Modeling Loss Risk in Loan Portfolios with Various Heterogeneity Factors

Osadchiy, Maksim

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Maksim Osadchiy[1]

This paper extends the classical Vasicek credit risk model by introducing a comprehensive multi-factor framework that simultaneously incorporates key sources of portfolio heterogeneity – namely, variations in asset weights, recovery rates, default probabilities, and asset correlations. By modeling the complex interactions among these factors, our approach provides a more realistic and nuanced assessment of loss distributions and risk measures. Monte Carlo simulations demonstrate that the extended Vasicek-style model yields accurate approximations of portfolio Value at Risk (VaR) across portfolios with diverse recovery profiles and moderate concentration levels. This advancement improves the precision of credit risk measurement, addresses current regulatory gaps, and offers a solid foundation for more sophisticated risk management of heterogeneous credit portfolios.

Keywords

Heterogeneous Credit Portfolios; Granularity Adjustment; Vasicek Model; Value at Risk; Monte Carlo Simulation

1 Introduction

The advancement of option pricing theory through the Black-Scholes model, along with the subsequent development of the firm liability valuation framework by Merton, paved the way for the first credit risk models that explicitly incorporated asset correlations – most notably the Vasicek (1987) model. This model, grounded in the Law of Large Numbers (LLN), assumes a perfectly granular portfolio composed of homogeneous exposures with equal weights and identical risk profiles across all assets. Under these idealized assumptions, the portfolio loss converges almost surely to its conditional expectation given the systematic risk factor, enabling a tractable analytical framework for credit risk assessment. However, in practical settings, the assumption of a perfectly homogeneous portfolio is unrealistic. Real-world credit portfolios are inherently heterogeneous, exhibiting variations in loan sizes, recovery rates, unconditional default probabilities, and asset correlations. Relying on models based on such oversimplified assumption leads to significant misestimations of credit risk. Despite these limitations, the Vasicek model has become a cornerstone of the AIRB (Advanced Internal Ratings-Based) approach under the Basel II and Basel III regulatory frameworks.

The overreliance of industry on the oversimplified Vasicek model, along with its closely connected Li copula model, contributed to the severity of the 2008 financial crisis. This dependence on simplified modeling frameworks contributed to a false sense of security regarding credit risk assessment and capital adequacy. This reliance significantly impacted the Vasicek model's reputation within the industry. In practice, banks do not employ the Vasicek model directly for portfolio construction due to its inability to capture heterogeneity effectively; instead, it is primarily used for regulatory reporting purposes.

Despite its compromised reputation, the Vasicek model can be substantially enhanced. Notably, extensions that relax the assumption of homogeneity and explicitly incorporate portfolio heterogeneity – such as varying exposure weights – offer promising avenues for more accurate risk modeling. The simplest such modification, introduced in the early 2000s, involves

[1] Independent researcher. E-mail: max1us@mail.ru

accounting for different exposure sizes within the portfolio, providing a more realistic representation of credit risk dynamics. The theoretical foundation for this approach is laid out in Gordy (2003), which demonstrates that, under mild regularity conditions, the loss in a large heterogeneous portfolio converges almost surely to its conditional expectation given the market factor. Gordy also highlighted the importance of the Herfindahl-Hirschman Index (HHI) as a key measure of granularity adjustment (GA). The mathematical basis for calculating GA to VaR was provided by Gouriéroux et al. (2000).

Emmer & Tasche (2005) obtained GA to VaR for both the general case of loss distribution and for the case of the Vasicek model.

Gordy & Lütkebohmert (2013) extended the CreditRisk+ model to account for recovery variability. They applied the formula of GA to VaR in the general case of the loss distribution, specifically addressing idiosyncratic LGD (Loss Given Default) uncertainty. In their approach, they assume that the default indicator variable follows a Poisson distribution. This extension allows for a more accurate modeling of risk by incorporating the variability in recoveries, enhancing the assessment of portfolio losses.

Voropaev (2011) then moved on to studying the behavior of the portfolio loss PDF and granularity adjustments to VaR and ES, using a moment-based method.

Osadchiy (2025) applied Lyapunov's Central Limit Theorem (CLT) for capturing granularity effects. This approach leads to the same formula for GA to VaR, as it was obtained by Gouriéroux et al. (2000) approach. Besides, this new approach provides a convenient platform for extensions of the portfolio loss models to take into account risks, connected with variability of unconditional default probabilities and asset correlations, and also differences in recovery rates.

This paper is organized as follows:

• Section 2 introduces our core model for heterogeneous credit portfolios, providing foundational background and detailing the construction of the loss distribution under portfolio heterogeneity.

• Section 3 delves into the classical Vasicek model framework, extending it to incorporate multiple sources of heterogeneity, such as asset weights, recovery rates, default probabilities, and asset correlations.

• Section 4 concludes the paper by summarizing key findings, discussing implications for risk management and regulation, and outlining potential avenues for future research.

2 Heterogeneous Portfolio Loss Model

Consider a portfolio comprising n loans, with weights $\{w_k\}_{k=1}^{n}$, satisfying the normalization constraints:

$$\sum_{k=1}^{n} w_k = 1, w_k \geq 0, \forall k = 1, \dots, n. \tag{2.1}$$

The degree of portfolio concentration is qualified by the Herfindahl-Hirschman Index (HHI):

$$h_n := \sum_{k=1}^{n} w_k{}^2, \tag{2.2}$$

where a higher HHI indicates greater concentration, thereby influencing the overall portfolio risk exposure.

The total portfolio loss L is modeled as a weighted sum of individual default indicators modulated by their respective loss severities (LGD):

$$L := \sum_{k=1}^{n} w_k \lambda_k l(X_k, Y; \vec{\omega}_k), \tag{2.3}$$

where the components are defined as follows:

• $l(X_k, Y; \vec{\omega}_k)$: indicator function of default for asset k, which equals 1 if asset k defaults, and 0 otherwise.

• Y: a systematic risk factor, capturing macroeconomic or sectoral influences. It is a random variable with CDF $F_Y(x)$ and PDF $f_Y(x)$.

• X_k: the idiosyncratic (asset-specific) risk for asset k, assumed independent across assets, with identical distribution across all k. It is a random variable with CDF $F_X(x)$ and PDF $f_X(x)$.

• $\vec{\omega}_k$: vector of indicator function parameters for asset k.

• λ_k: the LGD coefficient for asset k.

It is important to note that the LGD coefficients λ_k are latent variables (unobservable ex ante) and become observable only ex post. The indicator function parameters $\vec{\omega}_k$ are never directly observable. Distributions of λ_k and $\vec{\omega}_k$ can be inferred from econometric data. We assume that the random variables λ and $\vec{\omega}$ are assumed mutually independent and independent of other model random variables.

As the portfolio size $n \to \infty$, the distribution of the portfolio loss conditioned on the systematic factor Y converges, according to the Lyapunov Central Limit Theorem (Billingsley, 1995), to a systematic-plus-residual form:

$$L|Y \xrightarrow{d} \mathbb{E}[\lambda]\{V + \sqrt{h} c \sigma(V/c) Z\}, \tag{2.4}$$

where:

• $V := G(Y) := \mathbb{E}[l(X, Y; \vec{\omega})|Y]$ is a random variable with CDF F_V and PDF f_V supported on $[0,1]$. The function $G(Y)$ assumed to be monotonic in Y.

• $\sigma(x) := \sqrt{x(1-x)}$ captures binomial variance.

• $c := 1 + \frac{var[\lambda]}{(\mathbb{E}[\lambda])^2}$ reflects heterogeneity in LGD.

• $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable, independent of V, representing *granularity risk*.

• $h = \lim_{n\to\infty} h_n$: the asymptotic Herfindahl-Hirschman Index.

Proof details are provided in Appendix 1.

In the particular case where all LGDs are unity (no recovery), $\lambda = 1$, the asymptotic loss distribution simplifies to:

$$L|Y \xrightarrow{d} V + \sqrt{h}\sigma(V)Z, \tag{2.5}$$

recovering the base model described in Osadchiy (2025).

2.1 The CDF of the Portfolio Loss

The CDF of the portfolio loss is given by the convolution:

$$F(x; h, \mathbb{E}[\lambda], c) = \mathbb{P}[L < x] = \mathbb{P}\left[V + \sqrt{h}c\sigma(V/c)Z < \frac{x}{\mathbb{E}[\lambda]}\right] = \int_0^1 \Phi\left(\frac{\frac{x}{\mathbb{E}[\lambda]} - v}{\sqrt{h}c\sigma(v/c)}\right) dF_V(v) \tag{2.6}$$

Given that $0 < z \ll 1$, we expand $\Phi\left(u/\sqrt{z}\right)$ into a Taylor series around $z = 0$:

$$\Phi\left(u/\sqrt{z}\right) = \theta(u) + \sum_{k=1}^{\infty} (z/2)^k \frac{\delta^{(2k-1)}(u)}{k!} \tag{2.7}$$

where:

• $\delta^{(n)}(u)$ denotes the n-th derivative of the Dirac delta function,

• $u = \frac{x}{\mathbb{E}[\lambda]} - v$,

• $z = hc^2\sigma^2(v/c)$.

See proof in Osadchiy (2025).

Hence, the CDF can be written as:

$$\begin{aligned} F(x; h, \mathbb{E}[\lambda], c) &= \left\{F_V(v) + \sum_{k=1}^{\infty} \frac{(hc^2/2)^k}{k!} \frac{\partial^{2k-1}}{\partial v^{2k-1}}\left(\sigma^{2k}(v/c) f_V(v)\right)\right\}\Bigg|_{v=\frac{x}{\mathbb{E}[\lambda]}} \\ &= \left\{F_V(v) + \frac{hc^2}{2}\frac{\partial}{\partial v}\left(\sigma^2(v/c) f_V(v)\right)\right\}\Bigg|_{v=\frac{x}{\mathbb{E}[\lambda]}} + o(hc^2/2) \end{aligned} \tag{2.8}$$

Proof details are provided in Appendix 2.

First-order approximation yields:

$$F(x; h, \mathbb{E}[\lambda], c) = \left\{F_V(v) + \frac{hc^2}{2}\frac{\partial}{\partial v}\left(\sigma^2(v/c) f_V(v)\right)\right\}\Big|_{v=\frac{x}{\mathbb{E}[\lambda]}} + o(hc^2/2).$$

$$(2.9)$$

2.2 The Value at Risk

The Value at Risk (VaR) at confidence level α, denoted as $VaR_\alpha(L) = x(\alpha, h, \mathbb{E}[\lambda], c)$, is defined as the $(1-\alpha)$-quantile of the loss distribution:

$$1 - \alpha = F(x(\alpha, h, \mathbb{E}[\lambda], c); h, \mathbb{E}[\lambda], c). \tag{2.10}$$

Since a closed-form solution generally does not exist, we approximate $x(\alpha, h, \mathbb{E}[\lambda], c)$ via a first order Taylor expansion around $h = 0$:

$$VaR_\alpha(L) = x(\alpha, h, \mathbb{E}[\lambda], c) = \mathbb{E}[\lambda]x(\alpha) + \left.\frac{\partial x(\alpha,h,\mathbb{E}[\lambda],c)}{\partial h}\right|_{h=0} h + o(h), \tag{2.11}$$

where the homogenous portfolio loss VaR, with $\mathbb{E}[\lambda] = 1$, is given by:

$$x(\alpha) = x(\alpha, h = 0, \mathbb{E}[\lambda] = 1, c). \tag{2.12}$$

This quantity is the root of:

$$1 - \alpha = F_V\big(x(\alpha)\big), \tag{2.13}$$

and relates to the general case as:

$$x(\alpha, h = 0, \mathbb{E}[\lambda], c) = \mathbb{E}[\lambda]x(\alpha). \tag{2.14}$$

Using technics from Osadchiy (2025), the derivative at $h = 0$ is derived as:

$$\left.\frac{\partial x(\alpha,h,\mathbb{E}[\lambda],c)}{\partial h}\right|_{h=0} = -\frac{c^2}{2}\mathbb{E}[\lambda]\left.\frac{\frac{\partial}{\partial v}[\sigma^2(v/c)f_V(v)]}{f_V(v)}\right|_{v=x(\alpha)}. \tag{2.15}$$

Detailed proof is provided in Appendix 3.

Using the above result, the granularity adjustment to VaR is:

$$GA^{VaR} = -\frac{hc^2}{2}\mathbb{E}[\lambda]\left.\frac{\frac{\partial}{\partial v}[\sigma^2(v/c)f_V(v)]}{f_V(v)}\right|_{v=x(\alpha)}. \tag{2.16}$$

Applying a change of measure, as discussed in Osadchiy (2025), yields an alternative form:

$$GA^{VaR} = -\frac{hc^2}{2}\mathbb{E}[\lambda]\left.\frac{\frac{\partial}{\partial y}\left[\sigma^2(G(y)/c)\frac{f_Y(y)}{|G'(y)|}\right]}{f_Y(y)}\right|_{y=y(\alpha)}, \tag{2.17}$$

where:

$$y(\alpha) = \begin{cases} F_Y{}^{-1}(1-\alpha), & \text{if } G(y) \text{ is increasing} \\ F_Y{}^{-1}(\alpha) & \text{if } G(y) \text{ is decreasing} \end{cases}$$

(2.18)

3 The Vasicek-Style Model

In the Vasicek-style portfolio loss model, there are 4 heterogeneity risk factors (if number of assets $n \to \infty$):

- Variation in asset weights w_i,
- Variation in LGD coefficients λ_i,
- Variation in default probabilities p_i,
- Variation in correlation coefficients ρ_i.

The classical Vasicek model assumes a *homogeneous* portfolio with parameters w_i, λ_i, p_i, and ρ_i identical across assets and $\lambda_i = 1$ for each i. These assumptions are highly idealized, making such portfolios practically unattainable. Still, this model underpins the AIRB (Advanced Internal Ratings-Based) approach in Basel II and Basel III.

The literature has extensively studied the risk of granularity when weights w_i differ, but default probabilities p_i and correlation coefficients ρ_i are assumed uniform.

The purpose of our research is to construct the unified model for evaluation of all 4 heterogeneity risks in credit portfolios.

The portfolio loss L in the Vasicek-style model

$$L = \sum_{k=1}^{n} w_k \lambda_k \theta \left(\Phi^{-1}(p_k) - \left(\sqrt{\rho_k} Y + \sqrt{1-\rho_k} X_k \right) \right)$$

(3.1)

where:

- $\theta(\cdot)$ is the Heaviside step function, which equals 1 if its argument is positive, and 0 otherwise.

$Y \sim N(0,1)$: a systematic risk factor, capturing macroeconomic or sectoral influences.

- $X_k \sim N(0,1)$: an idiosyncratic (asset-specific) risk factor for asset k, assumed independent across assets.

Rearranging L, we have:

$$L = \sum_{k=1}^{n} w_k \lambda_k \theta \left(\frac{\Phi^{-1}(p_k) - \sqrt{\rho_k} Y}{\sqrt{1-\rho_k}} - X_k \right)$$

(3.2)

Assuming that the default probabilities p and correlation coefficients ρ are independent random variables, we can model their distributions accordingly. For example, they may follow Beta distributions, as the support for both variables is the interval [0,1].

Consequently, it holds:

$$V = G(Y) = \mathbb{E}_{p,\rho}\left[\Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}Y}{\sqrt{1-\rho}}\right)\right], \tag{3.3}$$

where the expectation is taken over the joint distribution of p and ρ.

3.1 Special Case: Constant p and ρ

If p and ρ are constants, the distribution of V reduces to the Vasicek loss PDF. Specifically,

$$f_V(v;p,\rho) := \sqrt{\frac{1-\rho}{\rho}}exp\left\{-\frac{1}{2\rho}\left(\sqrt{1-\rho}\Phi^{-1}(v)-\Phi^{-1}(p)\right)^2+\frac{1}{2}\left(\Phi^{-1}(v)\right)^2\right\}. \tag{3.4}$$

The derivative of its logarithm with respect to v is:

$$\frac{\partial lnf_V(v;p,\rho)}{\partial v} = \frac{(2\rho-1)\Phi^{-1}(v)+\sqrt{1-\rho}\Phi^{-1}(p)}{\rho\varphi\left(\Phi^{-1}(v)\right)}, \tag{3.5}$$

where $\varphi(\cdot)$ is the standard normal PDF.

The VaR of homogenous portfolio loss with no recovery at level α:

$$x(\alpha) = \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right). \tag{3.6}$$

The GA to VaR, incorporating the effect of heterogeneity and recovery variability, is given by:

$$GA^{VaR} = -\frac{h\mathbb{E}[\lambda]}{2}\left\{c-2v+[(c-v)v]\frac{(2\rho-1)\Phi^{-1}(v)+\sqrt{1-\rho}\Phi^{-1}(p)}{\rho\varphi\left(\Phi^{-1}(v)\right)}\right\}\Bigg|_{v=x(\alpha)}. \tag{3.7}$$

Figure 1 highlights how incorporating recovery variability influences the tail risk measure of the portfolio. The accuracy of the first-order Taylor series VaR approximation across different levels of HHI up to 0.3, noted by Osadchiy (2025), remains consistent when considering the randomness of λ. The downward branches of outliers of the simulated data result from the presence of large loans; for example, when h=0.5, the weight of the largest loan is 0.2, and when h=0.8, it is approximately 0.099. The details of the simulation methodology are described in Osadchiy (2025).

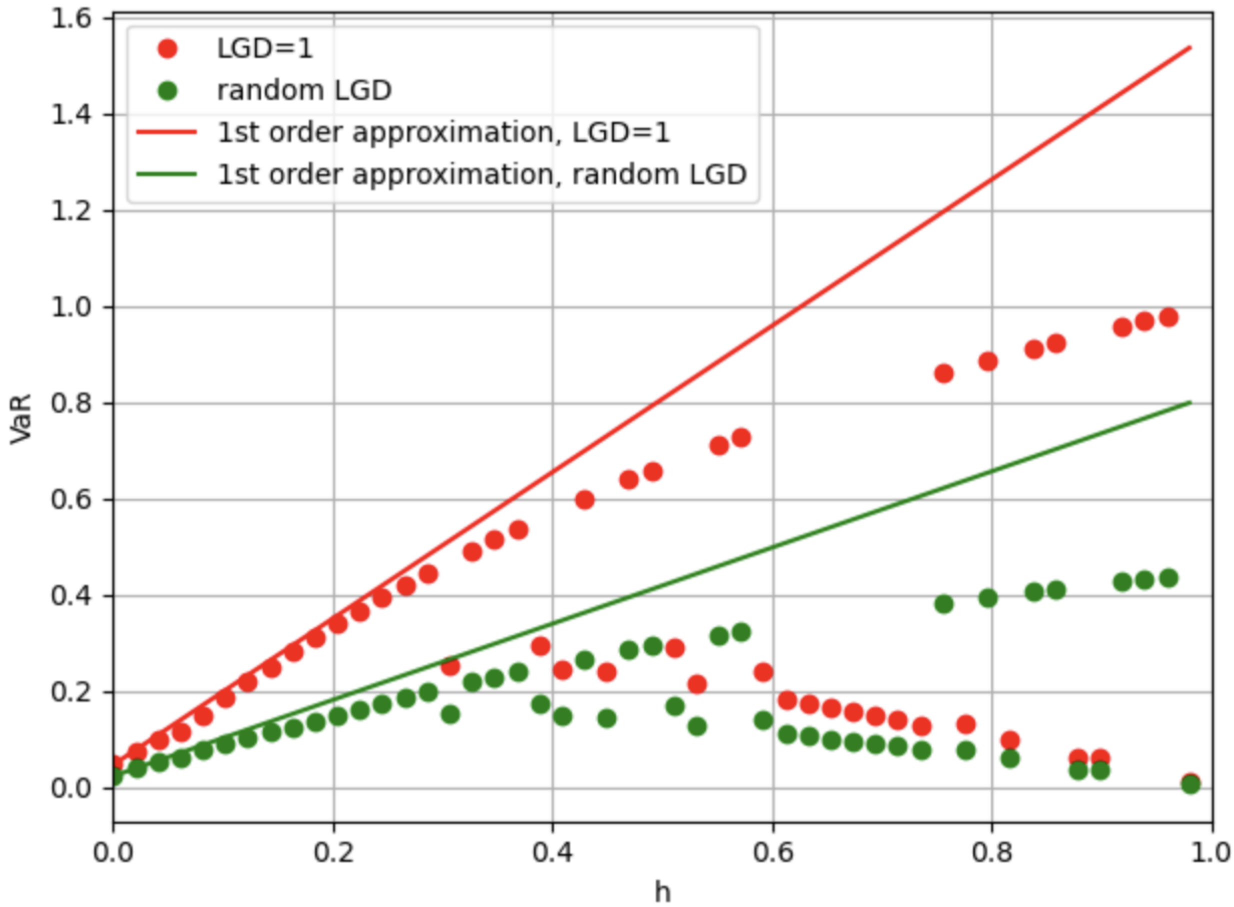

Figure 1. Impact of recovery risk on the VaR of a heterogeneous portfolio loss.

Red circles represent the simulated $VaR_{\alpha}(L)$ assuming no recovery. Green circles show the simulated $VaR_{\alpha}(L)$ when the LGD coefficient λ is modeled as a random variable with a $Beta(12, 12)$ distribution, which has a mean of 0.5 and variance of 0.01.

The red line shows the first-order Taylor series approximation of VaR in the no-recovery scenario. The green line presents the analogous approximation considering the randomness of λ. Monte Carlo simulation involved 20 000 runs, with parameters set at default probability $p = 0.01$, correlation $\rho = 0.1$, portfolio size $n = 20\,000$, and confidence level $\alpha = 0.01$.

4 Conclusion

This paper critically revisits the classical Vasicek model, which assumes a homogeneous credit portfolio characterized by identical exposure weights, recovery rates, default probabilities, and correlation parameters across all assets. While this simplification facilitates analytical solutions, it inadequately reflects the heterogeneity inherent in real-world portfolios, potentially leading to misestimated credit risk. Despite its limitations, the Vasicek model remains a cornerstone of the AIRB (Advanced Internal Ratings-Based) approach under Basel II and Basel III frameworks.

Building upon this foundation, prior research has identified four primary sources of heterogeneity risk: asset weight variability (granularity), differences in recovery rates, default probabilities, and correlation coefficients. However, existing studies predominantly focus on the impact of granularity and, to a lesser extent, recovery rates, leaving the combined effects of multiple heterogeneity factors largely unexamined.

This study extends the work of Osadchiy (2025), which utilized Lyapunov's Central Limit Theorem to examine granularity effects in portfolios characterized by heterogeneous loan sizes. While Osadchiy (2025) focused solely on a single heterogeneity factor – specifically, variations in asset weights – this research broadens the scope significantly. It addresses the more general case of heterogeneous indicator function parameters and enhances the Vasicek model to incorporate multiple sources of heterogeneity, including variations in recovery rates, default probabilities, and correlation parameters. By doing so, this work offers a more comprehensive framework for understanding the complex interplay of diverse portfolio characteristics and their impact on credit risk assessment.

This study introduces, for the first time, a closed-form formula for the granularity adjustment (GA) to the Value-at-Risk (VaR) of portfolio losses within the Vasicek framework, explicitly accounting for heterogeneity in both asset weights and LGD coefficients. Monte Carlo simulations demonstrate that, up to HHI h values of approximately 0.2, the first-order Taylor series expansion of the portfolio loss VaR around $h = 0$ provides highly accurate estimates. Furthermore, this approximation continues to give reasonable results even as h approaches 0.3 (see details in the caption of Figure 1).

These findings underscore the importance of accounting for multiple sources of heterogeneity in credit risk models and offer practical tools for improved risk quantification in regulatory and risk management contexts.

5 Appendix 1

Lyapunov CLT

In this section, we analyze the asymptotic distribution of the portfolio loss function conditional on an external shock Y:

$$L|Y := \sum_{k=1}^{n} w_k \lambda_k l(X_k, Y; \vec{\omega}_k), \tag{5.1}$$

by applying Lyapunov's Central Limit Theorem (CLT). Our goal is to establish the conditions under which $L|Y$ converges in distribution to a Gaussian random variable as $n \to \infty$.

Let $\{\xi_k\}_{k=1}^{n}$ be a sequence of independent random variables, where each ξ_k with finite mean μ_k and variance ${\sigma_k}^2$. Define the total variance:

$${s_n}^2 := \sum_{k=1}^{n} {\sigma_k}^2. \tag{5.2}$$

Lyapunov's CLT states that if, for some $\delta > 0$,

$$\lim_{n\to\infty} \frac{1}{{s_n}^{2+\delta}} \sum_{k=1}^{n} \mathbb{E}\left[|\xi_k - \mu_k|^{2+\delta}\right] = 0, \tag{5.3}$$

then the normalized sum tends toward a standard normal distribution:

$$\frac{1}{s_n} \sum_{k=1}^{n} (\xi_k - \mu_k) \xrightarrow{d} \mathcal{N}(0,1). \tag{5.4}$$

In our model, the summands are:

$$\xi_k = w_k \lambda_k l(X_k, Y; \vec{\omega}_k). \tag{5.5}$$

The conditional mean of ξ_k given Y is:

$$\mu_k = w_k \mathbb{E}[\lambda_k l(X_k, Y; \vec{\omega}_k)|Y] = w_k \mathbb{E}[\lambda] G(Y), \tag{5.6}$$

where

$$G(Y) := \mathbb{E}[l(X_k, Y; \vec{\omega}_k)|Y] \tag{5.7}$$

The conditional variance of ξ_k given Y is:

$$\sigma_k{}^2 = w_k{}^2 \{\mathbb{E}[\lambda_k{}^2] \mathbb{E}[l^2(X_k, Y; \vec{\omega}_k)|Y] - (\mathbb{E}[\lambda_k l(X_k, Y; \vec{\omega}_k)|Y])^2\} = w_k{}^2 \{\mathbb{E}[\lambda^2] V - (\mathbb{E}[\lambda] V)^2\} \tag{5.8}$$

where

$$V := G(Y) \tag{5.9}$$

The total variance sum can be expressed as:

$$s_n{}^2 = h(\mathbb{E}[\lambda^2] V - (\mathbb{E}[\lambda] V)^2) = h(\mathbb{E}[\lambda])^2 \left(\frac{\mathbb{E}[\lambda^2]}{(\mathbb{E}[\lambda])^2} V - V^2 \right) = h(\mathbb{E}[\lambda])^2 c^2 \sigma^2 (V/c) \tag{5.10}$$

where

$$h := \lim_{n \to \infty} h_n$$

- $h := \lim_{n \to \infty} h_n$,
- $c := 1 + \frac{var[\lambda]}{(\mathbb{E}[\lambda])^2}$.

Applying Lyapunov's CLT, the standardized loss conditioned on Y converges in distribution:

$$\frac{L|Y - \mathbb{E}[\lambda] V}{\sqrt{h_n} c \mathbb{E}[\lambda] \sigma(V/c)} \xrightarrow{d} \mathcal{N}(0,1) \tag{5.11}$$

Rearranging, the asymptotic distribution of $L|Y$ is:

$$L|Y \xrightarrow{d} \mathbb{E}[\lambda] \{V + \sqrt{h} c \sigma(V/c) Z\} \tag{5.12}$$

where $V, Z \sim \mathcal{N}(0,1)$ are independent standard normal variables.

Q.E.D.

Proof of the Lyapunov Condition

The applicability of our approach is constrained by the limits of the Lyapunov CLT. Let $\delta = 1$. We need to verify:

$$\lim_{n\to\infty} \frac{1}{{s_n}^3} \sum_{i=1}^{n} \mathbb{E}[|\xi_i - \mu_i|^3] = 0$$
(5.13)

which is equivalent to:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}} = 0$$
(5.14)

Proof.

We start with the expression:

$$\begin{aligned}\mathbb{E}[|\xi_i - \mu_i|^3|Y] &= {w_i}^3\mathbb{E}[\lambda^3]\mathbb{E}[|l(X_k, Y; \vec{\omega}_k) - G(Y)|^3|Y] \\ &= {w_i}^3\mathbb{E}[\lambda^3]G(Y)(1 - G(Y))\big((1 - G(Y))^2 + G^2(Y)\big)\end{aligned}$$
(5.15)

Thus, we have:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{{s_n}^3} \sum_{i=1}^{n} \mathbb{E}\,[|\xi_i - \mu_i|^3|Y] &= \frac{\mathbb{E}[\lambda^3]}{(\mathbb{E}[\lambda^2])^{3/2}} \frac{\mathbb{E}[|l(X_k, Y; \vec{\omega}_k) - G(Y)|^3|Y]}{\sigma^3(G(Y))} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{(\sum_{i=1}^{n} {w_i}^2)^{3/2}} \\ &= \frac{\mathbb{E}[\lambda^3]}{(\mathbb{E}[\lambda^2])^{3/2}} \frac{(1 - G(Y))^2 + G^2(Y)}{\sqrt{G(Y)(1 - G(Y))}} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} {w_i}^3}{\left(\sum_{j=1}^{n} {w_j}^2\right)^{3/2}}\end{aligned}$$
(5.16)

Q.E.D.

6 Appendix 2

Approximation of the Portfolio Loss Distribution

The cumulative distribution function (CDF) of the portfolio loss L is given by:

$$\mathbb{P}[L < x] = F(x; h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]) = \mathbb{P}\left[V + \sqrt{h}c\sigma(V/c)Z < \frac{x}{\mathbb{E}[\lambda]}\right] = \int_0^1 \Phi\left(\frac{\frac{x}{\mathbb{E}[\lambda]} - v}{\sqrt{h}c\sigma(v/c)}\right) dF_V(v)$$
(6.1)

Given that $0 < z \ll 1$, expand $\Phi\big(u/\sqrt{z}\big)$ into a Taylor series around $z = 0$:

$$\Phi\big(u/\sqrt{z}\big) = \theta(u) + \sum_{k=1}^{\infty} \left(\frac{z}{2}\right)^k \frac{\delta^{(2k-1)}(u)}{k!},$$
(6.2)

where:

- $\delta^{(n)}(\cdot)$ denotes the n-th derivative of the Dirac delta function,

- $\theta(\cdot)$ is the Heaviside step function, which equals 1 if its argument is positive, and 0 otherwise,

• $u = \frac{x}{\mathbb{E}[\lambda]} - v$,

• $z = hc^2\sigma^2(v/c)$.

Substituting the expansion into (6.1), we obtain:

$$F = F_V\left(\frac{x}{\mathbb{E}[\lambda]}\right) + \sum_{k=1}^{\infty} \frac{\left(hc^2/2\right)^k}{k!} \int_0^1 \delta^{(2k-1)}\left(\frac{x}{\mathbb{E}[\lambda]} - v\right) \sigma^{2k}(v/c) f_V(v) dv. \tag{6.3}$$

Utilizing the properties of derivatives of the delta distribution, specifically:

$$\delta^{(k)}(a - v) = (-1)^k \delta^{(k)}(v - a) \tag{6.4}$$

and the integral identity:

$$\int_{-\infty}^{+\infty} \delta^{(k)}(v - a) f(v) dv = (-1)^k f^{(k)}(a), \tag{6.5}$$

we arrive at the simplified form:

$$F = \left\{ F_V(v) + \sum_{k=1}^{\infty} \frac{\left(hc^2/2\right)^k}{k!} \frac{\partial^{2k-1}}{\partial v^{2k-1}} \left(\sigma^{2k}(v/c) f_V(v)\right) \right\} \Bigg|_{v=\frac{x}{\mathbb{E}[\lambda]}}. \tag{6.6}$$

7 Appendix 3

Granularity Adjustment of VaR

The Value at Risk at confidence level α, denoted as $VaR_\alpha(L) = x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])$, is defined as the quantile $q_{1-\alpha}(L)$ satisfying:

$$1 - \alpha = F(x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]); h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]). \tag{7.1}$$

The numerical value of this root can be readily determined using known parameters.

Since a closed-form solution generally does not exist, we approximate the VaR via a first-order Taylor expansion around $h = 0$:

$$VaR_\alpha(L) = x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]) = \mathbb{E}[\lambda] x(\alpha) + \frac{\partial x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial h}\Bigg|_{h=0} h + o(h), \tag{7.2}$$

where $x(\alpha)$ is the VaR in the no-granularity limit:

$$x(\alpha) = x(\alpha, h = 0, \mathbb{E}[\lambda] = 1, \mathbb{E}[\lambda^2]) \tag{7.3}$$

solving

$$1 - \alpha = F_V\big(x(\alpha)\big) \tag{7.4}$$

and where

$$x(\alpha, h = 0, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]) = \mathbb{E}[\lambda]x(\alpha). \tag{7.5}$$

Indeed,

$$1 - \alpha = \mathbb{P}[L < x; h = 0] = F(x; h = 0, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]) = \mathbb{P}\left[V < \frac{x}{\mathbb{E}[\lambda]}\right] = F_V\left(\frac{x}{\mathbb{E}[\lambda]}\right). \tag{7.6}$$

Differentiating the equation (7.1) with respect to the parameter h, we obtain:

$$\frac{\partial x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial h} \frac{\partial}{\partial v} F(v; h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])\bigg|_{v=x(\alpha,h,\mathbb{E}[\lambda],\mathbb{E}[\lambda^2])} + \frac{\partial}{\partial h} F(v; h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])\bigg|_{v=x(\alpha,h,\mathbb{E}[\lambda],\mathbb{E}[\lambda^2])} = 0 \tag{7.7}$$

Since

$$\frac{\partial F(v; h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial v}\bigg|_{h=0} = \frac{1}{\mathbb{E}[\lambda]} f_V\left(\frac{v}{\mathbb{E}[\lambda]}\right) \tag{7.8}$$

$$\frac{1}{\mathbb{E}[\lambda]} \frac{\partial x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial h}\bigg|_{h=0} f_V\left(\frac{v}{\mathbb{E}[\lambda]}\right)\bigg|_{v=x(\alpha,h=0,\mathbb{E}[\lambda],\mathbb{E}[\lambda^2])} + \frac{c^2}{2} \frac{\partial}{\partial v}[\sigma^2(v/c) f_V(v)]\bigg|_{v=\frac{x(\alpha,h=0,\mathbb{E}[\lambda],\mathbb{E}[\lambda^2])}{\mathbb{E}[\lambda]}} = 0 \tag{7.9}$$

$$x(\alpha, h = 0, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2]) = \mathbb{E}[\lambda]x(\alpha) \tag{7.10}$$

$$\frac{1}{\mathbb{E}[\lambda]} \frac{\partial x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial h}\bigg|_{h=0} f_V\big(x(\alpha)\big) + \frac{c^2}{2} \frac{\partial}{\partial v}[\sigma^2(v/c) f_V(v)]\bigg|_{v=x(\alpha)} = 0 \tag{7.11}$$

$$\frac{\partial x(\alpha, h, \mathbb{E}[\lambda], \mathbb{E}[\lambda^2])}{\partial h}\bigg|_{h=0} = -\frac{c^2}{2} \mathbb{E}[\lambda] \frac{\frac{\partial}{\partial v}[\sigma^2(v/c) f_V(v)]}{f_V(v)}\bigg|_{v=x(\alpha)} \tag{7.12}$$

Using the above result, the granularity adjustment to VaR is:

$$GA^{VaR} = -\frac{hc^2}{2} \mathbb{E}[\lambda] \frac{\frac{\partial}{\partial v}[\sigma^2(v/c) f_V(v)]}{f_V(v)}\bigg|_{v=x(\alpha)} \tag{7.13}$$

Expressed explicitly, using $\sigma^2(v/c) = \frac{v}{c}\left(1 - \frac{v}{c}\right)$:

$$GA^{VaR} = -\frac{hc^2}{2}\mathbb{E}[\lambda]\frac{\frac{\partial}{\partial v}\left[\frac{v}{c}\left(1-\frac{v}{c}\right)f_V(v)\right]}{f_V(v)}\Bigg|_{v=x(\alpha)} \tag{7.14}$$

Further expanding derivatives yields the final, explicit form:

$$GA^{VaR} = -\frac{h\mathbb{E}[\lambda]}{2}\left\{\frac{\partial}{\partial v}\left[\left(\frac{\mathbb{E}[\lambda^2]}{(\mathbb{E}[\lambda])^2}-v\right)v\right]+\left[\left(\frac{\mathbb{E}[\lambda^2]}{(\mathbb{E}[\lambda])^2}-v\right)v\right]\frac{\partial lnf_V(v)}{\partial v}\right\}\Bigg|_{v=x(\alpha)} \tag{7.15}$$

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