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Heller, Claudia

Universidade Estadual Paulista - UNESP

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Claudia Heller [*]

This paper focuses on Keynes's exposition of the Principle of Effective Demand and its generalised mathematical representation – the basis of a *Z-D* type model. It elaborates on Keynes's algebraic formulation in the *General Theory*, relying on interpreters who contributed to the generalisation of his most restrictive hypotheses on competition and returns to scale as well as on those who developed the algebraic argumentation that Keynes left only indicated. Instead of correcting Keynes's mathematics (which is right), the paper concludes that there has been a "slip of the pen" in his own description of these concepts on the footnote to page 55 of the *General Theory*. Keynes's employment function, the inverse of his aggregate supply curve is not the same thing as his aggregate supply function. Therefore, in the controversial footnote, it is not the aggregate supply function but the employment function that is linear with a slope given by the reciprocal of the money-wage.

Key words: Principle of effective demand, D/Z-model, Aggregate demand, Aggregate supply
JEL classification: B2, B22, B31

Address for correspondence:
Departamento de Economia, Faculdade de Ciências e Letras, Universidade Estadual Paulista, UNESP. Rodovia Araraquara-Jaú km 1. Araraquara 14800-109. SP Brazil. email: hellerc@fclar.unesp.br
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1. Introduction

Three cross-shaped graphical interpretations sprung up from John Maynard Keynes's *General Theory of Employment, Interest, and Money*: the IS-LM, the 45^{o} and the *Z-D* models. Although all seek to achieve graphical and algebraic formalisations of Keynes's theory, the last two are more specific, being directly related to the Principle of Effective Demand.

According to King (1994) the initial attempts to diagrammatically represent Keynes's Principle of Effective Demand were made by Tarshis (1947), Dillard (1948), Patinkin (1949), Weintraub (1951) and Hansen (1953). No consensus on Keynes's aggregate supply and demand analysis has emerged from them nor from the controversy that began in 1954 in the *Economic Journal* – after Patinkin´s (1949) paper – nor from several others.[1] It is in fact an enduring controversy, having led to the recent contributions by Hayes (2008) and Hartwig and Brady (2008).

A majority of these studies feature the discussion of Keynes's propositions on the supply side, as opposed to the most popular understanding that his 1936 book deals only or mainly with the demand side. They have tried to point out and eventually criticize or develop Keynes's ideas on supply. Many also deal with Keynes's propositions that aggregate demand and effective demand are different concepts. While trying to mathematically depict the Principle of Effective Demand, some have arrived at Keynes's employment function but failed to realize it; others arrived at diagrams close to Keynes's literary and mathematical description of the function but did not get it totally right.

This paper does not intend to survey all the contributions on the field, but it takes into account those that assessed Keynes's employment function through a mathematical and/or a graphic representation. Thus, the paper has only one section besides this introduction and the conclusion. It focuses on Keynes's exposition of the Principle of Effective Demand and its generalised mathematical representation – the basis of a *Z-D* type model. It elaborates on Keynes's algebraic formulation as presented in the *General Theory*, relying on interpreters who contributed to the generalisation of his most restrictive hypotheses on competition and returns to scale as well as on

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[1] In the *Economic Journal* the debate initially involved De Jong (1954A, 1954B, 1955, 1956), Hawtrey (1954, 1956), Robertson and Johnson (1955), Robertson (1956), Weintraub (1957), Wells (1960, 1961), Marty (1961), Veendorp and Werkema (1961), Kooros (1961), Neissen (1961) and Davidson (1962). Twenty years later the issue was reassessed in the same journal by Casarosa (1981, 1984) and Torr (1984), while, in addition to working papers and book chapters by Weintraub (1961), Tarshis (1979), Casarosa (1982), Ferreira and Michel (1988, 1991), Amadeo (1989), Pasinetti (1997), Brady (2004), and Heller and Dessotti (2007), papers on the subject were also published in several different journals such as the *South African Journal of Economics* (Soper 1956, Torr 1981), *Metroeconomica* (Marty 1959), *Canadian Journal of Economics and Political Sciences* (Wells 1962), *Rivista di Politica Economica* (Casarosa 1978), *Economic Enquiry* (Patinkin 1979, Marty 1981), *Journal of Post Keynesian Economics* (Parrinello 1980, Ambrosi 1981, Vickers 1987, Nevile 1992, Chick 1992, King 1994), *Australian Economic Papers* (Asimakopulos, 1982), *History of Economics Review* (Brady 1996), *Indian Journal of Applied Economics* (Brady 1999) and *European Journal for the History of Economic Thought* (Hartwig 2007). Other assessments on the *Z-D* type Keynesian cross went on in the *History of Political Economy*, initially (in 1974) between Wells, Leijonhufvud and Weintraub, followed by Patinkin's (1976) "Keynes´s monetary thought", with further contributions by Roberts (1978), Patinkin (1977, 1978, 1989), Fusfeld (1985, 1989), Davidson (1989A, 1989B) and Brady (1990).

those who developed the algebraic argumentation that Keynes left only indicated.

Instead of correcting Keynes's mathematics (which is right), the paper concludes that there has been a "slip of the pen" (Ferreira and Michel, 1991) in Keynes's description of the afore mentioned concepts on the footnote to page 55 of the *General Theory*. Therefore, it is not the aggregate supply function but the employment function that is linear with a slope given by the reciprocal of the money-wage – though subject to restrictive assumptions.

2. Keynes's employment function

For our purpose a complete exposition of the *General Theory* or a discussion of its role for Keynes's or Keynesian theory is unnecessary. But, to understand Keynes's employment function as an algebraic description of the principle of effective demand, it is necessary to explicate his definitions of aggregate demand, aggregate supply, effective demand, production function, employment function and aggregate supply curve, including his own algebraic formalisations, for they are the foundations of the graphic and algebraic representations proposed by some of his interpreters. Here N represents employment, O is output, D is demand (and D^{EF} is effective demand), $C(=D_1)$ is consumption, $I(=D_2)$ is investment, p is price, P is profit, W is nominal wages and Y is income. Subscript W indicates values measured in wage-units ($Y_w = Y/W$) whereas subscript r indicates some degree of disaggregation. Particularly:

- $Z = \phi(N)$ is the aggregate supply function (Keynes, 1936, p. 25);
- $D = f(N)$ is the aggregate demand function with $D = D_1 + D_2$, where $D_1 = \chi(N) = C$ is the consumption function, χ is the propensity to consume and $D_2 = I$ is the investment function (pp. 25-9);
- $O = \psi(N)$ is the production function (p. 44);
- $N = F(D^{EF})$ or $N = F(Z = D)$ or $N = F(Z^*)$ is the employment function; (p. 280, although in a different notation from Keynes's, as justified ahead);
- $p = \frac{Z}{O} = \frac{\phi(N)}{\psi(N)}$ or $pO = Z = \frac{\phi(N)}{\psi(N)}$ is the aggregate supply curve, i.e. for the industry as a whole with fully vertically integrated firms (p. 44).

Keynes's initial references to the aggregate supply and demand functions are related to the assumption that businessmen maximise profits, which is their criterion to decide the level of employment to be hired. Z is the aggregate supply price - the expectations of proceeds required to hire that level of employment - and the aggregate supply function is $Z = \phi(N)$, the relationship between Z and N. The expected proceeds are constituted by profit P which the entrepreneur expects to maximise, and the factor cost, i.e., the production factor(s) income WN. It is, in Amadeo's

(1989, p. 91) words, a "simple behavioural rule [that] can be formally represented by the conventional profit maximization exercise".

Keynes warns in a footnote that the aggregate supply function is tightly related to the employment function presented in chapter 20 in the *General Theory*. He distinguishes the aggregate supply function $Z = \phi(N)$ from the employment function $N = F(D^{EF})$, and both of them from the aggregate supply curve $p = Z/O = \phi(N)/\psi(N)$, which depends jointly on the aggregate supply function $Z = \phi(N)$ and on the production function $O = \psi(N)$.

In its turn, D is the amount of proceeds the entrepreneur expects to receive from the sales of the production resulting from the employment of N men (p. 25). The relationship between D and N is $D = f(N)$ and is called aggregate demand function.

If for a given level of employment the expected proceeds (from sales) is higher than the supply price (which makes labour hiring worthwhile), the entrepreneurs will be motivated to increase the level of employment, even if it causes an increase in costs (due to competition for inputs that become increasingly scarce), up to the point where Z equals D. According to Keynes the equilibrium employment level is established at the intersection of these functions, for it is at this point that entrepreneurs maximise their expected profits (pp. 24-5). This is the effective demand point, i.e. the value of D (and of course, of Z) where the aggregate supply function intersects the aggregate demand function (p. 25). [2]

Keynes develops part of his formalisation in the third chapter of the *General Theory*, stressing that it is just a summary where he assumes (to ease exposition without jeopardising the core of his argument) that both nominal wages and other factor costs are constant by unit of employment. He also assumes that when the level of employment increases, both real income and consumption increase, but consumption increases less than income. [3] Thus, given the propensity to consume (the relation χ between income growth and consumption growth), the equilibrium level of employment depends on the amount of current investment – which, in turn, depends on other factors such as the relation between the schedule of the marginal efficiency of capital and the structure of interest rates on loans of different maturities and risks – which are not dealt with at this point of Keynes's reasoning. As it is well known, according to Keynes, although the equilibrium level of employment cannot be higher than full employment, nothing ensures that it will be equal to it (p. 28).

Keynes's own summary of the above reasoning and initial formalisation is presented in eight

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[2] Under the supposition of a non-linear increasing aggregate demand function combined with a linear increasing aggregate supply function, the "intersect" becomes a "tangent point" (according, for instance, to Ferreira & Michel's and Brady's *Z-D* graphic models, but not to Chick's).

[3] Consumption is a stable function of income, the marginal propensity to consume is positive and less than unity and both the average propensity to consume C/Y and the marginal propensity to consume dC/dY fall when income rises. Under these assumptions the aggregate demand function is non-linear.

propositions (pp. 28-30). They are quite well-known but two of them deserve comments. In the third proposition he states that the sum of the amount expected for a community to spend in consumption ($C = D_1$) to the amount that is expected to be invested ($I = D_2$) determines total employment N that entrepreneurs will decide to hire; this sum is D, "what we have called above the *effective demand*" (p. 29, italics in the original). Clearly, Keynes's wording here is confusing, for he uses D both for aggregate demand $D_1 + D_2 = C + I$ and for effective demand concurrently. [4] The fourth proposition is even more confusing. He writes that "since $D_1 + D_2 = D = \phi(N)$, where ϕ is the aggregate supply function and ... D_1 is a function of N, which we may write $\chi(N)$ ... it follows that $\phi(N) - \chi(N) = D_2$". The trouble here is that he puts several ideas together into one single expression: a definition which accounts for the components of aggregate demand ($D = D_1 + D_2$) and an equilibrium condition ($D = \phi(N)$) describing the point of effective demand through the equality between aggregate demand D and aggregate supply $Z = \phi(N)$. This led to the understanding that Keynes formulated two different demand functions, an aggregate demand function and an expected proceeds function (Wells 1973 and 1978, Casarosa 1982). Similarly, as the expression $D_2 = \phi(N) - \chi(N)$ describes the volume investment should have so that the level of employment is that of full employment, it conveys investment as a "residue" of the equilibrium condition where supply equals demand. [5]

Henceforth, D^{EF} will denote effective demand and D aggregate demand. Alternatively, once effective demand implies $D = Z$, it will be also denoted by Z^*, where $Z^* = (Z = D)$ to distinguish it from aggregate supply Z. On the same ground, the employment function relating the level of employment to effective demand will be described by $N = F(D^{EF})$ or $N = F(Z = D)$ or $N = F(Z^*)$.

Stressing the distinction between Z and Z^* is highly important to understand Keynes's formalisation of the idea that entrepreneurs decide the level of employment following the criterion of maximisation of expected profits, which he presents in a footnote in chapter 6. His reasoning begins with the aggregate supply function Z measured in wage-units, i.e., $Z_W = Z/W$ where $Z_W = \phi(N)$ and therefore $Z = W \cdot \phi(N)$. Assuming that (i) the aggregate supply function for each firm or industry does not depend on the number of workers hired in other firms or industries; (ii) the number of firms or industries does not change; (iii) nominal wages do not vary; and (iv) other factor costs keep a constant proportion to the wage-bill, Keynes comes to the conclusion that Z is linear with a slope that is reciprocal to the nominal wage. He also assumes equality between marginal

[4] This issue is also pointed out by Ferreira & Michel (1991).

[5] As pointed out by Heller and Dessotti (2007). Keynes states that for each value of N there is a corresponding marginal product of labour in the consumption goods sector, which determines real wage. Therefore, neither marginal product of labour nor real wage are constant when the level of employment changes. It also implies that not all changes in D (and in N) are compatible with the (temporary) supposition that nominal wages are constant. Thus, he eventually casts that supposition aside, but this issue will not be discussed in this paper.

revenue and marginal cost in each point of the aggregate supply curve. Since this footnote is still a controversial issue, it is worth being fully quoted. [6] While Keynes refers to the aggregate supply function and to the aggregate supply curve as different concepts, this is a distinction that apparently no one has paid attention to.

> For example, let us take $Z_W = \phi(N)$, or alternatively $Z = W \cdot \phi(N)$ as the aggregate supply function (where W is the wage-unit and $W \cdot Z_W = Z$). Then, since the proceeds of the marginal product is equal to the marginal factor-cost at every point on the aggregate supply curve, we have
> $$\Delta N = \Delta A_W - \Delta U_W = \Delta A Z_W = \Delta\phi(N)$$
> that is to say $\phi' = 1$; provided that factor cost bears a constant ratio to wage-cost, and that the aggregate supply function for each firm (the number of which is assumed to be constant) is independent of the number of men employed in other industries, so that the terms of the above equation, which hold good for each individual entrepreneur, can be summed for the entrepreneurs as a whole. This means that, if wages are constant and other factor costs are a constant proportion of the wages-bill, the aggregate supply function is linear with a slope given by the reciprocal of the money-wage. (Keynes, 1936, footnote 2, p. 55, added underlinings). [7]

Keynes's conception that the aggregate supply function is linear with a slope given by the reciprocal of the nominal wage is key to the discussion intended in this paper. However, its demonstration requires a preliminary presentation of both the production function and the employment function, which will be done immediately. It needs the assumptions of perfect competition and of a linear production function. [8]

The production function is brought up at the end of the fourth chapter, where it is directly related to the supply curve**:**

> ... the aggregate supply function for a given firm (and similarly for a given industry or for industry as a whole) is given by
> $$Z_r = \phi_r(N_r)$$
> where Z_r is the return the expectation of which will induce a level of employment N_r. If, therefore, the relation between employment and output is such that an employment N_r results in an output O_r, where $O_r = \psi_r(N_r)$, it follows that
> $$p_r = \frac{Z_r + U_r(N_r)}{O_r} = \frac{\phi_r(N_r) + U_r(N_r)}{\psi_r(N_r)}$$
> is the ordinary supply curve. (Keynes, 1936, p. 44, added underlinings). [9]

The employment function is defined in chapter 20. Its relevance also vindicates its whole transcription in order to call the reader's attention not only to the first part of Keynes's definition where he states that the employment function is the inverse of the aggregate supply function – thus, its usual specification as $N = F(Z)$, the inverse of $Z = \phi(N)$ – , but also to the second part, where he specifies that it relates the level of employment N to the level of effective demand (hence

[6] See King (1994) and Brady (1999) for an account of past controversies.
[7] In Keynes's notation (1936, pp. 52-3), A is the value of the output sold to consumers and/or to other entrepreneurs, U is the user cost pertinent to A and Z is (again) the aggregate supply function.
[8] R. S. Ferreira has called my attention to the fact that these assumptions contradict what is generally assumed in the *General Theory*, and I thank him for that. Nevertheless, these are indeed Keynes's assumptions to the footnote, which Ferreira also recognizes.
[9] If firms are fully integrated the aggregate supply curve becomes $p = Z/O = \phi(N)/\psi(N)$.

$N = F(D^{EF})$ or $N = F(Z^{*})$ as suggested in this paper, instead of $N = F(D)$ or $N = F(Z)$.

> In Chapter 3 (p. 25) we have defined the aggregate supply <u>function</u> $Z = \phi(N)$, which relates the employment N with the aggregate supply price of the corresponding output. The *employment function* only differs from the aggregate supply function in that it is, in effect, its inverse function and is defined in terms of the wage-unit; the object of the employment function being to relate the amount of the effective demand measured in terms of the wage-unit, directed to a given firm or industry or to industry as a whole with the amount of employment, the supply price of the output of which will compare to that amount of effective demand. Thus if an amount of effective demand D_{W_r} measured in wage-units, directed to a firm or industry calls forth an amount of employment N_r in that firm or industry, the employment function is given by $N_r = F_r(D_{W_r})$. Or, more generally, if we are entitled to assume that D_{W_r} is a unique function of the total effective demand D_W, the employment function is given by $N_r = F_r(D_W)$. That is to say, N_r men will be employed in industry r when effective demand is D_W. (Keynes, 1936, p. 280, italics in the original; added underlining). [10]

According to Keynes the role of the employment function dwells in the fact that it is the starting point to discuss the effects of a monetary expansion on prices and/or on output (and therefore on the level of employment), which he uses to generalise the quantity theory of money. Although it is not this paper's subject, some passages from Keynes's reasoning have to be mentioned, for they contain his formulation of the aggregate supply function (curve?), the essential component of the different algebraic and/or graphical representations of the principle of effective demand. Keynes's association of perfect output elasticity to constant returns to scale is crucial to the definition as well as to the graphic description of the aggregate supply function (curve?), be it in the 45° version or in the Z function version of the *Z-D* models. These passages are also particularly important to distinguish the aggregate supply <u>function</u> from the aggregate supply <u>curve</u>, as stressed by Brady (2004, pp. 463-90). Besides, they deal with issues that are in the core of the afore mentioned controversies that began circa 1954 and have not yet ended.

Keynes defines the elasticity of employment e_e as the coefficient that measures the proportional change in the number of labour units hired due to the change in the amount of wage units that are expected to be spent on the purchase of its output, i.e., the effective demand. [11] Assuming that output is measurable, he also defines its elasticity e_{O_r} for each industry, which measures the proportional change in output due to a proportional change in effective demand (measured in wage-units). [12] Given that price equals marginal prime cost, which is implicitly the

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[10] Again, it must be observed that Keynes uses D or D_W although he refers to effective demand, not to aggregate demand. To certify this statement see his definitions of output elasticity, expected price-elasticity, money-prices elasticity and nominal-wages elasticity (pp. 282-5). Note also that although Keynes describes the production function at a disaggregate level, he also assumes it is possible to use it in an aggregate way. His reasoning (p. 280) is that if $N = \sum N_r$ and if $N_r = F_r(D_W)$ then $N = \sum N_r = \sum F_r(D_W)$ and the aggregate employment function is $N = F(D_W)$ which in our suggested notation for effective demand is $N = F(D_W^{EF})$.

[11] It is equivalent to the income-elasticity of employment defined as the proportional change in the level of employment due to a proportional change in income (Keynes, 1936, footnote 1, p. 116). Here the proportional change in the level of employment is related to a proportional change in effective demand: $e_e = \frac{dN}{dD_W} \times \frac{D_W}{N}$ for the industry as a whole in Keynes's notation (p. 282) or $e_e = \frac{dN}{dD^{EF}{}_W} \times \frac{D^{EF}{}_W}{N}$ in ours.

[12] $e_O = \frac{dO}{dD_W} \times \frac{D_W}{O}$ for the industry as a whole in Keynes's notation (p. 283) or $e_O = \frac{dO}{dD^{EF}{}_W} \times \frac{D^{EF}{}_W}{O}$ in ours.

assumption of perfect or pure competition, Keynes relates changes in expected profits to changes in effective demand, a relation implying that if supply is perfectly inelastic ($e_O = 0$) any increase in effective demand is absorbed by profits ($\Delta D_{W_r} = \Delta P_{W_r}$ in Keynes's original notation) and if supply is perfectly elastic ($e_O = 1$) there is no profit increase due to an increase in effective demand (which will be totally absorbed by the components of prime cost). [13] Keynes (first footnote, p. 283) also assumes (i) equilibrium between supply and demand; (ii) discrete variations are equivalent to continuous variations and (iii) change in profit is the difference between change in revenue and in cost. According to him, the perfectly elastic supply condition is associated to constant returns to scale. The proof of this relation depends on another algebraic formulation of the same condition, which enables to show that supply is perfectly elastic if there are constant returns to scale – that is to say, if $\psi''(N) = 0$, then $\frac{1-e_O}{e_e} = 0$ and therefore, $e_O = 1$ (pp. 282-4). This second formulation is an outcome from the previous one but it supposes that output O is a function of the level of employment N, i.e., it incorporates the production function. [14]

It is finally possible to demonstrate the algebraic description of Keynes's aggregate supply function. The demonstration assumes an economy in the short run (implying that employment is the only variable input), where nominal wages do not change and firms are in an imperfect competition environment (and hence, price is not equal to marginal revenue) with no changes in the output composition or in the demand composition. Our assumptions are inspired by Tarshis (1979, pp. 364-76), Chick (1983, p. 88) and Ferreira & Michel (1991, pp. 176-8) and are compatible with Keynes's (1936) propositions.[15]

We start from Keynes's concept on page 44 that defines $Z = pO$. In Brady's terms – the expected proceeds are constituted by profit which the entrepreneur expects to maximise and the production factor cost – it means equalising $Z(= P + WN)$ to $D(= pO)$, notably the effective demand. The equalisation renders a parameter of the employment function $N = F(Z = D)$. As explained by Marty (1959, p. 181), "the employment function records the results of the following

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[13] $\Delta D_{W_r} = \frac{1}{1-e_{O_r}} \Delta P_{W_r}$ where P_{W_r} is the expected profit in each industry measured in wage-units (Keynes, 1936, p. 283). In our suggested and abridged notation $\Delta D^{EF}{}_W = \frac{1}{1-e_O} \Delta P_W$. The proof of this expression is based on Brady's writings and can be read in Appendix I.

[14] $\frac{1-e_O}{e_e} = -\frac{N\phi''(N)}{pw\left[\phi'(N)\right]^2}$ for the industry as a whole (p. 283). Keynes certainly does not help his readers, for besides living out several intermediate steps, he also changes his writing for the production function which in chapter 4 is $O = \psi(N)$ and in chapter 20 is $O = \phi(N)$. The proof of this expression is also supported by Brady's writings and can be read in Appendix II.

[15] Tarshis (1979) and Chick (1983) conceive the most general and simplest case of imperfect competition and the ensuing relation between price, marginal revenue and price-elasticity of demand. Ferreira & Michel (1991) conceive a disaggregated system and in a slightly different notation assume not only the case where the expected price p is a function ρ of the output level O which in turn is a function F of the level of employment N – so that $p_r = \rho_r(O_r)$, where $O_r = F_r(N_r)$ – but also the case where the expected price of inputs depend on the demand for such inputs (as well as on different possible values for the price-elasticity of that demand). Here we take Tarshis's and Chick's simplest versions.

experiment: assume an arbitrary change in the level of effective demand and ask what level of employment will be associated with this change. It relates various points of effective demand to their corresponding levels of employment".

Henceforth, for the sake of simplicity Z^* will stand for $Z = D$, although acknowledging that Z^* is a tricky denotation because it is not only the point of effective demand but it simultaneously describes what this point depends on. [16]

If, then, $(Z = D) = (Z = pO) = Z^*$, it is possible to re-write the function in terms of the price-elasticity of demand, the nominal wage, the marginal product of labour, the average product of labour and the level of employment, so that $Z = pO$ may be written as

$$Z^* = \frac{\eta}{\eta-1} \cdot \frac{W}{\psi'(N)} \cdot \psi(N) \quad \text{[I]}$$ [17]

Hence we get equations II and III - the slope and the curvature of Z^*.

$$\frac{dZ^*}{dN} = \frac{\eta}{\eta-1} \cdot W \cdot \left[1 - \frac{\psi(N) \cdot \psi''(N)}{[\psi'(N)]^2}\right] \quad \text{[II]}$$

$$\frac{d^2Z^*}{dN^2} = \frac{\eta}{\eta-1} \cdot W \cdot \left[\frac{2\,\psi(N)\,\psi(N)\psi(N)}{\psi'(N)\,\psi'(N)\,\psi'(N)} - \left(\frac{\psi'(N)\,\psi''(N) + \psi(N)\,\psi'''(N)}{\psi'(N)\,\psi'(N)}\right)\right] \quad \text{[III]}$$

Both the slope and the curvature of Z^* depend on the price-elasticity of demand (the degree of market (im)perfection), the nominal-wage level and the features of the production function (the assumptions about physical returns on scale, i.e., the average and marginal product of labour). As remarked by Ferreira and Michel (1991, footnote 2, p. 160), Z^* will have a rising slope [$\frac{dZ^*}{dN} > 0$] if $\psi''(N) \leq 0$ in the production function and its curvature (the sign of $\frac{d^2Z^*}{dN^2}$) depends on the assumptions on the third derivative of the production function [$\psi'''(N)$]. In other words, it may be assumed that

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[16] Our next reasoning is conceptually different from Brady's (2004, pp. 470-1), who arrives at $dO/dN = W/p$ starting by postulating that if $D = Z$ then $dD/dN = dZ/dN$ (where $D = pO$ and $Z = P + WN$), assuming perfect competition both in the goods and input markets (so that neither profit P nor price p nor the wage level W change due to changes in employment N). It should be noted that equalizing $dD/dN = dZ/dN$ only calculates a common slope to both functions in a specific point (the point of tangency, which in this case would be the point of effective demand), but it may also calculate a common slope for two parallel functions.

[17] If competition is not perfect, marginal revenue is $MR = \frac{d(pO)}{dO} = p\frac{dO}{dO} + O\frac{dp}{dO} = p + O\frac{dp}{dO} = p\left(1 - \frac{1}{\eta}\right)$. Assume the most general case where the price-elasticity of demand is $\eta = -\frac{\frac{dO}{O}}{\frac{dp}{p}} = -\frac{dO}{dp}\frac{p}{O} \neq 1$. Thus, $p = \frac{\eta}{\eta-1}MR$. In the short run labour is the only variable input and nominal-wages are fixed, even when there are changes in the volume of employment (and therefore in the volume of produced output), so that $\frac{dW}{dO} = 0$. Therefore, marginal cost of labour is $MC = \frac{d(NW)}{dO} = N\frac{dW}{dO} + W\frac{dN}{dO} = 0 + W\frac{dN}{dO} = W\frac{1}{MP}$, and marginal product of labour is $MP = \frac{dO}{dN}$. In equilibrium marginal revenue MR is equal to marginal cost MC, therefore $MR = W\frac{1}{MP}$. In addition, average product of labour is $AP = \frac{O}{N}$, so that $O = AP \cdot N$. After due substitutions in $Z = pO$ we arrive at $Z^* = \frac{\eta}{\eta-1} \cdot \frac{W}{MP} \cdot AP \cdot N$. Finally, replacing $MP = \frac{dO}{dN}$ by $\psi'(N)$ and $AP = \frac{O}{N}$ by $\frac{\psi(N)}{N}$, and remembering that $O = \psi(N)$ is the production function, $Z^* = \frac{\eta}{\eta-1} \cdot \frac{W}{MP} \cdot AP \cdot N$ may be written as $Z^* = \frac{\eta}{\eta-1} \cdot \frac{W}{\psi'(N)} \cdot \psi(N)$.

Z^* has a positive slope, but with an undetermined curvature. [18]

What is most striking is that some early papers dedicated to Keynes's supply functions – such as Robertson and Johnson (1955), Wells (1960, 1961, 1962, 1974), Veendorp and Verkema (1961) – did arrive at the correct curvature of the graph but failed to notice that the graph is the employment function – the one that relates points of effective demand to levels of employment. Robertson and Johnson (1955), for instance, plotted the value of output as a function of employment and arrived at a measure of curvature given by $\frac{d^2\left(\frac{x}{m}\right)}{dn^2} = \frac{1}{m^3}\left[2x\left(\frac{dm}{dn}\right)^2 - m^2\frac{dm}{dn} - mx\frac{d^2m}{dn^2}\right]$ (in their original notation), which is the same as our equation [III]. [19] Even Marty (1959), whose "experiment" correctly defines the employment function, has not been able to distinguish Keynes's references to the aggregate supply <u>function</u> and to the aggregate supply <u>curve</u> as different concepts. [20]

3. Conclusion

We have just mentioned few examples of literary or mathematical descriptions of the equation this paper argues to be Keynes's employment function, which is the inverse of his aggregate supply <u>curve</u>, not his aggregate supply <u>function</u>. It is obvious that this is the function Keynes describes at the end of the famous second footnote on page 55 of the *General Theory*, although the wording of the footnote needs correction.

As far as this paper's author knows, only Tarhis, Ferreira & Michel and Brady came close to this conclusion. Patinkin began the controversy back in 1949 but more than thirty years later kept referring to Keynes's (1936) note 2 on page 55 as ambiguous (Patinkin 1979, p. 170) and an error

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[18] Under a pure mathematical point of view Z^* will be positively sloped if the production function $O = \psi(N)$ has a positive or negative first derivative $\psi'(N)$. If it is positive, the marginal product of labour increases; if it is negative, it decreases. Chick (1983, p. 66) assumes that the marginal product of labour increases at decreasing rates and without explaining why, she ascribes to the third derivative a value that is greater than or equal to zero; she also assumes that the marginal product of labour may be constant (and equal to the average product of labour). In other words, she assumes that $\psi'(N) > 0$, that $\psi''(N) < 0$ and that $\psi'''(N) \geq 0$. According to Brady (2004, pp. 528-9), Keynes assumes two alternatives: constant or increasing marginal product of labour at decreasing rates, which means $\psi'(N) =$ constant or $\psi'(N) > 0$ with $\psi''(N) < 0$. Although there is no space to detail it here, it is worth registering Tarshis's (1979) several different assumptions about marginal cost: constant, increasing at constant rates ($\psi'(N) > 0$ with $\psi''(N) = 0$) and decreasing at increasing rates ($\psi'(N) < 0$ with $\psi''(N) > 0$). Besides Tarshis's numeric examples and graphic representations the most interesting feature about his contribution is the formulation of three kinds of "aggregate supply functions", one related to the level of output *O* (*ASF-O*), one to the level of employment *N* (*ASF-N*) and one to the level of income *Y* (*ASF-Y*). The third one is the 45° model. Unfortunately, despite drawing the *ASF-O* and the *ASF-Y* figures (as well as the aggregate demand function *ADF*), Tarshis discusses the *ASF-N* features by means of a detailed verbal description only, without any graphical representation.

[19] In order to verify the similarity, consider that $n = N$; $x = O = \psi(N)$; $m = \psi'(N)$; $\frac{dm}{dn} = \psi''(N)$; and $\frac{d^2m}{dn^2} = \psi'''(N)$. Consider also that $\frac{\eta}{\eta-1} \cdot W = 1$.

[20] As a matter of fact, since Johnson's mathematical appendix to Robertson and Johnson (1955), there have been several "duplications" of the equation by different authors using different notations – but none of them recognized it as the employment function. See Brady (1999).

(Patinkin 1984, p. 145), while Chick (1983, footnote 9, p. 80) contended that the note remained and open but quite unimportant matter.

Tarshis (1979, p. 379) stated that Keynes was referring to what he (Tarshis) had identified as *ASF-N* (the aggregate supply function against employment), but did not provide a drawing of the *ASF-N* function. Ferreira & Michel (1991, footnote 3, pp. 160-1) recognized that "In fact, [Keynes's (1936) note 2, pp. 55-6] states that 'the aggregate supply function is linear with a slope given by the *reciprocal* of the money-wage'. This must be a slip of the pen. Keynes had in mind the *inverse* of the aggregate supply function, the employment function" (italics in the original). They do draw a curve that is equivalent to Keynes's aggregate supply curve but fail to recognize it as the employment function. The same applies to Brady (1996) who also does not realize that his aggregate supply curve (which he correctly insists on distinguishing from the aggregate supply function) is nothing else but Keynes's employment function.

Last but not least, in a recently published paper, Hartwig & Brady (2008) write that "the last sentence of [Keynes's (1936) note 2, pp. 55-6], in which Keynes states that the slope of the aggregate supply function was given by the reciprocal of the money-wage remains odd, however. Obviously, the slope of Z_W cannot be equal to 1 and to $1/w$ at the same time".

What, then, is the solution? This paper argues that when Keynes writes that the aggregate supply function is linear with a slope given by the reciprocal of the nominal wage (footnote 2, p. 55) he assumes:

- perfect competition so that $p = MR$ and $\frac{\eta}{\eta-1} = 1$;
- that the marginal product of labour is not only constant, but equal to unity (i.e., that each additional hired worker produces one unit of additional output), so that $\psi'(N) = 1$ and $\psi''(N) = 0$.

In this case $Z^* = \frac{\eta}{\eta-1} \cdot \frac{W}{\psi'(N)} \cdot \psi(N)$ becomes $Z^* = W \cdot \psi(N)$, which measured in wage-units (Keynes's hypothesis in the already mentioned footnote) gives ${Z^*}_W = \psi(N)$. Referring to the slope of Z^*, which is $\frac{dZ^*}{dN} = \frac{\eta}{\eta-1} \cdot W \cdot \left[1 - \frac{\psi(N) \cdot \psi''(N)}{[\psi'(N)]^2}\right]$, Keynes's implicit assumptions result in $\frac{dZ^*}{dN} = W$ and therefore $\frac{d^2Z^*}{dN^2} = 0$. Measured in wage units these assumptions result in $\frac{d{Z^*}_W}{dN} = \frac{W}{W} = 1$ and $\frac{d^2{Z^*}_W}{dN^2} = 0$ as well.

However, for the function slope to be the reciprocal of the nominal wage, it is necessary to consider that Keynes had in mind not the aggregate supply function $Z = \phi(N)$ (as he indeed writes it) but its inverse, that is to say, the employment function where N is a function not of Z but of $Z = D$ and therefore $N = F(Z^*)$, so that instead of $\frac{dZ^*}{dN} = W$ we have $\frac{dN}{dZ^*} = \frac{1}{W}$, exactly the reciprocal to the nominal wage. Furthermore, if $Z^*{}_W = \frac{Z^*}{W}$, then $\frac{dN}{dZ^*{}_W} = 1$ for the employment function, which has to be Keynes's corrected proposition. Therefore, in the controversial footnote, it is not the aggregate supply function but the employment function that is linear with a slope given by the reciprocal of the money-wage.

Remarkably, as stated by Ferreira and Michel, it is indeed "a slip of the pen".

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Appendix I: $dD_{W_r} = \frac{1}{1-e_{O_r}} dP_{W_r}$ (Keynes, 1936, footnote 1, p. 283)

Assumptions: (i) discrete variations are equivalent to continuous variation; (ii) all values are measured in wage-units; (iii) all calculations are for the industry as a whole; (iv) aggregate supply $Z(= P + WN)$ is equal to aggregate demand $D(= pO)$.

If $Z = D$, then

$$(Z = D) = D^{EF} = Z^{*} \quad (1)$$

If $D^{EF} = pO$ then

$$p = \frac{D^{EF}}{O} \quad (2)$$

and

$$\frac{dD^{EF}}{dO} = \frac{d(pO)}{dO} = p\frac{dO}{dO} + O\frac{dp}{dO} \quad (3)$$

Substitute (2) into the 1st term on the right side of (3)

$$\frac{dD^{EF}}{dO} = \frac{D^{EF}}{O}\frac{dO}{dO} + O\frac{dp}{dO} \quad (4)$$

Put the last term of the right side of (4) apart

$$O\frac{dp}{dO} = \frac{dD^{EF}}{dO} - \frac{D^{EF}}{O} \cdot \frac{dO}{dO} \quad (5)$$

By definition (Keynes, 1936, p. 283)

$$e_O = \frac{dO}{dD^{EF}} \cdot \frac{D^{EF}}{O} \quad (6)$$

or

$$dO \cdot \frac{D^{EF}}{O} = e_O \cdot dD^{EF} \quad (7)$$

Substitute the right side of (7) into the last term on the right side of (5)

$$O\frac{dp}{dO} = \frac{dD^{EF}}{dO} - e_O\frac{dD^{EF}}{dO} \quad (8)$$

or

$$O\frac{dp}{dO} = \frac{dD^{EF}}{dO}(1 - e_O) \quad (9)$$

Isolate $\frac{dD^{EF}}{dO}$ in (9)

$$\frac{dD^{EF}}{dO} = \frac{dp}{dO} \cdot \frac{O}{(1 - e_O)} \quad (10)$$

Isolate $O\frac{dp}{dO}$ in (3)

$$O\frac{dp}{dO} = \frac{dD^{EF}}{dO} - p\frac{dO}{dO} = \frac{dD^{EF}}{dO} - p \quad (11)$$

Note that p is the price of one unity of output and that $p\frac{dO}{dO} = p$ (the last term on the right side of equation 11) measures the marginal revenue which is supposed to be equal to marginal cost. Note also that $\frac{dD^{EF}}{dO}$ (the first term on the right side of equation 11]) is the variation of effective demand. Therefore, the difference between the first and the last term in equation 11 is the variation of expected profit P (=dP) and equation 11 may be written as

$$O\frac{dp}{dO} = \frac{dP}{dO} \quad (12)$$

Substitute (12) into (10)

$$\frac{dD^{EF}}{dO} = \frac{dP}{dO} \cdot \frac{1}{(1 - e_O)} \quad (13)$$

Equation (13) is the same as Keynes's (in a different notation and omitting dO):

$$dD_{W_r} = \frac{1}{1 - e_{O_r}} dP_{W_r}$$

Appendix II: $\frac{1-e_{O_r}}{e_{e_r}} = -\frac{N_r\psi''(N_r)}{[\psi'(N_r)]^2 p_{W_r}}$ (Keynes, 1936, footnote 2, p. 283)

Assume (i) aggregate demand is equal to aggregate supply (the point of effective demand); (ii) output depends on the level of employment, i.e., there is a production function relating output to employment; (iii) the production function is $O = \psi(N)$ as in chapter 4 of the *General Theory*; (iv) all calculations are for the industry as a whole; (v) at the point of effective demand aggregate demand $D(= pO)$ is equal to aggregate supply $Z(= P + WN)$ and therefore $(Z = D) = D^{EF} = Z^*$ and $Z^*{}_W = p_W \cdot O = D^{EF}{}_W$.

$$D^{EF}{}_W - p_W \cdot O = 0 \quad (14)$$

Derivate (14) in relation to $D^{EF}{}_W$

$$1 - \left[p_W \frac{dO}{dD^{EF}{}_W} + O\frac{dp_W}{dD^{EF}{}_W}\right] = 1 - p_W \frac{dO}{dD^{EF}{}_W} - O\frac{dp_W}{dD^{EF}{}_W} = 0 \quad (15)$$

From (14):

$$p_W = \frac{D^{EF}{}_W}{O} \quad (16)$$

Or

$$O = \frac{D^{EF}{}_W}{p_W} \quad (17)$$

Substitute (16) into the 2nd term of (15) and (17) into the 3rd term of (15)

$$1 - \frac{D^{EF}{}_W}{O}\frac{dO}{dD^{EF}{}_W} - \frac{D^{EF}{}_W}{p_W}\frac{dp_W}{dD^{EF}{}_W} = 0 \quad (18)$$

By definition (Keynes, 1936, p. 283) the 2nd term of (18) is

$$\frac{D^{EF}{}_W}{O}\frac{dO}{dD^{EF}{}_W} = e_O \quad (19)$$

Substitute (19) into (18)

$$1 - e_O - \frac{D^{EF}{}_W}{p_W}\frac{dp_W}{dD^{EF}{}_W} = 0 \quad (20)$$

Multiply and divide the 3rd term of (20) by $\frac{dN}{N}$

$$1 - e_O - \left[\frac{D^{EF}{}_W}{p_W}\frac{dp_W}{dD^{EF}{}_W}\right]\left[\frac{dN}{N}\frac{N}{dN}\right] = 0 \quad (21)$$

Rewrite equation (21)

$$1 - e_O = \left[\frac{D^{EF}{}_W}{N}\frac{dN}{dD^{EF}{}_W}\right]\left[\frac{dp_W}{p_W}\frac{N}{dN}\right] \quad (22)$$

The 1st term on the right side of (22) is by definition (Keynes, 1936, p. 282)

$$\left[\frac{D^{EF}{}_W}{N}\frac{dN}{dD^{EF}{}_W}\right] = e_e \quad (23)$$

Substitute (23) into (22)

$$1 - e_O = e_e\left[\frac{dp_W}{p_W}\frac{N}{dN}\right] \Rightarrow \frac{1-e_O}{e_e} = \frac{N}{p_W}\frac{dp_W}{dN} \quad (24)$$

By construction

$$p_W = \frac{P}{W} \quad (25)$$

By definition the real wage is

$$\frac{W}{p} \quad (26)$$

and the production function is

$$O = \psi(N) \quad (27)$$

By definition the marginal product of labour is

$$\frac{dO}{dN} = \frac{\psi(N)}{dN} = \psi'(N) \quad (28)$$

Assume that marginal product of labour $\psi'(N)$ is equal to real wage $\frac{W}{p}$ in equilibrium

$$\frac{W}{p} = \psi'(N) \Rightarrow \frac{p}{W} = p_W = \frac{1}{\psi'(N)} \quad (29)$$

Derivate $p_W = \frac{1}{\psi'(N)}$ in relation to N

$$\frac{dp_W}{dN} = \frac{d\left(\frac{1}{\psi'(N)}\right)}{dN} = \left(\frac{d}{dN}\right)\left(\frac{1}{\psi'(N)}\right) = \frac{0 \cdot \psi'(N) - 1 \cdot \psi''(N)}{[\psi'(N)]^2} = -\frac{\psi''(N)}{[\psi'(N)]^2} \quad (30)$$

Substitute (30) into (24)

$$\frac{1 - e_O}{e_e} = -\frac{N}{p_W}\frac{\psi''(N)}{[\psi'(N)]^2}$$