Existence of Singularity Bifurcation in an Euler-Equations Model of the United States Economy: Grandmont was Right

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Abstract: Grandmont (1985) found that the parameter space of the most classical dynamic general-equilibrium macroeconomic models are stratified into an infinite number of subsets supporting an infinite number of different kinds of dynamics, from monotonic stability at one extreme to chaos at the other extreme, and with all forms of multiperiodic dynamics between.

But Grandmont provided his result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. As a result, Barnett and He (1999, 2001, 2002) investigated a Keynesian structural model, and found results supporting Grandmont's conclusions within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That prototypical Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant. In addition, results by Barnett and Duzhak (2008,2009) demonstrate the existence of Hopf and flip (period doubling) bifurcation within the parameter space of recent New Keynesian models.

Lucas-critique criticism of Keynesian structural models has motivated development of Euler equations models having policy-invariant deep parameters, which are invariant to policy rule changes. Hence, we continue the investigation of policy-relevant bifurcation by searching the parameter space of the best known of the Euler equations general-equilibrium macroeconometric models: the path-breaking Leeper and Sims (1994) model. We find the existence of singularity bifurcation boundaries within the parameter space. Although never before found in an economic model, singularity bifurcation may be a common property of Euler equations models, which often do not have closed form solutions. Our results further confirm Grandmont's views.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, to New Keynesian models, and now to Euler equations macroeconomic models having deep parameters.

Keywords:
Bifurcation, inference, dynamic general equilibrium, Pareto optimality, Hopf bifurcation, Euler equations, Leeper and Sims model, singularity bifurcation, stability.

JEL Codes:
C14, C22, E37, E32.

1. Introduction

1.1. The History

Grandmont (1985) found that the parameter space of even the simplest, classical general-equilibrium macroeconomic models are stratified into bifurcation regions. This result changed the prior common view that different kinds of economic dynamics can only be produced by different kinds of structures. But he provided that result with a model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. Years of controversy followed, as evidenced by papers appearing in Barnett, Deissenberg, and Feichtinger (2004) and Barnett, Geweke, and Shell (2005). The econometric implications of Grandmont's findings are particularly important, if bifurcation boundaries cross the confidence regions surrounding parameter estimates in policy-relevant models. Stratification of a confidence region into bifurcated subsets seriously damages robustness of dynamical inferences.[1]

The dramatic transformation of views precipitated by Grandmont's paper was criticized for lack of policy relevance. As a result, Barnett and He (1999, 2001, 2002) investigated a continuous-time traditional Keynesian structural model and found results supporting Grandmont's conclusions. Barnett and He found transcritical, codimension-two, and Hopf bifurcation boundaries within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. That highly regarded Keynesian model was produced from a system of second order differential equations. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy's data, and is clearly policy relevant. See Bergstrom and Wymer (1976), Bergstrom (1996), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom, Nowman, and Wymer (1992), and Bergstrom and Nowman (2006). Barnett and He found that bifurcation boundaries cross confidence regions of parameter estimates in that model, such that both stability and instability are possible within the confidence regions.

Barnett and Duzhak (2008,2009) have explored bifurcation within the more

[1] We assume that parameters are fixed and focus on the implications of bifurcation for robustness of inferences. But if parameters can move over time, as in Swamy, Tavlas, and Chang (2005), the implications of bifurcation are even more serious.

recent class of New Keynesian models. Those two papers included forward-looking and current-looking models, as well as hybrid models having both forward and current-looking features. They found Hopf and flip (period doubling) bifurcation, with the setting of the policy parameters influencing the existence and location of the bifurcation boundary. No other forms of bifurcation were found within the three-equations log-linearized New Keynesian models. One surprising result is the finding that a common setting of a parameter in the future-looking New-Keynesian model can put the model directly onto a Hopf bifurcation boundary.

The Lucas critique has motivated development of Euler-equations general-equilibrium macroeconometric models. Hence, we continue the investigation of policy relevant bifurcation by searching the parameter space of the best known of the policy relevant Euler-equations macroeconometric models: the path-breaking Leeper and Sims (1994) model. The results further confirm Grandmont's views, but with the finding of an unexpected form of bifurcation: singularity bifurcation. Although known in engineering and mathematics, singularity bifurcation has not previously been encountered in economics. Barnett and He (2004, 2006) have made clear the mathematical nature of singularity bifurcation and why it is likely to be common in the class of modern Euler equation models rendered important by the Lucas critique.

Leeper and Sims' model consists of differential equations with a set of algebraic constraints. Our analysis reveals the existence of a singularity bifurcation boundary within a small neighborhood of the estimated parameter values. When the parameter values approach the singularity boundary, one eigenvalue of the linearized part of the model moves rapidly to infinity, while other eigenvalues remain bounded. On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Such change in the order of dynamics has not previously been found with macroeconometric models. But we find from the relevant theory that singularity bifurcation may be a common property of Euler equations models. The dramatic implications of singularity bifurcation are not limited to the change in the dimension of the dynamics on the bifurcation boundary. The nature of the dynamics on one side of a singularity bifurcation boundary is very different from the nature of the dynamics on the other side, although of the dimension of the dynamics is the same on both sides. Knowledge of the location of a bifurcation boundary is very important, even if there is no chance that the economy will drop into the lower dimensional dynamics directly on that boundary.

Beginning with Grandmont's findings with a classical model, we continue to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, to New Keynesian macroeconometric models, and now to Euler equations models having deep parameters. At this stage of our research, we believe that Grandmont's conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest.

1.2 The Leeper and Sims Model

Various relevant dynamic macroeconometric models have been established in the literature.[2] Of particular importance is the Leeper and Sims (1994) Euler equations stochastic-dynamic general-equilibrium model intended to address such issues as the Lucas critique (Lucas (1976)) for the US economy. Similar models are developed in Kim (2000) and others, but the Leeper and Sims model was the seminal model in that literature.

The dimension of the state space in the Leeper and Sims model is substantially lower than in the Bergstrom, Norman, and Wymer UK model. However, the dimension is still too high for complete analysis by generally available analytical approaches. By numerical methods complementing theoretical analysis, we find that the dynamics of the Leeper and Sims model is complicated by its structure as an Euler equations model, since such models usually have no closed form algebraic solution.

In this paper, we are interested in how the dynamic behavior of the model is affected by its parameter settings. We find that the order of the dynamics of the Leeper and Sims model can change within a small neighborhood of the estimated parameter values. As parameters change within that neighborhood, one eigenvalue of the linearized part of the model can move quickly from finite to infinite and back again to finite. A large stable eigenvalue characterizes the case in which some variables can respond rapidly to changes of other variables, while a large unstable

[2] Among those models that have direct relevance to this research are the high-dimensional continuous-time macroeconometric models of Bergstrom, Nowman and Wymer (1992), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom and Wymer (1976), Bergstrom and Nowman (2006), Grandmont (1998), Leeper and Sims (1994), Powell and Murphy (1997) and Kim (2000). Surveys of relevant macroeconomic models are available in Bergstrom (1996) and in several textbooks such as Gandolfo (1996) and Medio (1992). General theory of economic dynamics is provided, in Boldrin and Woodford (1990) and Gandolfo (1996). Various bifurcation phenomena are reported in Bala (1997), Benhabib (1979), Medio (1992), Gandolfo (1996), and Nishimura and Takahashi (1992). Focused studies of stability are conducted in Grandmont (1998), Scarf (1960), and Nieuwenhuis and Schoonbeek (1997). Barnett and Chen (1988) empirically found chaotic dynamics in economics. Bergstrom, Nowman, and Wandasiewicz (1994) investigate stabilization of macroeconomic models using policy control. Wymer (1997) describes several mathematical frameworks for the study of the structural properties of macroeconometric models.

eigenvalue corresponds to the case in which rapid diversion occurs of one variable from other variables. Infinity eigenvalue implies existence of pure algebraic relationships among the variables. This sensitivity to the setting of the parameters presents serious challenges to the robustness of dynamical inferences. The source of the problem is the nature of the mapping from the Euclidean parameter space to the function space of dynamical solutions.

Change in the order of the dynamic part of the system in response to small changes in parameter settings is a fundamental property of the Leeper and Sims model and corresponds to a class of bifurcations known to engineers and mathematicians as "singularity" bifurcations. To our knowledge, this is the first discovery of singularity bifucation in macroeconometric models; but appears to be closely connected with the structure of Euler equations models.

2. The Model

The Leeper and Sims (1994) Euler-equations, stochastic, general-equilibrium model includes the dynamic behavior of consumers, firms, and government. With the parameters of consumer and firm behavior being the deep parameters of tastes and technology, those parameters are invariant to government policy rule changes.[3] These models contain dynamic subsystems consisting of ordinary differential equations and algebraic constraints. Such systems are called differential/algebraic systems in systems theory.

In the Leeper and Sims model, both consumers and firms maximize their respective objective functions. The government provides monetary and tax policies to satisfy an intertemporal government budget constraint and to the pursuit of countercyclical policy objectives. The detailed derivation of the models is available in Leeper and Sims (1994). The resulting model is summarized in this section.

The model contains the following 12 state variables.

L = labor supply

C^* = consumption net of transactions costs

M = consumer demand for non-interest-bearing money

D = consumer demand for interest-bearing money

K = capital

[3] Several similar models have been developed in Kim (2000) and in Binder and Pesaran (1999).

Y = factor income from capital and labor, excluding interest on government debt.

C = gross consumption

Z = investment

X = consumer goods aggregate price

Q = investment goods price

V = income velocity of money

P = general price level

The model assumes that the consumer maximizes[4]

$$E[\int_0^\infty exp(-\int_0^t \beta(s)ds)\frac{(C^{*\,\pi}(1-L)^{1-\pi})^{1-\gamma}}{1-\gamma}dt]$$

subject to

$$XC + QZ + \tau + \frac{\dot{M} + \dot{D}}{P} = Y + \frac{iD}{P},$$

$$XC^* + \phi VY = XC,$$

$$\dot{K} = Z - \delta K,$$

$$Y = rK + wL + S,$$

$$V = \frac{PY}{M},$$

where $\pi \in (0,1)$ and $\gamma > 0$ are parameters; $0 \leq \beta(s) \leq 1$ is the subjective rate of time preference at time s, τ is the level of lump-sum taxes paid by the representative consumer; i is the nominal rate of return earned on government bonds; S is the sum of dividends received by the representative consumer, w is the wage rate; $\varphi > 0$ is the transaction cost per unit of VY; $\delta \geq 0$ is the rate of depreciation of capital; and r = rental rate of return on capital.[5] As we shall see below, parameters in this stochastic

[4] Leeper and Sims describe the model's consumer as a "representative consumer" maximizing utility subject to constraints in total consumption of goods and leisure. This convention is unusual, since in aggregation theory, Gorman's representative consumer makes decisions in per capita variables, not totals. But as used empirically by Leeper and Sims, the resulting Euler equations are equivalent to those that would have resulted from a per capita decision for the representative consumer.

[5] Transactions are assumed by Leeper and Sims to be proportional to V and Y, with φ being the proportionality constant. The overdot is used throughout to designate time derivative.

dynamic general-equilibrium model are not necessarily assumed to be constant or deterministic.

The firms' optimization problem is

$$max\{X(C+g)+QI^{*}+A(\alpha K^{\sigma}+L^{\sigma})^{1/\sigma}-rK-wL-((C+g)^{\mu}+\theta I^{*\,\mu})^{1/\mu}\},$$

where g is the level of government purchases. The following are parameters: A>0, α>0, θ>0, $\mu \geq 0$, and $0 \leq \sigma \leq 1$. Investment goods produced by the firm, I^{*}, include both those bought by the existing population, Z, and those purchased by the government for distribution to the newborn. Thus, a market-clearing condition is $I^{*}=Z+nK$, where n = the fraction of existing capital purchased by the government for distribution to the newborn.[6]

In this model, the state variables satisfy the following differential equations:

$$\frac{1}{P}(\dot{M}+\dot{D})=Y-XC-QZ+\frac{iD}{P}+\tau\,, \tag{1}$$

$$\dot{K}=Z-\delta K\,, \tag{2}$$

$$(1-\pi(1-\gamma))\frac{\dot{C}^{*}}{C^{*}}+(1-\gamma)(1-\pi)\frac{\dot{L}}{1-L}+\frac{\dot{X}}{X}+\frac{\dot{P}}{P}=i-\beta+\frac{\dot{\pi}}{\pi}+\dot{\pi}(1-\gamma)\log(\frac{C^{*}}{1-L})\,, \tag{3}$$

$$\frac{\dot{P}}{P}+\frac{\dot{Q}}{Q}=i+\delta-(1-2\phi V)\frac{r}{Q}, \tag{4}$$

where equation (1) represents the consumers' budget constraint, (2) is the law of motion for capital, and (3)and (4) are first-order conditions from the consumers' optimization decision.

In addition to satisfying the four dynamic equations, the state variables satisfy the following algebraic constraints:

$$X=\left(\frac{Y}{C+g}\right)^{1-\mu}, \tag{5}$$

$$Q=\theta\left(\frac{Y}{Z+nK}\right)^{1-\mu}, \tag{6}$$

[6] We are using Leeper and Sims' definitions, which we ourselves are not advocating. Another view could equate n with population growth rate so that nK could be interpreted to include capital endowed to the young generation by the old.

$$r = A^{\sigma}\alpha\left(\frac{Y}{K}\right)^{1-\sigma}, \tag{7}$$

$$w = A^{\sigma}\left(\frac{Y}{L}\right)^{1-\sigma}, \tag{8}$$

$$XC^{*} + \phi VY = XC, \tag{9}$$

$$Y = rK + wL + S, \tag{10}$$

$$V = \frac{PY}{M}, \tag{11}$$

$$X(C+g) + Q(Z+nK) = Y, \tag{12}$$

$$(1-2\phi V)\frac{w}{X} = \frac{1-\pi}{\pi}\frac{C^{*}}{1-L}, \tag{13}$$

$$i = \phi V^{2}. \tag{14}$$

The relations (5)-(8) are obtained from the first-order conditions by maximizing the firms' objective function. Equation (9) defines consumption net of transactions costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (10) defines income. Equation (11) is the income velocity of money. Equation (12) is the social resources constraint. Equations ((13),(14)) are obtained from the first-order conditions for the consumers' decision.

The control variables are the government policy variables, consisting of the nominal rate of return on government bonds, i, and the level of lump-sum taxes, τ. Leeper and Sims (1994) introduced the following monetary and tax policies into the model. The monetary policy rule is

$$\frac{1}{i}\frac{di}{dt} = a_p \log\left(\frac{P}{\bar{P}}\right) + a_{int}\left(\frac{\dot{P}}{P}\right) + a_i \log\left(\frac{i}{\beta}\right) + a_L \log\left(\frac{L}{\bar{L}}\right) + \varepsilon_i, \tag{15}$$

and the tax policy is

$$\frac{d}{dt}\frac{\tau}{C} = b_{\tau}\left(\frac{\tau}{C} - \frac{\bar{\tau}}{\bar{C}}\right) + b_L \log\left(\frac{L}{\bar{L}}\right) + b_{inf}\frac{\dot{P}}{P} + b_x\left(\frac{D}{PY} - \frac{\bar{D}}{\bar{P}\bar{Y}}\right) + \varepsilon_{\tau}. \tag{16}$$

The overscored variables denote steady state values, so that $\bar{D}/\bar{Y}$ is the steady state debt-to-income level, where income is measured by Leeper and Sims as GNP. The free parameters are $\bar{D}/\bar{Y}$, the steady state price level, $\bar{P}$, the a's, and the b's. The disturbance noises are ε_i and ε_τ.

In this model, it is conventional to use $\tau_c = \tau/C$, rather than τ, as a control. Therefore, the two control variables are i and τ_c. The parameters and exogenous variables, n, g, π, δ, θ, α, A, and ϕ, are specified by Leeper and Sims to follow logarithmic first-order autoregressive (AR) processes in continuous time, while β is specified to be a logarithmic first-order AR in unlogged form. However, we analyze the structural properties of (1)-(14) without external disturbances. As a result, in equation (3), we set $\dot{\pi} = 0$ and treat π as a fixed parameter, along with the model's other parameters, which are all treated as fixed. We treat the exogenous variables as realized at their measured values. The extension of our analysis to the case of stochastic bifurcation is a subject for future research.

The original form (1)-(14) has 12 state variables and 14 equations. For analytical investigation, it is best to have as few state variables as possible. For this purpose, we next reduce the dimension of the problem by temporarily eliminating some state variables. We contract to the following 7 state variables

$$\mathbf{x} = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \end{bmatrix}. \tag{17}$$

The remaining state variables can be written as unique functions of **x**.

By eliminating M, C^*, V, Q, X from the independent state variables, we can determine directly from (1)-(14) that **x** satisfies the following equations.

$$\begin{aligned} &\frac{1}{P}\dot{D} + \frac{Y\sqrt{\phi/i}}{P}\dot{P} + (\sqrt{\phi/i})\dot{Y} \\ &= Y + \frac{iD}{P} - \left(\frac{Y}{C+g}\right)^{1-\mu} C - \theta\left(\frac{Y}{Z+nK}\right)^{1-\mu} L - \tau_c C + \frac{Y\sqrt{i/\phi}}{2V^2\phi}\dot{i}, \end{aligned} \tag{18}$$

$$(1-\pi(1-\gamma))\left(\frac{1-\phi VY^{\mu}(1-\mu)(C+g)^{-\mu}}{C-\phi VY^{\mu}(C+g)^{1-\mu}} - \frac{1-\mu}{C+g}\right)\dot{C}$$

$$-\left(\frac{(1-\pi(1-\gamma))\phi V \mu Y^{\mu-1}(C+g)^{1-\mu}}{C-\phi V Y^{\mu}(C+g)^{1-\mu}}+\frac{1-\mu}{Y}\right)\dot{Y}+\frac{\dot{P}}{P}+\frac{(1-\gamma)(1-\pi)}{1-L}\dot{L}$$

$$=i-\beta+\frac{Y^{\mu}(C+g)^{1-\mu}}{C-\phi V Y^{\mu}(C+g)^{1-\mu}}\frac{1}{2\sqrt{i\phi}}\dot{i}, \tag{19}$$

$$\frac{\dot{P}}{P}+(1-\mu)\left(\frac{\dot{Y}}{Y}-\frac{\dot{Z}+n\dot{K}}{Z+nK}\right)=-(1-2\phi V)\frac{a^{\sigma}\alpha}{\theta}Y^{\mu-\sigma}(Z+nK)^{1-\mu}K^{\sigma-1}+i+\delta, \tag{20}$$

$$\dot{K}=Z-\delta K, \tag{21}$$

$$0=(C+g)^{\mu}+\theta(Z+nK)^{\mu}-Y^{\mu}, \tag{22}$$

$$0=\alpha K^{\sigma}+L^{\sigma}-a^{-\sigma}Y^{-\sigma}, \tag{23}$$

$$0=(1-2\phi V)\frac{a^{\sigma}Y^{\mu-\sigma}(C+g)^{1-\mu}}{L^{1-\sigma}}+\frac{1-\pi}{\pi}\frac{\phi V}{1-L}Y^{\mu}(C+g)^{1-\mu}-\frac{1-\pi}{\pi}\frac{C}{1-L}. \tag{24}$$

For the ease of notation, we denote equations (18)-(24) as

$$\mathbf{h}(\mathbf{x},\mathbf{u})\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{u}), \tag{25}$$

$$\mathbf{0}=\mathbf{g}(\mathbf{x},\mathbf{u}), \tag{26}$$

Where equations (25) contain the four equations, (18) – (21), and equations (26) contain the three equations, (22), (23), and (24). We define **x** to be the 7-dimensional state vector, and **u** to be the 2-dimensional control vector. The functions **h**(**x**,**u**) comprise a matrix having dimension 4×7, while **f**(**x**,**u**) is a 4×1 vector of functions. The dimension of the vector of functions **g**(**x**,**u**) is 3 × 1. Equation (25) describes the nonlinear dynamical behavior of the model, and (26) represents the algebraic constraints, which are nonlinear. Many systems can be described in the form of (25) and (26). Models in that form are called nonlinear descriptor systems in the mathematical literature on nonlinear dynamics.[7]

We shall use m, m_1, m_2, and l (with $m=m_1+m_2$), to denote respectively the dimension of **x**, the number of differential equations in (25), the number of algebraic constraints in (26), and the dimension of the vector of control variables, **u**.

[7] The model developed in Kim (2000) is also in that form.

With the Leeper and Sims model, $m=7$, $m_1=4$, $m_2=3$, and $l=2$.

The steady state of the system ((25),(26)) for the 7 state variables, **x**, conditionally on the setting of the controls, **u**, can be solved from the following system of 7 equations equations:

$$\mathbf{0}=\mathbf{f}(\mathbf{x},\mathbf{u}), \tag{27}$$

$$\mathbf{0}=\mathbf{g}(\mathbf{x},\mathbf{u}). \tag{28}$$

The existence and uniqueness of the steady state follows from Leeper and Sims (1994). We denote the steady states of **x** and **u** by $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$, respectively, where $\bar{\mathbf{u}}$ is found from (15) and (16) in the steady state to be

$$\begin{aligned} \bar{i} &= \beta, \\ \bar{\dot{i}} &= 0, \\ \bar{\tau}_c &= \frac{\bar{\tau}}{\bar{C}}. \end{aligned} \tag{29}$$

In particular, the first equation of (29) is found from (15) in the steady state, the second equation from the definition of steady state, and the third equation from (16) in the steady state. The joint values of $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are solutions to (27)-(28), and (29). The resulting steady state is the equilibrium of (25)-(26), when the control variables are set at their steady state.

The vector of 10 parameters in the steady state system is

$$\mathbf{p}=[\pi\ \beta\ \theta\ \alpha\ a\ \phi\ \delta\ \mu\ \gamma\ \sigma]',$$

where the prime denotes transpose. Leeper and Sims (1994) estimate the parameters with quarterly data from 1959 to 1992. Although g is not a parameter of tastes or technology, it is taken as a fixed value by the private sector at its setting by the government.

The constraints on the parameter values and g are:

$$0<\pi<1,\ \gamma>0,\ 0\le\sigma\le 1,\ \mu\ge 1,\ \delta\ge 0,\ 0\le\beta\le 1,\ \delta>0\ \ g\ge 0. \tag{30}$$

3. Singularity Bifurcation in the Leeper and Sims Model

We explore the structural properties of the Leeper and Sims model in a small neighborhood of the steady state, $(\bar{\mathbf{x}},\bar{\mathbf{u}})$, by using local linearization around the steady state. The linearized version of the system ((25),(26)) is

$$\mathbf{E}_1\dot{\mathbf{x}}=\mathbf{A}_1\mathbf{x}+\mathbf{B}_1\mathbf{u} \tag{31}$$

$$\mathbf{0} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} \tag{32}$$

where

$$\mathbf{E}_1 = \mathbf{h}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{m_1 \times m} = \mathbb{R}^{4\times 7},$$
$$\mathbf{A}_1 = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}|_{\mathbf{x}=\bar{\mathbf{x}},\mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_1 \times m} = \mathbb{R}^{4\times 7},$$
$$\mathbf{A}_2 = \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}|_{\mathbf{x}=\bar{\mathbf{x}},\mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_2 \times m} = \mathbb{R}^{3\times 7},$$
$$\mathbf{B}_1 = \frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}|_{\mathbf{x}=\bar{\mathbf{x}},\mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_1 \times l} = \mathbb{R}^{4\times 2},$$
$$\mathbf{B}_2 = \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}|_{\mathbf{x}=\bar{\mathbf{x}},\mathbf{u}=\bar{\mathbf{u}}} \in \mathbb{R}^{m_2 \times l} = \mathbb{R}^{3\times 2}.$$

Equation (31) has 4 equations, and equation (32) has 3 equations.

The linearized system ((31),(32)) is solvable if it is regular. Using the relevant regularity condition from Gantmacher (1974), we have the following solvability condition, which must hold for some values of the determinant's parameter, s:

$$\det\left(\begin{bmatrix} s\mathbf{E}_1 - \mathbf{A}_1 \\ -\mathbf{A}_2 \end{bmatrix}\right) \neq 0.$$

If that regularity condition is violated for all s, the linearized system either has multiple solutions or no solution. We randomly chose parameter values within the theoretically feasible region and observed that the Leeper and Sims model, as expected, is regular.

To study the structural properties of the Leeper and Sims model, we further transform the linearized system ((31),(32)) into the following form.

Definition 3.1 Two systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{33}$$

and

$$\tilde{\mathbf{E}}\dot{\mathbf{y}} = \tilde{\mathbf{A}}\mathbf{y} + \tilde{\mathbf{B}}\mathbf{u} \tag{34}$$

are said to be restricted system equivalent (r.s.e.) if there exist two nonsingular matrices, $\mathbf{T}_1$ and $\mathbf{T}_2$, such that

$$\mathbf{T}_1\mathbf{E}\mathbf{T}_2 = \tilde{\mathbf{E}},$$
$$\mathbf{T}_1\mathbf{A}\mathbf{T}_2 = \tilde{\mathbf{A}},$$
$$\mathbf{T}_1\mathbf{B} = \tilde{\mathbf{B}},$$
$$\mathbf{x} = \mathbf{T}_2\mathbf{y}.$$

The transformed system (34) can be obtained by substituting the coordinate transform, $\mathbf{x} = \mathbf{T}_2\mathbf{y}$, into (33) and then multiplying both sides by $\mathbf{T}_1$ from the left. The relationship of r.s.e. permits transforming a system into a convenient form, while preserving important properties.

We next transform (31)-(32) into a suitable r.s.e. form. First, denote

$$r_E = rank(\mathbf{E}_1),$$

where $r_E \in \{1,2,3,4\}$. Then there exist nonsingular matrices $\mathbf{T}_1 \in \mathbb{R}^{4\times 4}$ and $\mathbf{T}_2 \in \mathbb{R}^{7\times 7}$ such that

$$\mathbf{T}_1\mathbf{E}_1\mathbf{T}_2 = \begin{bmatrix} \mathbf{I}_{r_E} & 0 \\ 0 & 0 \end{bmatrix},$$

which is a 4×7 matrix. Consider the following coordinate transform:

$$\mathbf{x} = \mathbf{T}_2 \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix},$$

where $\boldsymbol{y}_1 \in \mathbb{R}^{r_E}$ and $\boldsymbol{y}_2 \in \mathbb{R}^{m-r_E} = \mathbb{R}^{7-r_E}$.

Substituting that form of $\mathbf{x}$ into (31)-(32) and then multiplying both sides of (31) by $\mathbf{T}_1$, we find that (31)-(32) is r.s.e. to

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \tag{35a}$$

$$0 = \mathbf{A}_{21}\mathbf{y}_1 + \mathbf{A}_{22}\mathbf{y}_2 + \mathbf{B}_{12}\mathbf{u}, \tag{35b}$$

$$0 = \mathbf{A}_{31}\mathbf{y}_1 + \mathbf{A}_{32}\mathbf{y}_2 + \mathbf{B}_{2}\mathbf{u}, \tag{35c}$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{T}_1\mathbf{A}_1\mathbf{T}_2, \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix} = \mathbf{T}_1\mathbf{B}_1, [\mathbf{A}_{31} \quad \mathbf{A}_{32}] = \mathbf{A}_2\mathbf{T}_2.$$

Note that in acquiring (35c) from (32), we did not premultiply by $\mathbf{T}_1$. Differential equation system, (35a), contains r_E equations, (35b) contains 4-r_E equations, and (35c) contains 3 equations. Also note that $\mathbf{A}_{11} \in \mathbb{R}^{r_E\times r_E}$, $\mathbf{A}_{12} \in \mathbb{R}^{r_E\times(7-r_E)}$, $\mathbf{A}_{21} \in \mathbb{R}^{(4-r_E)\times r_E}$, $\mathbf{A}_{22} \in \mathbb{R}^{(4-r_E)\times(7-r_E)}$, $\mathbf{A}_{31} \in \mathbb{R}^{3\times r_E}$, $\mathbf{A}_{32} \in \mathbb{R}^{3\times(7-r_E)}$, $\mathbf{B}_{11} \in \mathbb{R}^{r_E\times 2}$, and $\mathbf{B}_{12} \in \mathbb{R}^{(4-r_E)\times 2}$, while $\mathbf{y}_1$ is an r_E dimensional vector and $\mathbf{y}_2$ is a 7 - r_E dimensional vector.

Combining equations (35a) and (35b), we have

$$\dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \tag{36a}$$

$$0 = \widetilde{\mathbf{A}}_{21}\mathbf{y}_1 + \widetilde{\mathbf{A}}_{22}\mathbf{y}_2 + \widetilde{\mathbf{B}}_{12}\mathbf{u}, \tag{36b}$$

where

$$\widetilde{\mathbf{A}}_{21} = \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{31} \end{bmatrix}, \widetilde{\mathbf{A}}_{22} = \begin{bmatrix} \mathbf{A}_{22} \\ \mathbf{A}_{32} \end{bmatrix}, \widetilde{\mathbf{B}}_{12} = \begin{bmatrix} \mathbf{B}_{12} \\ \mathbf{B}_{2} \end{bmatrix}.$$

Note that $\widetilde{\mathbf{A}}_{22}$ is a square matrix of dimension $(7 - r_E) \times (7 - r_E)$.

If $\widetilde{\mathbf{A}}_{22}$ is nonsingular, it is possible to solve for $\mathbf{y}_2$ from the algebraic constraint equation (36b). In this case, we have

$$\boldsymbol{y}_2 = -(\widetilde{\mathbf{A}}_{22})^{-1}(\widetilde{\mathbf{A}}_{21}\mathbf{y}_1 + \widetilde{\mathbf{B}}_{12}\mathbf{u}).$$

Substituting this form of $\mathbf{y}_2$ into (36a), we obtain

$$\dot{\boldsymbol{y}}_1 = \left(\mathbf{A}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{A}}_{21}\right)\boldsymbol{y}_1 + \left(\mathbf{B}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{B}}_{12}\right)\mathbf{u},$$

or equivalently,

$$\dot{\boldsymbol{y}}_1 = \mathbf{C}\boldsymbol{y}_1 + \mathbf{D}\mathbf{u}, \tag{37}$$

where $\mathbf{C} = \mathbf{A}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{A}}_{21} \in \mathbb{R}^{r_E \times r_E}$ and $\mathbf{D} = \mathbf{B}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{B}}_{12} \in \mathbb{R}^{r_E \times 2}$. Hence, if $\widetilde{\mathbf{A}}_{22}$ is nonsingular, the dynamics of $\mathbf{y}_1$ can be explained entirely in terms the system of ordinary differential equations, (37). The algebraic relationship between $\mathbf{y}_1$ and $\mathbf{y}_2$ in equation (36b) is needed solely to determine the derived dynamics of $\mathbf{y}_2$.

However, this transformation would not be possible, if $\widetilde{\mathbf{A}}_{22}$ were singular. Hence, it also is true that the untransformed linear system ((31),(32)) is equivalent to ((37),(36c)), only when $\widetilde{\mathbf{A}}_{22}$ is nonsingular. Settings of the parameters of $\widetilde{\mathbf{A}}_{22}$ that cause that matrix to become singular produce a "singularity bifurcation" boundary within the parameter space, as we demonstrate and explore further below.

As explained in Barnett and He (2001,2004,2006), the dimension of dynamics change, when parameters move onto a singularity bifurcation boundary. Even if the parameters do not move onto the bifurcation boundary, but instead cross that boundary between two regions within which the matrix is nonsingular, the nature of the dynamics will change dramatically. Consequently, the dynamics of the system ((31),(32)) could be dramatically different from those of ordinary linear differential equations, if $\widetilde{\mathbf{A}}_{22}$ were singular. The dynamics also would change substantially, if $\widetilde{\mathbf{A}}_{22}$ moves between two settings located on opposite sides of a singularity bifurcation boundary. But the dimension of the dynamics will change, only if $\widetilde{\mathbf{A}}_{22}$ becomes exactly singular, putting the model directly on a singularity bifurcation boundary.

To see further what could happen when $\widetilde{\mathbf{A}}_{22}$ is singular, we rewrite the linearized system ((36a),(36b)) as

$$\begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} \\ \widetilde{\mathbf{B}}_{12} \end{bmatrix} \mathbf{u}. \tag{38}$$

If the Leeper and Sims model is regular, so is the matrix pair

$$\left(\begin{bmatrix}\mathbf{I}_{r_E} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22}\end{bmatrix}\right),$$

which is in the form of a matrix pencil.

For a regular matrix pencil, there exist nonsingular matrices $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{T}}_2$ such that[8]

$$\widetilde{\mathbf{T}}_1\begin{bmatrix}\mathbf{I}_{r_E} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\widetilde{\mathbf{T}}_2 = \begin{bmatrix}\mathbf{I}_{\widetilde{m}_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{N}\end{bmatrix} \text{ and } \widetilde{\mathbf{T}}_1\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22}\end{bmatrix}\widetilde{\mathbf{T}}_2 = \begin{bmatrix}\widetilde{\mathbf{A}}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{\widetilde{m}_2}\end{bmatrix},$$

where $\tilde{m}_1 + \tilde{m}_2 = m$ and **N** is a nilpotent matrix. By the definition of nilpotent matrix, there exists a positive integer $d > 1$ such that

$$\mathbf{N}^d = \mathbf{0}.$$

The smallest such integer d is called the nilpotent index of **N**.

Clearly $\mathbf{N} = \mathbf{0}$ satisfies the definition of nilpotence. The following is another example of a nilpotent matrix:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ & .. & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1\\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{39}$$

Consider the coordinate transform

$$\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = \widetilde{\mathbf{T}}_2\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix}.$$

Substituting for **y** in equation (38) and multiplying both sides of (38) by $\tilde{\mathbf{T}}_1$ from the left, we have another r.s.e. form of ((31),(32)),

$$\dot{\mathbf{z}}_1 = \widetilde{\mathbf{A}}_1\mathbf{z}_1 + \widetilde{\mathbf{B}}_1\mathbf{u}, \tag{40}$$

$$\mathbf{N}\dot{\mathbf{z}}_2 = \mathbf{z}_2 + \widetilde{\mathbf{B}}_2\mathbf{u}, \tag{41}$$

where

$$\begin{bmatrix}\widetilde{\mathbf{B}}_1\\ \widetilde{\mathbf{B}}_2\end{bmatrix} = \widetilde{\mathbf{T}}_1\begin{bmatrix}\mathbf{B}_{11}\\ \widetilde{\mathbf{B}}_{12}\end{bmatrix}.$$

The solutions to (40) and (41) are respectively,

$$\mathbf{y}_1 = e^{\tilde{\mathbf{A}}_1(t-t_0)}\mathbf{y}_1(0) + \int_{t_0}^{t} e_1^{\tilde{\mathbf{A}}}(t-\xi)\tilde{\mathbf{B}}_1\mathbf{u}(\xi)d\xi$$

[8] See Gantmacher (1974).

$$\mathbf{y}_2 = -\sum_{k=1}^{d-1} \delta^{(k-1)}(t)\mathbf{N}^k \mathbf{y}_2(0) - \sum_{k=0}^{d-1} \mathbf{N}^k \tilde{\mathbf{B}}_{12} \mathbf{u}^{(k)}(t),$$

where $t_0 \geq 0$ is the initial time, $\delta^{(k-1)}$(t) is the derivative of order k-1 of the Dirac delta function, and $\mathbf{u}^{(k)}$ denotes the k-th order derivative of $\mathbf{u}$.[9]

Unless $\mathbf{N} = \mathbf{0}$ or the initial state $\mathbf{y}_2(0) = \mathbf{0}$, there exist the impulse terms in the first summation in the solution for $\mathbf{y}_2$, as well as the smooth derivative terms of $\mathbf{u}$ in the second summation. In fact when $\mathbf{N} = \mathbf{0}$, the above solution for $\mathbf{y}_2$ does not apply, although the solution for $\mathbf{y}_1$ above remains valid. This solution structure with nonzero $\mathbf{N}$ is very different from that of ordinary differential equations, such as (40), for $\mathbf{y}_1$.

The first summation in the solution for $\mathbf{y}_2$ could produce shock effects to the state response of $\mathbf{y}_2$. The Dirac delta, which is $\delta^{(k-1)}$ when k=1, is often called the unit impulse function.[10] But if $\mathbf{N} = \mathbf{0}$, we have from (41) that

$$\mathbf{z}_2 = -\tilde{\mathbf{B}}_2 \mathbf{u},$$

which is a smooth algebraic relationship between $\mathbf{y}_2$ and $\mathbf{u}$. This bifurcation phenomenon at $\mathbf{N} = \mathbf{0}$ is consistent with the following theorem, proving equivalence between bifurcation at $\mathbf{N} \neq \mathbf{0}$ and at singularity of $\tilde{\mathbf{A}}_{22}$.

Theorem 3.1. If both systems ((40),(41)) and ((36a),(36b)) are r.s.e. forms of the same linearized system ((31),(32)), then

$$\mathbf{N} = \mathbf{0},$$

if and only if $\tilde{\mathbf{A}}_{22}$ is nonsingular, i.e., $det(\tilde{\mathbf{A}}_{22}) \neq 0$.

Proof. If $\mathbf{N} = \mathbf{0}$, then ((40),(41)) and ((36a),(36b)) are r.s.e. forms with $\tilde{\mathbf{A}}_{22} = \mathbf{I}_{\tilde{m}_2}$, which is nonsingular.

---

[9] We use e raised to a matrix power to designate the matrix of e to the power of each element of the matrix power. Regarding the form of the solutions to (40) and (41), see Cobb (1982, 1983). The discrete analog of the delta function is the Kronecker delta.

[10] The Lebesgue integral of the Dirac delta function from minus infinity to plus infinity is 1.0. Formally the Dirac delta, δ(t), is not a function but the limit of a sequence of functions (the nascent delta functions). In that limit, the Dirac delta is a measure with unit mass at the origin and is often called the unit impulse function. The antiderivative of the Dirac delta is the Heaviside (unit) step function, so that the Dirac delta can be viewed as the derivative of the step function. Since the Dirac delta is a measure, its derivatives require careful definition. Those derivatives are higher order "singularity functions" called "doublets," "triplets," etc. It can be shown that the n'th derivative of δ(t) is $\delta^{(n)} = (-1)^n n!\ \delta(t)/t^n$. Note that if δ(x) is the unit impulse at t=0, then $\delta^{(n)}$(t) is a rescaled impulse at t=0.

Conversely, assume $\widetilde{\mathbf{A}}_{22}$ is nonsingular. Then choose

$$\widetilde{\mathbf{T}}_1 = \begin{bmatrix} \mathbf{I}_{\widetilde{m}_1} & -\mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_{\widetilde{m}_2} \end{bmatrix}, \widetilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{A}}_{21} & \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_{\widetilde{m}_2} \end{bmatrix}. \quad (42)$$

Direct verification confirms that

$$\widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{I}_{r_E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \widetilde{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{I}_{\widetilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } \widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \end{bmatrix} \widetilde{\mathbf{T}}_2 = \begin{bmatrix} \widetilde{\mathbf{A}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\widetilde{m}_2} \end{bmatrix},$$

with

$$\widetilde{\mathbf{A}}_1 = \mathbf{A}_{11} - \mathbf{A}_{12}\widetilde{\mathbf{A}}_{22}^{-1}\widetilde{\mathbf{A}}_{21}.$$

Therefore, we have that $\mathbf{N} = \mathbf{0}$. This completes the proof. ∎

With the linearized model ((31),(32)) singularity of $\widetilde{\mathbf{A}}_{22}$ results in completely different dynamical solution behavior. As a result, we say a singularity bifurcation occurs, when

$$det(\widetilde{\mathbf{A}}_{22}) \neq 0. \quad (43)$$

The preceding condition has another form in terms of the original coefficient matrices of ((31),(32)), as shown in the following theorem.

Theorem 3.2. Assume that $\mathbf{E}_1$ has full row rank, so that

$$rank(\mathbf{E}_1) = m_1. \quad (44)$$

Then $\widetilde{\mathbf{A}}_{22}$ is nonsingular if and only if the m×m (i.e., 7×7) matrix

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

is nonsingular, so that

$$rank(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}) = m. \quad (45)$$

Proof. Denote

$$\hat{\mathbf{T}}_1 = \begin{bmatrix} \tilde{\mathbf{T}}_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}}_1 \end{bmatrix}, \hat{\mathbf{T}}_2 = \begin{bmatrix} \tilde{\mathbf{T}}_2 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}}_2 \end{bmatrix},$$

where $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{T}}_2$ are defined as in (42). Then both $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{T}}_2$ are non-singular.

Consider the following matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then we have

$$\widehat{\mathbf{T}}_1 \mathbf{\Lambda} \widehat{\mathbf{T}}_2 = \begin{bmatrix} \widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{0} \end{bmatrix} \widetilde{\mathbf{T}}_2 & \widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \widetilde{\mathbf{T}}_2 \\ \mathbf{0} & \widetilde{\mathbf{T}}_1 \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{0} \end{bmatrix} \widetilde{\mathbf{T}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\widetilde{m}_1} & \mathbf{0} & \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{A}}_{21} & \widetilde{\mathbf{A}}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\widetilde{m}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

with

$$rank(\widehat{\mathbf{T}}_1 \mathbf{\Lambda} \widehat{\mathbf{T}}_2) = 2\widetilde{m}_1 + rank(\widetilde{\mathbf{A}}_{22}). \tag{46}$$

But if $\mathbf{E}_1$ has full row rank, $\tilde{m}-1 = m_1$, then

$$rank(\mathbf{E}_1) = \tilde{m}_1 = m_1,$$

and

$$rank(\mathbf{\Lambda}) = rank(\begin{bmatrix} \mathbf{E}_1 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}) = rank(\mathbf{E}_1) + rank(\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{E}_1 \end{bmatrix})$$

$$= m_1 + rank(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}).$$

Combining the previous equation with (42), we obtain

$$rank \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix} = m_1 + rank(\widetilde{\mathbf{A}}_{22}). \tag{47}$$

Note that $\widetilde{\mathbf{A}}_{22} \in \mathbb{R}^{\widetilde{m}_2 \times \widetilde{m}_2}$ and $\widetilde{m}_2 = m_2$. Hence equation (47) says that $\widetilde{\mathbf{A}}_{22}$ is nonsingular if and only

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

is nonsingular. ∎

Therefore, the following condition for singularity bifurcation is provided by Theorem 3.2:

$$\det(\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}) = 0. \tag{48}$$

Note that $\mathbf{x}_2$ is solvable from (37) alone if $\widetilde{\mathbf{A}}_{22}$ is nonsingular. Therefore, singularity condition implies the case in which $\mathbf{x}_2$ is not readily solvable from the algebraic (37) alone. We need to take into account of the dynamic constraint (36).

We next introduce another property to have a closer look at the singularity condition.

Corollary 3.1. Consider the following system describing the dynamics of ($\mathbf{x}$,$\mathbf{v}$), where $\mathbf{v} \in \mathbb{R}^{m_3}$ for arbitrary m_3.

$$\mathbf{E}_1\dot{\mathbf{x}} + \mathbf{E}_{1\mathbf{v}}\dot{\mathbf{v}} = \mathbf{A}_1\mathbf{x} + \mathbf{A}_{1\mathbf{v}}\mathbf{v} + \mathbf{B}_1\mathbf{u}, \tag{49a}$$

$$\dot{\mathbf{v}} = \mathbf{A}_\mathbf{v}\mathbf{v} + \mathbf{B}_\mathbf{v}\mathbf{u}, \tag{49b}$$

$$\mathbf{0} = \mathbf{A}_2\mathbf{x} + \mathbf{A}_{2\mathbf{v}}\mathbf{v} + \mathbf{B}_2\mathbf{u}, \tag{49c}$$

where $\mathbf{E}_{1\mathbf{v}}$, $\mathbf{A}_{1\mathbf{v}}$, $\mathbf{A}_\mathbf{v}$, $\mathbf{B}_\mathbf{v}$, $\mathbf{A}_{2\mathbf{v}}$ are arbitrary matrices of dimension $m_1 \times m_3$, $m_1 \times m_3$, $m_3 \times m_3$, $m_3 \times l$, and $m_2 \times m_3$, respectively, and the other matrices are as defined above. Then the singularity condition for ((49a),(49b),(49c)) is the same as that for ((31),(32)).

Proof. According to Theorem 3.2, the singularity condition for ((49a),(49b),(49c)) is

$$\det\begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_{1\mathbf{v}} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{A}_2 & \mathbf{A}_{2\mathbf{v}} \end{bmatrix}.$$

By eliminating the second column, that determinant condition is equivalent to (48), which is the singularity condition for ((31),(32)). ∎

Corollary 3.1 says that adding (or deleting) state variables that can be modeled by ordinary differential equations does not change the singularity condition. This property is useful in reducing the dimension of the problem under consideration. For example, we could drop the Leeper and Sims' model's state variable, K, from its state vector, (17), in the system ((31),(32)), without affecting the singularity condition.

By thereby dropping the state variable, K, the singularity condition becomes

$$\det(\begin{bmatrix} \mathbf{E}'_1 \\ \mathbf{A}'_2 \end{bmatrix}) = 0, \tag{50}$$

where

$$\mathbf{E}'_1 = \begin{bmatrix} \frac{1}{P} & \frac{Y}{PV} & 0 & 0 & 0 & \frac{1}{V} \\ 0 & \frac{1}{P} & e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & 0 & e_{26} \\ 0 & \frac{1}{P} & 0 & 0 & -\frac{1-\mu}{Z+nK} & \frac{1-\mu}{Y} \end{bmatrix}$$

and

$$\mathbf{A}'_2 = \begin{bmatrix} 0 & 0 & \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & \mu Y^{\mu-1} \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{26} \\ 0 & 0 & 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1} \end{bmatrix},$$

with

$$e_{23} = \frac{1-\pi(1-\gamma)}{C^*}[1-\phi VY^{\mu}(\mu-1)(C+g)^{\mu-2}] - \frac{1-\mu}{C+g},$$

$$e_{26} = \frac{1-\pi(1-\gamma)}{C^*}[-\phi VY^{\mu}\mu(C+g)^{\mu-1}] + \frac{1-\mu}{Y},$$

$$a_{23} = (1-2\phi V)A^{\sigma}Y^{\mu-\sigma}L^{\sigma-1}(1-\mu)(C+g)^{-\mu} - \frac{1-\pi}{\pi}\frac{1}{1-L},$$

$$a_{24} = (1-2\phi V)A^{\sigma}Y^{\mu-\sigma}(\sigma-1)L^{\sigma-2}(C+g)^{1-\mu} - \frac{1-\pi}{\pi}\frac{C}{(1-L)^2},$$

$$a_{26} = (1-2\phi V)A^{\sigma}(\mu-\sigma)Y^{\mu-\sigma-1}L^{\sigma-1}(C+g)^{1-\mu}.$$

Note that the prime does not designate transpose but rather deletion of the state variable, K, from the vector $\mathbf{x}$ in equation (17) and deletion of equation (21), which is the corresponding differential equation for capital.

Direct calculation shows that (50) is equivalent to

$$0 = \det\left(\begin{bmatrix} e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & \frac{1-\mu}{Z+nK} & e'_{26} \\ \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & -\mu Y^{\mu-1} \\ a_{23} & a_{24} & 0 & a_{26} \\ 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1} \end{bmatrix}\right), \qquad (51)$$

where

$$e'_{26} = \frac{1-\pi(1-\gamma)}{C^*}[-\phi V Y^{\mu} \mu (C+g)^{\mu-1}].$$

As we shall demonstrate later, singularity does occur within the theoretically feasible parameter regions.

In systems theory, bifurcation is said to occur if change of structural dynamic solution properties occurs, when a parameter crosses a certain value. Such a critical value is called a bifurcation point. Many types of bifurcation are known, such as saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. Bifurcation analysis is particularly useful in locating subsets of the parameter space supporting various dynamical behaviors of a system, such as the existence of limit cycles, multiperiodic instability, monotonic stability, or damped stability.

We find that the Leeper and Sims model has structural changes in its dynamics, and the boundary determined by (51) is a singularity-induced bifurcation boundary. To the best of our knowledge, this is the first time that this type of bifurcation has been found in a macroeconometric model.

Leeper and Sims (1994) proposed government policy control using the monetary policy (15) and the tax policy (16). To investigate bifurcation of the closed-loop system under the control of government policies, let us augment the state variable to include the two controls, as follows:

$$\mathbf{x}_c = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_C \end{bmatrix}. \tag{52}$$

With this new augmented state vector, the linearized system ((31),(32)) becomes

$$\mathbf{E}_1^c \dot{\mathbf{x}}_c = \mathbf{A}_1^c \mathbf{x}_c, \tag{53}$$

$$\mathbf{0} = [\mathbf{A}_2 \quad \mathbf{0}]\mathbf{x}_c, \tag{54}$$

where $\mathbf{E}_1^c \in \mathbb{R}^{m_1^c \times m^c} = \mathbb{R}^{6\times 9}$, $\mathbf{A}_1^c \in \mathbb{R}^{m_1^c \times m^c} = \mathbb{R}^{6\times 9}$, $m_1^c = m_1 + 2$, and $m^c = m + 2$.

4. Numerical Results

In this section, we numerically locate the singularity-induced bifurcation boundaries. We use the condition (51) applied to the closed-loop system (54).[11]

We first test all pairs of parameters to determine those pairs that reach bifurcation boundaries, when the pair is varied with all other parameters set at their point estimates.[12] Pairs of parameters permitted to vary about their point estimates are allowed to take values within the intersection of their theoretically feasible ranges and their 95% confidence intervals of their estimated values. In particular, the intersection, $\mathcal{H}$, of (30) and

$$p(i) \in [\bar{p}(i) - \bar{c}\sigma_i \, , \, \bar{p}(i) + \bar{c}\sigma_i],$$

where $\bar{p}(i)$ is the estimated value of parameter $p(i)$, σ_i is the standard error of the estimate, and $\bar{c}$ is the critical value of the 95th-percentile confidence interval for N(0,1).[13]

Figures 4.1 and 4.2 show some of the sections of the singularity-induced bifurcation boundary that we located. Figure 4.1 displays 2-dimensional sections with the other parameters set at their point estimates, while figure 4.2 displays 3-dimensional sections with the other parameters set at their point estimates.[14] In the first section of figure 4.2, we display a section varying μ and g, while in the second section, we display μ versus β. The range of the plots' axes are within the $\mathcal{H}$ intervals about each parameter's estimate. Table 1 provides the point estimates, standard errors, and $\mathcal{H}$ intervals used in producing figures 4.1 and 4.2.[15]

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[11] Regarding numerical stability, we limited our computations to the theoretical procedure mentioned earlier. We did not use additional algorithms to check for numerical stability. But since we encountered no stability problems with MatLab software, we had no indication of the need for redundant checks of numerical stability.

[12] Hyperplanes along which only two parameters vary can fail to intersect bifurcation boundaries, even if they exist at other settings of some parameters.

[13] For some parameters, standard errors are not provided in Leeper and Sims (1994). In such unfortunate cases, we permitted parameter values to take values within 50% of the point estimates. Such a range of parameter values keep the parameters well within the theoretically feasible region. Another complication is produced by the fact that Leeper and Sims did not report covariances of parameter estimators. Hence, in our three dimensional searches we do not have 3-dimensional confidence regions, but rather use the Cartesian products of the pairwise confidence intervals.

[14] Since the parameter space is a high dimensional space, we investigated many sections of the space within $\mathcal{H}$. We display only the sections that we found to be particularly informative about the location and nature of the singularity boundary.

[15] While β and μ are parameters, g is an exogenous variable. What we display as "estimate" and "standard error" for g is the sample mean and standard deviation of g.

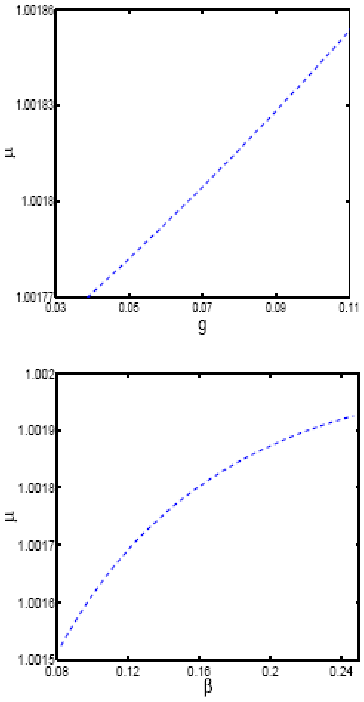

Figure 4.1. Two-dimensional sections of a singularity-induced bifurcation boundary.

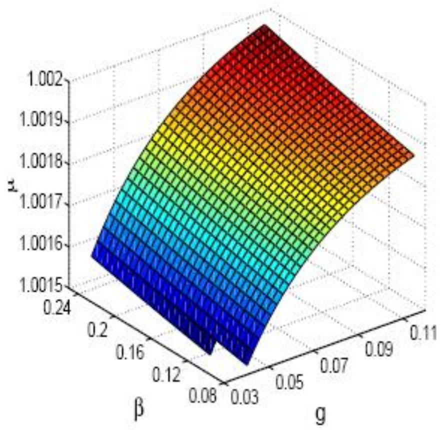

Figure 4.2. **Three-dimensional sections of a singularity-induced bifurcation boundary.**

The estimation information for the parameters μ, g, and β used in figures 4.1 and 4.2 are in Table 1. All estimation information is taken directly from the Leeper and Sims paper. We make no changes in their models, in their reported point estimates, or their reported standard errors. Our experiments are conditional upon what Leeper and Sims have published, without modification.

Table 1. Estimation of μ, g, and β

parameter	estimate	standard error	$\mathcal{H}$ interval
μ	1.0248	0.324	[1,1.6598]
g	0.0773*	0.292*	[0, 0.6496]
β	0.1645	0.288	[0, 0.7290]

*Since g is an exogenous variable, rather than a parameter, the "estimate" is the sample mean and the "standard error" is the sample standard deviation.

To illustration what happens when parameter values cross the singularity boundary, consider the parameter β. Table 2 displays the changes of finite eigenvalues, $\lambda_1,...,\lambda_8$, when β varies.

Table 2. Eignevalue changes

β	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_1	1.002	1.002	1.002	1.002	1.002	1.002	1.002
λ_2	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_3	-0.303	-0.262	-0.220	-0.215	-0.210	-0.178	-0.135
λ_4	-3.558	-3.559	-3.561	-3.561	-3.561	-3.563	-3.566
λ_5	-0.098	-0.084	-0.077	-0.076	-0.075	-0.072	-0.069
λ_6	-0.002	-0.003	-0.003	-0.003	-0.003	-0.004	-0.004
λ_7	3.101	5.177	8.237	8.682	9.254	13.416	28.401
λ_8	-117.790	-204.703	-1811.413	∞	1456.294	195.888	58.059

The first row in Table 2 contains the settings of β that we explore. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model at each setting of β. There are three more eigenvalues, which are not shown in the table. Those eigenvalues are infinite. The table shows that when the value of β increases and crosses the bifurcation boundary, λ_8 decreases rapidly to $-\infty$, spikes suddenly from $-\infty$ to $+\infty$, and then decreases from $+\infty$.

Table 2 clearly shows that the Leeper and Sims model has a structural change in dynamics, when β crosses the singularity-induced bifurcation boundary. The two regions separated by the boundary exhibit drastically different dynamical behaviors. Also note the very small range of values of μ displayed along its axis in figures 4.1 and 4.2. That displayed range consists of a small subset of values within the interval $\mathcal{H}$. Clearly, very small changes in μ can cause bifurcation, independently of the settings of g or β.

As shown by Table 2, such singularity bifurcations can have dramatic effects. The number of dynamic equations and the number of algebraic equations change, when the singularity-induced bifurcation boundary is reached.

5. Conclusions

The Leeper and Sims Euler-equations macroeconometric model is representative of a larger class of systems, designed to address the Lucas critique. The most distinguishing characteristic of this class of system is the models' form,

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

in which the matrix **E** could become singular at some settings of the parameters. In this paper, we have examined the basic properties of such a model: the important Leeper and Sims model. We propose an approach for bifurcation analysis of such models, and most importantly discover the existence of singularity-induced bifurcations for the first time in a macroeconometric model.

Within a small region of the estimated parameter values, we locate and characterized the nature of the singularity-induced bifurcation. The dynamic order of the system changes in a dramatic manner, when parameter values reach or cross the bifurcation boundary. In a theoretical survey paper of various types of bifurcation, Barnett and He (2006) have argued that singularity bifurcation may not be unusual in Euler equations models; and we have, in the current paper, illustrated that theoretical speculation in a well known model from that class.

With the policy-relevant Bergstrom Keynesian second-order differential-equations macroeconometric model of the UK economy, Barnett and He (1998,1999, 2001, 2002) found three types of bifurcation boundaries within the parameters' confidence regions. Subsequently with New Keynesian models, Barnett and Duzhak (2008,2009) have found Hopf and flip (period doubling) bifurcation boundaries. Now in the current paper, we have found a particularly dramatic type of bifurcation with an Euler equations model, having deep parameters that are invariant to policy rule changes and thereby immune to the Lucas critique.

In all of these studies, the models used are highly policy-relevant and were not modified from their influential previously-published forms. While Grandmont's model has been criticized for its lack of policy relevance, we believe from the accumulating evidence that Grandmont's conclusions are correct and are highly relevant to policy. In particular, these results cast into doubt the dynamical inferences acquired in the traditional manner by simulating macroeconometric models solely at their parameter point-estimates. To be able to achieve robustness of dynamical inferences, such simulations should be made at various settings throughout the parameters' confidence region.

Although the Leeper and Sims model is specified as a closed economy model, it is implicitly open economy as estimated by Leeper and Sims, since the US data used in the model includes imported and exported goods. A logical extension of this experiment would be to apply our procedure to an explicitly open-economy Euler-equations model. Because of the connection between Euler-equation implicit functions and singularity bifurcation, we would expect similar results with an explicitly open-economy Euler-equations model.

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