpha_C (1 - \alpha_I) \delta}{\rho + \delta (1 - \alpha_I)}\right)^{\frac{1}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{1}{1-\alpha_I}} \tag{23}$$

$$K_C^* = \frac{S_I\left(\rho + \delta\left(1 - \alpha_I\right)\right)}{\delta\left(\rho + \delta\right)} \left(\frac{\alpha_C\left(1 - \alpha_I\right)\delta}{\rho + \delta\left(1 - \alpha_I\right)}\right)^{\frac{1}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{24}$$

Eq.s (23) and (24) reveal that whenever $\alpha_I \in (0,1)$, K_I^* and K_C^* are both positive.[4] Therefore, the hypothesis of constant returns to scale in the sector of investment goods – or the convexity of the technology prevailing in that industry – leads certainly to an economically meaningful equilibrium sectoral allocation of capital by excluding the reasonableness of considering the case in which α_I is equal to 1.[5]

Fifth, adding the two expressions in eq.s (23) and (24) returns the equilibrium level of the total stock of capital, that is

$$K^* = \frac{S_I}{\delta} \left(\frac{\alpha_C\left(1 - \alpha_I\right)\delta}{\rho + \delta\left(1 - \alpha_I\right)}\right)^{\frac{1}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{25}$$

Eq.s (23) $-$ (25) straightforwardly imply that $\Phi^* \equiv \alpha_I \delta / \left(\rho + \delta\right)$ is the equilibrium share of capital allocated in the sector of investment goods, whereas $1 - \Phi^* = \left(\rho + \delta\left(1 - \alpha_I\right)\right) / \left(\rho + \delta\right)$ is the corresponding share of capital allocated in the sector of consumption goods.

Sixth, eq.s (23) and (23) together with eq.s (2) and (3) conveys, respectively, the equilibrium production of investment goods and the equilibrium production of consumption goods, namely

$$Y_I^* = S_I \left(\frac{\alpha_C\left(1 - \alpha_I\right)\delta}{\rho + \delta\left(1 - \alpha_I\right)}\right)^{\frac{1}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{26}$$

$$Y_C^* = S_C \left(\frac{S_I\left(\rho + \delta\left(1 - \alpha_I\right)\right)}{\delta\left(\rho + \delta\right)}\right)^{\alpha_C} \left(\frac{\alpha_C\left(1 - \alpha_I\right)\delta}{\rho + \delta\left(1 - \alpha_I\right)}\right)^{\frac{\alpha_C}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho + \delta}\right)^{\frac{\alpha_I \alpha_C}{1-\alpha_I}} \tag{27}$$

At this stage of analysis, it is worth noticing that the equilibrium capital-labour ratio in the sector of investment goods, namely, K_I^*/L_I^*, is independent from the technology prevailing in the consumption good industry that, on the contrary, matters for the determination of the equilibrium level of employment, the equilibrium value of the total stock of capital in the economy and the equilibrium levels of production of both sectors (cf. Senouci, 2011).

Seventh, eq.s (17), (19), (20) and (27) allows us to derive the equilibrium values of the Pareto optimal relative price of investment goods and the real wage, in detail

$$\frac{p_I^*}{p_C^*} = \frac{\alpha_C S_C}{\left(\rho + \delta\right) \left(\frac{S_I(\rho+\delta(1-\alpha_I))}{\delta(\rho+\delta)} \left(\frac{\alpha_C(1-\alpha_I)\delta}{\rho+\delta(1-\alpha_I)}\right)^{\frac{1}{1+\gamma}} \left(\frac{\alpha_I S_I}{\rho+\delta}\right)^{\frac{\alpha_I}{1-\alpha_I}}\right)^{1-\alpha_C}} \tag{28}$$

[4] Notice that according to eq. (23), K_I^* could be positive even for values of α_I larger than $1 + \rho/\delta$. However, given the expression in eq. (24), the same does not hold for K_C^*.

[5] Since the utility arising from consumption goods satisfies the Inada conditions, the optimizing agent will always choose to produce this kind of goods by allocating capital in the relative industry (cf. Drabicki and Takayama, 1975; Senouci, 2011).

$$\frac{w^*}{p_C^*} = \frac{\alpha_C\left(1-\alpha_I\right)S_C\left(\frac{\alpha_I S_I}{\rho+\delta}\right)^{\frac{1-\alpha_I(1-\alpha_C)}{1-\alpha_I}}}{\alpha_I\left(\frac{S_I(\rho+\delta(1-\alpha_I))}{\delta(\rho+\delta)}\left(\frac{\alpha_C(1-\alpha_I)\delta}{\rho+\delta(1-\alpha_I)}\right)^{\frac{1}{1+\gamma}}\right)^{1-\alpha_C}} \tag{29}$$

Furthermore, setting $\dot{K}(t) = 0$ in eq. (4) by taking into account the result in eq. (2) and (6) allows us to find the golden-rule on capital accumulation that in the present context is represented by the ratio between K and L_I that maximizes the production and the consumption of Y_C when the total stock of capital achieves its stationary solution. Straightforward algebra reveals that such a maximum is defined for all the eligible values of α_C whenever K/L_I is given by the following expression:

$$\left(\frac{K}{L_I}\right)_{\text{GR}} = \frac{S_I}{\delta}\left(\frac{\alpha_I S_I}{\delta}\right)^{\frac{\alpha_I}{1-\alpha_I}} \tag{30}$$

As illustrated in Figure 1, eq.s (22) and (25) imply that as long as the social planner that solves the problem in (7) discounts future utility and disutility streams at a positive rate, the long-run equilibrium value of K/L_I will be lower than the level implied by eq. (30), i.e., $K^*/L_I^* < (K/L_I)_{\text{GR}}$ and it also fall short of its maximum possible level $(K/L_I)_{\max} \equiv (S_I/\delta)^{1/(1-\alpha_I)}$ that would prevail if the equilibrium value of the production of consumption goods were set to zero. Interestingly, the ratio between the total stock of capital and the level of employment in the sector of investment goods conveyed by eq. (30) is completely unrelated from the technology available in the sector of consumption goods (cf. Senouci, 2011).

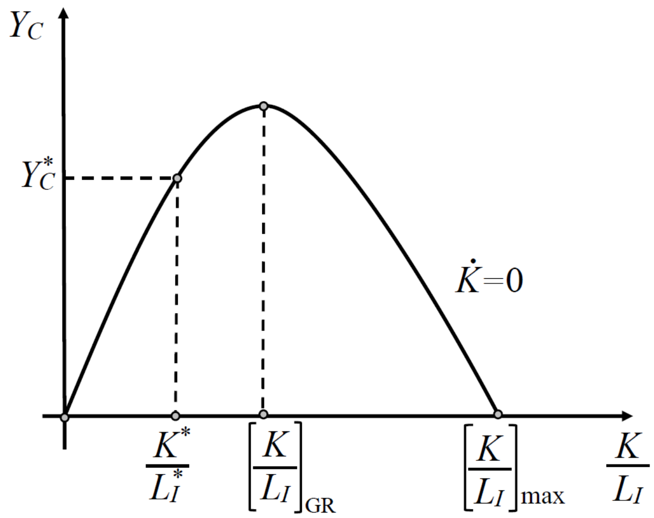

Figure 1: Golden rule

2.4 Local dynamics

The local dynamics of Y_C, L_I and K around the unique element of $\mathcal{S}$ is given by the following 3×3 linear system:

$$\begin{pmatrix} \dot{Y}_C(t) \\ \dot{L}_I(t) \\ \dot{K}(t) \end{pmatrix} = \boldsymbol{J} \begin{pmatrix} Y_C(t) - Y_C^* \\ L_I(t) - L_I^* \\ K(t) - K^* \end{pmatrix} \tag{31}$$

where $\boldsymbol{J} \in \mathbb{R}^{3\times3}$ is the Jacobian matrix.

Considering the expression in eq.s (15), (16), (4), (2), (6), (22), (25), (27), exploiting the results in Appendix and recalling the definition of Φ^*, $\boldsymbol{J}$ can be written as

$$\begin{bmatrix} \frac{(1-\alpha_I)(\rho+\delta(1-\alpha_I))(\rho+\delta)}{(1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)} & \frac{\alpha_C(1-\alpha_I)\delta(\alpha_I+\gamma)(\rho+\delta)Y_C^*}{\left((1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)\right)L_I^*} & -\frac{\alpha_C(1-\alpha_I)(\rho+\delta(1+\gamma))(\rho+\delta)Y_C^*}{\left((1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)\right)K^*} \\ -\frac{(\rho+\delta(1-\alpha_I))(\rho+\delta)(\rho+\delta-\alpha_I(\rho+\delta(1-\alpha_I)))L_I^*}{\alpha_C\alpha_I\delta\left((1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)\right)Y_C^*} & -\frac{(1-\alpha_I)(\rho+\delta(1-\alpha_I))(\rho+\delta)}{(1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)} & \frac{(\rho+\delta)\left(\alpha_I^2\delta\rho+(1-\alpha_I)(\rho+\delta)^2\right)L_I^*}{\alpha_I\delta\left((1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)\right)K^*} \\ -\frac{(\rho+\delta(1-\alpha_I))K^*}{\alpha_C Y_C^*} & \frac{(1-\alpha_I)\delta K^*}{L_I^*} & \rho \end{bmatrix} \tag{32}$$

Given that the trace of the Jacobian matrix in (32) is equal to $\rho > 0$, its characteristic equation is the following:

$$\lambda^3 - \rho\lambda^2 + \mathcal{M}\lambda - \mathcal{D} = 0 \tag{33}$$

where the three solutions of eq. (33) – say λ_1, λ_2 and λ_3 – are the eigenvalues of $\boldsymbol{J}$, $\mathcal{M} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ is the sum of its principal minors of order two and $\mathcal{D} = \lambda_1\lambda_2\lambda_3$ is its determinant.

The straightforward implementation of Sarrus' rule reveals that $\mathcal{D}$ is equal to zero no matter the actual values of the involved parameters. Obviously, given that the trace of $\boldsymbol{J}$ is certainly higher than zero for the hypothesis of positive discounting, this means that at least one of its three eigenvalues is equal to zero as well by confirming that the motion of the model economy – in principle – could be framed in a bidimensional system (cf. Bosi and Desmarchelier, 2019). By contrast, $\mathcal{M}$ is equal to

$$-\frac{\delta(1-\alpha_I)(\rho+\delta(1-\alpha_I))(\rho+\delta)(1+\gamma)}{(1+\gamma)\alpha_I^2\delta+(1-\alpha_I)\gamma(\rho+\delta)} \tag{34}$$

For given values of ρ, δ and γ, the expression in eq. (34) consists in the ratio between two parabolas whose independent variable is given by α_I. The actual magnitude taken by this ratio conveys the value of the product between the two eigenvalues which are not necessarily equal to zero, a value that together with the positive one of the trace allows us to characterize the local dynamics of the model economy in a precise manner. On the one hand, since α_C does not enter eq. (34), the local dynamics of the model economy does not depend on the shape of the technology prevailing in the sector of consumption goods which may even be the one that produces the largest share of output. This feature of the model follows from the fact that Y_C – differently from Y_I – does not contribute to capital accumulation but only to the stream of utilities of the optimizing agent (cf. Guerrazzi, 2025a-b). On the other hand, the parabola at the numerator of eq. (34) has two distinct roots, namely $\overline{\alpha}_I = 1$ and $\overline{\overline{\alpha}}_I = 1 + \rho/\delta$ and it is positive for external values, whereas whenever $4(1+\gamma)\delta/\gamma(\rho+\delta)$ is higher than 1, the parabola at the numerator has a negative discriminant so that it is always positive.

Since the estimated values of ρ, δ and γ are usually consistent with the inequality recalled above, we can conclude that whenever α_I belongs to the interval $(0,1)$ the product between the two egeinvalues which are different from zero is negative and all the components of the stationary solution take economically meaningful positive values. Consequently, as long as the technology prevailing in the sector of investment goods is convex, i.e., as long as $\alpha_I \in (0,1)$, the unique element of $\mathcal{S}$ will be a saddle point. In this case, no matter the shape of the technology prevailing in the industry of consumption goods, $\boldsymbol{J}$ will have three real eigenvalues, one positive, one equal to zero and one negative, say $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_3 < 0$. Therefore, as it would happen in a companion one-sector model with logarithmic preferences and Cobb-Douglas technology with decreasing returns to scale, $\begin{pmatrix} K^* & Y_C^* & L_I^* \end{pmatrix} \in \mathbb{R}_{++}^{1\times 3}$ will be characterized by a determinate dynamics as in the conventional one-sector model (cf. Cass, 1965). In other words, given the value of $\overline{K}$, there will be a unique value of the vector $\begin{pmatrix} Y_C(0) & L_I(0) \end{pmatrix} \in \mathbb{R}_{++}^{1\times 2}$ that places the system in (31) on the stable branch of the saddle path whereas all the others trajectories will tend to diverge by violating the resource constraint implied by eq.s (2), (4) and (6), or the transversality condition in (12).

In addition to the saddle path, it is widely recognized in the one-sector model of optimal growth the first element of the last row of the Jacobian matrix, namely $\partial \dot{K}(t)/\partial Y_C(t)\big|_{\mathcal{S}}$, is equal to -1 because in that setting in which the final output is a homogenous good the consumption of an additional unit of Y_C reduces saving – and investment – at the same pace (cf. Cass, 1965). Recalling the result is eq. (5), the corresponding value of $j_{3,1} = \partial Y_I(t)/\partial Y_C(t)|_{\mathcal{S}}$ in (32) is equal to -1 whenever

$$K_C^* = \left(\frac{\alpha_C S_C}{\rho+\delta}\right)^{\frac{1}{1-\alpha_C}} \tag{35}$$

The equality in eq. (35) is the one that prevails when the marginal propensity to consume of the optimizing agent is complementary to its marginal propensity to save and invest, and – recalling the result in eq. (20) – it corresponds to the case in which the equilibrium level of capitalization prevailing in the sector of consumption is determined in the same way as the equilibrium capital intensity in the sector of investment goods (K_I^*/L_I^*), i.e., by equating the marginal productivity of capital in the sector of consumption goods to the opportunity cost of capital which is given by the sum between the discount and the depreciation rates.

Suppose now that $\mathbb{V}(\lambda_3) \in \mathbb{R}^{3\times 1}$ is the column-eigenvector associated to the negative convergent eigenvalue. Thereafter, whenever the unique element of $\mathcal{S}$ is a saddle point, the dynamics of the output in the sector of consumption goods, the one of the labour input employed in the sector of investment goods and the one of the total stock of capital is conveyed by

$$\begin{pmatrix} Y_C(t) \\ L_I(t) \\ \dot{K}(t) \end{pmatrix} = \begin{pmatrix} Y_C^* \\ L_I^* \\ K^* \end{pmatrix} + \begin{pmatrix} \frac{v_1(\lambda_3)}{v_3(\lambda_3)} \\ \frac{v_2(\lambda_3)}{v_3(\lambda_3)} \\ 1 \end{pmatrix} \exp(\lambda_3 t)\left(\overline{K} - K^*\right) \tag{36}$$

where $v_i(\lambda_3) \in \mathbb{R}$ is the i-th element of $\mathbb{V}(\lambda_3)$.

Relying on a suitable discretization and a tailored calibration, the expression in (36) will be used to examine the numerical properties of the two-sector model economy under scrutiny.

3 Numerical properties

In order to assess its numerical properties, the theoretical model developed above is parametriezed, calibrated and simulated on an annual basis by taking as reference the US economy. Specifically, following Christiano and Fisher (2003), the values of the two sectorial elasticities of output with respect to capital – respectively, α_C and α_I – are taken from the estimates of Hornstein and Praschnick (1997) which are consistent with convex technologies in both industries. Moreover, the common value of the depreciation rate of capital (δ) is fixed as in Kydland and Prescott (1982), the value of the discount rate (ρ) is set at the point value suggested by Itskhoki and Moll (2019), whereas the value of the inverse elasticity of labour supply (γ) is set as suggested by Benhabib and Farmer (1996). Such a triplet implies that $4(1+\gamma)\delta/\gamma(\rho+\delta) = 6.154$, a value which is largely consistent with the positivity of the denominator of $\mathcal{M}$ in eq. (34). Thereafter, after having normalized to 1 the value of S_I, the value of S_C is calibrated to match the average ratio between the share of consumption and investment expenditures over GDP, a figure that according to historical data amounts to 4.566.[6] The values of each parameter together with their description is summarized in Table 1.

Parameter	Description	Value
S_C	Productivity in the sector of consumption goods	1.497
α_C	Elasticity of capital in the sector of consumption goods	0.450
S_I	Productivity in the sector of investment goods	1.000
α_I	Elasticity of capital in the sector of investment goods	0.260
δ	Capital depreciation rate	0.100
ρ	Discount rate	0.030
γ	Inverse elasticity of labour supply	1.000

Table 1: Calibration

[6] Data on US national account can be retrieved at https://fred.stlouisfed.org.

The figures in Table 1 imply that in the steady-state equilibrium the share of capital employed in the sector of consumption goods (K_C^*/K^*) amount to 80%. Consequently, in the calibrated two-sector economy the consumption goods industry is more capital-intensive than the investment goods one. In addition, under the baseline calibration the equality in eq. (35) is not verified. Specifically, the adopted parameter values imply that the first element on the third row of $\boldsymbol{J}$ in eq. (32) is less than one in absolute value. To be precise, $j_{3,1} = -0.506 > -1$. This means that according to the parameter values in Table 1 the decision to produce and consume an additional unit of Y_C restrains capital accumulation but less than one-to-one as it happens in the conventional one-sector model of optimal growth (cf. Cass, 1965). In other words, given the values of the involved parameters, in the two-sector model developed above the marginal productivity of the stock of capital employed in the sector of consumption goods is so high that the production and the consumption of an additional unit of Y_C is possible by slowing capital accumulation by a lower amount.[7] Therefore, in the calibrated model, the sum between the marginal propensity to consume and the marginal propensity to save – and invest – is larger than 1 (cf. Senouci, 2011).

On a visual perspective, according to the expression in (36), the parameters values in Table 1 lead to the saddle-path trajectories of Y_C, L_I and K illustrated in the diagram of Figure 1.[8]

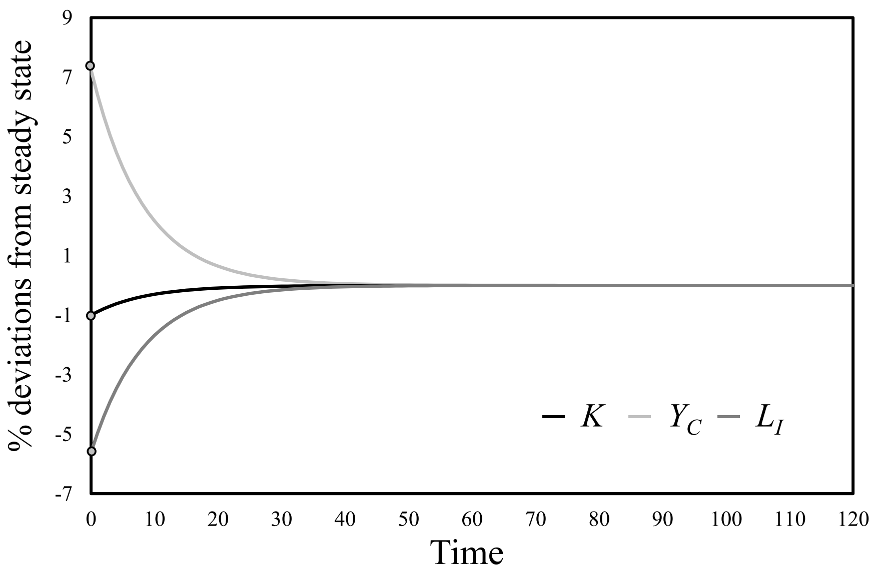

Figure 2: Saddle-path dynamics of Y_C, L_I and K

The plot in Figure 2 shows that when the stock of capital undershoots its steady-state value by 1% the production and the enjoyment of consumption goods overshoots its equilibrium value by about

[7]Obviously, given that the parametrized value of α_C is lower than 1, this follows from the fact that K_C^* fall short of the amount appointed by eq. (35).

[8]The time step of the simulations is set to 1. MAT LAB codes are available from the author upon reasonable request.

7%, whereas the labour supply for the production of investment goods undershoots its steady-state reference by about 6%. This pattern is definitely different from the one observed in the one-sector model – where consumption tends to move in the same direction of the stock of capital – and reveals that along the optimal path the optimizing agent trades-off the consumption of goods with labour provision in the sector of investment goods. Unsurprisingly, as I noticed in commenting the result in eq. (22), the decision to offer labour for the production of investment goods traces out the outcomes of a typical saving decision that leads to a reduction of consumption.

Another interesting aspect of the model economy under scrutiny is certainly the way in which capital is optimally allocated over time in the two existing sectors. The implied adjustments of K_C and K_I obtained when K initially undershoots its steady-state value by 1% as it does in Figure 2 are tracked in Figure 3.

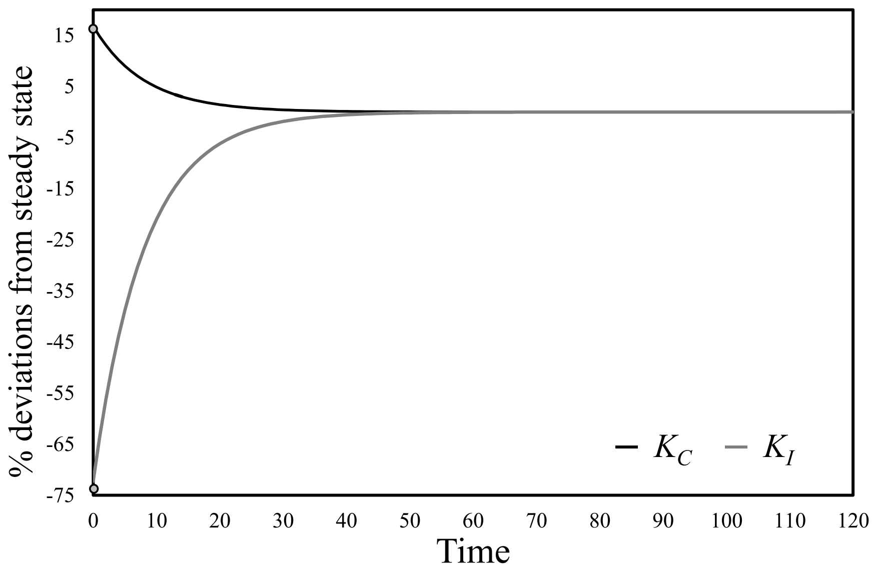

Figure 3: Capital adjustments in the two sectors

Consistently with the optimal choices documented above, the plot in Figure 3 shows that the stock of capital allocated in the sector of consumption (investment) goods initially overshoots (undershoots) its equilibrium value by about 17% (72%) and then it converges towards its steady-state reference. Nevertheless, given the involved magnitudes, along the optimal path there is no factor reversal so that the implied share of capital allocated in sector of consumption goods is persistently above the one allocated in the sector of investment goods.

Furthermore, recalling the expressions in eq.s (17) and (19), the allocation of labour and capital in the two sectors is also the determinant of the relative prices that in a decentralized environment implement the centralized solution of the problem in (7). Specifically, the competitive paths of the ratios p_I/p_C and w/p_C implied by the adjustments of K, Y_C and L_I plotted in Figure 2 are illustrated in the two panels of Figure 4.

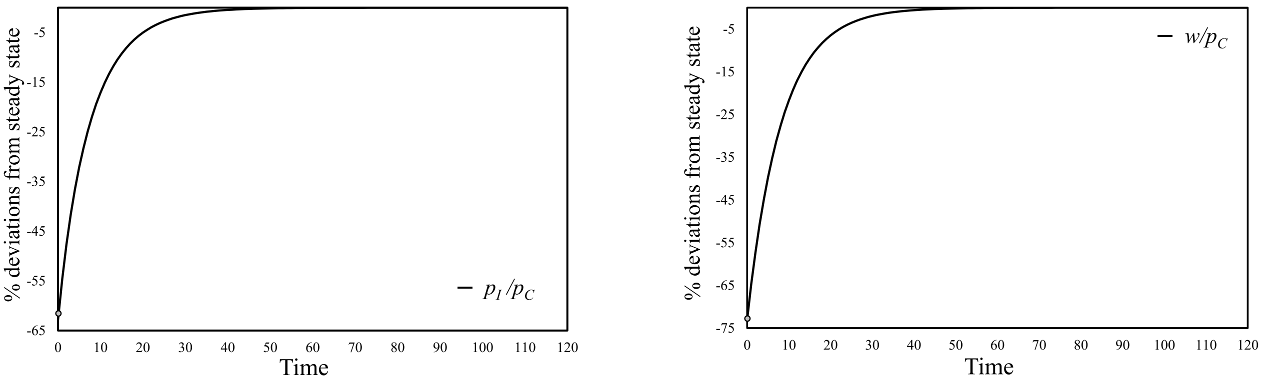

Figure 4: The implied paths of the relative price of investment goods and the real wage

The plot on the left-hand-side of Figure 4 reveals the price of investment goods measured in units of the price of consumption goods which is consistent with the solution of the central planner problem in (7) follows an increasing path towards its steady-state value. Since the quantity adjustments plotted in Figures 2 and 3 imply that the production of Y_C follows a decreasing path whereas the one of Y_I follows an increasing path, this pattern is perfectly consistent with upward-sloped supply functions for the two types of goods.[9] Similarly, the plot on the right-hand-side shows that the profit-maximizing level of the real wage follows an increasing path towards its steady-state reference which is driven by the accumulation of capital and the employment of labour in the sector of investment goods and by the path followed by the relative price of investment goods. Even in this case, recalling the path of L_I tracked in Figure 2, the dynamic behaviour of the wage is consistent with an upward-sloped labour supply schedule.

From an empirical point of view, the increasing path of w/p_C is consistent with the well-recognized procyclical pattern displayed by the US value of hourly compensation – which includes not only wage and salary compensation, but also all the other entries of labour remuneration such as fringe benefits and contributions for social insurance programs – deflated with the consumer price index for urban consumers (cf. Guerrazzi, 2025a). According to actual US data, such a measure of the real wage is also the one that closely follows the observed path of labour productivity (cf. Sallivan, 1997). By contrast, the increasing path followed by p_I/p_C tracked on the left-hand-side of Figure 4 is at odds with respect to the counter-cyclical pattern of the price of tradable capital goods retrieved with US data by a number of scholars and often explained by referring to technical progress and/or market integration (cf. Justiniano et al. 2011; Hergovich and Merz, 2018; Lian et al. 2020). Nevertheless, especially when the sign of the correlation with working hours is concerned, Beaudry et al. (2015) show that focusing on US data starting from the second half of the 1960s, the actual behaviour of the relative price of many capital goods appears mainly procyclical as the one generated by the theoretical model developed above.

[9]Denoting with $r(t)$ the rental rate paid by firms in a decentralized environment for each unit of employed capital, the supply functions of each goods would be, respectively, given by $p_I(t) = (1/S_I)(r(t)/\alpha_I)^{\alpha_I}(w(t)/(1-\alpha_I))^{1-\alpha_I}$ and $p_C(t) = r(t)(Y_C(t))^{1/(1-\alpha_C)}/\alpha_C S_C^{1/\alpha_C}$.

4 Concluding remarks

In this paper, I developed a frictionless two-sector optimal growth model with endogenous labour supply in which the non-reproducible factor can be employed only in the industry that produces investment goods. From an analytical point of view, I showed that no matter the shape of the technology prevailing in the sector of consumption goods, a convex technology in the industry of investment goods is necessary and sufficient for an economically meaningful stationary solution as well as for the saddle-path of the equilibrium trajectories followed by the stock of capital, the amount of consumption goods which are produced and enjoyed, and the labour input (cf. Guerrazzi, 2025a-b). In addition, tailoring model to the US economy, I showed that the present framework relaxes the complementarity between the propensities to consume and save of the optimizing agent, and it is also able to provide a rationale for the procyclical patterns of the relative price of capital goods and the real wage rate (cf. Sallivan, 1997; Beaudry et al. 2015).

An interesting development of the analysis carried out in this paper would be certainly the exploration of the possibility to optimally adjust the labour input even in the sector of consumption goods, i.e., to consider the situation in which the non reproducible factor becomes malleable as capital so that it can be actually employed in both industries (cf. Benhabib and Farmer, 1996; Benhabib and Nishimura, 1998; Harrison and Weder, 2002; Zhang, 2005; Dufourt et al. 2015). In the context of the labour market segmentation triggered by the existence of different sectors of production, such an option would allow to disentangle the two distinct trade-offs implied by the possibility to supply labour services for the production of different types of final goods and it would likely require the existence of wage differentials (cf. Reilly and Zanchi, 2003; Plasman et al. 2007). Within the theoretical framework developed in this paper, the existence of wage differentials is possible only by assuming different separate CRRA disutilities with respect to labour provision built on distinct values of the inverse elasticity of (sectoral) labour supply.[10] In this case, relying on the findings presented in Guerrazzi (2025a), we will obtain a constant labour provision in the sector of consumption goods whereas all the results achieved in this paper would remain the qualitatively same. Consequently, as illustrated in the RHS panel of Figure 4, there will be a procyclial real wage rate paid by the sector of investment goods together with a countercyclial wage paid by the sector of consumption goods driven by the decumulation of capital occurring in that industry as depicted in Figure 3.

More promising developments could be achieved by moving in different directions. First, the model could be augmented with stochastic shocks that randomly hit the production functions of the two final goods. This could be done by assuming that S_C and S_I follows distinct erratic processes. In this way, it could be possible to analyse how sectoral supply shock affect and propagate the model economy. In addition, an even more interesting extension could be also the introduction of sectoral matching frictions through the assumption that in each industry the employment of labour requires some unproductive recruiting effort (cf. Yedid-Levi, 2016). This would allow to explore how

[10] Under perfect substitutability of labour disutility, competitive labour supply would always require the same wage in each industry.

supply shocks interact with matching shocks and – on the policy side – it could also provide some guidance for the scope of sectoral reforms of the labour market. These could be intriguing avenues for additional deepening.

Appendix

According to eq. (11), the optimal growth rate of the costate variable associated to the problem in (7) can be written as

$$\frac{\dot{q}(t)}{q(t)} = \rho + \delta - \frac{\alpha_I S_I}{\left(\frac{K(t) - \left(\frac{Y_C(t)}{S_C}\right)^{\frac{1}{\alpha_C}}}{L_I(t)}\right)^{1-\alpha_I}} \tag{A1}$$

Considering the expressions in eq.s (22), (25) and (27), and recalling the definition of Φ^*, the derivatives of (A1) with respect to $Y_C(t)$, $L_I(t)$ and $K(t)$ evaluated in the stationary solution, i.e., in the unique element of $\mathcal{S}$, are, respectively, given by

$$\left.\frac{\frac{\dot{q}(t)}{q(t)}}{\partial Y_C(t)}\right|_{\mathcal{S}} = -\frac{(1-\alpha_I)(\rho+\delta)(\rho+\delta(1-\alpha_I))}{\alpha_I \alpha_C \delta Y_C^*} \tag{A2}$$

$$\left.\frac{\frac{\dot{q}(t)}{q(t)}}{\partial L_I(t)}\right|_{\mathcal{S}} = -\frac{(1-\alpha_I)(\rho+\delta)}{L_I^*} \tag{A3}$$

$$\left.\frac{\frac{\dot{q}(t)}{q(t)}}{\partial K(t)}\right|_{\mathcal{S}} = \frac{(1-\alpha_I)(\rho+\delta)^2}{\alpha_I \delta K^*} \tag{A4}$$

References

[1] Beaudry, P., Moura, A., Portier, F. (2015), Reexamining the Cyclical Behavior of the Relative Price of Investment, *Economics Letters*, Vol. 135, No. C, pp. 108-111.

[2] Benhabib, J., Farmer, R. E. A. (1996), Indeterminacy and Sector-Specific Externalities, *Journal of Monetary Economics*, Vol. 37, No. 3, pp. 421-442.

[3] Benhabib, J., Nishimura, K. (1998), Indeterminacy and Sunspots with Constant Returns, *Journal of Economic Theory*, Vol. 81, No. 1, pp. 58-96.

[4] Bosi, S., Desmarchelier, D. (2019), Local Bifurcations of Three and Four-Dimensional Systems: A Tractable Characterization with Economic Applications, *Mathematical Social Sciences*, Vol. 97, No. C, pp. 38-50.

[5] Cai, D. (2006), A Two-Sector Economic Growth Model with Optimal Labor and Capital Allocation, *Applied Mathematics and Computation*, Vol. 183, No. 2, pp. 1359-1377.

[6] Cardi, O., Restout, R., Claeys, P. (2020), Imperfect Mobility of Labor Across Sectors and Fiscal Transmission, *Journal of Economic Dynamics and Control*, Vol. 111, No. 103815.

[7] Cass, D. (1965), Optimum Growth in an Aggregative Model of Capital Accumulation, *Review of Economic Studies*, Vol. 32, No. 3, pp. 233-240.

[8] Christiano, L. J., Fisher, J. D. M. (2003), Stock Market and Investment Goods Prices: Implications for Macroeconomics, *NBER Working Paper Series*, No. 10031.

[9] Dufourt, F., Nishimura; K., Venditti, A. (2015), Indeterminacy and Sunspots in Two-Sector RBC Models with Generalized No-Income-Effect Preferences, *Journal of Economic Theory*, Vol. 157, No. C, pp. 1056-1080.

[10] Drabicki, J. Z., Takayama, A. (1975), On the Optimal Growth of the Two Sector Economy, *Keio Economic Studies*, Vol. 12, No. 1, pp. 1-36.

[11] Ferreira, P. C., de Abreu Pessoa, A., Veloso, F. A. (2014), Distortions in the Investment Goods Sector and Productivity Decline, *FGV EPGE Economics Working Papers*, No. 755.

[12] Galor, O. (1992), A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System, *Econometrica*, Vol. 60, No. 6, pp. 1351-1386.

[13] Growiec, J. (2022), Automation, Partial and Full, *Macroeconomic Dynamics*, Vol. 26, No. 7, pp. 1731-1755.

[14] Guerrazzi, M. (2025a), A Two-Sector Model of Optimal Growth in which Labour is Employed only in the Industry of Consumption Goods: A Complete Characterization of Equilibrium Paths, *MPRA Paper*, No. 126101.

[15] Guerrazzi, M. (2025b), A Complete Characterization of Equilibrium Paths in a Labour-Less Two-Sector Model of Optimal Growth, *MPRA Paper*, No. 124225.

[16] Harrison, S. H., Weder, M. (2002), Tracing Externalities as a Source of Indeterminacy, *Journal of Economic Dynamics & Control*, Vol. 26, No. 5, pp. 851-867.

[17] Hashimoto, H., Sakuragawa, M. (1999), A Characterization of the Dynamical System in a Two-Sector Overlapping-Generations Model: The Case of Unbounded Growth, *Japanese Economic Review*, Vol. 50, No. 2, pp. 138-147.

[18] Hergovich, P., Merz, M. (2018), The Price of Capital, Factor Substitutability, and Corporate Profits, *IZA Discussion Papers*, No. 11791.

[19] Hornstein, A., Prashinick, J. (1997), Intermediate Inputs and Sectoral Comovement in the Business Cycle, *Journal of Monetary Economics*, Vol. 40, No. 3, pp. 573-595.

[20] Itskhoki, O., Moll, B. (2019), Optimal Development Policies with Financial Frictions, *Econometrica*, Vol 87, No. 1, pp. 139-173.

[21] Justiniano, A., Primiceri, G.E., Tambalotti, A. (2011), Investment Shocks and the Relative Price of Investment, *Review of Economic Dynamics*, Vol. 14, No. 1, pp. 102-121.

[22] Kydland, F. E., Prescott, E. C. (1982), Time to Build and Aggregate Fluctuations, *Econometrica*, Vol. 50, No. 6, pp. 1345-1370.

[23] Lian, W., Novta, N., Pugacheva, E., Timmer, Y., Topalova, P. (2020), The Price of Capital Goods: A Driver of Investment under Threat, *IMF Economic Review*, Vol. 68, No. 3, pp. 509-549.

[24] Meade, J. E. (1961), *A Neo-Classical Theory of Economic Growth*, London: Allen & Unwin.

[25] Plasman, R. A., Rycx, R., Tojerow, I. D. (2007), Wage Differentials in Belgium: The Role of Worker and Employer Characteristics, *Brussels Economic Review*, Vol. 50, No. 1, pp. 11-40.

[26] Reilly, K. T., Zanchi, L. (2003), Industry Wage Differentials: How Many, Big and Significant are They?, *International Journal of Manpower*, Vol. 24, No. 4, pp. 367-398.

[27] Ryder, H. E. (1969), Optimal Accumulation in a Two-Sector Neoclassical Economy with Non-Shiftable Capital, *Journal of Political Economy*, Vol. 77, No. 4, pp. 665-683.

[28] Sallivan, D. G. (1997), Trends in Real Wage Growth, *Chicago Fed Letter*, No. 115.

[29] Senouci, M. (2011), Optimal Growth and the Golden Rule in a Two-Sector Model of Capital Accumulation, *PSE Working Papers*, No. 00572510.

[30] Srinivasan, T. N. (1964), Optimal Savings in a Two-Sector Model of Growth, *Econometrica*, Vol. 32, No. 3, pp. 358-373.

[31] Uzawa, H. (1961), On a Two-Sector Model of Economic Growth, *Review of Economic Studies*, Vol. 29, No. 1, pp. 40-47.

[32] Zhang, W-B. (2005), A Two-Sector Model with Labour Supply, *Australian Journal of Labour Economics*, Vol. 8, No. 3, pp. 245-260.

[33] Yedid-Levi, Y. (2016), Why does Employment in all Major Sectors Move Together over the Business Cycle?, *Review of Economic Dynamics*, Vol. 22, No. pp. 131-156.