Sustainability of public debt, investment subsidies, and endogenous growth with heterogeneous firms and financial frictions

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Abstract

This study investigates the effect of public debt on growth, interest rates, and fiscal sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. Increases in public debt lead to higher real interest rates through financial markets, increase the cost of repaying public debt, and reduce private investment, resulting in lower long-run growth. Thus, large public debt is less sustainable. This study also examines the effect of investment subsidies financed by public debt and finds that they can hinder economic growth in the long run unless the financial market is close to perfect. Therefore, increases in investment subsidies should be financed not only by issuing public bonds, but also through tax increases. Moreover, the impact of these fiscal policies on inequality among agents is briefly discussed.

JEL classification: E62; H20; H60

Keywords: Sustainability of public debt, Financial frictions, Firm heterogeneity, Investment subsidies

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1 Introduction

Public debt has increased significantly in many developed countries, raising concerns among policymakers about how to ensure its sustainability.[1] Higher economic growth and lower interest rates are important factors in stabilizing the debt-to-GDP ratio. However, economic growth, interest rates, and public debt accumulation are determined by dynamic processes and are interrelated, making it difficult to assess the sustainability of public debt. Indeed, no consensus exists on what constitutes sustainable debt (e.g., D'Erasmo et al., 2016). Ramsey-type models with representative infinitely lived agents show that public debt cannot be sustainable if the government violates its transversality condition (e.g., Greiner, 2007, 2011, 2012, 2015; Kamiguchi and Tamai, 2012). Many empirical studies have tested whether the transversality condition holds (e.g., Afonso, 2005; Bohn, 1998; Hamilton and Flavin, 1986). Meanwhile, overlapping generations (OLG) models show that the government removes this constraint and can run a Ponzi game. Acknowledging the possibility of a public Ponzi game, many recent studies have analyzed fiscal sustainability in OLG models and defined fiscal sustainability as whether the ratio of public debt to GDP (or capital) converges to a stable level in the long run (Agénor and Yilmaz, 2017; Arai, 2011; Bräuninger, 2005; Chalk, 2000; Maebayashi and Konishi, 2021; Teles and Mussolini, 2014; Yakita, 2008, 2014).

Despite this large literature, to the best of my knowledge, few studies have examined long-run economic growth, interest rates, and the sustainability of public debt simultaneously. Moreover, although the abovementioned studies on the sustainability of public debt employ different types of models, they all assume that agents are homogeneous and financial markets are perfect.

This study first contributes to the literature by investigating the effect of public debt on growth, interest rates, and sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. Drawing on the recent literature, we also incorporate the Pareto distribution of firms' productivity into the impact of fiscal policy on growth when firms are heterogeneous (e.g., Arawatari et al., 2023; Jaimovich and Rebelo, 2017; Mino, 2015, 2016).

Under financial frictions and firm heterogeneity, only high-productivity firms can borrow in the financial market. Low-productivity firms cannot borrow because of their credit constraints. Low-productivity agents then become lenders to high-productivity entrepreneurs (borrowers) and buyers of government bonds. A rise in government debt increases the issuance of government bonds and can have the following opposing effects on growth. First, it reduces the aggregate supply of credit in the financial market, thereby raising the interest rate. A rise in the interest rate reduces the number of less productive firms by increasing the cost of borrowing for entrepreneurs, which causes reallocations towards more efficient firms (e.g., Melitz, 2003). We refer to this as the

[1] One way to address this problem would be fiscal consolidation efforts, especially in the European Union (EU).

positive firm selection effect. It enhances aggregate productivity and economic growth. Second, an increase in government bonds increases lenders' assets through bequests from their parents. These assets then flow into investments in high-productivity firms and promote economic growth. We refer to this as the *liquidity effect*. Finally, a rise in government debt crowds out private investments and negatively affects economic growth. Therefore, the overall impact of government debt on economic growth depends on the relative strength of these factors.

The second contribution of this study is to examine the growth effect of investment subsidies financed by public debt. Governments offer various investment subsidies, including investment cost subsidies, research and development (R&D) subsidies, investment tax reductions, and direct investment grants (Kang, 2022). These investment subsidies are recognized as instruments for boosting economic growth and improving the fiscal situation.[2] The World Bank (2023) shows that R&D expenditure in the United States increased from 2.67% to 3.46% of GDP between 2012 and 2021.[3] The public finances of these developed countries have relied heavily on public debt. However, to the best of my knowledge, the growth-enhancing effect of investment subsidy policy with heterogeneous firms has been studied only in the case of a balanced government budget. Therefore, it is important to examine whether investment subsidies financed by public debt can promote growth and improve the fiscal situation.

Herein, I develop an OLG model with endogenous growth, firm heterogeneity, and financial frictions. In the young period, agents earn wage income, receive bequests from their parents, and draw productivity levels from the Pareto distribution. Agents who draw higher (lower) productivity than the cutoff level become entrepreneurs (lenders) and engage in (refrain from) production in old age. Additionally, the government issues public bonds to finance investment subsidies. The cutoff level of productivity is endogenously determined and it depends on fiscal policy.

The main findings of this study are as follows. First, an increase in the public debt-to-capital ratio has opposing effects on economic growth as mentioned above. One is the negative growth effect through the crowding-out effect of public debt on investment. Next is the positive growth effect resulting from the liquidity effect. The last is the positive effect stemming from the positive firm selection effect. The negative growth effect dominates the positive effects. Therefore, an increase in the public debt-to-capital ratio hinders economic growth.

Second, a rise in the interest rate due to an increase in public debt raises interest payments and the growth in public debt. When the public debt-to-capital ratio is sufficiently high, the negative effect of increased public debt on economic growth combined with the effect of rising interest

[2]Policy instruments to support young, growing, and innovative companies are also expected to boost investment and economic growth (European Commission, 2017).

[3]The government budget allocations for R&D across the EU reached 0.74% of GDP, showing an increase of 5.4% compared with 2021 and an increase of 49.2% compared with 2012 (Eurostat Statistics Explained, 2024).

rates on further increasing public debt makes fiscal policy unsustainable.

Third, investment subsidies to firms financed by public debt makes public debt less sustainable. Investment subsidies encourage investment by each firm and promote economic growth. However, they also lower the barrier to becoming an entrepreneur and increase the number of less productive firms, decreasing aggregate productivity and economic growth through the *negative firm selection effect*. Moreover, the increase in these firms increases aggregate demand for credit in the financial market, putting upward pressure on the interest rate. This raises the cost of repaying public debt and accelerates its accumulation. Investment subsidies financed by debt itself also increase the issuance of public bonds and accelerate the accumulation of public debt. When the ratio of public debt to capital is large, the positive effects of investment subsidies on the growth in public debt surpass those on economic growth, making public debt less sustainable.

Finally, increases in the investment subsidy financed by public debt decrease (resp. increase) long-run economic growth when the financial market is not so perfect (resp. is so perfect). Increases in the investment subsidy financed by public debt encourage investment and increase the aggregate demand for credit in the financial market. This raises the interest rate and enhances economic growth through the positive firm selection effect. However, increases in the interest rate raise the cost of repaying public debt, crowding out private investment and decreasing economic growth. Investment subsidies also lower economic growth through the negative firm selection effect, as explained in the results presented previously. The former positive growth effects strengthen when the financial market is more perfect. This is because a larger degree of financial perfection leads to a looser credit constraint and makes borrowing easier (i.e., an increase in the aggregate demand of credit) in the financial market. This reinforces the positive firm selection effect. Accordingly, when the financial market is (resp. not) close to perfect, the positive (resp. negative) growth effects dominate the negative (resp. positive) ones. This advocates that increases in investment subsidies should be financed not only by issuing public bonds but also through tax increases unless the financial market is close to perfect. This is an important policy implication for many developed countries that have increased investment subsidies while relying heavily on public debt for fiscal management.

In addition to these main analyses, this study also briefly explores and discusses how fiscal policy, including government debt, affects wealth inequality. An increase in the public debt-to-capital ratio raises interest rates, producing a positive firm selection effect, which raises barriers to entry for entrepreneurship. This reduces the number of borrowers and increases the number of lenders. Furthermore, while rising interest rates increase lenders' wealth, they decrease borrowers' wealth. Therefore, an increase in the public debt-to-capital ratio lowers the wealth ratio between entrepreneurs and non-entrepreneurs, reducing wealth inequality between the two groups. Moreover, when the positive firm selection effect from rising public debt-to-capital ratios increases firm productivity, less productive firms exit the market, and profits are distributed only to more productive firms. Consequently, wealth disparities among entrepreneurs narrow. Thus, an increase in the public debt-to-capital ratio also reduces wealth inequality among entrepreneurs.

Investment subsidies financed by public debt also decrease inequality for the following reasons. First, they raise the interest rate, which increases lenders' (non-entrepreneur's) wealth. Second, a rise in the interest rate reduces the number of less productive firms and allocates larger firms' profits among active entrepreneurs.

Related Literature

Chalk (2000), de la Croix and Michel (2002), and Yakita (2014) examine the sustainability of public debt in OLG models and conclude that a Ponzi game by governments is possible.[4] Fiscal sustainability in OLG models is often defined as the convergence of public debt to a sustainable level in the long run. Chalk (2000) and Maebayashi (2023) examine this issue under specific fiscal policy rules in OLG models. Chalk (2000) employs a constant deficit rule, while Maebayashi (2023) implements the fiscal consolidation rule based on the Stability and Growth Pact in the EU. However, as these studies are based on exogenous growth models, they ignore the long-run endogenous growth effect from a non-decreasing return to capital (Romer, 1986).

This study is closely related to those of Bräuninger (2005), Yakita (2008), Arai and Kunieda (2010), Arai (2011), Teles and Mussolini (2014), Agénor and Yilmaz (2017), Maebayashi and Konishi (2021), and Futagami and Konishi (2023), which also investigate the sustainability of public debt when the government plays a Ponzi game in OLG models with an endogenous growth structure.[5]

Bräuninger (2005), Arai (2011), Teles and Mussolini (2014), and Maebayashi and Konishi (2021) find that public debt has a negative effect on long-run growth (Saint-Paul, 1992); however, they all assume a constant interest rate over time because they use the standard AK model (Romer, 1986). The present study differs from these previous studies because it considers endogenous movements of interest rates through the financial market structure.

Yakita (2008) and Agénor and Yilmaz (2017) analyze public capital accumulation—following the tradition of Futagami et al. (1993)—financed through public debt. Under the assumption of perfect financial markets, they show that public capital generates positive externalities for economic growth and the interest rate through its effects on factor markets in final-good production.

[4] Tirole (1985) uses Diamond's (1965) OLG model and show that positive bubbles can exist when the economy is dynamically inefficient. This result can also imply that the government's Ponzi game is possible.

[5] Greiner (2007, 2011, 2012, 2015), Kamiguchi and Tamai (2012), and Miyazawa et al. (2019) investigate the sustainability of public debt in representative infinitely lived agent models with an endogenous growth structure in which a Ponzi game by the government is impossible (according to the transversality condition).

This study differs from the previous literature in three main ways. First, it examines the growth effects of investment subsidies that act directly on firms rather than through public capital. Second, the interest rate is endogenously determined by the structure of the financial market. Third, we show that the growth impact of investment subsidies depends critically on the degree of financial frictions. When financial markets are imperfect, subsidies may even reduce growth by lowering entry barriers, increasing the number of low-productivity firms, and generating the negative firm selection effect. Moreover, greater financial imperfections weaken credit demand, dampening the positive firm selection effect. These negative growth effects can surpass the positive direct effect of investment subsidies unless the degree of financial friction is very small.

Minami and Horii (2025) examine the growth effects of debt-financed R&D subsidies in a continuous-time OLG model (Blanchard 1985). They show that debt-financed R&D subsidies do not enhance long-run growth unless R&D productivity is sufficiently high. Unlike Minami and Horii (2025), this study considers heterogeneous firms and financial frictions and finds that the long-run growth effect of debt-financed investment subsidies depends on the degree of financial frictions.

Although Arai and Kunieda (2010) similarly consider financial market imperfection with heterogeneous agents, they assume a uniform distribution of individual productivity as well as risk neutrality, and ignore the investment subsidy policy. In contrast to Arai and Kunieda (2010), this study incorporates more realistic growth effects under investment subsidy policies through microfoundations with risk-averse utility and a Pareto distribution of firms' productivity, as mentioned in the following literature.

Recent trends in the growth theory literature incorporate the heterogeneity of individuals or firms into endogenous growth models (e.g., Arawatari et al., 2023; Jaimovich and Rebelo, 2017; Mino, 2015, 2016). Mino (2016), Jaimovich and Rebelo (2017), and Arawatari et al. (2023) consider the effect of tax and fiscal policies when firms differ in their productivity under a Pareto distribution. These studies show that the effects of such public policies on growth significantly differ from that in the (homogeneous) representative agent economy. To the best of my knowledge, studies on investment subsidy policies in this context are somewhat limited. Morimoto (2018) studies R&D subsidy policy under heterogeneity in individual productivity and shows that R&D subsidies increase economic growth when they are not very large. However, Morimoto (2018) considers a balanced budget to finance the subsidies. This study contributes to the literature by considering the effect of investment subsidies financed by public debt and showing that the growth effect of subsidies can be negative unless the financial market is close to perfect.

2 Model

2.1 Entrepreneurial households

Consider an economy comprising two types of households living in two periods. The number of households is normalized to one. Entrepreneurial households can invest in capital, hire young labor, and use capital to produce goods. The production technology follows the form used in Mino (2016):

$$y_{i,t} = \mathcal{F}(z_{i,t-1}k_{i,t}, n_{i,t}K_t) = A(z_{i,t-1}k_{i,t})^{\alpha}(n_{i,t}K_t)^{1-\alpha}, \quad A > 0 \quad i \in [0,1], \tag{1}$$

where $y_{i,t}$, $k_{i,t}$, $n_{i,t}$, and K_t denote output, capital, labor, and aggregate capital, respectively. Aggregate capital has a positive external effect on production (e.g., Romer, 1986). We assume that capital $k_{i,t}$ is broadly viewed to include both ICT capital (knowledge capital) related to innovation and development (R&D) and non-ICT capital.[6] Here, $z_{i,t}$ is the production efficiency of the firm owned by the type i entrepreneurial household.

In the young period, each entrepreneurial household draws $z_{i,t}$ from a Pareto distribution whose cumulative distribution is given by

$$F(z) = 1 - z^{-\varphi}, \quad 1 \leq z < \infty, \quad \varphi > 1. \tag{2}$$

Here, a lower (higher) value of φ means a higher (lower) degree of heterogeneity in the production technology. Following Itskhoki and Moll (2014), Liu and Wang (2014), and Mino (2015), we assume that $z_{i,t}$ is independent and identically distributed (iid) both over time and across agents.

After realizing z_i, each entrepreneurial household maximizes its lifetime utility:

$$U^j_{i,t} = (1-\beta)\ln c^{y,j}_{i,t} + \beta\left[(1-\gamma)\ln c^{o,j}_{i,t+1} + \gamma \ln x^j_{i,t+1}\right], \quad j \in \{e, l\} \tag{3}$$

subject to the budget and credit constraints. Here, $c^{y,e}_{i,t}$, $c^{o,e}_{i,t+1}$, and $x^e_{i,t+1}$ represent consumption in the young period and old age and bequests by entrepreneurial households that produce goods (active entrepreneurs, hereafter), while $c^{y,l}_{i,t}$, $c^{o,l}_{i,t+1}$, and $x^l_{i,t+1}$ represent consumption by those who do not produce goods (non-active entrepreneurs hereafter), respectively.

In the young period, active entrepreneurs supply one unit of labor inelastically, earn wage income w_t, and inherit from parents $x_{i,t}$.[7] With this income, they can borrow from non-active

[6] ICT capital includes hardware, communication, and software, whereas non-ICT capital includes transport equipment and non-residential construction; agricultural products, metal products, and machinery other than hardware and communication equipment; and other products of non-residential gross fixed capital formation (see OECD, 2010).

[7] Even with endogenous labor supply, our main results are robust if we do not consider the bequest motive.

entrepreneurs and invest in capital $k_{i,t+1}$. Here, let us denote $d_{i,t}$ as private debt. Then, the net worth of active entrepreneurs is $a_{i,t+1} = k_{i,t+1} - d_{i,t}$. Investment $k_{i,t+1}$ is subsidized by the government at the rate of σ_k. Since total income is allocated to consumption and net worth, the budget constraint of active entrepreneurs in the young period is represented as

$$c_{i,t}^{y,e} = w_t + x_{i,t} + \sigma_k k_{i,t+1} - a_{i,t+1}, \qquad a_{i,t+1} = k_{i,t+1} - d_{i,t}. \tag{4}$$

Note that $x_{i,t}$ depends on whether parents are active or non-active entrepreneurs, as we see next. However, this is not critical to the macroeconomy when we aggregate all agents, as we see later.[8]

In old age, active entrepreneurs produce final goods with the production function (1). They hire young labor $n_{i,t+1}$ in generation $t+1$ and use capital $k_{i,t+1}$ installed in period t for production. Profit $\pi_{i,t+1}$ is sales $y_{i,t}$ minus both wage payments and private debt repayments. Active entrepreneurs in old age allocate it into consumption $c_{i,t+1}^{o,e}$ and bequest to their children $x_{i,t+1}^{e}$. Thus, the budget constraint of active entrepreneurs in old age is written as

$$c_{i,t+1}^{o,e} = \pi_{i,t+1} - x_{i,t+1}^{e}, \tag{5}$$

$$\pi_{i,t+1} = y_{i,t+1} - w_{t+1} n_{i,t+1} - R_{t+1} d_{i,t}, \tag{6}$$

where $R_t (= 1 + r_t)$ is the gross interest rate when r_t denotes the interest rate. We assume full capital depreciation because we consider a period to be about 30 years.[9] Furthermore, the financial market is assumed to be imperfect in the following sense. Entrepreneurs face a credit constraint such that

$$d_{i,t} \leq \lambda k_{i,t+1}. \tag{7}$$

If $\lambda = 1$, the financial market is perfect, while no borrowing is available if $\lambda = 0$, meaning that λ is the degree of (im)perfection of the financial market.

From (1), (3), (4), (5), (6), and (7), the first-order conditions (FOCs) with respect to $n_{i,t+1}$,

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[8]Even without the bequest motive $x_{i,t}$, our main results remain robust. However, without $x_{i,t}$, the investment levels of all firms becomes the same, which is somewhat unrealistic. Furthermore, through bequests $x_{i,t}$, public debts held by lenders can have a crowding-in effect on investment. For these reasons, and for future reference, we allow for the presence of the bequest motive.

[9]Without full capital depreciation ($\delta \neq 1$), the first term on the right-hand side of (6) is replaced by $y_{i,t+1} + (1 - \delta) k_{i,t+1}$ if we denote $\delta \in [0, 1]$ as capital depreciation.

$d_{i,t}$, and $k_{i,t+1}$ are given by

$$n_{i,t+1}; \qquad w_{t+1} = (1-\alpha)\frac{y_{i,t+1}}{n_{i,t+1}}, \tag{8}$$

$$d_{i,t}; \qquad \frac{1-\beta}{c_{i,t}^{y,e}} = \frac{\beta(1-\gamma)R_{t+1}}{c_{i,t+1}^{o,e}} + \mu_{i,t}, \tag{9}$$

$$k_{i,t+1}; \qquad \frac{(1-\beta)(1-\sigma_k)}{c_{i,t}^{y,e}} = \frac{\beta(1-\gamma)}{c_{i,t+1}^{o,e}}\frac{\partial \pi_{i,t+1}}{\partial k_{i,t+1}} + \lambda\mu_{i,t}, \tag{10}$$

$$\mu_{i,t}(\lambda k_{i,t+1} - d_{i,t}) = 0, \;\; \mu_{i,t} \geq 0, \;\; \lambda k_{i,t+1} - d_{i,t} \geq 0, \tag{11}$$

$$\mu_{i,t} = \frac{\beta(1-\gamma)}{1-\sigma_k-\lambda}\frac{\alpha(y_{i,t+1}/k_{i,t+1}) - (1-\sigma_k)R_{t+1}}{c_{i,t+1}^{o,e}}, \tag{12}$$

where $\mu_{i,t}$ is the Lagrangian multiplier associated with the debt constraint and represents the investment wedge between the marginal product of capital $\alpha(y_{i,t+1}/k_{i,t+1})$ and marginal cost $(1-\sigma_k)R_{t+1}$ in (12). Note that (12) is derived from (9) and (10) with (1), and (6). If the credit constraint is not binding, $\mu_{i,t} = 0$ holds from (11). We assume that entrepreneurs produce goods as long as their profits are not negative. Thus, from (11) and (12), the credit constraint (7) binds when

$$\alpha(y_{i,t+1}/k_{i,t+1}) \geq (1-\sigma_k)R_{t+1}. \tag{13}$$

From (1) and (8), we obtain

$$y_{i,t} = Az_{i,t-1}k_{i,t}\left[\frac{(1-\alpha)AK_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}, \tag{14}$$

which together with (13) yields the cutoff value of z as

$$z_t^* = \frac{(1-\sigma_k)R_{t+1}}{\alpha A}\left[\frac{w_{t+1}}{(1-\alpha)AK_{t+1}}\right]^{\frac{1-\alpha}{\alpha}}. \tag{15}$$

Substituting (14) and (15) into the left-hand side (LHS) and the right-hand side (RHS) of (13) respectively, we can rewrite (13) as

$$z_{i,t} \geq z_t^*.$$

This indicates that entrepreneurs who draw $z_{i,t} \geq z_t^*$ produce and become active entrepreneurs. They also become borrowers because the debt constraint (7) is binding. Conversely, credit constraints are ineffective for entrepreneurs who draw $1 \leq z_{i,t} < z_t^*$.[10] In the competitive final

[10]From (2), the minimal value of z is 1.

good market, firms with $z_{i,t} < z_t^*$ cannot compete with those with z^*. Thus, entrepreneurs who own firms with $z_{i,t} < z_t^*$ give up production (i.e., become non-active entrepreneurs) and become lenders. We show in Lemma 1 that the indifference between becoming borrowers and lenders holds for agents with $z_{i,t} = z_t^*$.

Moreover, (15) indicates that investment subsidies reduce the barriers to becoming an entrepreneur z^* and increase the number of borrowers for given interest rates R_{t+1}.

Utility maximization with respect to $x_{i,t+1}^e$ yields

$$x_{i,t+1}^e = \gamma \pi_{i,t+1}. \tag{16}$$

Substituting (8) and $d_{i,t} = \lambda k_{i,t+1}$ into (6), we obtain

$$\pi_{i,t+1} = \alpha y_{i,t+1} - R_{t+1} \lambda k_{i,t+1}. \tag{17}$$

This together with (14) yields

$$\pi_{i,t+1} = \left\{ A z_{i,t} \left[\frac{(1-\alpha) A K_{t+1}}{w_{t+1}} \right]^{\frac{1-\alpha}{\alpha}} - R_{t+1} \lambda \right\} k_{i,t+1} \tag{18}$$

More productive firms (with larger values of $z_{i,t}$) reap more profits and gains in market share while less productive firms lose both. Then, a larger cutoff value of z_t^* in (15) causes reallocations toward more efficient firms (e.g., Melitz, 2003) and can increase aggregate productivity as we will see in Section 3.

From (4), (5), (9), (12), (16), (17), and $d_{i,t} = \lambda k_{i,t+1}$, we obtain

$$k_{i,t+1} = \frac{\beta}{1 - \sigma_k - \lambda} (w_t + x_{i,t}). \tag{19}$$

(19) indicates that a higher degree of financial market imperfection (a lower value of λ) reduces the investment of each firm. Conversely, a larger investment subsidy to firms increases the investment of each firm.

We move onto the case of lenders (non-active entrepreneurs). Lenders (entrepreneurs who draw $z_{i,t} < z_t^*$ and do not engage in production) maximize their utility (3) subject to the following budget constraints:

$$c_{i,t}^{y,l} = w_t + x_{i,t} - l_{i,t} - q_t b_{i,t+1}, \tag{20}$$

$$c_{i,t+1}^{o,l} = R_{t+1} l_{i,t} + b_{i,t+1} - x_{i,t+1}^l, \tag{21}$$

where $l_{i,t}$ is the loan to active entrepreneurs, while $b_{i,t+1}$ is the quantity of government bonds purchased and q_t is the price of government bonds. We assume that government bonds are discount bonds with a maturity of 1 period and a face value of 1 (Maebayashi and Tanaka, 2022). The no-arbitrage condition between lending to active entrepreneurs and buying treasuries equates these rates of return as

$$R_{t+1} = 1/q_t. \tag{22}$$

(20) indicates that in the young period, lenders supply one unit of labor inelastically, earn wage income w_t, and inherit from parents $x_{i,t}$, as active entrepreneurs do. They allocate this income toward loans to active entrepreneurs $l_{i,t}$, purchases of government bonds $q_t b_{i,t+1}$, and consumption $c_{i,t}^{y,l}$. (21) implies that the total return from lending to the private sector $R_{t+1}l_{i,t}$ and public sector $(1/q_t)q_t b_{i,t+1}$ in old age is divided into consumption $c_{i,t+1}^{o,l}$ and bequests $x_{i,t+1}^{l}$.

Lenders' FOCs with respect to $l_{i,t} + q_t b_{i,t+1}$ and $x_{i,t+1}^{l}$ result in

$$l_{i,t} + q_t b_{i,t+1} = \beta \left(w_t + x_{i,t} \right), \tag{23}$$

$$x_{i,t+1}^{l} = \gamma (R_{t+1} l_{i,t} + b_{i,t+1}). \tag{24}$$

From (23) and (24) with (22), we obtain

$$x_{i,t+1}^{l} = \gamma \beta R_{t+1} \left(w_t + x_{i,t} \right). \tag{25}$$

We summarize the discussion so far in the following lemma.

Lemma 1. *The indifference between being an active entrepreneur (borrower) and an non-active entrepreneur (lender) holds for households whose productivity is z_t^* (see Appendix A for this proof). Households with $z_{i,t} \geq z_t^*$ become active entrepreneurs (borrowers) while those with $z_{i,t} < z_t^*$ become non-active entrepreneurs (lenders) in period t.*

Hereafter, we omit the index i for simplicity.

2.2 Government

The government owes a given amount of debt (denoted by B_t) at the beginning of period t. It repays this debt and finances investment subsidies by issuing new bonds, indicating that it is playing a Ponzi game, as in Yakita (2014) and Maebayashi and Tanaka (2022).[11] This simple

[11]Recall that a Ponzi game by the government is possible in OLG models as noted in the Introduction. Previous studies (e.g., Maebayashi and Tanaka, 2022; Yakita, 2014) investigate fiscal sustainability in a Ponzi game played

approach to the government budget aims to clearly study the growth effect of investment subsidies financed by public debt. A more comprehensive approach, including income tax revenue, is provided in Section 7.1. Here, recall that investment subsidies are provided only to active entrepreneurs ($z_t \geq z_t^*$) and government bonds are purchased by lenders ($1 \leq z_t < z_t^*$). Thus, the government's budget constraint in period t is given by

$$q_t B_{t+1} = B_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t), \tag{26}$$

$$B_{t+1} = \int_1^{z_t^*} b_{t+1} dF(z_t) \quad \left(B_t = \int_1^{z_{t-1}^*} b_t dF(z_{t-1}) \right). \tag{27}$$

Here, let us define $\tilde{B}_{t+1} \equiv q_t B_{t+1}$. Then, we obtain $\tilde{B}_{t+1} = B_{t+1}/R_{t+1}$ from (22). Accordingly, (26) is transformed into

$$\tilde{B}_{t+1} = R_t \tilde{B}_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t). \tag{28}$$

3 Equilibrium

To aggregate the economic variables, we note that individual heterogeneity depends not only on productivity z_t, but also on bequests from parents x_t. (19) shows that the investment of active entrepreneurs (borrowers) depends on this pair. For convenience, we then represent k_{t+1} as $k_{t+1}(z_t, x_t)$. This together with (16) and (17) represent x_{t+1}^e and π_{t+1} as $x_{t+1}^e(z_t, x_t)$ and $\pi_{t+1}(z_t, x_t)$, respectively. Similarly, for non-active entrepreneurs (lenders), by (23) and (25), l_t, $q_t b_{t+1}$, and x_{t+1}^l are expressed as $l_t(z_t, x_t)$, $q_t b_{t+1}(z_t, x_t)$, and $x_{t+1}^l(z_t, x_t)$, respectively. Finally, the bequest from parent x_t is either $x_t^e(z_{t-1}, x_{t-1})$ or $x_t^l(z_{t-1}, x_{t-1})$, regardless of the agent type and z_t in period t (i.e., $x_t \in \{x_t^e(z_{t-1}, x_{t-1}), x_t^l(z_{t-1}, x_{t-1})\}$) because z is iid over time and across agents. Therefore, z_t is independent of x_t for all t. We summarize these points in the following remark.

Remark 1. *Individual heterogeneity depends not only on productivity z_t, but also on bequests from parents x_t. x_t can be either $x_t^e(z_{t-1}, x_{t-1})$ or $x_t^l(z_{t-1}, x_{t-1})$. z_t is independent of x_t for all t.*

by the government. Maebayashi and Tanaka (2022) consider public debt finance only for the repayment of debt ($q_t B_t = B_t$), while Yakita (2014) considers both the repayment of debt and net income transfers to young labor ($q_t B_t = B_t +$ net transfers).

3.1 Aggregation

Aggregate capital K_t is held by active entrepreneurs (borrowers) who draw $z_{t-1} \geq z^*_{t-1}$ in period $t-1$; therefore, it is represented by $\int_{z^*_{t-1}}^{\infty} k_t(z_{t-1}, x_{t-1})dF(z_{t-1}) = K_t$ (Mino, 2015, 2016). Using (2) and given that z is iid, we can rewrite aggregate capital as[12]

$$K_t \left(= \int_{z^*_{t-1}}^{\infty} k_t(z_{t-1}, x_{t-1})dF(z_{t-1}) \right) = (z^*_{t-1})^{-\varphi}\bar{k}_t. \tag{29}$$

where, $\bar{k}_t$ is the average level of k_t. Aggregating the credit constraint $d_t = \lambda k_{t+1}$, the equilibrium condition of the financial market is represented as

$$\int_{1}^{z^*_t} l_t(z_t, x_t)dF(z_t) = \int_{z^*_t}^{\infty} d_t(z_t, x_t)dF(z_t) = \lambda K_{t+1}. \tag{30}$$

The labor market clears as

$$N_t = \int_{z^*_{t-1}}^{\infty} n_t dF(z_{t-1}) = 1, \tag{31}$$

which indicates that total labor demand equals total labor supply, whose aggregate level is unity ($\int_1^{\infty} 1dF(z_{t-1}) = 1$).

Using (2) and (29), we can also aggregate production function (14) as

$$\begin{aligned} Y_t \left(= \int_{z^*_{t-1}}^{\infty} y_t(z_{t-1}, x_{t-1})dF(z_{t-1}) \right) &= \int_{z^*_{t-1}}^{\infty} Az_{t-1}k_t(z_{t-1}, x_{t-1}) \left[\frac{(1-\alpha)AK_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}} dF(z_{t-1}) \\ &= \frac{A\varphi}{\varphi - 1} z^*_{t-1} \left[\frac{(1-\alpha)AK_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}} K_t. \end{aligned} \tag{32}$$

Substituting (15) into (32), we obtain

$$Y_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi - 1)} R_t K_t. \tag{33}$$

From (8), (31), and (33), we obtain $w_t N_t = w_t \cdot 1 = (1-\alpha) \int_{z^*_{t-1}}^{\infty} y_t(z_{t-1}, x_{t-1})dF(z_{t-1})$, leading

[12] Since z_t is iid across agents and over time,

$$K_t = \int_{z^*_{t-1}}^{\infty} k_t(z_{t-1}, x_{t-1})dF(z_{t-1}) = \int_{z^*_{t-1}}^{\infty} dF(z_{t-1})\bar{k}_t,$$

where, $\bar{k}_t$ is the average level of k_t. From (2), this is reduced to (29).

to

$$w_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_tK_t. \tag{34}$$

Substituting (34) into (15), we obtain

$$z_t^* = \left(\frac{\varphi}{\varphi-1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\sigma_k)R_{t+1}}{\alpha A}\right)^{\frac{1}{\alpha}}. \tag{35}$$

Here, a larger z_t^* represents a higher barrier to becoming active entrepreneurs (see the explanation below (15)), indicating a decrease in less productive firms. As (18) shows, this causes reallocations toward more efficient firms. We refer to this as the positive *firm selection effect* (e.g., Melitz, 2003). Then, (35) shows that an increase in R_{t+1} causes the positive firm selection effect (i.e., $\partial z_t^*/\partial R_{t+1} > 0$), because it increases the cost of borrowing for entrepreneurs and reduces the number of less productive firms. This increases aggregate productivity and output Y_{t+1} by (33), as well as the wage rate w_{t+1} by (34).

In addition, investment subsidies σ_k lower the barrier to becoming an entrepreneur z_t^* and increase the number of less productive firms (i.e., $\partial z_t^*/\partial \sigma_k < 0$). This lowers aggregate productivity and reduces both output, Y_{t+1} by (33), and the wage rate, w_{t+1} by (34).

We now continue the aggregation of the other elements. Aggregating (19), using (29) and (34), and keeping in mind that x_t is independent of z_t (Remark 1), we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_tK_t + X_t\right], \tag{36}$$

where $X_t \equiv \int_1^\infty x_t(z_{t-1}, x_{t-1})dF(z_{t-1}) = \int_1^{z_{t-1}^*} x_t^l(z_{t-1}, x_{t-1})dF(z_{t-1}) + \int_{z_{t-1}^*}^\infty x_t^e(z_{t-1}, x_{t-1})dF(z_{t-1})$. Here, X_t is derived using (16) and (24) associating with (2), (17), (27), (29), (30), (33), and (35) as

$$X_t = \gamma\left[\frac{\varphi(1-\sigma_k)}{\varphi-1}R_tK_t + B_t\right]. \tag{37}$$

See Appendix B for the derivations of (36) and (37). From (36) and (37), we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z_t^*)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)R_tK_t + \gamma B_t\right]. \tag{38}$$

Through bequests X_t, public debt held by lenders (γB_t) has a crowding-in effect on investment. In other words, public debt reallocates resources from low-productivity firms to high-productivity ones. We refer to this crowding-in effect as the *liquidity effect* (e.g., Farhi and Tirole, 2012; Hirano and Yanagawa, 2017).[13] An increase in government bonds grows lenders' assets through bequests from low-productivity agents. These assets then flow into investments in high-productivity firms.

Aggregating (23) and using (27) and (34), we obtain

$$\int_{1}^{z_t^*} [l_t(z_t, x_t) + q_t b_{t+1}(z_t, x_t)] dF(z_t) = \beta \left[1 - (z_t^*)^{-\varphi}\right] \left[\frac{(1-\sigma_k)\varphi}{\alpha(\varphi - 1)}(1-\alpha) R_t K_t + X_t\right]. \tag{39}$$

See Appendix B for the derivation of (39). Substituting (27), (30) and (37) into (39), we obtain

$$\lambda K_{t+1} = \beta \left[1 - (z_t^*)^{-\varphi}\right] \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi - 1)}(1-\alpha+\alpha\gamma) R_t K_t + \gamma B_t\right] - q_t B_{t+1}. \tag{40}$$

The RHS of (40) represents the aggregate supply of credit in the financial market, whereas the LHS shows aggregate demand for it. Increases in public borrowing (public debt issuance), $q_t B_{t+1}$, crowd out the aggregate supply of credit. This leads to upward pressure on the interest rate R_{t+1} in the financial market, as we see in Section 5.[14]

Associating (38) and (40) with (22) and $\tilde{B}_{t+1} \equiv q_t B_{t+1}$, we obtain the following asset market-clearing condition:

$$K_{t+1} + \tilde{B}_{t+1} = \beta R_t \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi - 1)}(1-\alpha+\alpha\gamma) K_t + \gamma \tilde{B}_t\right] + \sigma_k K_{t+1}. \tag{41}$$

(41) indicates that an increase in public debt $\tilde{B}_{t+1}$ decreases private investment K_{t+1}. This is the familiar crowding-out effect of public debt $\tilde{B}_{t+1}$ on private investment K_{t+1}.

From (28) and (29), we obtain

$$\tilde{B}_{t+1} = R_t \tilde{B}_t + \sigma_k K_{t+1}. \tag{42}$$

An increase in outstanding public debt increases the issuance of public bonds and worsens the fiscal condition. This is also the case for increases in investment subsidies σ_k for a given investment K_{t+1}.

The definition of this competitive equilibrium is as follows.

[13] Farhi and Tirole (2012) and Hirano and Yanagawa (2017) consider the liquidity effect through bubbles rather than public debt.

[14] There are two types of asset holdings: lending to firms and purchasing public bonds. Following Walras' law, we only have to focus on one of them (financial market or public bond market). This study focuses on the financial market and explains the findings based on it. However, we also briefly mention the public bond market. In equilibrium, the supply of bonds $q_t B_{t+1}$ by the government equals its demand by households for a given value of λK_{t+1}. An increase in the issuance of bonds by the government leads to an excess supply of bonds, which lowers the price of bonds q_t and leads to upward pressure on the interest rate R_{t+1} from (22).

Definition 1. *Given the initial state* $(k_0, b_0, z_{-1}, z^*_{-1}, d_{-1}, l_{-1}, x^e_0, x^l_0, K_0, B_0, R_0, q_{-1})$ *where* $z^*_{-1} = \left(\frac{\varphi}{\varphi-1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\sigma_k)R_0}{\alpha A}\right)^{\frac{1}{\alpha}}$ *by* (35), $d_{-1} = \lambda k_0$ *by* (7), $\int_1^{z^*_{-1}} l_{-1} dF(z_{-1}) = \int_{z^*_{-1}}^{\infty} d_{-1} dF(z_{-1}) = \lambda K_0$ *by* (30), $x^l_0 = \gamma(R_0 l_{-1} + b_0)$ *by* (24), $x^e_0 = \gamma[(z_{-1}/z^*_{-1})(1-\sigma_k) - \lambda] R_0 k_0$ *by* (14), (15), (16), (17), *and* (34), $B_0 = \int_1^{z^*_{-1}} b_0 dF(z_{-1})$ *by* (27), *and* $q_{-1} = 1/R_0$ *by* (22), *a competitive equilibrium in the economy where* σ_k *is exogenous is a sequence of*

$$\left\{ c^{y,e}_t, c^{o,e}_t, c^{y,l}_t, c^{o,l}_t, y_t, n_t, \pi_t, d_t, l_t, x_{t+1}, x^e_{t+1}, x^l_{t+1}, k_{t+1}, b_{t+1}, z^*_t, Y_t, X_t, K_{t+1}, B_{t+1} \right\}_{t=0}^{\infty}$$

and prices $\{w_t, R_{t+1}, q_t\}_{t=0}^{\infty}$ *such that (a) taking prices, the investment subsidy rate* σ_k*, and the distribution of* z_t *as given, active and non-active entrepreneurs optimize their solutions (from* (8) *to* (12), (16), (23), *and* (24)*); (b) government's budget is balanced (*(26)*); (c) the cutoff value* z^*_t *satisfies* (35)*; and (d) markets clear with* (30), (31), *and* (41).

3.2 Characteristics of the economy without government intervation

This subsection briefly explains the characteristics of the economy without government intervention ($\sigma_k = 0$ and $b_t = 0$ ($B_t = 0$) for all t) to capture the fundamental properties of the model. Applying $\sigma_k = 0$ and $B_t = B_{t+1} = 0$ into (38) and (40), and using (35) and $Y_{t+1}/Y_t = (R_{t+1}/R_t)(K_{t+1}/K_t)$ by (33), we obtain

$$z^*_t = z^*_{t-1} = \left(\frac{1}{1-\lambda}\right)^{\frac{1}{\varphi}}, \quad R_{t+1} = R_t = \alpha A \left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha} \left(\frac{1}{1-\lambda}\right)^{\frac{\alpha}{\varphi}},$$
$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \beta A \left(\frac{\varphi}{\varphi-1}\right)^{\alpha} \left(\frac{1}{1-\lambda}\right)^{\frac{\alpha}{\varphi}} (1-\alpha+\alpha\gamma),$$

all of which are constant over time. Furthermore, a decrease (resp. increase) in λ (i.e., larger (resp. smaller) financial frictions) lowers (resp. raises) the interest rate R_{t+1} and increases (resp. reduces) the number of firms (i,e., a decrease (resp. an increase) in z^*_t).[15] This is because a smaller (resp. larger) λ leads to a tighter (resp. looser) credit constraint (see (7)), making borrowing more difficult (resp. easier). In other words, there is a decrease (resp. an increase) in the aggregate demand for credit in the financial market. This decreases (resp. increases) the interest rate $\partial R_{t+1}/\partial\lambda > 0$ and reduces (resp. raises) the hurdles to becoming an entrepreneur $\partial z^*_t/\partial\lambda > 0$.

A decrease (resp. increase) in z^*_t due to a fall (resp. rise) in λ (i.e., larger (resp. smaller) financial frictions) causes reallocations toward less (resp. more) efficient firms (i.e., the negative

[15]This result is in line with Farhi and Tirole (2012), who show that larger financial friction (a low pledgeability in their model) leads to a low interest rate.

(resp. positive) firm selection effect) and decreases (resp. increases) aggregate productivity (see the explanation below (35)). Thus, a decrease (resp. an increase) in λ (i.e., larger (resp. smaller) financial frictions) hinders (resp. promotes) economic growth. We summarize these results in the following Remark 2.

Remark 2. *A decrease (resp. increase) in λ (i.e., larger (resp. smaller) financial friction) lowers (resp. raises) the interest rate R_{t+1} and increases (resp. reduces) z_t^* (i.e., $\partial R_{t+1}/\partial\lambda > 0$ and $\partial R_{t+1}/\partial\lambda > 0$). Moreover, a decrease (resp. increase) in λ hinders (resp. promotes) economic growth through the negative (resp. positive) firm selection effect ($\partial(Y_{t+1}/Y_t)/\partial\lambda > 0$).*

In the following sections, we show that the properties in Remark 2 are robust even in an economy with fiscal policy.

4 Dynamic system and (un)sustainable paths of the economy

In this section, we derive the dynamic system of the economy and check whether public debt is sustainable.

Let us define $\theta_t \equiv \tilde{B}_t/K_t$ as the ratio of public debt to capital. From (41) and (42), we obtain

$$\frac{K_{t+1}}{K_t} = R_t\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t\right], \tag{43}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} = R_t\left[\frac{\varphi\sigma_k(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\theta_t^{-1}+[1-\sigma_k(1-\beta\gamma)]\right]. \tag{44}$$

Here, to ensure $K_{t+1}/K_t > 0$, we assume the following condition:

$$\theta_t < \bar{\theta} \equiv \frac{\varphi(1-\sigma_k)\beta(1-\alpha+\alpha\gamma)}{\alpha(\varphi-1)(1-\beta\gamma)}. \tag{45}$$

The last term of the RHS of (43) ($-(1-\beta\gamma)\theta_t$) indicates that the crowding-out effect of public debts on investment (see (41)) is stronger than the crowding-in effect public debts throuth bequests (see (43)).

From (43) and (44), we obtain

$$\theta_{t+1} = \frac{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\sigma_k(1-\sigma_k)+[1-\sigma_k(1-\beta\gamma)]\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k)-(1-\beta\gamma)\theta_t} \equiv \Lambda(\theta_t;\sigma_k). \tag{46}$$

(46) with (45) characterizes the dynamic system of the economy. The LHS of (46) represents the 45 degree line, while the RHS satisfies $\Lambda(0;\sigma_k) = \sigma_k > 0$, $\Lambda'(\theta_t;\sigma_k) > 0$, and $\Lambda''(\theta_t;\sigma_k) > 0$ ($\Lambda(0;0) = 0$, $\Lambda'(\theta_t;0) > 0$, and $\Lambda''(\theta_t;0) > 0$) obviously. The RHS of (46) intersects the LHS

twice at the steady states S and U, as in Figure 1, if and only if

$$\left[1-\sigma_k(1-\beta\gamma)-\tfrac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k)\right]^2 > 4(1-\beta\gamma)\tfrac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\sigma_k(1-\sigma_k),$$
$$\text{and} \quad \frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)(1-\sigma_k)-[1-\sigma_k(1-\beta\gamma)]>0. \tag{47}$$

Let us denote the two stationary values of θ_t as θ_S^* at S and θ_U^* at U. From (45) and (46), $\theta_S^* < \theta_U^* < \bar{\theta}$ is satisfied if and only if

$$\Lambda(\bar{\theta};\sigma_k) > \bar{\theta}. \tag{48}$$

Thus, we arrive at the following lemma.

Lemma 2. *Two steady states, as represented by S and U in Figure 1, exist under* (47) *and* (48). *The steady state S is stable, while U is unstable.*

[Figure 1]

Lemma 2 indicates that the ratio of public debt to capital θ_t converges to the stable steady-state value θ_S^* as long as the initial value is $\theta_0 < \theta_U^*$. Otherwise (the case of $\theta_0 > \theta_U^*$), θ_t continues to grow and eventually violates (45), making fiscal policy with debt financing unsustainable. Therefore, θ_U^* represents the maximum ratio of public debt to capital to ensure the sustainability of public debt. Thus, we obtain the following proposition.

Proposition 1. *Public debt is (not) sustainable if the initial debt to capital ratio θ_0 is smaller (larger) than θ_U^*. On sustainable transition paths, θ_t converges to the stable steady-state value θ_S^*.*

The definition of $\theta_t = \tilde{B}_t/K_t$ indicates that the growth in public debt ($\tilde{B}_{t+1}/\tilde{B}_t$) is larger (smaller) than that in private capital (K_{t+1}/K_t) for the fiscally unsustainable (sustainable) region $\theta_t > \theta_U^*$ ($\theta_S^* < \theta_t < \theta_U^*$). To see the intuition behind this, in the next section we examine how an increase in θ_t affects the interest rate as well as the growth in capital, GDP, and public debt.

5 Interest rates and growth in capital, GDP, and public debt

We first derive the relationships among the behavior of entrepreneurs z_t^*, the (gross) interest rate R_{t+1}, and θ_t. Using (38) and (43) with $\tilde{B}_t \equiv B_t/R_t$ and $\theta_t \equiv \tilde{B}_t/K_t$ yields

$$z_t^* = \left\{\left(\frac{\beta}{1-\sigma_k-\lambda}\right)\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\sigma_k)(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t}\right\}^{1/\varphi} \equiv z^*(\theta_t;\sigma_k). \tag{49}$$

Substituting (49) into (35), we obtain

$$R_{t+1} = \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\left(\frac{\beta}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}}\left[\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\sigma_k)(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t}\right]^{\frac{\alpha}{\varphi}}$$
$$\equiv \mathcal{R}(\theta_t;\sigma_k). \tag{50}$$

The effects of λ on z_t^* and R_{t+1} are robust (see Remark 2 in Subsection 3.2). Recall that a smaller (resp. larger) λ leads to a tighter (resp. looser) credit constraint (see (7)) and makes borrowing more difficult (resp. easier). In other words, there is a decrease (resp. increase) in the aggregate demand for credit, in the financial market. This decreases (resp. increases) the interest rate R_{t+1} ($\partial R_{t+1}/\partial\lambda > 0$) and reduces (resp. raises) the hurdles to becoming an entrepreneur z_t^* ($\partial z_t^*/\partial\lambda > 0$).

Next, we examine the effect of θ_t on $z^*(\theta_t;\sigma_k)$ and $\mathcal{R}(\theta_t;\sigma_k)$. From (49) and (50), we derive the following lemma.

Lemma 3. *$z^{*\prime}(\theta_t;\sigma_k) > 0$ and $\mathcal{R}'(\theta_t;\sigma_k) > 0$.*

Increases in the public debt-to-capital ratio θ_t raise the interest rate, $\mathcal{R}'(\theta_t;\sigma_k) > 0$.[16] This is because an increase in government debt increases the issuance of government bonds (see (42)) and decreases the aggregate supply of credit in the financial market (see (40)). The rise in $\mathcal{R}(\theta_t;\sigma_k)$ (through an increase in θ_t) causes the positive firm selection effect (i.e., $\partial z_t^*/\partial R_{t+1} > 0$), because it increases the cost of borrowing for entrepreneurs and reduces the number of less productive firms (see (35)). Thus, $z^{*\prime}(\theta_t;\sigma_k) > 0$. This increase in $z^*(\theta_t;\sigma_k)$ due to a rise in θ_t causes reallocations toward more efficient firms (the positive firm selection effect) and increases aggregate productivity (see the explanation below (35)). Figure 2(b) shows this in a numerical example with the reasonable set of parameter values in Table 1 (see Appendix C for the choice of parameter values), while Figure 2(a) reproduces the dynamic system of θ_t presented in Section 4.

Furthermore, from (46) and (50), we obtain the following relationship between θ_t and the current interest rate R_t:

$$R_t = \Psi(\theta_t;\sigma_k) \ \text{ and } \ \Psi'(\theta_t;\sigma_k) = \frac{\mathcal{R}'(\theta_{t-1};\sigma_k)}{\Lambda'(\theta_{t-1};\sigma_k)} > 0, \tag{51}$$

indicating that the current interest rate R_t is increasing in θ_t. Figure 2(c) shows this relationship (51) numerically. This result is different from that in previous studies on the growth effect of public debt (e.g., Bräuninger, 2005; Futagami and Konishi, 2023; Saint-Paul, 1992) because

[16]This result is in line with Bernanke and Gertler (1989), who show that bubbles increase the rate of return on savings and improve borrowers' net worth, which crowds in their future investments. Here, bubbles are similar to an increase in government debt in the sense that they crowd out private investments.

these assume that the interest rate is constant over time because they adopt the standard AK model without firm heterogeneity and financial frictions.

[Table 1 and Figure 2]

Next, we derive the relationship between θ_t and the growth in capital, GDP, and public debt, shown as K_{t+1}/K_t, Y_{t+1}/Y_t, and $\tilde{B}_{t+1}/\tilde{B}_t$, respectively. Applying (51) to (43) and (44) and using $\frac{Y_{t+1}}{Y_t} = \frac{R_{t+1}}{R_t}\frac{K_{t+1}}{K_t}$ (from (33)) and (50), we obtain

$$\frac{K_{t+1}}{K_t} \equiv g^K(\theta_t;\sigma_k) = \Psi(\theta_t;\sigma_k)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma) - (1-\beta\gamma)\theta_t\right], \tag{52}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} \equiv g^B(\theta_t;\sigma_k) = \Psi(\theta_t;\sigma_k)\left[\frac{\varphi\sigma_k(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)\theta_t^{-1} + 1 - \sigma_k(1-\beta\gamma)\right], \tag{53}$$

$$\frac{Y_{t+1}}{Y_t} \equiv g^Y(\theta_t;\sigma_k) = \underbrace{\mathcal{R}(\theta_t;\sigma_k)}_{(\#1)}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma) - \underbrace{(1-\beta\gamma)}_{(\#2)(\#3)}\theta_t\right]. \tag{54}$$

An increase in the public debt-to-capital ratio θ_t has opposing effects on $g^K(\theta_t;\sigma_k)$ and $g^Y(\theta_t;\sigma_k)$.

First, the crowding-out effect of public debt on investment has a negative effect on growth (see (41) and the term $(\#2)$: $-\theta_t$ in (54)). Second, a positive growth effect results from the liquidity effect (see (38) and the term $(\#3)$: $\beta\gamma\theta_t$ in (54)).[17] Finally, a positive growth effect emerges from $(\#1)$ in (54). This effect is owing to a rise in the interest rate $\mathcal{R}'(\theta_t,\sigma_k) > 0$ by Lemma 3.[18][19] The rise in $\mathcal{R}(\theta_t;\sigma_k)$ (through an increase in θ_t) causes the positive firm selection effect (i.e., $\partial z_t^*/\partial R_{t+1} > 0$), because it increases the cost of borrowing for entrepreneurs and reduces the number of less productive firms $z^{*\prime}(\theta_t;\sigma_k) > 0$ by Lemma 3. This increase in $z^*(\theta_t;\sigma_k)$ causes reallocations toward more efficient firms (the positive firm selection effect) and increases aggregate productivity (see the explanation below (35)). The negative growth effect dominates the two positive growth effects in the numerical example in Figure 2(d), indicating that both

[17] From the terms $(\#2)$ and $(\#3)$ in (54), the positive liquidity effect cannot dominate the negative crowding-out effect. This result contrasts with that of Hirano and Yanagawa (2017), who show that the opposite occurs with bubbles when λ is relatively low. The difference lies in the stronger crowding-out effect of debt in OLG models with finite-lived households compared to Ramsey-type models with infinitely lived Ricardian households. Hirano and Yanagawa (2017) use a Ramsey-type model.

[18] $R_t = \Psi(\theta_t;\sigma_k)$ in (51) is increasing in $\mathcal{R}(\theta_t;\sigma_k)$ because of $\Psi'(\theta_t;\sigma_k) > 0$ in (51) and $\mathcal{R}'(\theta_t;\sigma_k) > 0$ in Lemma 3.

[19] Using a log-linear utility function means that the income and substitution effects of interest rates cancel each other out. Thus, higher interest rates only have a level effect on saving, with no impact on the propensity to save. However, the impact of interest rates on savings is limited, even when considering other utility functions, for the following reason. Based on the literature of economic growth (e.g., Jones et al., 1993) and empirical findings (e.g., Guvenen, 2006), the inverse of intertemporal substitution elasticity is greater than 1. Therefore, the substitution effect is less than the income effect, and an increase in the interest rate reduces savings.

$g^K(\theta_t; \sigma_k)$ and $g^Y(\theta_t; \sigma_k)$ are decreasing in θ_t.[20][21]

The non-monotonic relationship between the growth in the interest rate R_{t+1}/R_t and θ_t creates the disparity between $g^K(\theta_t; \sigma_k)$ and $g^Y(\theta_t; \sigma_k)$. R_{t+1} is convex in θ_t (because so is z_t from Figure 2(b)), while R_t is concave in θ_t from Figure 2(c), resulting in $g^K(\theta_t; \sigma_k) > (<) g^Y(\theta_t; \sigma_k)$ for $\theta_S^* \leq \theta_t \leq \theta_U^* (\theta_t < \theta_S^*$ and $\theta_t > \theta_U^*)$.

We next examine the relationship between $g^B(\theta_t; \sigma_k)$ and θ_t. An increase in the public debt to capital ratio θ_t also has two opposite effects on $g^B(\theta_t; \sigma_k)$. First, an increase in this ratio θ_t raises the interest rate by decreasing the aggregate supply of credit in the financial market (see (40)). This rise in the interest rate boosts interest payments and $g^B(\theta_t; \sigma_k)$. Second, an increase in public debt crowds out private investment. Therefore, the burden of public spending on investment subsidies $\sigma_k K_{t+1}$ shrinks as the ratio of public debt to capital θ_t increases, leading to a reduction in $g^B(\theta_t; \sigma_k)$.[22] The negative (positive) effects on $g^B(\theta_t; \sigma_k)$ dominate the positive (negative) effects on $g^B(\theta_t; \sigma_k)$ when θ_t is small (large), as Figure 2(d) shows.

We summarize these results in the following proposition.

Proposition 2. *An increase in θ_t increases the interest rate and decreases the growth rate of K_t and Y_t. The non-monotonic relationship between the growth in the interest rate R_{t+1}/R_t and θ_t creates the disparity between $g^K(\theta_t; \sigma_k)$ and $g^Y(\theta_t; \sigma_k)$. An increase in θ_t decreases the growth rate of B_t when θ_t is small, whereas it increases the growth rate of B_t when θ_t is large.*

Finally, we address the values of $\mathcal{R}(\theta_t; \sigma_k)$, $z^*(\theta_t; \sigma_k)$, $\Psi(\theta_t; \sigma_k)$, $g^K(\theta_t; \sigma_k)$, $g^Y(\theta_t; \sigma_k)$, and $g^B(\theta_t; \sigma_k)$ in the stable steady state S. From (46), (49) (with Lemma 2), (50), (51), (52), (53), and (54), we obtain the following proposition.

Proposition 3. *In the steady state S, the interest rate and cutoff value of z take the constant values of $\mathcal{R}(\theta_S^*; \sigma_k)(= \Psi(\theta_S^*; \sigma_k))$ and $z^*(\theta_S^*; \sigma_k)$ over time. Furthermore, K_t, Y_t, and $\tilde{B}_t$ grow at the same rate of $g(\theta_S^*; \sigma_k) \equiv g^K(\theta_S^*; \sigma_k) = g^Y(\theta_S^*; \sigma_k) = g^B(\theta_S^*; \sigma_k)$.*

Propositions 2 and 3 together with Figure 2 show the reason why fiscal policy with debt financing is unsustainable (sustainable) for $\theta_t > \theta_U^*$ $(0 < \theta_t < \theta_U^*)$.

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[20]This result is in line with Hirano and Yanagawa (2017). They show that when the degree of financial imperfection is low (λ is relatively high), bubbles lower growth.

[21]Appendix D discusses the case in which the liquidity effect is large, with high values of β and γ (e.g., $\beta = \gamma = 0.5$). The relationship between the growth in capital $g^K(\theta_t; \sigma_k)$ and the debt-to-capital ratio θ_t is inverted U-shaped. However, the growth in GDP $g^Y(\theta_t; \sigma_k)$ remains monotonically decreasing in θ_t even with large values of λ (i.e., large positive firm selection effects by Remark 2) and reasonable changes in other parameter values. To understand this, recall that $Y_{t+1}/Y_t = (R_{t+1}/R_t)(K_{t+1}/K_t)$ holds. The growth in the interest rate R_{t+1}/R_t is decreasing in θ_t when θ_t is small, because θ_t is likely to converge to the stable level θ_S^*.

[22]This is because the subsidy rate is assumed fixed. Given a scenario in which a fixed proportion of government borrowing is allocated to subsidies, the crowding out of investment by bond issuance does not reduce total subsidies, thus preserving the growth effect of subsidies. This point is discussed in Subsection 7.2.

6 Investment subsidies to firms

In this section, we examine the effects of introducing investment subsidies to firms on growth and fiscal sustainability. In this study, as noted in the Introduction, investment subsidies are financed through the issuance of public bonds rather than through tax finance as in previous studies (e.g., Morimoto, 2018).

6.1 The effects of investment subsidies to fiscal sustainability

From (46), we obtain

$$\left.\frac{\partial \Lambda(\theta_t;\sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} = 1 + \frac{\beta\varphi(1-\alpha+\alpha\gamma)}{[\beta\varphi(1-\alpha+\alpha\gamma)+\alpha(\varphi-1)(1-\beta\gamma)\theta_t]^2} > 0, \tag{55}$$

indicating that $\Lambda(\theta_t;\sigma_k)$ shifts upward in response to a marginal increase in σ_k evaluated at $\sigma_k = 0$, as Figure 3 shows. Appendix E studies the case of an increase in σ_k from 0.01 to 0.04 numerically. It shows that increases in σ_k (i) make public debt less sustainable and (ii) increase θ_S^*.Thus, we arrive at the following proposition.

Proposition 4. *Providing an investment subsidy for entrepreneurs (a) makes public debt less sustainable and (b) increases the ratio of public debt to capital θ_S^* in the steady state S.*

[Figure 3]

Investment subsidies to firms encourage investment by each firm ((19)) and promote economic growth $g^Y(\theta_t;\sigma_k)$. However, investment subsidies σ_k lower the barrier to becoming an entrepreneur z^* and increase the number of less productive firms, decreasing aggregate productivity (the negative firm selection effect in (35)) and $g^Y(\theta_t;\sigma_k)$. Moreover, the increase in these firms increases aggregate demand for credit in the financial market ((40)), putting upward pressure on the interest rate ((50)). This raises the cost of repaying public debt and accelerates its accumulation $g^B(\theta_t;\sigma_k)$. Investment subsidies financed by public debt itself also increase the issuance of public bonds and accelerate the accumulation of public debt $g^B(\theta_t;\sigma_k)$((42)).

The positive effects of σ_k on $g^B(\theta_t;\sigma_k)$ dominate those on $g^Y(\theta_t;\sigma_k)$, shifting $\theta_{t+1} = \Lambda(\theta_t;\sigma_k)$ upward; hence, θ_S^* increases, while θ_U^* decreases. Consequently, in the long run, investment subsidies to firms (i) make public debt less sustainable and (ii) increase the public debt-to-capital ratio in the steady state S.

6.2 The long-run effects of σ_k on growth $g^Y(\theta_S^*; \sigma_k)$

Next, we investigate the long-run growth effect of investment subsidies σ_k in the steady state S. In the long run, θ_S^* increases as σ_k rises (see Proposition 4(b)). As shown in Section 5, an increase in θ_t increases the interest rate $\mathcal{R}(\theta_S^*; \sigma_k)$ and decreases the growth rates of K_t and Y_t (recall Proposition 2). Keeping this in mind, we calculate the long-run growth effect of an increase in σ_k evaluated at $\sigma_k = 0$. Using (54), and noting that $\theta_S^* = 0$ for $\sigma_k = 0$ and that $\frac{\partial \ln g^Y(\theta_S^*;\sigma_k)}{\partial \sigma_k}|_{\sigma_k=0} = \frac{1}{g^Y(\theta_S^*;\sigma_k)} \frac{\partial g^Y(\theta_S^*;\sigma_k)}{\partial \sigma_k}|_{\sigma_k=0}$, we obtain

$$\left.\frac{\partial \ln g^Y(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} = \left.\frac{\partial \ln \mathcal{R}(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} - \frac{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) + (1-\beta\gamma)\frac{\partial \theta_S^*}{\partial \sigma_k}|_{\sigma_k=0}}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right)}. \tag{56}$$

From (46), (50), and $\theta_S^* = 0$ for $\sigma_k = 0$, we obtain

$$\left.\frac{\partial \theta_S^*}{\partial \sigma_k}\right|_{\sigma_k=0} = \frac{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)-1} > 0, \quad \left.\frac{\partial \ln \mathcal{R}(\theta_S^*; \sigma_k)}{\partial \sigma_k}\right|_{\sigma_k=0} = 1 + \frac{\alpha}{\varphi}\frac{1}{1-\lambda} + \frac{\alpha}{\varphi}\frac{\frac{\partial \theta_S^*}{\partial \sigma_k}|_{\sigma_k=0}}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)} > 0. \tag{57}$$

Here, note that $\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) - 1 > 0$ from Condition (47). Substituting (57) into (56) yields the following proposition.

Proposition 5. *Introducing investment subsidies to firms increases (decreases) long-run growth* $\partial \ln g^Y(\theta_S^*; \sigma_k)/\partial \sigma_k|_{\sigma_k=0} > (<)0$ *if and only if*

$$\frac{\alpha}{\varphi}\frac{1}{1-\lambda} - \frac{1 - \beta\gamma - \frac{\alpha}{\varphi}}{\beta \frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha} + \gamma\right) - 1} > (<)0. \tag{58}$$

Increases in the investment subsidy rate σ_k encourage investment and increase the aggregate demand for credit in the financial market. This raises the interest rate $\mathcal{R}(\cdot)$(see (57)). Such an interest rate rise increases the cost of borrowing for entrepreneurs and reduces the number of less productive firms ($\partial z_t^*/\partial R_{t+1}$). This increases aggregate productivity through the positive firm selection effect. This works positively for economic growth, as shown in the first term of (58). However, an increase in θ_t decreases the growth rate of $g^Y(\theta_S^*; \sigma_k)$ (Proposition 2), as shown in the second term of (58). Investment subsidies also lower the barrier to becoming an entrepreneur and decrease $g^Y(\theta_S^*; \sigma_k)$ through the negative firm selection effect.

The former positive growth effect of σ_k becomes stronger when the financial market is more perfect (λ is larger) by (19) and (50). This is because a larger λ leads to a looser credit constraint (see (7)) and makes borrowing easier (i.e., an increase in the aggregate demand for credit) in the financial market. This reinforces the positive firm selection effect: $\partial \mathcal{R}(\cdot)/\partial \lambda > 0$ (see Remark

2). Accordingly, condition (58) shows that when the financial market is close to perfect, $\lambda \approx 1$, the positive growth effects dominate the negative ones. By contrast, when the financial market is somewhat imperfect (λ is low), the negative growth effects dominate the positive ones because a smaller λ mitigates the positive firm selection effect.

Thus, investment subsidies financed by public debt σ_k decrease (increase) long-run economic growth when λ is not so large (very large). Figure 4 uses a numerical exercise to show that $g^Y(\theta_S^*; \sigma_k)$ is decreasing (increasing) in σ_k in wide ranges of λ (in the case of 0.9), allowing us to check the result of Proposition 5. Interestingly, we find an inverted U-shaped relationship between σ_k and growth when $\sigma_k = 0.8$, as shown on the LHS of Figure 4, yielding the growth-maximizing subsidy rate σ_k^{GM}.[23] These results advocate that investment subsidies should not be financed by public debt unless the financial market is not so perfect.

[Figure 4]

7 Some extensions and discussion

7.1 Investment subsidies financed through tax increases

The previous section showed that investment subsidies fully financed by public debt issuance hinder economic growth when λ is not so large (i.e., the financial market is not so perfect). The main purpose of this section is to incorporate income tax and briefly discuss how investment subsidies financed by increases in income tax can enhance economic growth.

The government issues public bonds ($q_t B_{t+1}$) and collects income tax revenue (T_t) to finance investment subsidies and repay its debt. Specifically, the government imposes a flat rate of tax $\tau \in (0, 1)$ on wage income, the profits of active entrepreneurs, and the asset income of lenders. Therefore, the government budget constraints are given by

$$q_t B_{t+1} + T_t = B_t + \sigma_k \int_{z_t^*}^{\infty} k_{t+1} dF(z_t), \tag{59}$$

$$T_t = \tau \left[w_t N_t + \int_{z_{t-1}^*}^{\infty} \pi_t dF(z_{t-1}) + R_t \int_{1}^{z_{t-1}^*} (l_{t-1} + q_{t-1} b_t) dF(z_{t-1}) \right]. \tag{60}$$

Appendix F fully explains this extension of the model and shows the following. Increases in income tax have a distortionary effect on growth. Furthermore, a distortionary effect of tax on capital accumulation decreases demand for credit in the financial market and lowers interest rates. Lower interest rates decrease borrowing costs and increase the number of less productive

[23] This is also likely to happen when λ is around 0.80 and 0.85. Moreover, we find that when λ is larger, so is σ_k^{GM}.

firms, which affects economic growth negatively through the negative firm selection effect. By contrast, increases in income tax rates boost tax revenue and can reduce the issuance of public bonds, thereby lowering the ratio of public debt to capital θ_S^*. This reduces the costs of repaying public debt and issuing new public bonds, which crowds in capital accumulation and enhances economic growth. The latter positive growth effects dominate the former negative effects. Thus, investment subsidies financed by increases in income tax can enhance economic growth.

7.2 A fixed proportion of government borrowing for investment subsidies

Thus far, we have assumed a constant investment subsidy rate σ_k because it is very common setting in the literature on economic growth. Under this setting, as Sections 5 and 6 show, the crowding-out effect of public debt on private investment (i.e., a decrease in K_{t+1}) reduces public expenditure on investment subsidies $\sigma_k K_{t+1}$ and causes the following tradeoff. While a smaller $\sigma_k K_{t+1}$ decreases the growth in public debt, it also reduces its positive growth effect.

Now, let us consider an alternative government budget rule that allows for a fixed proportion $v \in (0,1)$ of government borrowing $q_t B_{t+1} (= \tilde{B}_{t+1})$ for investment subsidies $\sigma_{k,t} K_{t+1}$ as follows:

$$v\tilde{B}_{t+1} = \sigma_{k,t} K_{t+1}.$$

This indicates that $\sigma_{k,t}$ is endogenously adjusted to satisfy

$$\sigma_{k,t} = v\theta_{t+1}.$$

In this case, the crowding-out effect of public debt on private investment does not reduce public investment subsidy spending, but rather increases public debt, thereby decreasing the positive growth effect of investment subsidies. Appendix G shows that such a trade-off under the endogenous investment subsidy rate is qualitatively similar to the trade-off under the constant investment subsidy rate above, indicating that a large θ_t cannot be sustainable.

7.3 Inequalities between active and non-active entrepreneurs, and among active entrepreneurs

Because the wage income is common between active and non-active entrepreneurs, we focus on the wealth distribution between these two in this subsection. Let us denote $X_{t+1}^e \equiv \int_{z_t^*}^{\infty} x_{t+1}^e(z_t, x_t) dF(z_t)$ and $X_{t+1}^l \equiv \int_1^{z_t^*} x_{t+1}^l(z_t, x_t) dF(z_t)$. From the definition of X_t, we obtain $X_t = X_t^e + X_t^l$ (see the paragraph including (36)). From (16), (17), (24) (with (22)), (33), (36), and (39), we obtain

$$\frac{X^e_{t+1}}{X^l_{t+1}} = \left(\frac{\varphi(1-\sigma_k)}{\varphi-1} - \lambda\right) \frac{(z^*(\theta_t;\sigma_k))^{-\varphi}}{(1-\sigma_k-\lambda)\left[1-(z^*(\theta_t;\sigma_k))^{-\varphi}\right]}. \tag{61}$$

See Appendix H for the details of this derivation. As Lemma 3 shows, increases in the public debt-to-capital ratio θ_t raise the interest rate $\mathcal{R}(\theta_t;\sigma_k)$. The rise in $\mathcal{R}(\theta_t;\sigma_k)$ causes the positive firm selection effect (i.e., $\partial z^*_t/\partial R_{t+1} > 0$), and thus, increases $z^*(\theta_t;\sigma_k)$. An increase in z^*_t creates a higher barrier to becoming an active entrepreneur and therefore, decreases (resp. increases) the number of borrowers (resp. lenders). Furthermore, an increase in the interest rate increases (resp. decreases) lenders' (resp. borrowers') wealth. Thus, an increase in θ_t reduces the wealth ratio between active and non-active entrepreneurs (X^e_{t+1}/X^l_{t+1}) and decreases the wealth inequality between these two types of agents.

Next, we move onto the wealth inequality among active entrepreneurs. From (16) and (17) together with (14), (15), and (34), we obtain

$$x^e_{t+1} = \gamma\left(\frac{z_t}{z^*_t}(1-\sigma_k) - \lambda\right) R_{t+1}k_{t+1}. \tag{62}$$

Aggregating (62) for $z_t \geq z_{p,t} > z^*_t$ and using (49), we obtain

$$\frac{X^e_{p,t+1}}{X^e_{t+1}} = \frac{(z_{p,t}/z^*(\theta_t;\sigma_k))^{\frac{\varphi(1-\sigma_k)}{\varphi-1}} - \lambda}{\frac{\varphi(1-\sigma_k)}{\varphi-1} - \lambda} \left(\frac{z_{p,t}}{z^*(\theta_t;\sigma_k)}\right)^{-\varphi}, \tag{63}$$

where $X^e_{p,t+1}$ represents the total assets among agents with $z_t \geq z_{p,t}$ in period $t+1$. See Appendix H for the derivation of (63). An increase in z^*_t due to a rise in θ_t reduces the number of less productive firms. Then, larger firms' profits by the positive firm selection effect are allocated among active entrepreneurs, which decreases the wealth gap between active entrepreneurs. This is explained by (i) $\partial(X^e_{p,t+1}/X^e_{t+1})/\partial\theta_t > 0$ (i.e., the increases in the proportion of wealth held by households with productivity $z_t \geq z_{p,t} > z^*_t$ relative to the total wealth of all active entrepreneurs) and (ii) $\partial(X^e_{p,t+1}/X^e_{t+1})/\partial z_{p,t} < 0$ (i.e., this ratio is decreasing in $z_{p,t}$) in (63). Appendix H derives $\partial(X^e_{p,t+1}/X^e_{t+1})/\partial\theta_t > 0$ and $\partial(X^e_{p,t+1}/X^e_{t+1})/\partial z_{p,t} < 0$. Thus, increases in θ_t decrease the wealth inequality among active entrepreneurs as well.[24]

Economic growth slows (accelerates) and inequality decreases (increases) when the public debt-to-capital ratio θ_t rises (falls). Therefore, a trade-off arises between economic growth and

[24]With the different discount factor β among agents, increases in θ_t can increase wealth inequality as demonstrated by Maebayashi and Konishi (2021). This is because the rich with large β save more than the poor with low β, absorbing public debt as their assets, according to Maebayashi and Konishi (2021). Therefore, further expansion, including the introduction of differences in β, may be necessary in future research to obtain a more detailed understanding of the relationship between the public debt-to-capital ratio θ_t and economic inequality.

inequality.

Finally, we investigate the effects of σ_k on the steady state values of X^e/X^l and X^e_p/X^e. Appendix H shows that $\partial \ln\left(X^e/X^l\right)/\partial\sigma_k|_{\sigma_k=0} < 0$ and $\partial \ln\left(X^e_p/X^e\right)/\partial\sigma_k|_{\sigma_k=0} > 0$ (with $\partial(X^e_p/X^e)/\partial z_p < 0$), and therefore, investment subsidies financed by public debt decrease inequality. Increases in the investment subsidy rate σ_k encourage investment and increase the aggregate demand for credit in the financial market. This raises the interest rate, increases the cost of borrowing for entrepreneurs, and reduces the number of less productive firms. Then, non-active entrepreneurs (resp. active entrepreneurs) increase (resp. decrease). Moreover, increases in the interest rate increase (resp. decrease) lenders' (resp. borrowers') wealth, reducing the wealth ratio between active and non-active entrepreneurs (X^e/X^l). Additionally, the positive firm selection effect associated with a rise in the interest rate allocates larger firms' profits among active entrepreneurs, which decreases the wealth gap between active entrepreneurs as well.

7.4 Endogenous labor supply

Let us incorporate endogenous labor supply into the baseline model. We modify production function (1) into $y_t = \mathcal{F}(z_{t-1}k_t, n_t\Gamma_t) = A(z_{t-1}k_t)^\alpha(n_t\Gamma_t)^{1-\alpha}$, where Γ_t is labor augmenting technology and it is specified as $\Gamma_t = K_t/N_t$ (e.g., Romer, 1986; Miyazawa, 2021).

In the young period, each individual is endowed with 1 unit of time and allocates it between working h^j_t and leisure $1 - h^j_t$ ($j \in \{e, l\}$). Then, lifetime utility (3) and the budget constraint in the young period are replaced by $U^j_t = (1-\beta)\left[\ln c^{y,j}_t + \epsilon\ln(1-h^j_t)\right] + \beta\left[(1-\gamma)\ln c^{o,j}_{t+1} + \gamma \ln x^j_{t+1}\right]$ ($j \in \{e, l\}$) and $c^{y,e}_t = w_t h^e_t + x_t + \sigma_k k_{t+1} - a_{t+1}$ with $a_{t+1} = k_{t+1} - d_t$ (resp. $c^{y,l}_t = w_t h^l_t + x_t - l_t - q_t b_{t+1}$) for borrowers (resp. lenders). The parameter $\epsilon(>0)$ represents the preference for leisure.

Appendix I explains this extension of the model in detail and shows the following. Households whose bequests from parents x_t are greater than $w_t/[\epsilon(1-\beta)]$ supply no labor, regardless of whether they are borrowers or lenders ($h^j_t = 0$, $j \in \{e, l\}$). Additionally, their bequests (assets) include public bonds when their parents are lenders. Thus, total labor supply decreases with the public debt-to-capital ratio. However, the main results thus far are robust, because we can derive the same equations as (41) and (42).

8 Conclusion

This study examines the effect of public debt on growth, interest rates, and fiscal sustainability using a simple endogenous growth model with financial frictions and firm heterogeneity. An increase in public debt leads to higher real interest rates through financial markets increase the cost of repaying public debt, and reduce private investment, leading to lower long-run growth. Thus, large public debt is less sustainable. This study also examines the effect of investment subsidies financed by public debt. We finds that they hamper economic growth in the long run unless the credit market is close to perfect. Therefore, increases in investment subsidies should be financed not only by issuing public bonds, but also through tax increases. This is an important policy implication for many developed countries that have increased investment subsidies while relying heavily on public debt for fiscal management.

This study also briefly analyzes and discusses how fiscal policy, including government debt, affects wealth inequality. An increase in the public debt-to-capital ratio reduces inequality between active entrepreneurs and non-active entrepreneur as well as among active entrepreneurs. Investment subsidies financed by public debt also reduce economic inequality in the steady state.

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Appendices

A Proof of $U_t^e = U_t^l$ for $z_{i,t} = z_t^*$

Let us denote the items $(c_t^{y,j}, c_{t+1}^{o,j}, x_{t+1}^j, x_t, k_{t+1}, y_{t+1})$, and $j \in \{e, l\}$ of households whose productivity is z_t^* as $(\bar{c}_t^{y,j}, \bar{c}_{t+1}^{o,j}, \bar{x}_{t+1}^j \bar{x}_t, \bar{k}_{t+1}, \bar{y}_{t+1})$, respectively. First, consider the case in which households with $z_{i,t} = z_t^*$ become entrepreneurs.

Substituting $d_{i,t} = \lambda k_{i,t+1}$ and (19) into (4) yields

$$\bar{c}_t^{y,e} = (1-\beta)(w_t + \bar{x}_t), \tag{A.1}$$

while inserting (16) and (17) into (5) yields

$$\bar{c}_{t+1}^{o,e} = (1-\gamma)(\alpha \bar{y}_{i,t+1} - \lambda R_{t+1} \bar{k}_{i,t+1}). \tag{A.2}$$

From (14), $\bar{y}_{t+1} = A z_t^* \bar{k}_{t+1} \left[\frac{(1-\alpha)AK_t}{w_t} \right]^{\frac{1-\alpha}{\alpha}}$. Substituting (15) into this, we obtain $\bar{y}_{t+1} = (1 - \sigma_k) R_{t+1} \bar{k}_{t+1}$. Hence, (A.2) can be rewritten as

$$\begin{aligned} \bar{c}_{t+1}^{o,e} &= (1-\gamma)(1-\sigma_k-\lambda) R_{t+1} \bar{k}_{i,t+1} \\ &= \beta(1-\gamma) R_{t+1}(w_t + \bar{x}_t), \end{aligned} \tag{A.3}$$

where we use (19). Substituting $\bar{y}_{t+1} = (1-\sigma_k) R_{t+1} \bar{k}_{t+1}$ and (17) into (16), we obtain

$$\bar{x}_{t+1}^e = \beta \gamma R_{t+1}(w_t + \bar{x}_t). \tag{A.4}$$

Second, consider the case in which households with $z_{i,t} = z_t^*$ become lenders. Substituting (22), (23), and (24) into (20) and (21), we obtain

$$\bar{c}_t^{y,l} = (1-\beta)(w_t + \bar{x}_t), \tag{A.5}$$

$$\bar{c}_{t+1}^{o,l} = \beta(1-\gamma) R_{t+1}(w_t + \bar{x}_t). \tag{A.6}$$

From (23) and (24), we obtain

$$\bar{x}_{t+1}^l = \beta \gamma R_{t+1}(w_t + \bar{x}_t). \tag{A.7}$$

Owing to $\bar{c}_t^{y,e} = c_{i,t}^{y,l}$, $\bar{c}_{t+1}^{o,e} = c_{t+1}^{o,l}$, and $\bar{x}_{t+1}^e = \bar{x}_{t+1}^l$ and (3), $U_t^e = U_t^l$ holds for $z_{i,t} = z_t^*$.

B Derivations of (36), (37), and (39)

By aggregating (19), we obtain

$$\int_{z_t^*}^{\infty} k_{t+1}(z_t, x_t) dF(z_t) \int_1^{\infty} dF(z_{t-1})$$
$$= \frac{\beta}{1-\sigma_k-\lambda} \int_{z_t^*}^{\infty} dF(z_t) \left[w_t \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) \right]. \tag{B.1}$$

Here, we used the facts that (i) z is iid and (ii) x_t is independent of z_t (see Remark 1 for (ii)). (B.1) together with (29), (34), and $X_t \equiv \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1})$ yields (36).

Next, we derive (37). From (16) and (17), we obtain $x_{t+1}^e(z_t, x_t) = \gamma(\alpha y_{t+1}(z_t, x_t) - R_{t+1}\lambda k_{t+1}(z_t, x_t))$. Aggregating this using (29) and (33), we obtain

$$\int_{z_t^*}^{\infty} x_{t+1}^e(z_t, x_t) dF(z_t) = \gamma \left(\frac{\varphi(1-\sigma_k)}{\varphi - 1} - \lambda \right) R_{t+1} K_{t+1}. \tag{B.2}$$

Next, aggregating $x_{i,t+1}^l$ in (24) leads to

$$\int_1^{z_t^*} x_{t+1}^l(z_t, x_t) dF(z_t) = \gamma \left[R_{t+1} \int_1^{z_t^*} l_t(z_t, x_t) dF(z_t) + \int_1^{z_t^*} b_{t+1}(z_t, x_t) dF(z_t) \right].$$

This, together with (2), (27), and (30), yields

$$\int_1^{z_t^*} x_{t+1}^l(z_t, x_t) dF(z_t) = \gamma(R_{t+1}\lambda K_{t+1} + B_{t+1}). \tag{B.3}$$

Associating $X_{t+1} \equiv \int_1^{\infty} x_{t+1}(z_t, x_t) dF(z_t) = \int_1^{z_t^*} x_{t+1}^l(z_t, x_t) dF(z_t) + \int_{z_t^*}^{\infty} x_{t+1}^e(z_t, x_t) dF(z_t)$ with (35), (B.2), and (B.3) yields

$$X_{t+1} = \gamma \left[\frac{\varphi(1-\sigma_k)}{\varphi - 1} R_{t+1} K_{t+1} + B_{t+1} \right]. \tag{B.4}$$

Thus, we obtain (37).

Finally, we derive (39). By aggregating (23) as

$$\left[\int_1^{z_t^*} l_t(z_t, x_t) dF(z_t) + \int_1^{z_t^*} q_t b_{t+1}(z_t, x_t) dF(z_t) \right] \int_1^{\infty} dF(z_{t-1})$$
$$= \beta \int_1^{z_t^*} dF(z_t) \left[w_t \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) \right], \tag{B.5}$$

and substituting (27), (34), and $X_t \equiv \int_1^\infty x_t(z_{t-1}, x_{t-1}) dF(z_{t-1})$ into (B.5), we obtain (39).

C Choice of the parameter values for the numerical example

We set $\beta = 0.3$ because the discount factor should be $\beta/(1-\beta) = 0.97^{30} \approx 0.4$. The parameter α is set to 0.4, which is near the average values of the United States (US) ($\alpha = 0.35$), the EU ($\alpha = 0.38$), and Japan ($\alpha = 0.38$).[25] We select the value of γ to satisfy $\gamma = \beta$ as the benchmark. The scale parameter $A = 5$ yields positive plausible values for the long-run growth rates, as shown in Figures 2, 4, and 6. We select $\lambda = 0.7$ so that the degree of financial market imperfection serves as the benchmark case. Finally, following Diamond and Saez (2011), Jaimovich and Rebelo (2017), and Mino (2015), we set $\varphi = 1.5$. This choice of φ suggests that the right tail of the income distribution implied by the model is the same as that estimated by Diamond and Saez (2011) for the US economy.

D Numerical studies on growth effects when the liquidity effect is large

We discuss the case when the liquidity effect is large, with high values of β and γ (e.g., $\beta = \gamma = 0.5$). Figure 5 shows the following. The relationship between the growth in capital $g^K(\theta_t; \sigma_k)$ and the debt-to-capital ratio θ_t is inverted U-shaped, indicating that when θ_t is small (large), increases in θ_t raise (lower) $g^K(\theta_t; \sigma_k)$. However, the growth in GDP $g^Y(\theta_t; \sigma_k)$ remains monotonically decreasing in θ_t.

To understand this, recall that $Y_{t+1}/Y_t = (R_{t+1}/R_t)(K_{t+1}/K_t)$ holds. Figure 5 shows that the growth in the interest rate R_{t+1}/R_t is decreasing (resp. increasing) in θ_t when θ_t is small (resp. large), because θ_t is likely to converge (resp. deverge from) to the stable level θ_S^*. The negative effect of an increase in θ_t on R_{t+1}/R_t dominates the positive effect of an increase in θ_t on $g^K(\theta_t; \sigma_k)$. Thus, $g^Y(\theta_t; \sigma_k)$ remains monotonically decreasing in θ_t.

[Figure 5]

Even under large values of λ (e.g., $\lambda = 0.9$ and 0.95), which reinforces the positive firm selection effects (see Remark 2), the growth in GDP $g^Y(\theta_t; \sigma_k)$ remains monotonically decreasing in θ_t.

[25] See Trabandt and Uhlig (2011) for the values of the United States and the EU, and Hansen and İmrohoroğlu (2016) for the value of Japan.

E Numerical studies on the effects of investment subsidies on fiscal sustainability

Figure 6(a) uses the parameter values in Table 1 and illustrates the case of an increase in σ_k from 0.01 to 0.04. It shows that increases in σ_k (i) make public debt less sustainable and (ii) raise the ratio of public debt to capital θ_S^* in the steady state S.

[Figure 6]

F Investment subsidies financed by public debt and income tax

Entrepreneurs' budget constraints with a constant income tax rate τ are given by

$$c_t^{y,e} = (1-\tau)w_t h_t^e + x_t + \sigma_k k_{t+1} - a_{t+1}, \qquad a_{t+1} = k_{t+1} - d_t, \tag{F.1}$$

$$c_{t+1}^{o,e} = (1-\tau)\pi_{t+1} - x_{t+1}^e \qquad \text{with (6).} \tag{F.2}$$

The FOC with respect to n_{t+1} is given by (8) and those with respect to d_t and k_{t+1} are replaced by

$$d_t; \qquad \frac{1-\beta}{c_t^{y,e}} = \frac{\beta(1-\gamma)(1-\tau)R_{t+1}}{c_{t+1}^{o,e}} + \tilde{\mu}_t, \tag{F.3}$$

$$k_{t+1}; \qquad \frac{(1-\beta)(1-\sigma_k)}{c_t^{y,e}} = \frac{\beta(1-\gamma)(1-\tau)}{c_{t+1}^{o,e}}\frac{\partial \pi_{t+1}}{\partial k_{t+1}} + \lambda\tilde{\mu}_t, \tag{F.4}$$

$$\tilde{\mu}_t(\lambda k_{i,t+1} - d_t) = 0, \;\; \tilde{\mu}_t \geq 0, \;\; \lambda k_{t+1} - d_t \geq 0, \tag{F.5}$$

$$\tilde{\mu}_t = \frac{\beta(1-\gamma)(1-\tau)}{1-\sigma_k-\lambda}\frac{\alpha(y_{t+1}/k_{t+1}) - (1-\sigma_k)R_{t+1}}{c_{t+1}^{o,e}}. \tag{F.6}$$

Note that (13), (14), and (15) remain unchanged. Maximizing utility with respect to x_{t+1}^e yields

$$x_{t+1}^e = \gamma(1-\tau)\pi_{t+1}. \tag{F.7}$$

From (F.1), (F.2), (F.3), (F.6), (F.7), (17) and $d_t = \lambda k_{t+1}$, we obtain

$$k_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}[(1-\tau)w_t + x_t]. \tag{F.8}$$

The budget constraints of lenders with a balanced budget and without public debt are

$$c_t^{y,l} = (1-\tau)w_t + x_t - l_t - q_t b_{t+1}, \tag{F.9}$$

$$c_{t+1}^{o,l} = (1-\tau)(R_{t+1}l_t + b_{t+1}) - x_{t+1}^l. \tag{F.10}$$

The FOCs of the lenders with respect to $l_t + q_t b_{t+1}$ and x^l_{t+1} result in

$$l_t + q_t b_{t+1} = \beta\left[(1-\tau)w_t + x_t\right], \tag{F.11}$$

$$x^l_{t+1} = \gamma(1-\tau)(R_{t+1}l_t + b_{t+1}). \tag{F.12}$$

The equations (29) to (35) remain unchanged. Aggregating (F.8) and using (34), we obtain

$$K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}(z^*_t)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)(1-\tau)R_tK_t + X_t\right]. \tag{F.13}$$

From (F.7) and (17), we obtain $x^e_{t+1}(z_t,x_t) = \gamma(1-\tau)[\alpha y_{t+1}(z_t,x_t) - R_{t+1}\lambda k_{t+1}(z_t,x_t)]$. Aggregating this using (29) and (33), we obtain

$$\int_{z^*_t}^{\infty} x^e_{t+1}(z_t,x_t)dF(z_t) = \gamma\left(\frac{\varphi(1-\sigma_k)}{\varphi-1} - \lambda\right)(1-\tau)R_{t+1}K_{t+1}. \tag{F.14}$$

Next, aggregating x^l_{t+1} in (F.12) with (2), (27), and (30), we obtain

$$\int_1^{z^*_t} x^l_{t+1}(z_t,x_t)dF(z_t) = \gamma(1-\tau)(R_{t+1}\lambda K_{t+1} + B_{t+1}). \tag{F.15}$$

Associating $X_{t+1} = \int_1^{z^*_t} x^l_{t+1}(z_t,x_t)dF(z_t) + \int_{z^*_t}^{\infty} x^e_{t+1}(z_t,x_t)dF(z_t)$ with (35), $\tilde{B}_{t+1} = q_tB_t$, $R_{t+1} = 1/q_t$, (F.14), and (F.15) yields

$$X_{t+1} = \gamma(1-\tau)R_{t+1}\left[\frac{\varphi(1-\sigma_k)}{\varphi-1}K_{t+1} + \tilde{B}_{t+1}\right]. \tag{F.16}$$

From (F.13) and (F.16), we obtain

$$K_{t+1} = \frac{\beta(1-\tau)}{1-\sigma_k-\lambda}(z^*_t)^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)R_tK_t + \gamma R_t\tilde{B}_t\right]. \tag{F.17}$$

Aggregating (F.11) and using (34), (27), $\tilde{B}_{t+1} = q_tB_{t+1}$, and $R_{t+1} = 1/q_t$, we obtain

$$\int_1^{z^*_t} l_t(z_t,x_t)dF(z_t) = \beta\left[1-(z^*_t)^{-\varphi}\right]\left[\frac{(1-\alpha)\varphi}{\alpha(\varphi-1)}(1-\tau)R_tK_t + X_t\right] - \tilde{B}_{t+1}. \tag{F.18}$$

Substituting (30) and (F.16) into (F.18), we obtain

$$\lambda K_{t+1} = \beta\left[1-(z^*_t)^{-\varphi}\right]\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)(1-\tau)R_tK_t + \gamma\tilde{B}_t\right] - \tilde{B}_{t+1}. \tag{F.19}$$

From (F.17) and (F.19), we obtain

$$(1-\sigma_k)K_{t+1} = \beta(1-\tau)R_t\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)K_t + \gamma\tilde{B}_t\right] - \tilde{B}_{t+1}. \tag{F.20}$$

Substituting (17) and (31) into (60), we obtain

$$\begin{aligned} T_t =& \tau\Bigg[w_t + \alpha\int_{z_{t-1}^*}^{\infty} y_t(z_{t-1},x_{t-1})dF(z_{t-1}) - R_t\lambda\int_{z_{t-1}^*}^{\infty} k_t(z_{t-1},x_{t-1})dF(z_{t-1}) \\ &+ R_t\int_1^{z_{t-1}^*} l_{t-1}(z_{t-1},x_{t-1})dF(z_{t-1}) + q_{t-1}\int_1^{z_{t-1}^*} b_t(z_{t-1},x_{t-1})dF(z_{t-1})\Bigg]. \end{aligned} \tag{F.21}$$

Substituting (27) with $\tilde{B}_{t+1} = q_t B_t$ and $R_{t+1} = 1/q_t$, (30), $\int_{z_{t-1}^*}^{\infty} k_t(z_{t-1},x_{t-1})dF(z_{t-1}) = K_t$, $\int_{z_{t-1}^*}^{\infty} y_t(z_{t-1},x_{t-1})dF(z_{t-1}) = Y_t$ with (33), and (34) into (F.21), we obtain

$$T_t = \tau\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}R_tK_t + R_t\tilde{B}_t\right]. \tag{F.22}$$

Substituting (29) and (F.22) into (59) and using $\tilde{B}_{t+1} = q_tB_t$ and $R_{t+1} = 1/q_t$, we obtain

$$\tilde{B}_{t+1} = R_t\tilde{B}_t - \tau\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}R_tK_t + R_t\tilde{B}_t\right] + \sigma_kK_{t+1}. \tag{F.23}$$

From (F.20) and (F.23), we obtain

$$\frac{K_{t+1}}{K_t} = R_t\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\tau)(1-\alpha+\alpha\gamma) + \tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - (1-\tau)(1-\beta\gamma)\theta_t\right], \tag{F.24}$$

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} = R_t\left[\sigma_k\beta(1-\tau)\left\{\tfrac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)\theta_t^{-1} + \gamma\right\} + (1-\sigma_k)\left\{1-\tau-\tau\tfrac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\theta_t^{-1}\right\}\right]. \tag{F.25}$$

To ensure the positive growth in capital $K_{t+1}/K_t > 0$, we assume

$$\theta_t < \bar{\theta}' \equiv \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau]}{(1-\tau)(1-\beta\gamma)}. \tag{F.26}$$

From (F.24) and (F.25), we obtain

$$\begin{aligned}\theta_{t+1} &= \frac{(1-\sigma_k)\left[(1-\tau)\theta_t - \tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\right] + \sigma_k\beta(1-\tau)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)+\gamma\theta_t\right]}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\beta(1-\tau)(1-\alpha+\alpha\gamma) + \tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - (1-\tau)(1-\beta\gamma)\theta_t} \\ &= \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\sigma_k\beta(1-\tau)(1-\alpha+\alpha\gamma) - (1-\sigma_k)\tau\right] + (1-\tau)(1-\sigma_k+\sigma_k\gamma\beta)\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_t} \\ &\equiv \hat{\Lambda}(\theta_t;\sigma_k,\tau). \end{aligned} \tag{F.27}$$

From (F.27), we obtain the following lemma.

Lemma 4. *Two steady states, S and U, as shown in Lemma 2, exist under the following conditions:*

$$\begin{aligned}&\left[(1-\tau)\{1-\sigma_k(1-\beta\gamma)\} - \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\{\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\}\right]^2 \\ &> 4(1-\tau)(1-\beta\gamma)\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\sigma_k\beta(1-\tau)(1-\alpha+\alpha\gamma) - (1-\sigma_k)\tau\right], \end{aligned} \tag{F.28}$$

$$\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\{\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\} > (1-\tau)\{1-\sigma_k(1-\beta\gamma)\}, \tag{F.29}$$

$$\hat{\Lambda}(\bar{\theta}';\sigma_k,\tau) > \bar{\theta}'. \tag{F.30}$$

From (F.17) and (F.24), we obtain

$$\begin{aligned}z_t^* &= \left\{\left(\frac{\beta(1-\tau)}{1-\sigma_k-\lambda}\right)\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_t}\right\}^{1/\varphi} \\ &\equiv \hat{z}^*(\theta_t;\sigma_k,\tau). \end{aligned} \tag{F.31}$$

Substituting (F.31) into (35) yields

$$\begin{aligned}R_{t+1} =\ & \frac{\alpha A}{1-\sigma_k}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\left(\frac{\beta(1-\tau)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}} \\ & \times\left[\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-\sigma_k)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_t}\right]^{\frac{\alpha}{\varphi}} \\ \equiv\ & \hat{\mathcal{R}}(\theta_t;\sigma_k,\tau). \end{aligned} \tag{F.32}$$

From (F.31) and (F.32), we obtain the same properties as in Lemma 3 in the benchmark model.

Lemma 5. $\hat{z}^{*\prime}(\theta_t;\sigma_k,\tau) > 0$ *and* $\hat{\mathcal{R}}'(\theta_t;\sigma_k,\tau) > 0$.

Substituting (F.32) into (F.24), we obtain the following growth rate of GDP at the steady state:

$$
\begin{aligned}
\frac{Y_{t+1}}{Y_t} &\equiv \hat{g}^Y(\theta_S^*;\sigma_k,\tau) \left(= \frac{K_{t+1}}{K_t} \equiv \hat{g}^K(\theta_S^*;\sigma_k,\tau) \right) \\
&= \hat{\mathcal{R}}(\theta_S^*;\sigma_k,\tau) \left[\tfrac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} \left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau \right] - (1-\tau)(1-\beta\gamma)\theta_S^* \right]. \qquad \text{(F.33)}
\end{aligned}
$$

Before proceeding, considering the balanced budget case that does not incorporate government bonds ($\tilde{B}_t = 0$ for any period) helps explain the effects on economic growth when investment subsidies are funded through tax increases. Applying $\tilde{B}_t = 0$ (or $\theta_t = 0$) for any $t \geq 0$ induces (F.23) and (F.24) into

$$
\tau \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} R_t K_t = \sigma_k K_{t+1}, \qquad \text{(F.34)}
$$

$$
\frac{K_{t+1}}{K_t} = \beta \frac{\varphi}{\alpha(\varphi-1)} (1-\alpha+\alpha\gamma)(1-\tau) R_t, \qquad \text{(F.35)}
$$

both of which lead to

$$
\tau = \frac{\beta\sigma_k(1-\alpha+\alpha\gamma)}{1-\sigma_k+\beta\sigma_k(1-\alpha+\alpha\gamma)}. \qquad \text{(F.36)}
$$

Because (F.31) is reduced to $z_t^* = z^* = \left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{1}{\varphi}}$, (F.32) and (F.33) become

$$
\begin{aligned}
R_t = R &= \frac{\alpha A}{1-\sigma_k} \left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha} \left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}}, \\
\frac{Y_{t+1}}{Y_t} &= \frac{\beta A}{1-\sigma_k} \left(\frac{\varphi}{\varphi-1}\right)^{\alpha} \frac{1-\alpha+\alpha\gamma}{1-\sigma_k[1-\beta(1-\alpha+\alpha\gamma)]} \left(\frac{\beta(1-\sigma_k)}{1-\sigma_k-\lambda}\right)^{\frac{\alpha}{\varphi}},
\end{aligned}
$$

where we use (F.36) to derive the long-run growth rate. As $\varphi > 1$, $\partial(Y_{t+1}/Y_t)/\partial\sigma_k > 0$ for all $\sigma_k > 0$. Therefore, investment subsidies fully financed by income tax enhance economic growth. Even with public debt, the increase in σ_k with income tax financing enhances economic growth.

To see this, we return to the original topic. The total differentials of (F.33), (F.32), and (F.27) are given by

$$
d\hat{g}^Y = \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} d\hat{\mathcal{R}} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} d\tau + \frac{\partial \hat{g}^Y(\cdot)}{\partial \sigma_k} d\sigma_k + \frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} d\theta_S^*, \qquad \text{(F.37)}
$$

$$
d\hat{\mathcal{R}} = \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} d\tau + \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \sigma_k} d\sigma_k + \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} d\theta_S^*, \qquad \text{(F.38)}
$$

$$
d\theta_S^* = \frac{\partial \theta_S^*}{\partial \tau} d\tau + \frac{\partial \theta_S^*}{\partial \sigma_k} d\sigma_k. \qquad \text{(F.39)}
$$

Substituting (F.38) and (F.39) into (F.37), we obtain

$$\begin{aligned} d\hat{g}^Y = & \left(\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} \right) d\tau \\ & + \underbrace{\left(\frac{\partial \hat{g}^Y(\cdot)}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \sigma_k} \right)}_{(-)} d\sigma_k. \end{aligned} \tag{F.40}$$

The second term on the LHS of (F.40) is negative because we consider the case in which $d\hat{g}^Y(\cdot)/d\sigma_k < 0$ for $d\tau = 0$. Thus, $d\hat{g}^Y(\cdot)/d\sigma_k > 0$ if and only if

$$\frac{d\tau}{d\sigma_k} > -\frac{\frac{\partial \hat{g}^Y(\cdot)}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \sigma_k} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \sigma_k}}{\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau}}, \tag{F.41}$$

since the denominator on the RHS (the growth effects of income tax) takes a positive value for any λ numerically. (F.41) indicates that a marginal increase in the tax rate in response to a subsidy increase $d\tau/d\sigma_k$ must be larger than the the value in the RHS to ensure the positive growth effect of income tax rate. To see more, we divide the growth effects of income tax into the direct effects $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau}$ and the indirect effects (through changes in θ_S^*) $\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} + \frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau}$.

The direct effects are $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} > 0$ from (F.43) and $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} < 0$ from (F.42) and (F.48). The reasons for this are as follows. First, increases in income tax have a distortionary effect on growth, while they also reduce the issuance of public bonds, decrease the crowding-out effect of public debt on capital accumulation, and promote economic growth. The latter effects dominate the former, leading to $\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} > 0$. Second, increases in income tax have a distortionary effect on capital accumulation, which decreases demand for credit in the financial market and lowers interest rates. Lower interest rates decrease borrowing costs and increase the number of less productive firms, which negatively affects growth through the negative firm selection effect $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} < 0$.

The indirect effects are $\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} > 0$ ($\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} < 0$ from (F.45) and $\frac{\partial \theta_S^*}{\partial \tau} < 0$ numerically) and $\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} \frac{\partial \theta_S^*}{\partial \tau} < 0$ ($\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} \frac{\partial \hat{\mathcal{R}}}{\partial \theta_S^*} > 0$ from (F.42) and (F.47), and $\frac{\partial \theta_S^*}{\partial \tau} < 0$ numerically). The reasons for this are as follows. First, an increase in the income tax rate boosts tax revenue and can reduce the issuance of public bonds, which lowers the ratio of public debt to capital θ_S^*. This reduction in θ_S^* decreases the costs of repaying public debt and issuing new public bonds, thereby crowding in capital accumulation and enhancing economic growth. Second, a lower θ_S^* decreases the interest rate $\hat{\mathcal{R}}(\cdot)$ because it reduces both the costs of repaying public debt and the issuance of new public bonds, increasing the aggregate supply of credit in the financial market. This increase in the supply of credit promotes the entry of less productive firms, and lowers economic growth through the negative firm selection effect.

These positive direct and indirect effects on growth dominate the negative ones. Therefore, the total effect of income tax on growth is positive. To check this, we select the cases of $\lambda = 0.6, 0.7$ and 0.8 without changing the other parameter values and calculate a marginal increase in the tax rate from $\tau = 0$ in response to a subsidy increase for each value of σ_k to ensure the positive growth effects of σ_k. Here, recall that the growth effect of σ_k is always negative when $\lambda = 0.6$ and 0.7, while it exhibits an inverted U-shape when $\lambda = 0.8$ if investment subsidies are financed entirely by public debt.

The results in Table 2 show the following. Considering a marginal increase in the tax rate from $\tau = 0$ in response to a subsidy increase: $d\tau/d\sigma_k$, we find that the growth effect of σ_k turns positive, $d\hat{g}^Y(\theta_S^*; \sigma_k, \tau)/d\sigma_k|_{\tau=0} > 0$, if and only if $d\tau/d\sigma_k|_{\tau=0}$ is higher than the value in Table 2 for each value of σ_k.

In the case of $\lambda = 0.8$, a marginal increase in the tax rate in response to a subsidy increase $d\tau/d\sigma_k|_{\tau=0}$ takes negative values for low values of σ_k because the growth effects of σ_k even without tax increases are already positive for these values of σ_k (as we have seen previously) and no tax increases are necessary.

[Table 2]

The remainder of this appendix provides the partial derivatives that consist of (F.41). First, from (F.33), we obtain

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \hat{\mathcal{R}}} = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_S^* > 0 \text{ by (F.26),} \tag{F.42}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \tau} = \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau)\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\beta(1-\alpha+\alpha\gamma)) + (1-\beta\gamma)\theta_S^*\right] > 0, \tag{F.43}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \sigma_k} = -\frac{\varphi}{\alpha(\varphi-1)}\hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau)[\tau + \beta(1-\tau)(1-\alpha+\alpha\gamma)] < 0, \tag{F.44}$$

$$\frac{\partial \hat{g}^Y(\cdot)}{\partial \theta_S^*} = (1-\tau)\hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau)(1-\beta\gamma) > 0. \tag{F.45}$$

Second, from (F.32), we obtain

$$\begin{aligned}\frac{\partial \ln \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = \ & \frac{\alpha}{\varphi}\frac{\gamma}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)+\gamma\theta_S^*} \\ & + \frac{\alpha}{\varphi}\frac{(1-\tau)(1-\beta\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right]-(1-\tau)(1-\beta\gamma)\theta_S^*}.\end{aligned} \tag{F.46}$$

Because of $\frac{\partial \ln \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = \frac{1}{\hat{\mathcal{R}}} \frac{\hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*}$ and (F.46), we obtain

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \theta_S^*} = {} & \frac{\alpha}{\varphi} \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau) \Bigg[\frac{\gamma}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma) + \gamma\theta_S^*} \\ & + \frac{(1-\tau)(1-\beta\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_S^*} \Bigg] > 0 \text{ by (F.26).}\end{aligned} \tag{F.47}$$

Similarly, we have

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \tau} = {} & -\frac{\alpha}{\varphi} \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau) \\ & \times \left[\frac{1}{1-\tau} + \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\beta(1-\alpha+\alpha\gamma)) + (1-\beta\gamma)\theta_S^*}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_S^*} \right] < 0 \text{ by (F.26),}\end{aligned} \tag{F.48}$$

$$\begin{aligned}\frac{\partial \hat{\mathcal{R}}(\cdot)}{\partial \sigma_k} = {} & \hat{\mathcal{R}}(\theta_S^*; \sigma_k, \tau) \Bigg[\frac{1}{1-\sigma_k} + \frac{\alpha}{\varphi}\frac{1}{1-\sigma_k-\lambda} - \frac{\alpha}{\varphi} \frac{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma) + \gamma\theta_S^*} \\ & + \frac{\alpha}{\varphi} \frac{\frac{\varphi}{\alpha(\varphi-1)}[\tau + \beta(1-\tau)(1-\alpha+\alpha\gamma)]}{\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}\left[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau\right] - (1-\tau)(1-\beta\gamma)\theta_S^*} \Bigg].\end{aligned} \tag{F.49}$$

Finally, from (F.27), we obtain

$$\frac{\partial \theta_S^*}{\partial \tau} = \frac{\Theta(\theta_S^*; \sigma_k, \tau)}{\Omega(\theta_S^*; \sigma_k, \tau)}, \quad \frac{\partial \theta_S^*}{\partial \sigma_k} = \frac{\Upsilon(\theta_S^*; \sigma_k, \tau)}{\Omega(\theta_S^*; \sigma_k, \tau)}, \tag{F.50}$$

$$\begin{aligned}\Omega(\theta_S^*; \sigma_k, \tau) \equiv {} & -\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}[\beta(1-\tau)(1-\alpha+\alpha\gamma)+\tau] + (1-\tau)(1-\beta\gamma)\theta_S^* \\ & + (1-\tau)[1-\sigma_k(1-\beta\gamma)], \\ \Theta(\theta_S^*; \sigma_k, \tau) \equiv {} & \theta_S^* \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\beta(1-\alpha+\alpha\gamma)) + (1-\beta\gamma)\theta_S^* \right] + [1-\sigma_k(1-\beta\gamma)]\theta_S^* \\ & + \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}[1-\sigma_k+\sigma_k\beta(1-\alpha+\alpha\gamma)], \\ \Upsilon(\theta_S^*; \sigma_k, \tau) \equiv {} & -\frac{\varphi}{\alpha(\varphi-1)}[\tau+\beta(1-\tau)(1-\alpha+\alpha\gamma)]\theta_S^* + (1-\tau)(1-\beta\gamma)\theta_S^* \\ & - 2\tau\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} - \beta\frac{\varphi}{\alpha(\varphi-1)}(1-\tau)(1-2\sigma_k)(1-\alpha+\alpha\gamma).\end{aligned}$$

G A fixed proportion of government borrowing allocated to investment subsidies

Under the endogenous $\sigma_{k,t}$, equations (35), (38), (41) and (42) are replaced by

$$z_t^* = \left(\frac{\varphi}{\varphi-1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\sigma_{k,t})R_{t+1}}{\alpha A}\right)^{\frac{1}{\alpha}}. \tag{G.1}$$

$$K_{t+1} = \frac{\beta}{1-\sigma_{k,t}-\lambda}(z_t^*)^{-\varphi}R_t\left[\frac{\varphi(1-\sigma_{k,t-1})}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)K_t+\gamma\tilde{B}_t\right]. \tag{G.2}$$

$$K_{t+1}+\tilde{B}_{t+1} = \beta R_t\left[\frac{\varphi(1-\sigma_{k,t-1})}{\alpha(\varphi-1)}(1-\alpha+\alpha\gamma)K_t+\gamma\tilde{B}_t\right]+\sigma_{k,t}K_{t+1}. \tag{G.3}$$

$$\tilde{B}_{t+1} = R_t\tilde{B}_t+\sigma_{k,t}K_{t+1}. \tag{G.4}$$

From (G.3) and (G.4) together with $\sigma_{k,t} = v\theta_{t+1}$, we obtain

$$\frac{K_{t+1}}{K_t} = R_t\left[\frac{\varphi(1-v\theta_t)}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t\right]. \tag{G.5}$$

Here, to ensure $K_{t+1}/K_t > 0$, we assume the following condition:

$$\theta_t < \tilde{\tilde{\theta}} \equiv \frac{\varphi\beta(1-\alpha+\alpha\gamma)}{v\varphi\beta(1-\alpha+\alpha\gamma)+\alpha(\varphi-1)(1-\beta\gamma)}. \tag{G.6}$$

From $v\tilde{B}_{t+1} = \sigma_{k,t}K_{t+1}$ and (G.4), we obtain

$$\frac{\tilde{B}_{t+1}}{\tilde{B}_t} = \frac{R_t}{1-v}. \tag{G.7}$$

(G.1), (G.2), (G.5), and (G.7) together with $\sigma_{k,t} = v\theta_{t+1}$ yields

$$\theta_{t+1} = \frac{(1-v)^{-1}\alpha(\varphi-1)\theta_t}{\varphi\beta(1-\alpha+\alpha\gamma)-[v\varphi\beta(1-\alpha+\alpha\gamma)+\alpha(\varphi-1)(1-\beta\gamma)]\theta_t} \equiv \tilde{\Lambda}(\theta_t, v), \tag{G.8}$$

$$\begin{aligned} z_t^* &= \left\{\left(\frac{\beta}{1-v\tilde{\Lambda}(\theta_t,v)-\lambda}\right)\frac{\frac{\varphi}{\alpha(\varphi-1)}(1-v\theta_t)(1-\alpha+\alpha\gamma)+\gamma\theta_t}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-v\theta_t)(1-\alpha+\alpha\gamma)-(1-\beta\gamma)\theta_t}\right\}^{1/\varphi} \\ &\equiv \tilde{z}^*(\theta_t, v), \end{aligned} \tag{G.9}$$

$$R_{t+1} = \frac{\alpha A}{1-v\tilde{\Lambda}(\theta_t,v)}\left(\frac{\varphi-1}{\varphi}\right)^{1-\alpha}\tilde{z}^*(\theta_t,v)^{\alpha} \equiv \tilde{\mathcal{R}}(\theta_t, v). \tag{G.10}$$

The RHS of (G.8) obviously satisfies $\tilde{\Lambda}(0,v) = 0$, $\tilde{\Lambda}'(\theta_t,v) > 0$, and $\tilde{\Lambda}''(\theta_t,v) > 0$. Then, if the initial value θ_0 is larger than θ_U^* in Figure 7, public debt is not sustainable. Furthermore, an increase in v shifts $\tilde{\Lambda}(\theta_t, v) > 0$ upward, and therefore, makes public debt less sustainable.

In the stable steady state S, $\theta_S^* = 0$ always holds independently of λ (the degree of financial friction), as depicted in Figure 7. When λ is small, the relationship between σ_k and economic growth becomes negative. Therefore, this result is equivalent to reducing σ_k to zero under a fixed subsidy rate of $\sigma_k = 0$ (in Subsection 6.2). When λ is large, the positive constant σ_k can promote economic growth, whereas endogenous $\sigma_{k,t}$ remains at zero and cannot do so in the steady state.

[Figure 7]

H Derivations of (61) and (63) and the effect of σ_k on inequality in the steady state.

From (16), (17), (33), and (36), we obtain

$$X_{t+1}^e = \gamma \left(\frac{\varphi(1-\sigma_k)}{\varphi - 1} - \lambda \right) \frac{\beta R_{t+1}}{1-\sigma_k-\lambda} (z_t^*)^{-\varphi} \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_t K_t + X_t \right]. \tag{H.1}$$

From (24) (with (22)) and (39), we obtain

$$X_{t+1}^l = \gamma \beta R_{t+1} \left[1 - (z_t^*)^{-\varphi} \right] \left[\frac{(1-\sigma_k)\varphi}{\alpha(\varphi-1)}(1-\alpha)R_t K_t + X_t \right]. \tag{H.2}$$

(H.1), (H.2), and (49) yield (61).

$$\begin{aligned} &\int_{z_{p,t}^*}^{\infty} k_{t+1}(z_t, x_t) dF(z_t) \int_1^{\infty} dF(z_{t-1}) \\ &= \frac{\beta}{1-\sigma_k-\lambda} \int_{z_{p,t}^*}^{\infty} dF(z_t) \left[w_t \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) \right]. \end{aligned} \tag{H.3}$$

Here, we used the facts that (i) z is iid and (ii) x_t is independent of z_t (see Remark 1 for (ii)). (H.3), (34), and $X_t \equiv \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1})$ yield

$$\int_{z_{p,t}^*}^{\infty} k_{t+1}(z_t, x_t) dF(z_t) = \frac{\beta}{1-\sigma_k-\lambda} (z_{p,t})^{-\varphi} \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_t K_t + X_t \right]. \tag{H.4}$$

Integrating (62), we have

$$X^e_{p,t+1} \equiv \int_{z^*_{p,t}}^{\infty} x^e_{t+1}(z_t,x_t)dF(z_t) = \gamma R_{t+1}\bar{k}_{p,t+1}\int_{z^*_{p,t}}^{\infty}\left(\frac{z_t}{z^*_t}(1-\sigma_k)-\lambda\right)dF(z_t)$$
$$= \gamma R_{t+1}\bar{k}_{p,t+1}\left(\frac{\varphi}{\varphi-1}\frac{z_{p,t}}{z^*_t}(1-\sigma_k)-\lambda\right)(z_{p,t})^{-\varphi} \quad \text{(H.5)}$$

where $\bar{k}_{p,t+1}$ is the average level of k_{t+1} among agents with $z_t \geq z_{p,t}$ and

$$\int_{z^*_{p,t}}^{\infty} k_{t+1}(z_t,x_t)dF(z_t) = \bar{k}_{p,t+1}(z_{p,t})^{-\varphi}. \quad \text{(H.6)}$$

(H.4), (H.5), and (H.6) yield

$$X^e_{p,t+1} = \frac{\gamma\beta R_{t+1}}{1-\sigma_k-\lambda}\left(\frac{\varphi}{\varphi-1}\frac{z_{p,t}}{z^*_t}(1-\sigma_k)-\lambda\right)(z_{p,t})^{-\varphi}\left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha)R_tK_t+X_t\right]. \quad \text{(H.7)}$$

By (H.1), (H.7). and (49), we obtain (63). Moreover, from (63), we obtain

$$\frac{\partial(X^e_{p,t+1}/X^e_{t+1})}{\partial z^*_t} = \frac{\varphi z^{-1}_{p,t}}{\frac{\varphi(1-\sigma_k)}{\varphi-1}-\lambda}\left(\frac{z^*_t}{z_{p,t}}\right)^{\varphi-2}\left[1-\sigma_k-\lambda\left(\frac{z^*_t}{z_{p,t}}\right)\right] > 0, \quad \text{(H.8)}$$

$$\frac{\partial(X^e_{p,t+1}/X^e_{t+1})}{\partial z_{p,t}} = -\frac{\varphi(z^*_t)^{-1}}{\frac{\varphi(1-\sigma_k)}{\varphi-1}-\lambda}\left(\frac{z_{p,t}}{z^*_t}\right)^{-\varphi-1}\left[1-\sigma_k-\lambda\left(\frac{z_{p,t}}{z^*_t}\right)\right] < 0. \quad \text{(H.9)}$$

(H.9) with $z^{*\prime}(\theta_t;\sigma_k) > 0$ by Lemma 3 leads to $\partial(X^e_{p,t+1}/X^e_{t+1})/\partial\theta_t > 0$.

Next, from (61), we obtain

$$\left.\frac{\partial \ln\left(X^e/X^l\right)}{\partial\sigma_k}\right|_{\sigma_k=0} = \frac{\lambda\left(\frac{\varphi}{\varphi-1}-1\right)}{\left(\frac{\varphi}{\varphi-1}-\lambda\right)(1-\lambda)} - \varphi\left[1+\frac{z^*(0;0)^{-\varphi}}{1-z^*(0;0)^{-\varphi}}\right]\left.\frac{\partial\ln z^*(\theta^*_S;\sigma_k)}{\partial\sigma_k}\right|_{\sigma_k=0}, \quad \text{(H.10)}$$

where because $\theta^*_S = 0$ for $\sigma_k = 0$ by (46), applying it into (49), we have

$$z^*(0;0)^{-\varphi} = 1-\lambda. \quad \text{(H.11)}$$

From (49) and $\theta^*_S = 0$ for $\sigma_k = 0$, we obtain

$$\left.\frac{\partial\ln z^*(\theta^*_S;\sigma_k)}{\partial\sigma_k}\right|_{\sigma_k=0} = \frac{1}{\varphi}\left[\frac{1}{1-\lambda}+\frac{\frac{\partial\theta^*_S}{\partial\sigma_k}|_{\sigma_k=0}}{\frac{\varphi}{\alpha(\varphi-1)}\beta(1-\alpha+\alpha\gamma)}\right] > 0. \quad \text{(H.12)}$$

Recall that by (57), we have

$$\left.\frac{\partial\theta_S^*}{\partial\sigma_k}\right|_{\sigma_k=0} = \frac{\beta\frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)}{\beta\frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)-1} > 0. \tag{H.13}$$

From (H.10) to (H.13), we obtain

$$\left.\frac{\partial\ln\left(X^e/X^l\right)}{\partial\sigma_k}\right|_{\sigma_k=0} = -\frac{1}{\lambda}\left[\frac{(1+\lambda)\frac{\varphi}{\varphi-1}-\lambda}{\frac{\varphi}{\varphi-1}-\lambda} + \frac{1}{\beta\frac{\varphi}{\varphi-1}\left(\frac{1-\alpha}{\alpha}+\gamma\right)-1}\right] < 0. \tag{H.14}$$

Finally, from (63), we obtain

$$\left.\frac{\partial\ln\left(X_p^e/X^e\right)}{\partial\sigma_k}\right|_{\sigma_k=0}$$
$$= \left(\frac{\varphi}{\varphi-1}\frac{z_p}{z^*(0;0)}-\lambda\right)^{-1}\left[\frac{\frac{\varphi}{\varphi-1}\lambda\left(\frac{z_p}{z^*(0;0)}-1\right)}{\frac{\varphi}{\varphi-1}-\lambda} + \varphi\left(\frac{z_p}{z^*(0;0)}-\lambda\right)\left.\frac{\partial\ln z^*(\theta_S^*;\sigma_k)}{\partial\sigma_k}\right|_{\sigma_k=0}\right] > 0. \tag{H.15}$$

I Endogenous labor supply

We modify production function (1) into

$$y_t = \mathcal{F}(z_{t-1}k_t, n_t\Gamma_t) = A(z_{t-1}k_t)^{\alpha}(n_t\Gamma_t)^{1-\alpha}, \tag{I.1}$$

where Γ_t is labor augmenting technology, specified as $\Gamma_t = K_t/N_t$ (e.g., Romer, 1986; Miyazawa, 2021).

In the young period, each individual is endowded with 1 unit of time and allocates it between working h_t^j and leisure $1-h_t^j$ ($j\in\{e,l\}$). Then, lifetime utility (3) is replaced by

$$U_t^j = (1-\beta)\left[\ln c_t^{y,j} + \epsilon\ln(1-h_t^j)\right] + \beta\left[(1-\gamma)\ln c_{t+1}^{o,j} + \gamma\ln x_{t+1}^j\right], \quad j\in\{e,l\}, \tag{I.2}$$

where the parameter $\epsilon(>0)$ represents the preference for leisure. With elastic labor supply h_t^e, equation (4) is replaced by

$$c_t^{y,e} = w_t h_t^e + x_t + \sigma_k k_{t+1} - a_{t+1}, \qquad a_{t+1} = k_{t+1} - d_t. \tag{I.3}$$

By (I.2) and (I.3), the FOC with respect to h_t^e is given by

$$w_t(1-h_t^e) = \epsilon c_t^{y,e}. \tag{I.4}$$

(5) to (13) remain unchanged. From (8), (I.1) and (13), we replace (14) and (15) with

$$y_t = Az_{t-1}k_t \left[\frac{(1-\alpha)AK_t}{w_t N_t}\right]^{\frac{1-\alpha}{\alpha}}, \tag{I.5}$$

$$z_t^* = \frac{(1-\sigma_k)R_{t+1}}{\alpha A}\left[\frac{w_{t+1}N_{t+1}}{(1-\alpha)AK_{t+1}}\right]^{\frac{1-\alpha}{\alpha}}. \tag{I.6}$$

(16) and (17) remain unchanged. From (I.3), (5), (9), (12), (16), (17), and $d_{i,t} = \lambda k_{i,t+1}$, we replace (19) with

$$k_{t+1}(z_t, x_t) = \frac{\beta}{1-\sigma_k-\lambda}(w_t h_t^e + x_t). \tag{I.7}$$

From (I.3), (I.4), and (I.7) we obtain

$$\begin{aligned} 1 - [1+\epsilon(1-\beta)]h^e &= \frac{\epsilon(1-\beta)x_t}{w_t} \\ \Leftrightarrow h_t^e(z_t, x_t) &= \frac{1}{1+\epsilon(1-\beta)}\left[1 - \frac{\epsilon(1-\beta)x_t}{w_t}\right], \end{aligned} \tag{I.8}$$

from which h_t^e depends on x_t. Then, we represent h_t^e as $h_t^e(z_t, x_t)$.

We next consider the lender's case. With elastic labor supply h_t^l, (20) is replaced by

$$c_t^{y,l} = w_t h_t^l + x_t - l_t - q_t b_{t+1}, \tag{I.9}$$

By (I.2) and (I.9), the FOC with respect to h_t^e is given by

$$w_t(1-h_t^l) = \epsilon c_t^{y,l}. \tag{I.10}$$

Equation (24) remains unchanged whereas (23) and (25) are replaced by

$$l_t(z_t, x_t) + q_t b_{t+1}(z_t, x_t) = \beta\left(w_t h_t^l + x_t\right), \tag{I.11}$$

$$x_{t+1}^l(z_t, x_t) = \gamma\beta R_{t+1}\left(w_t h_t^l + x_t\right). \tag{I.12}$$

From (I.9), (I.10), and (I.11), we obtain

$$\begin{aligned} 1 - [1+\epsilon(1-\beta)]h^l &= \frac{\epsilon(1-\beta)x_t}{w_t} \\ \Leftrightarrow h_t^l(z_t, x_t) &= \frac{1}{1+\epsilon(1-\beta)}\left[1 - \frac{\epsilon(1-\beta)x_t}{w_t}\right], \end{aligned} \tag{I.13}$$

from which h_t^l depends on x_t. Then, we represent h_t^l as $h_t^l(z_t, x_t)$. By (I.8) and (I.13), households whose bequests from their parents x_t are greater than $w_t/[\epsilon(1-\beta)]$ supply no labor, regardless of whether they are borrowers or lenders ($h_t^j = 0$, $j \in e, l$).

Both Lemma 1 and Remark 1 are robust. Then, aggregating (I.8) and (I.13), and keeping in mind that z is iid and x_t is independent of z_t, we obtain

$$(z_t^*)^{-\varphi} - [1+\epsilon(1-\beta)] \int_{z_t^*}^{\infty} h_t^e(z_t, x_t) dF(z_t) = \frac{\epsilon(1-\beta)}{w_t} (z_t^*)^{-\varphi} \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}), \tag{I.14}$$

$$1 - (z_t^*)^{-\varphi} - [1+\epsilon(1-\beta)] \int_1^{z_t^*} h_t^l(z_t, x_t) dF(z_t) = \frac{\epsilon(1-\beta)}{w_t} \left[1 - (z_t^*)^{-\varphi}\right] \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}). \tag{I.15}$$

Let us define $H_t \equiv \int_{z_t^*}^{\infty} h_t^e(z_t, x_t) dF(z_t) + \int_1^{z_t^*} h_t^l(z_t, x_t) dF(z_t)$. Then, adding (I.14) to (I.15) yields

$$1 - [1+\epsilon(1-\beta)] H_t = \frac{\epsilon(1-\beta) X_t}{w_t}, \tag{I.16}$$

where $X_t \equiv \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) \left(= \int_{z_{t-1}^*}^{\infty} x_t^e(z_{t-1}, x_{t-1}) dF(z_{t-1}) + \int_1^{z_{t-1}^*} x_t^l(z_{t-1}, x_{t-1}) dF(z_{t-1})\right)$. The labor market clears as

$$N_t = \int_{z_{t-1}^*}^{\infty} n_t dF(z_{t-1}) = H_t, \tag{I.17}$$

which indicates that total labor demand N_t equals total labor supply H_t. Aggregating the credit constraint $d_t = \lambda k_{t+1}$, the equilibrium condition of the financial market is the same as (30):

$$\int_1^{z_t^*} l_t(z_t, x_t) dF(z_t) = \int_{z_t^*}^{\infty} d_t(z_t, x_t) dF(z_t) = \lambda K_{t+1}.$$

Using (2) and (29), we can aggregate the production function (I.5) as

$$\begin{aligned} Y_t \left(= \int_{z_{t-1}^*}^{\infty} y_t(z_{t-1}, x_{t-1}) dF(z_{t-1})\right) &= \int_{z_{t-1}^*}^{\infty} A z_{t-1} k_t(z_{t-1}, x_{t-1}) \left[\frac{(1-\alpha) A K_t}{w_t N_t}\right]^{\frac{1-\alpha}{\alpha}} dF(z_{t-1}) \\ &= \frac{A\varphi}{\varphi - 1} z_{t-1}^* \left[\frac{(1-\alpha) A K_t}{w_t N_t}\right]^{\frac{1-\alpha}{\alpha}} K_t. \end{aligned} \tag{I.18}$$

Substituting (I.6) into (I.18), we obtain

$$Y_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)} R_t K_t, \tag{I.19}$$

which is the same as (33).

From (8), (I.17), and (33), we obtain $w_t N_t = w_t H_t = (1-\alpha)\int_{z_{t-1}^*}^{\infty} y_t(z_{t-1}, x_{t-1}) dF(z_{t-1})$, leading to

$$w_t N_t = w_t H_t = \frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1-\alpha) R_t K_t. \tag{I.20}$$

Substituting (I.20) into (I.6), we obtain

$$z_t^* = \left(\frac{\varphi}{\varphi-1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\sigma_k)R_{t+1}}{\alpha A}\right)^{\frac{1}{\alpha}}, \tag{I.21}$$

which is the same as (35).

Aggregating (I.7), using (29) and (I.20), and keeping in mind that z is iid and x_t is independent of z_t (Remark 1), we obtain

$$\begin{aligned}
&\int_{z_t^*}^{\infty} k_{t+1}(z_t, x_t) dF(z_t) \int_1^{\infty} dF(z_{t-1}) \\
&= \frac{\beta}{1-\sigma_k-\lambda}\left[w_t \int_{z_t^*}^{\infty} h_t^e(z_t, x_t) dF(z_t) \int_1^{\infty} dF(z_{t-1}) + \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) \int_{z_t^*}^{\infty} dF(z_t)\right] \\
&\Leftrightarrow K_{t+1} = \frac{\beta}{1-\sigma_k-\lambda}\left[w_t \int_{z_t^*}^{\infty} h_t^e(z_t, x_t) dF(z_t) + (z_t^*)^{-\varphi} X_t\right],
\end{aligned} \tag{I.22}$$

where $X_t \equiv \int_1^{\infty} x_t(z_{t-1}, x_{t-1}) dF(z_{t-1}) = \int_1^{z_{t-1}^*} x_t^l(z_{t-1}, x_{t-1}) dF(z_{t-1}) + \int_{z_{t-1}^*}^{\infty} x_t^e(z_{t-1}, x_{t-1}) dF(z_{t-1})$. Here, X_t is derived using (16) and (24) associating with (2), (17), (27), (29), (30), (I.19), and (I.21) as

$$X_t = \gamma\left[\frac{\varphi(1-\sigma_k)}{\varphi-1} R_t K_t + B_t\right]. \tag{I.23}$$

See Appendix B for the derivation of (I.23) because (I.23) is the same as (37).

Aggregating (I.11) and using (27) and $X_t \equiv \int_1^\infty x_t(z_{t-1}, x_{t-1})dF(z_{t-1})$, we obtain

$$\begin{aligned}
&\int_1^{z_t^*} [l_t(z_t, x_t) + q_t b_{t+1}(z_t, x_t)]dF(z_t) \int_1^\infty dF(z_{t-1}) \\
&= \beta\left[w_t \int_1^{z_t^*} h^l(z_t, x_t)dF(z_t) \int_1^\infty dF(z_{t-1}) + \int_1^\infty x_t(z_{t-1}, x_{t-1})dF(z_{t-1}) \int_1^{z_t^*} dF(z_t)\right] \\
&\Leftrightarrow \int_1^{z_t^*} l_t(z_t, x_t)dF(z_t) = \beta\left[w_t \int_1^{z_t^*} h^l(z_t, x_t)dF(z_t) + \left\{1 - (z_t^*)^{-\varphi}\right\} X_t\right] - q_t B_{t+1}. \qquad \text{(I.24)}
\end{aligned}$$

Substituting (30), $\int_1^{z_t^*} l_t(z_t, x_t)dF(z_t) = \int_{z_t^*}^\infty d_t(z_t, x_t)dF(z_t) = \lambda K_{t+1}$, into (I.24), we obtain

$$\lambda K_{t+1} = \beta\left[w_t \int_1^{z_t^*} h^l(z_t, x_t)dF(z_t) + \left\{1 - (z_t^*)^{-\varphi}\right\} X_t\right] - q_t B_{t+1}. \qquad \text{(I.25)}$$

From (I.22), (I.25), and $H_t \equiv \int_{z_t^*}^\infty h_t^e(z_t, x_t)dF(z_t) + \int_1^{z_t^*} h_t^l(z_t, x_t)dF(z_t)$ (see (I.16)), we obtain

$$K_{t+1} + q_t B_{t+1} = \beta(w_t H_t + X_t) + \sigma_k K_{t+1}. \qquad \text{(I.26)}$$

Substituting (I.20) and (I.23) into (I.26) and using (22) and $\tilde{B}_{t+1} \equiv q_t B_{t+1}$, we obtain the following asset market-clearing condition:

$$K_{t+1} + \tilde{B}_{t+1} = \beta R_t \left[\frac{\varphi(1-\sigma_k)}{\alpha(\varphi-1)}(1 - \alpha + \alpha\gamma)K_t + \gamma \tilde{B}_t\right] + \sigma_k K_{t+1}, \qquad \text{(I.27)}$$

which is the same as (41). Furthermore, (42) derived from (28) and (29) remains unchanged. Thus, the main results thus far are robust.

From (I.16), (I.23), (22), and $\tilde{B}_{t+1} \equiv q_t B_{t+1}$, we obtain

$$[1 + \epsilon(1-\beta)]H_t = 1 - \frac{\gamma\left[\frac{\varphi(1-\sigma_k)}{\varphi-1} R_t K_t + R_t \tilde{B}_t\right]}{w_t}. \qquad \text{(I.28)}$$

Substituting (I.20) into (I.28) and using $\theta_t \equiv \tilde{B}_t / K_t$, we obtain

$$H_t = \left[1 + \epsilon(1-\beta) + \frac{\gamma\alpha}{1-\alpha}\left(1 + \frac{\varphi - 1}{\varphi(1-\sigma_k)}\theta_t\right)\right]^{-1}. \qquad \text{(I.29)}$$

As seen from (I.8) and (I.13), households whose bequests from their parents x_t are greater than $w_t/[\epsilon(1-\beta)]$ supply no labor, regardless of whether they are borrowers or lenders ($h_t^j = 0$, $j \in \{e, l\}$). Additionally, their bequests (assets) include public bonds when their parents are lenders. Thus, total labor supply decreases with the public debt-to-capital ratio.

Table 1: The benchmark numerical settings of the parameters

	Benchmark	Source
α	0.4	Average velues of the US, the EU, and Japan
β	0.3	$\beta/(1-\beta) = 0.97^{30}$
γ	0.3	$\beta = \gamma$
φ	1.5	Diamond and Saez (2011), Jaimovich and Rebelo (2017), and Mino (2015)
λ	0.7	Set
A	5	Set to yield positive plausible values for the long-run growth rates
σ_k	0.01	Set

Table 2: The magnitude of tax increases relative to investment subsidy increases for $d\hat{g}^Y(\cdot)/d\sigma_k > 0$

$\lambda = 0.8$						
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	-0.0411	-0.0282	-0.0120	0.0091	0.0375	0.0806
$\lambda = 0.7$						
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	0.0207	0.0328	0.0471	0.0646	0.0865	0.1174
$\lambda = 0.6$						
σ_k	0	0.01	0.02	0.03	0.04	0.05
$d\tau/d\sigma_k\|_{\tau=0}$	0.0515	0.0625	0.0750	0.0900	0.1083	0.1332

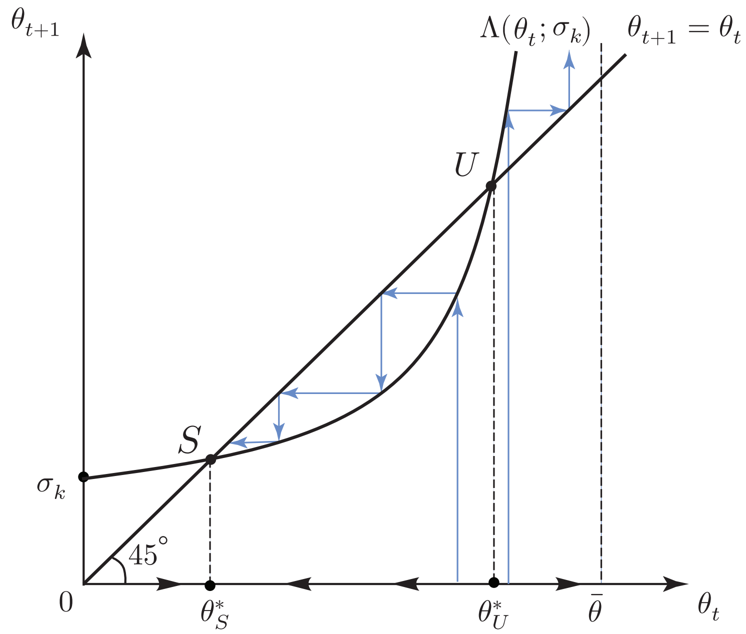

Figure 1: The dynamics of θ_t

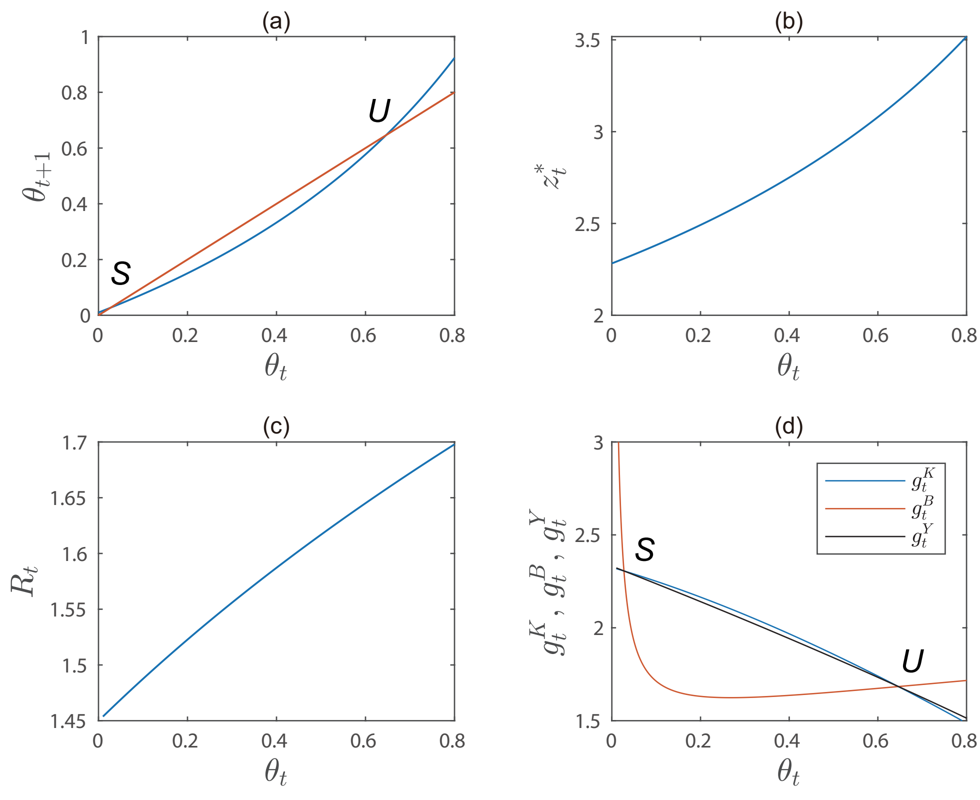

Figure 2: $\theta_{t+1} = \Lambda(\theta_t, \sigma_k)$, $R_t = \Psi(\theta_t, \sigma_k)$, $z(\theta_t, \sigma_k)$, $g^K(\theta_t, \sigma_k)$, $g^B(\theta_t, \sigma_k)$, and $g^Y(\theta_t, \sigma_k)$ in the numerical example

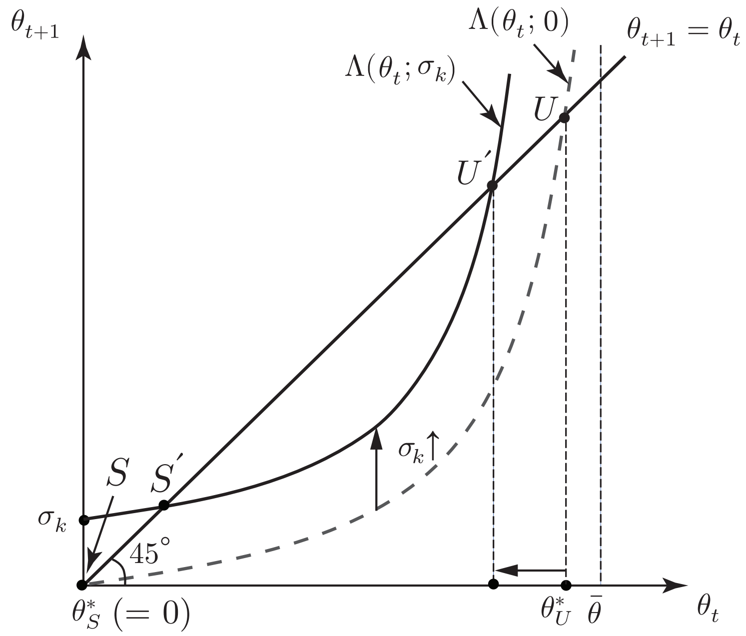

Figure 3: The effect of an increase in σ_k on θ_S^* and fiscal sustainability

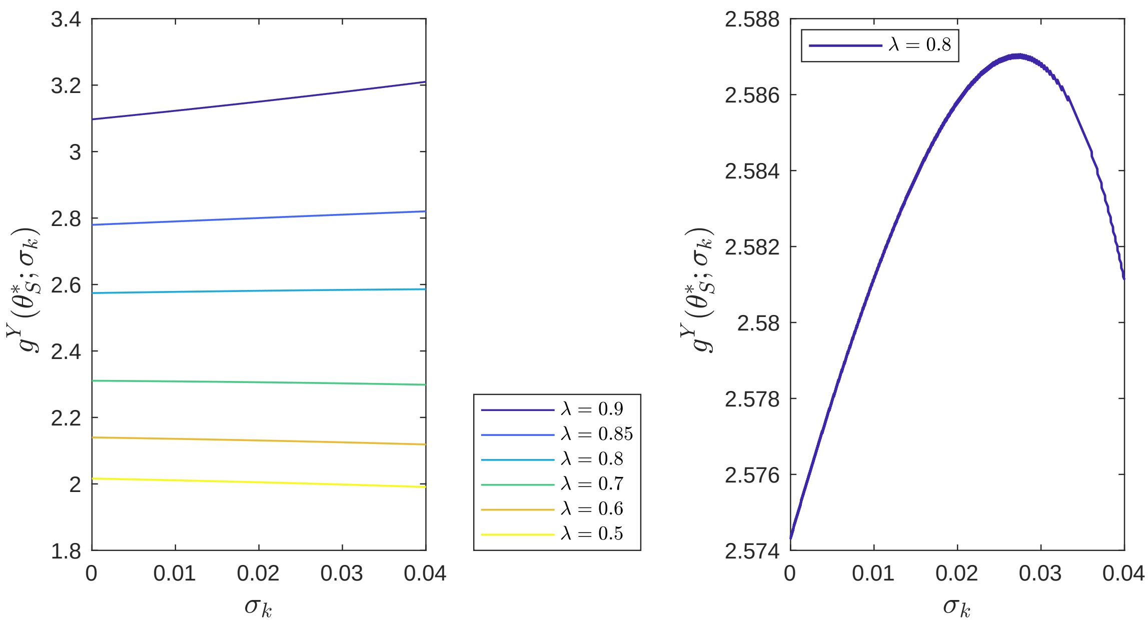

Figure 4: The relationship between λ and the growth effect of σ_k

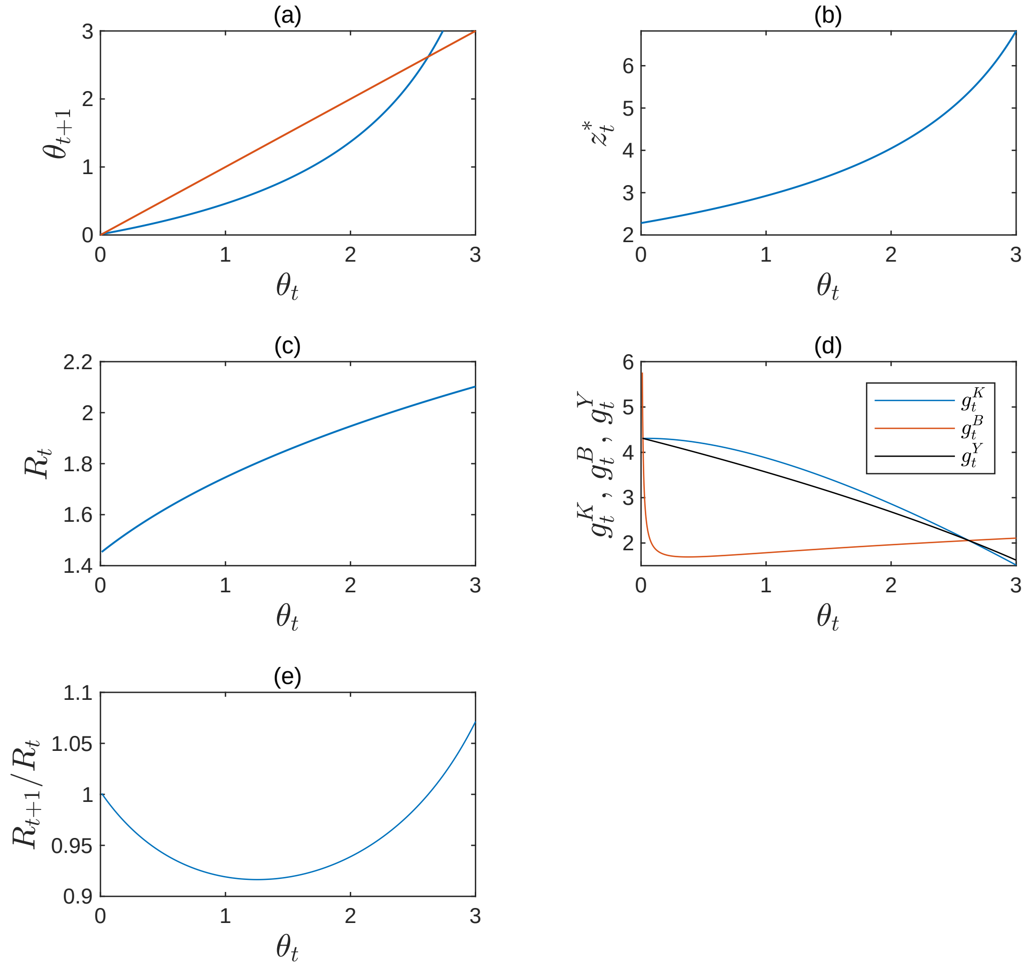

Figure 5: $\theta_{t+1} = \Lambda(\theta_t, \sigma_k)$, $R_t = \Psi(\theta_t, \sigma_k)$, $z(\theta_t, \sigma_k)$, $g^K(\theta_t, \sigma_k)$, $g^B(\theta_t, \sigma_k)$, $g^Y(\theta_t, \sigma_k)$, and R_{t+1}/R_t when the liquidity effect is large ($\beta = \gamma = 0.5$)

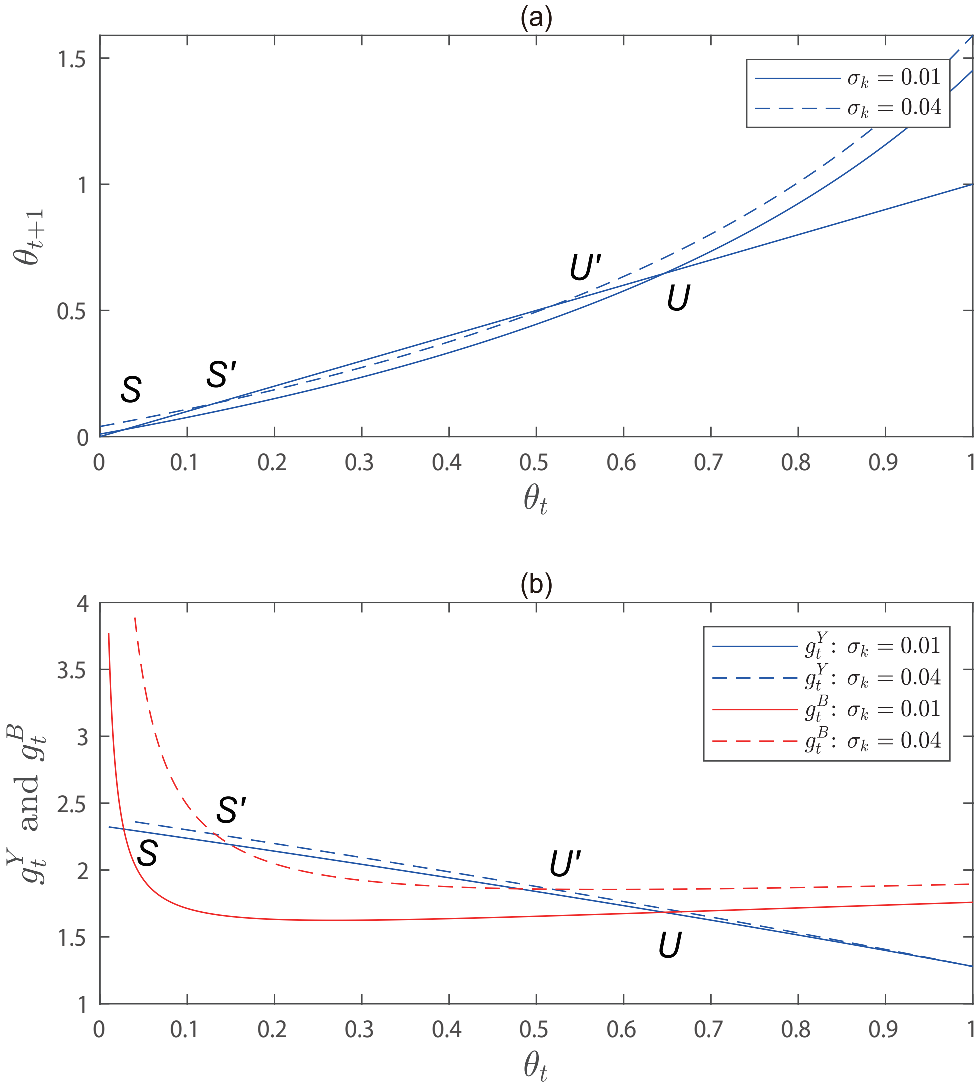

Figure 6: The effect of an increase in σ_k from 0.01 to 0.04 on θ_S^*, θ_U^*, $g^Y(\theta_t, \sigma_k)$, and $g^B(\theta_t, \sigma_k)$

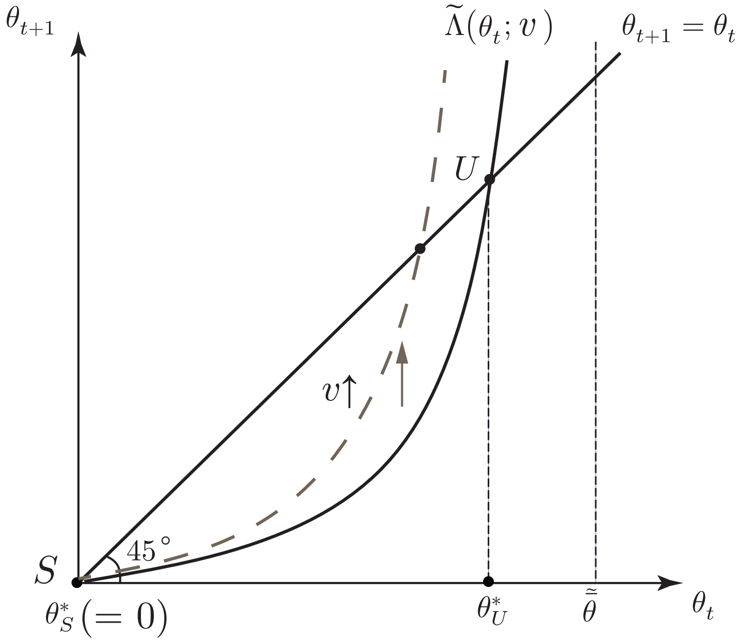

Figure 7: The effect of an increase in v on fiscal sustainability