A Note on Intrinsic Correlation

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Abstract. In this note we characterize the strategic implication of intrinsic correlation, introduced by Brandenburger and Friedenberg (2008), in the subjective correlated equilibrium setting of a complete information game. Intrinsic correlation restricts correlation devices to variables within the game, i.e. players's beliefs (and higher order beliefs) about each other's strategies, in contrast to signals or sunspots from the "outside." The characterization is a strengthening of best-response set with an injectivity condition for a certain subset identified by an iterative procedure. We also give an iterative procedure, analogous to the iterated removals of dominated strategies, that arrives at strategies consistent with our characterization, which always exist.

1. Introduction

Correlation is a natural and important concept in game theory. In non-cooperative game theory, the combined implication of correlation and rationality is first analyzed by Aumann (1974) with his solution concept of correlated equilibrium.

Very recently, Brandenburger and Friedenberg (2008) introduced a subtle classification on correlations in non-cooperative games: they can be intrinsic, coming from correlation devices that are players's beliefs (and higher order beliefs) about each other's strategies, or extrinsic, which means that there is no restriction on the correlation devices; Brandenburger and Friedenberg use the abjective extrinsic because

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such correlations are usually interpreted as correlations from signals and sunspots that are not explicitly specified in the game.

There is a large literature on the strategic implications of extrinsic correlation, for example the study of correlated equilibrium in one-shot games and in dynamic games (well-known and needs not to be cited), of correlations from observations in adaptive heuristics (Hart and Mas-Colell (2000)), correlations from robustness considerations in information structure (Kajii and Morris (1997), Morris and Ui (2005)), etc.

This note contributes toward understanding the strategic implications of intrinsic correlation in one-shot, complete information game. In particular, we provide an exact characterization of strategies played in a subjective correlated equilibrium when all correlations are restricted to be intrinsic. Our characterization is inspired by the injetivity condition in Brandenburger and Friedenberg (2008) and can be seen as a generalization of injectivity from first order beliefs to higher order bliefs. We also give an iterative procedure, analogous to the iterated removals of dominated strategies, that arrives at strategies consistent with our characterization, which always exist.

2. Set-up

From now on we fix a finite, complete information game: (u, A, N), where N is the set of players ($|N| \geq 2$), $A = \prod_{i \in N} A_i$ the set of strategy profiles, and $u_i : A \to \mathbb{R}$, $i \in N$, the payoffs. All of our definitions, constructions and results are stated with respect to this game.

We work with type spaces that captures strategic uncertainty: $(T_i, \lambda_i, \sigma_i)_{i \in N}$, where T_i is a finite or countably infinite set of types of player i, $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$ are i's belief, contingent on his type, about types and strategies of other players, and $\sigma_i : T_i \to A_i$ is i's pure strategy contingent on type.

We are interested in type space and strategies $(T_i, \lambda_i, \sigma_i)_{i \in N}$ that form a subjective correlated equilibrium, with all correlations being intrinsic; this is formalized by the following three conditions: for every $i \in N$ and every $t_i \in T_i$, (A-1) $\sigma_i(t_i) \in \operatorname{argmax}_{a'_i \in A_i} u_i(a'_i, \operatorname{marg}_{A_{-i}} \lambda_i(t_i))$

(A-2) $\lambda_i(t_i)[t_{-i}, a_{-i}] = (\operatorname{marg}_{T_{-i}} \lambda_i(t_i))[t_{-i}] \cdot \delta_{\sigma_{-i}(t_{-i})}(a_{-i})$, for[1] all $t_{-i} \in T_{-i}$ and $a_{-i} \in A_{-i}$.

(A-3) each $t_i \in T_i$ induces a distinct belief hierarchy; that is, there is no redundant type in T_i, using the terminology of Mertens and Zamir (1985).

Definition 2.1. $(T_i, \lambda_i, \sigma_i)_{i\in N}$ *is a* subjective correlated equilibrium *if (A-1) and (A-2) are satisfied.*

$(T_i, \lambda_i, \sigma_i)_{i\in N}$ *is a* subjective equilibrium with intrinsic correlation (s.e.i.c.) *if (A-1), (A-2) and (A-3) are satisfied.*

Condition (A-1) says that every type of every player is maximizing according to his belief. (A-2) says that this belief is compatible with others' type-contingent strategies. Since this belief can be correlated, (A-3) requires the source of correlation to be belief hierarchies, which formalize the notion of higher order beliefs.

Here is an example of redundant types, which (A-3) rules out.

Example 2.2. *Consider the following symmetric type space with two players. $N = \{1, 2\}$, with types $T_1 = T_2 = \{s, t\}$, and strategies $A_1 = A_2 = \{a, b\}$. Let $\sigma_1(s) = \sigma_2(s) = \sigma_1(t) = \sigma_2(t) = a$. And let λ_1 and λ_2 be such that $\lambda_1(s)[t, a] = \lambda_2(s)[t, a] = 1$ and $\lambda_1(t)[s, a] = \lambda_2(t)[s, a] = 1$.*

Notice that $\lambda_i(s) \neq \lambda_i(t)$ for each $i \in N$. Nevertheless, for both players, type s and type t have the same belief hierarchy: their first order beliefs are both with probability one on strategy a; and they both believe (with probability one) that strategy a is played and the other believes so, thus the same second order belief, and so on.

One interpretation in support of non-redundant types (and thus of intrinsic correlation) is that the players in an complete information game can only reason about non-redundant types, so an analysis relying on the presence of redundant types introduces an uncomfortable asymmetry between the analyst and the players in the game. Of course,

[1] $\delta_{\sigma_{-i}(t_{-i})}$ is the degenerate probability measure concentrated on $\sigma_{-i}(t_{-i})$, i.e. $\delta_{\sigma_{-i}(t_{-i})}(a_{-i}) = 1$ if $\sigma_{-i}(t_{-i}) = a_{-i}$, and 0 otherwise.

this is not the case if the redundant types are actually physical signals like stop lights, but such signals, as they are not modeled as a part of strategies, must come from outside of the game.

3. Characterization

It is well-known (e.g. Brandenburger and Dekel, 1987) that for any set of strategy $(Q_i)_{i\in N}$, there exists a subjective correlated equilibrium $(T_i, \lambda_i, \sigma_i)_{i\in N}$ such that $Q_i = \sigma_i(T_i)$ for every $i \in N$, if and only if $(Q_i)_{i\in N}$ is a best-response set; that is, if and only if for each $i \in N$ and $a_i \in Q_i$, there exists a (perhaps correlated) belief $\mu \in \Delta(Q_{-i})$ such that a_i is a player i's best response to μ. We now characterize the additional strategic implications of insisting on non-redundant types (condition (A-3)).

We first define the best-response correspondence,

$$\mathrm{BR}_i(\mu) := \{a_i \in A_i : u_i(a_i, \mu) \geq u_i(a_i', \mu)\ \forall a_i' \in A_i\},$$

for each $i \in N$ and $\mu \in \Delta(A_{-i})$.

For a fixed $(Q_i)_{i\in N}$, where each $Q_i \subseteq A_i$, let

$$\begin{aligned} \beta_i(a_i) &:= \{\mu \in \Delta(Q_{-i}) : a_i \in \mathrm{BR}_i(\mu)\}, a_i \in Q_i, \\ \beta_i(C_i) &:= \bigcup_{a_i \in C_i} \beta_i(a_i), C_i \subseteq Q_i, \end{aligned} \tag{1}$$

for every $i \in N$. $\beta_i(a_i)$ is simply the set of correlated beliefs that support strategy a_i (for which a_i is a best response for player i).

When $\beta_i(a_i) = \{\mu\}$, we simply write $\beta_i(a_i)$ for μ.

And for each $i \in N$, let

$$\begin{aligned} W_i^1 &:= \{a_i \in Q_i : |\beta_i(a_i)| = 1\}, \\ W_i^l &:= \{a_i \in W_i^1 : \beta_i(a_i)[W_{-i}^{l-1}] = 1\}, l \geq 2, \\ W_i &:= \bigcap_{l\geq 1} W_i^l. \end{aligned} \tag{2}$$

W_i^1 is the set of strategies in Q_i that has an unique supporting belief. W_i^2 is the subset of W_i^1 for which the unique supporting belief has support contained in $W_{-i}^1 = \prod_{j\neq i} W_j^1$, and so on.

Theorem 3.1. *For any $(Q_i)_{i \in N}$, where each $Q_i \subseteq A_i$, there exist a s.e.i.c. $(T_i, \lambda_i, \sigma_i)_{i \in N}$ such that $Q_i = \sigma_i(T_i)$ for each $i \in N$, if and only if for every $i \in N$, for each $a_i \in Q_i$, $\beta_i(a_i) \neq \emptyset$, and for each $a_i \neq a_i' \in W_i$, $\beta_i(a_i) \neq \beta_i(a_i')$.*

Brandenburger and Friedenberg (2008) points out that in games with generic payoffs, if Q_i is the set of player i's strategies that survive iterated deletions of strictly dominated strategies, then W_i^1 is empty for every player. Thus, as shown in their Proposition H.2 and H.3, in these generic games, the set of strategies played under some s.e.i.c. equals the set of strategies played under some subjective correlated equilibrium. This can also be seen in the statement of the theorem above: if W_i^1 is empty, so is every W_i, so each β_i is automatically injective over W_i, and the condition simply becomes best-response set. However, even in these generic games, the set of strategies played under a fixed subjective correlated equilibrium needs not to be exactly the set of strategies played under any s.e.i.c.; it may be a strict subset.

Example 3.2. *Consider the following symmetric two-person game:*

	A	B	C	D
A	*1, 1*	*3, 3*	*0, 0*	*0, 4*
B	*3, 3*	*1, 1*	*0, 4*	*0, 0*
C	*0, 0*	*4, 0*	*1, 1*	*1, 1*
D	*4, 0*	*0, 0*	*1, 1*	*1, 1*

First, note that $\{A, B, C, D\} \times \{A, B, C, D\}$ is a best-response set, so every strategy is under a subjective correlated equilibrium.

Let $Q_1 = Q_2 = \{A, B, C, D\}$. Then $\beta_1(A) = \beta_1(B) = \beta_2(A) = \beta_2(B) = \{1/2A{+}1/2B\}$. Thus, $W_1 = W_2 = \{A, B\}$, and the conditions of Theorem 3.1 fail for (Q_1, Q_2).

In fact, using Theorem 3.1 it is easy to see that A or B can be played by either player under no s.e.i.c. Thus, s.e.i.c. refines away a weakly dominated Nash equilibrium $(1/2A + 1/2B, 1/2A + 1/2B)$.

4. Iterative Procedure and Existence

In this section we give an iterative procedure that arrives at the set of strategies played under a s.e.i.c. We will show that this iterative procedure always gives a non-emtpy set, thus there always exists a s.e.i.c. in every finite game.

For each $i \in N$, let R_i^1 be the the set of player i's correlated rationalizable strategies, or equivalently, the set of player i's strategies surviving iterated removals of strictly dominated strategies.

Now inductively, for $l \geq 2$, let β_i^{l-1} and $W_i(l-1)$ be, respectively, the β_i and W_i obtained in Equations (1) and (2) when $Q_i = R_i^{l-1}$, $i \in N$. And for each $i \in N$ and $\gamma \in \beta_i^{l-1}(W_i(l-1))$, fixed an $a^{l-1}(\gamma) \in W_i(l-1)$ such that $\beta_i^{l-1}(a^{l-1}(\gamma)) = \gamma$; note that if β_i^{l-1} is injective, there is an unique choice for $a^{l-1}(\gamma)$.

For each $i \in N$, let

$$\begin{aligned} R_i^{l,1} &:= \big(R_i^{l-1} \setminus W_i(l-1)\big) \cup \{a^{l-1}(\gamma) : \gamma \in \beta_i^{l-1}(W_i(l-1))\}, \\ R_i^{l,k} &:= \{a_i \in R_i^{l,1} : \exists \mu \in \Delta(R_{-i}^{l,k-1}) \text{ s.t. } a_i \in \mathrm{BR}_i(\mu)\}, k \geq 2, \\ R_i^l &:= \bigcap_{k \geq 1} R_i^{l,k}. \end{aligned} \tag{3}$$

Finally, let

$$R_i := \bigcap_{l \geq 1} R_i^l$$

for each $i \in N$.

Notice that for each $i \in N$ we have

$$R_i^1 \supseteq R_i^2 \supseteq R_i^3 \supseteq \ldots \supseteq R_i.$$

Theorem 4.1. *For each $i \in N$, R_i is non-empty. Moreover, there exist a s.e.i.c. $(T_i, \lambda_i, \sigma_i)_{i \in N}$ such that $R_i = \sigma_i(T_i)$ for every $i \in N$.*

Proof. The second part in obvious, given Theorem 3.1.

To prove that R_i's are non-empty, we define a sequence of smaller iterating sets for each $i \in N$:

$$
\begin{aligned}
&S_i^1 := R_i^1, \\
&\text{for } l \geq 2: \\
&\quad S_i^{l,1} := S_i^{l-1} \setminus W_i(l-1), \\
&\quad S_i^{l,k} := \{a_i \in S_i^{l,1} : \exists \mu \in \Delta(S_{-i}^{l,k-1}) \text{ s.t. } a_i \in \mathrm{BR}_i(\mu)\}, k \geq 2, \\
&\quad S_i^{l} := \bigcap_{k \geq 1} S_i^{l,k}, \\
&\quad S_i := \bigcap_{l \geq 1} S_i^{l},
\end{aligned}
$$

where $W_i(l-1)$ (and β_i^{l-1}) here are obtained in Equations (1) and (2) when $Q_i = S_i^{l-1}$ (a slight abuse of notations).

Clearly, we have $S_i^1 \supseteq S_i^2 \supseteq \ldots \supseteq S_i$ and $S_i^l \subseteq R_i^l$ for each $l \geq 1$ and $i \in N$. Thus, it suffices to show that each S_i^l is non-empty.

It is well-known that each of $S_i^1 = R_i^1$ is non-empty.

Now for a fixed $l \geq 2$, suppose each of S_i^{l-1} is non-empty. By Lemma 4.2, for a fixed $i \in N$ we have an $\bar{a}_i \in A_i$ such that $\bar{a}_i$ is i's best response to multiple beliefs in S_{-i}^{l-1}. Clearly, $\bar{a}_i \in S_i^1$. And for $2 \leq k \leq l-1$, we have $\bar{a}_i \in S_i^k$ because $\bar{a}_i \notin W_i(k-1)$ by construction and $\bar{a}_i$ is i's best response to a belief with support in $S_{-i}^{l-1} \subseteq S_{-i}^{k,m}$ for any $m \geq 1$. Thus, we have $\bar{a}_i \in S_i^{l-1} \setminus W_i(l-1) = S_i^{l,1} \neq \emptyset$.

For a fixed $m \geq 2$, suppose each $S_i^{l,m-1}$ is non-empty. Fix an $i \in N$. Let $\mu \in \Delta(S_{-i}^{l,m-1})$ be an arbitrary belief, and choose any $\tilde{a}_i \in \mathrm{BR}_i(\mu)$. We will show that $\tilde{a}_i \in S_i^{l,m} \neq \emptyset$, which finishes the proof.

We have $\tilde{a}_i \notin W_i(k-1)$ for each $2 \leq k \leq l$; suppose otherwise, then we must have $\beta_i^{k-1}(\tilde{a}_i) = \mu$, which is impossible, because the support of μ is contained in $S_{-i}^{l,m-1}$, and the support of $\beta_i^{k-1}(\tilde{a}_i)$ is contained in $W_{-i}(k-1)$. And $\tilde{a}_i \in S_i^{k,n}$ (where $n \geq 2$ arbitrary when $k < l$, and $2 \leq n \leq m$ when $k = l$) because the support of μ is contained in $S_{-i}^{l,m-1} \subseteq S_{-i}^{k,n-1}$. $\square$

Lemma 4.2. *For a fixed player $i \in N$ and any non-empty $Q_j \subseteq A_j$, $j \neq i$, there exists an $\bar{a}_i \in A_i$ such that $\bar{a}_i$ is player i's best response to at least two distinct beliefs in Q_{-i}.*

Proof. Let C be the convex hull,

$$C := \{(u_i(\mu_i, a_{-i}))_{a_{-i} \in Q_{-i}} : \mu_i \in \Delta(A_i)\} \subseteq \mathbb{R}^{Q_{-i}}.$$

Let x be any extreme point of C that is not weakly dominated by any other point in C; that is, x is an extreme point of convex set C, and there is no $y \in C \setminus \{x\}$ such that $y(a_{-i}) \geq x(a_{-i})$ for all $a_{-i} \in Q_{-i}$; there exists a weakly undominated extreme point in C, because C is a convex hull of its extreme points, and $z \in C$ is not weakly dominated in C if and only if there exists some $\mu \in \Delta(Q_{-i})$ such that $\operatorname{supp} \mu = Q_{-i}$ and $\mu \cdot z \geq \mu \cdot y$ for all $y \in C$. Clearly, there exists an $\bar{a}_i \in A_i$ such that $x = (u_i(\bar{a}_i, a_{-i}))_{a_{-i} \in Q_{-i}}$. This $\bar{a}_i$ satisfies our desired conclusion because x is a weakly undominated extreme point of C, so there must be multiple hyperplanes separating $C - x = \{y - x : y \in C\}$ from the positive orthant $\mathbb{R}_+^{Q_{-i}}$. $\square$

We include $\{a^{l-1}(\gamma) : \gamma \in \beta_i^{l-1}(W_i(l-1))\}$ in $R_i^{l,1}$ because the resulting R_i is a maximal (in the set-inclusion order) set of strategies played under a s.e.i.c. Furthermore, they are canonical, in the sense that for any s.e.i.c., by some choice of $a^{l-1}(\gamma)$ for each l and γ, we will have $\sigma_i(T_i) \subseteq R_i$ for each $i \in N$.

5. Proof of Theorem 3.1

5.1. **Only if.** Fix a s.e.i.c. $(T_i, \lambda_i, \sigma_i)_{i \in N}$; let $Q_i := \sigma_i(T_i)$ for each $i \in N$.

Clearly for every $i \in N$ and $a_i \in Q_i$, $\beta_i(a_i) \neq \emptyset$.

If $W_i = \emptyset$ for every $i \in N$, then there is nothing else to prove. Thus, suppose otherwise; note that this implies that $W_i \neq \emptyset$ for all $i \in N$.

Our desired conclusion follows from the following lemma.

Lemma 5.1. *For any $i \in N$ and $a_i \in W_i^l$, $l \geq 1$, there is at most one l-th order belief in T_i mapped by σ_i to a_i; that is, if $\sigma_i(t_i) = \sigma_i(t_i') = a_i$, then t_i and t_i' has the same l-th order belief.*

Proof. If $\sigma_i(t_i) = a_i \in W_i^1, t_i \in T_i$, then $\operatorname{marg}_{A_{-i}} \lambda_i(t_i) = \beta(a_i)$ by condition (A-1) combined with $|\beta(a_i)| = 1$. Thus the lemma is true when $l = 1$.

Now suppose that $\sigma_i(t_i) = \sigma_i(t_i') = a_i \in W_i^2, t_i, t_i' \in T_i$. Then, $\operatorname{marg}_{A_{-i}} \lambda_i(t_i) = \operatorname{marg}_{A_{-i}} \lambda_i(t_i') = \beta(a_i)$ because $a_i \in W_i^1$. If $\beta_i(a_i)[a_{-i}] > 0$, $\lambda(t_i)[t_{-i}, a_{-i}] > 0$ and $\lambda(t_i')[t_{-i}', a_{-i}] > 0$, then we must have $\sigma_{-i}(t_{-i}) = \sigma_{-i}(t_{-i}') = a_{-i}$ because of condition (A-2); and $a_{-i} \in W_{-i}^1$ because $a_i \in W_i^2$. Thus, by the previous paragraph, t_j must have the same first order belief as t_j' for each $j \neq i$; this is precisely saying that t_i and t_i' have the same second order belief.

The general induction step is completely analogous to the above. □

Corollary 5.2. *For every $i \in N$ and $\mu \in \Delta(W_{-i})$, there can be at most one type in T_i having first order belief μ, i.e.* $\operatorname{marg}_{A_{-i}} \lambda_i(t_i) = \mu = \operatorname{marg}_{A_{-i}} \lambda_i(t_i') \Rightarrow t_i = t_i'$.

Proof. Suppose $\mu \in \Delta(W_{-i})$ and $\operatorname{marg}_{A_{-i}} \lambda_i(t_i) = \mu, t_i \in T_i$. The previous lemma together with conditions (A-2) and (A-3) implies that, for every $a_{-i} \in \operatorname{supp} \mu$, we have $\lambda_i(t_i)[t_{-i}, a_{-i}] = \mu(a_{-i})$, where t_j is the unique type in T_j such that $\sigma_j(t_j) = a_j$, $j \neq i$. Thus, by the non-redundancy of T_i, t_i is unique in T_i. □

Now, for each $i \in N$ and $a_i \neq a_i' \in W_i$, by the assumption of $Q_i = \sigma_i(T_i)$, there exists $t_i \neq t_i' \in T_i$ such that $\sigma_i(t_i) = a_i$ and $\sigma_i(t_i') = a_i'$. Because of condition (A-1), we have $\operatorname{marg}_{A_{-i}} \lambda_i(t_i) = \beta_i(a_i)$ and $\operatorname{marg}_{A_{-i}} \lambda_i(t_i') = \beta_i(a_i')$; and clearly $\beta_i(a_i)[W_{-i}] = \beta_i(a_i')[W_{-i}] = 1$. Then $\beta_i(a_i) \neq \beta_i(a_i')$, for otherwise the corollary above would imply that $t_i = t_i'$.

5.2. **If.** We will prove this direction by construction.

Suppose for each $i \in N$ and $a_i \in Q_i$, $\beta_i(a_i) \neq \emptyset$, and for each $i \in N$ and $a_i \neq a_i' \in W_i$, $\beta_i(a_i) \neq \beta_i(a_i')$.

For each $i \in N$, let

$$X_i = \{a_i \in Q_i \setminus W_i : |\beta_i(a_i)| = 1\}$$

$$Y_i = \{a_i \in Q_i \setminus W_i : |\beta_i(a_i)| > 1\}$$

Clearly, we have $Q_i = X_i \cup Y_i \cup W_i$. And note that we also have $X_i = W_i^1 \setminus W_i$ and $Y_i = Q_i \setminus W_i^1$.

For each $i \in N$ and $a_i \in Y_i$, fix $b(a_i) \neq c(a_i) \in \beta_i(a_i) \setminus \beta_i(W_i \cup X_i)$ such that $|\{b(a_i) : a_i \in Y_i\} \cup \{c(a_i) : a_i \in Y_i\}| = 2|Y_i|$. This is possible because $|\beta_i(a_i)| > 1 \Rightarrow |\beta_i(a_i)| = \infty$, for $\beta_i(a_i)$ is convex.

Notice that $\beta_i(X_i) \cap \beta_i(W_i) = \emptyset$: if $\beta_i(a_i) \in \beta_i(W_i)$ and $a_i \in X_i$, then clearly we must have $a_i \in W_i$, which cannot happen by construction.

For each $i \in N$, we let

$$T_i := \beta_i(W_i) \cup \bigcup_{\mu \in \beta_i(X_i)} \{\mu(1), \dots, \mu(K_\mu)\} \cup \bigcup_{a_i \in Y_i} \{b(a_i), c(a_i)\},$$

where $K_\mu = 2|\operatorname{BR}_i(\mu) \cap Q_i|$ for each $\mu \in \beta_i(X_i)$; and $\mu(1), \dots, \mu(K_\mu)$ are K_μ copies of μ.

The intuition and the essence of this type space is as follows. For every $a_i \in W_i$, type $\beta_i(a_i)$ will have first order belief $\beta_i(a_i)$. Likewise, for every $a_i \in Y_i$, type $b(a_i)$ (respectively, $c(a_i)$) will have first order belief $b(a_i)$ (respectively, $c(a_i)$). And for every $\mu \in \beta_i(X_i)$ and $1 \leq k \leq K_\mu$, type $\mu(k)$ will have first order belief μ.

By the way we picked these beliefs, we have that types in $T_i \setminus \bigcup_{\mu \in \beta_i(X_i)} \{\mu(1), \dots, \mu(K_\mu)\}$ are distinguished from each other by their first order beliefs. And for any fixed $\mu \in \beta_i(X_i)$, each $\mu(k)$, $1 \leq k \leq K_\mu$, is also distinguished from types in $T_i \setminus \{\mu(1), \dots, \mu(K_\mu)\}$ by its first order belief as well. We will define the belief of each $\mu(k)$ on other types such that $\mu(k)$'s are distinguished from each other by their higher order beliefs.

We first define the strategy $\sigma_i : T_i \to A_i$. For each $i \in N$, let $\sigma_i(\beta_i(a_i)) := a_i$ for every $a_i \in W_i$; this is where we used the assumption that β_i is injective among W_i. And let $\sigma_i(b(a_i)) = \sigma_i(c(a_i)) := a_i$ for every $a_i \in Y_i$. Finally, for every $\mu \in \beta_i(X_i)$, suppose $\operatorname{BR}_i(\mu) \cap Q_i = \{a(1), \dots, a(n)\}$ ($n = K_\mu/2$ by definition), let $\sigma_i(\mu(2k-1)) = \sigma_i(\mu(2k)) := a(k)$ for each $1 \leq k \leq K_\mu/2$.

Then clearly we have $Q_i = \sigma_i(T_i)$ for each $i \in N$.

And a final piece of construction before defining the beliefs λ_i. For every $i \in N$, let $t(a_i) := \beta_i(a_i)$ if $a_i \in W_i$, $t(a_i) := b(a_i)$ if $a_i \in Y_i$; let

$t(a_i) := \mu(k)$ and $s(a_i) := \mu(k+1)$ for $a_i \in X_i$, where $\mu = \beta_i(a_i)$ and $\sigma_i(\mu(k)) = \sigma_i(\mu(k+1)) = a_i$, $1 \leq k < K_\mu$.

For each $i \in N$, define the belief $\lambda_i : T_i \to \Delta(T_{-i} \times A_{-i})$ as follows.

For every $\alpha \in T_i \setminus \bigcup_{\mu \in \beta_i(X_i)} \{\mu(1), \ldots, \mu(K_\mu)\}$, let

$$\lambda_i(\alpha)[t_{-i}, a_{-i}] := \begin{cases} \alpha(a_{-i}) & t(a_j) = t_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases}$$

for all $t_{-i} \in T_{-i}$ and $a_{-i} \in Q_{-i}$. Then clearly, each type $\alpha \in T_i \setminus \bigcup_{\mu \in \beta_i(X_i)} \{\mu(1), \ldots, \mu(K_\mu)\}$ has a distinct first order belief α.

Now for any $\mu \in \beta_i(X_i)$, we must have $\mu(W_{-i}) < 1$, so there exists a smallest $l \geq 1$ such that $\operatorname{supp} \mu \not\subseteq W^l_{-i}$. Pick a $d_m \in Q_m$ such that $m \neq i$, $(\operatorname{marg}_{A_m} \mu)[d_m] > 0$ and $d_m \notin W^l_m$.

If $l = 1$ (i.e. $|\beta_m(d_m)| > 1$), then let

$$\lambda_i(\mu(k))[t_{-i}, a_{-i}] := \begin{cases} \mu(a_{-i}) & a_m \neq d_m, \text{ and } t(a_j) = t_j \text{ for all } j \neq i \\ \frac{k-1}{K_\mu - 1}\mu(a_{-i}) & a_m = d_m, t_m = b(a_m), \text{ and } t(a_j) = t_j \text{ for all } j \notin \{i, m\} \\ \frac{K_\mu - k}{K_\mu - 1}\mu(a_{-i}) & a_m = d_m, t_m = c(a_m), \text{ and } t(a_j) = t_j \text{ for all } j \notin \{i, m\} \\ 0 & \text{otherwise.} \end{cases}$$

for all $t_{-i} \in T_{-i}$, $a_{-i} \in Q_{-i}$ and $1 \leq k \leq K_\mu$.

If $l > 1$, let

$$\lambda_i(\mu(k))[t_{-i}, a_{-i}] := \begin{cases} \mu(a_{-i}) & a_m \neq d_m, \text{ and } t(a_j) = t_j \text{ for all } j \neq i \\ \frac{k-1}{K_\mu - 1}\mu(a_{-i}) & a_m = d_m, t_m = t(a_m), \text{ and } t(a_j) = t_j \text{ for all } j \notin \{i, m\} \\ \frac{K_\mu - k}{K_\mu - 1}\mu(a_{-i}) & a_m = d_m, t_m = s(a_m), \text{ and } t(a_j) = t_j \text{ for all } j \notin \{i, m\} \\ 0 & \text{otherwise.} \end{cases}$$

By induction on l, we can easily show that each $\mu(k)$, $1 \leq k \leq K_\mu$, has a distinct $(l+1)$th order belief.

Therefore, condition (A-3) is satisfied. It is readily checked that conditions (A-1) and (A-2) hold as well. And we have noted before that $Q_i = \sigma_i(T_i)$ for each $i \in N$.

6. Conclusion

We have characterized the set of strategies played under a s.e.i.c. and showed by an iterative procedure that it is always non-empty. Some interesting questions for the future include the relationship and interaction between intrinsic correlation and common prior, and intrinsic correlation in dynamic games.

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