Revisiting the Derivative: Implications on the Rate of Change Analysis

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Implications on the Rate of Change Analysis

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Abstract: The aim of this paper is to raise concerns with the mathematical concept of the derivative as we know it. It raises concerns of accuracy. The paper is kept as simple as possible, solutions are always meant to be as simple as possible to be easily understood. The paper looks at linear and polynomial functions to illustrate that the derivative is not as precise as it should be, and in some instances can be considered almost a relic, though the solutions that are derived consider the simple derivative. It is the nature of polynomial functions that lead to the derivative not to be accurate and this paper clearly shows the shortcomings. The paper ends with a derivative that is accurate and precise, a derivative that when broken down is so simple. The main lesson/ conclusion is that it is all in the function, complex derivatives are not always necessary. This has important implications to all researchers, scientists who use the derivative to predict.

Introduction:

To understand what we will be discussing in this paper it is important to start with a short but accurate definition of what is the derivative, especially what is the derivative in context with this paper. To get this accurate definition we must return to a reference source that will give us this definition, that source of reference being a dictionary. The internet dictionary, www.dictionary.com, has several definitions for a derivative, these including:.

adj

1. Resulting from or employing derivation: a derivative word; a derivative process.
2. Copied or adapted from others: a highly derivative prose style.

n.

1. Something derived.
2. *Linguistics* A word formed from another by derivation, such as electricity from electric.
3. *Mathematics*
   a. The limiting value of the ratio of the change in a function to the corresponding change in its independent variable.
   b. The instantaneous rate of change of a function with respect to its variable.
   c. The slope of the tangent line to the graph of a function at a given point. Also called differential coefficient, fluxion.
4. *Chemistry* A compound derived or obtained from another and containing essential elements of the parent substance.
5. *Business* An investment that derives its value from another more fundamental investment, as a commitment to buy a bond for a certain sum on a certain date.

We are dealing with a mathematical concept, therefore the definition that best suites our needs is definition number 3, in particularly definition 3b, "the instantaneous rate of change of a function with respect to its variable," after all that is what is important to scientists, be they natural scientists or social scientists, any scientist who must deal with rates of change and functions that determine the growth of variables. Being an economist, I shall endeavor to solve the issues that this paper raises in an economic fashion, but no doubt they imply to all sciences that deal with a rate of change. Accuracy is fundamental in any science, precision is what this paper attempts to get to.

1. Understanding the Rate of Change

It is commonly known that there are two types of functions, these being linear functions and non linear functions. A common type of linear function has the form $Y = aX + b$, whereby Y is the dependent variable and X the independent function, a determines the slope and b the Y intercept. If the intercept goes through the origin $b = 0$ and the function becomes $Y = aX$. This paper attempts to raise concerns with the derivative and to simplify as much as possible it shall discuss simple functions that pass through the origin.

An example of a non linear function is $Y = aX^n$, where n is not equal to 1 or 0. Again to simplify matters it is best we take the a and just remain with a function of the type $Y = X^n$ in order to put a point across without undue complications. This paper shall strictly deal with the two types of functions, that is to say, $Y = aX$ and $Y = X^n$, however the conclusions will deal with most types of functions.

Figure 1 illustrates a linear function of the type $Y = aX$, at this point we do not mind the value of a, the slope. Obviously the greater slope the greater the derivative should be, anybody reading this paper understands that the derivative is calculated as if $Y = aX^n$. Then the derivative is naX^{n-1}. For a linear function n =1, then the derivative will simply be a, the slope. Therefore the greater the slope the greater the rate of change. In a simple linear function as that illustrated in figure 1 the rate of change is determined by a. Simple concept but important to be mentioned if we are to successfully raise concerns

Given a function $f(X) = aXn$ (1)
Then the derivative $f'(X)$ is:
$f'(X) = naX^{n-1}$ (2)

Figure 2 illustrates a similar function as that shown in figure 1 except n > 1 as well as a whole number and the function is no longer linear, it becomes non linear. However the rules of deriving the derivative are the same as expressed in functions 1 and 2 above.

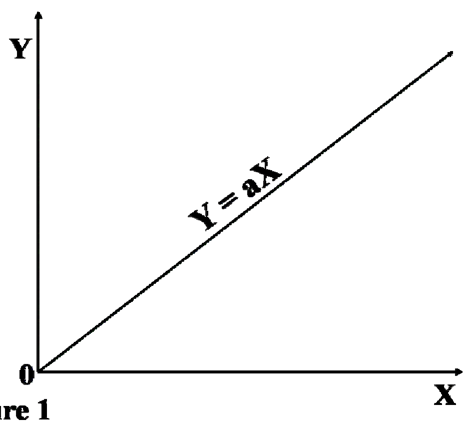

Figure 1

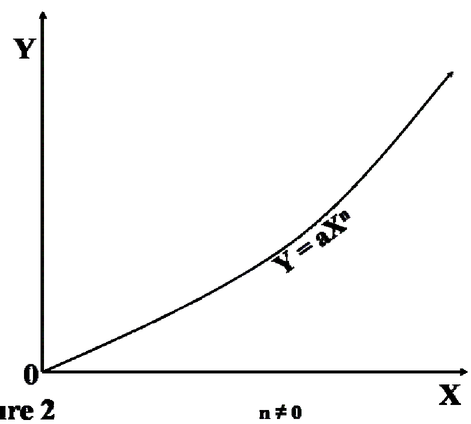

Figure 2

Just visually from figures 1 and 2 we can see that for a linear function the derivative, more precisely the rate of change is constant and in a non linear function the rate of change and the derivative are not constant. In the type of non linear function that is illustrated in figure 2, the rate of change increases, the derivative increases, this is easily understood by any high school mathematics student so we shall delve on that matter no more, instead we shall raise concerns with the derivative.

2. Concern No. 1

Accuracy is our main concern when we deal with science matters, in many instances not being precise can lead to one having a wrong impression, one can end up with a model that is far off the mark. For example calculating the marginal revenue wrong could lead to bad forecasting of profit, profit being a function of revenue and costs, in science it could lead to all sorts of wrong predictions, because of the lack of precision.

The first concern to be raised will consider simple linear functions as well as polynomial functions at the first unit, that is to say at $X = 1$. For simplicities sake we consider functions that go through the origin, meaning that when X the independent variable is equal to zero, Y the dependent variable is also equal to zero. This is just for us to easily grasp the concerns that are being raised. The concerns obviously will also apply to a function whereby Y is not equal to zero when X is equal to zero.

Figure 3 illustrates two simple functions, both pass through the origin. The first function is a linear function Y = X and the second function is a polynomial of degree 2, $Y = X^2$.

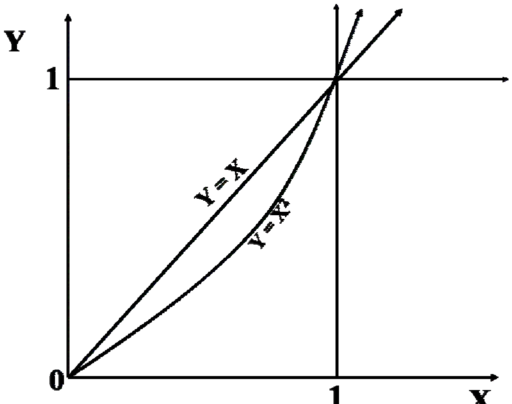

Figure 3

Note that the physical shape of the two functions are different, Y = X is obviously a straight line and $Y = X^2$ is non linear. Figure 3 is not accurate but it gives a good general illustration of what is going on. Though between X = 0 and X = 1 $Y = X^2$ is below Y =X, this is because it is about to have a steeper rise than Y = X after X = 1. At X = 1 both functions are equal, they are 1, that is the nature of polynomials when they go through the origin. Figure 3 gives an accurate view though the illustration is not to scale.

However when we look at the derivative of Y =X and $Y = X^2$ we get misleading results. We shall follow the universal rules of finding the derivative as set out by great minds like Newton, and these rules are set out in equation 2 above. The derivative for Y = X is 1, and that for $Y = X^2$ is 2X. Therefore when X = 1 given that we start at the origin, we would expect Y = X to add 1 to 0 and get 1. Therefore the derivative seemingly is working. However given that the derivative for $Y = X^2$ is 2X, at X = 1 passing through the origin we should get 2 but however the reality is at X = 1, $Y = X^2$ is 1 not 2. Looking at figure 4 this is true for all polynomials at their most simple form, the form Y = Xn, they all pass through 1, because $1 = 1^2 = 1^3 = 1^4 = \ldots = 1^\infty$. This is the first sign that there are concerns with the derivative.

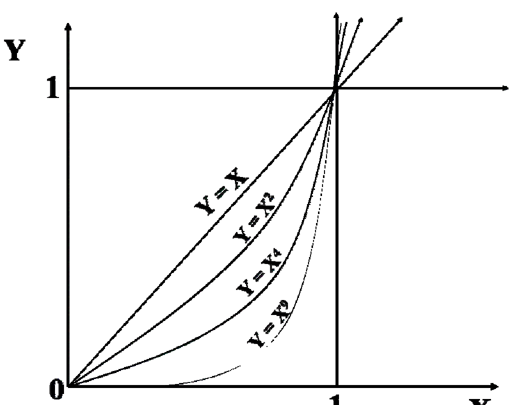

Figure 4

Like figure 3, figure 4 is not to scale, but it does give a clear illustration of what is happening between X = 0 and X = 1, clearly for any derivative where n is greater than 1, between X = 0 and X =1, only the derivative for the function where n = 1 works. By n, one is talking about the degree of the polynomial. Immediately from the onset the derivative fails. As will be seen the derivative is only ever accurate for linear functions, and the higher the degree the polynomial the more inaccurate the derivative becomes, is the accuracy reasonable, that depends, but for precision it becomes to inaccurate. Take $Y = X^9$, at X = 1, Y also is equal to 1. Clearly at this juncture the observant mind will see a problem with the derivative, they all add one but the tangents are all different. Theoretically speaking at X = 1, the function $Y = X^9$ Y should be

9, $Y = X^3$ Y should be 3, and at $Y = X^2$ Y should be 2, none of the corresponding Y values should be 1 at X = 1.

Figure 5 illustrates this conflict with the derivative being the rate of change visually more clear.

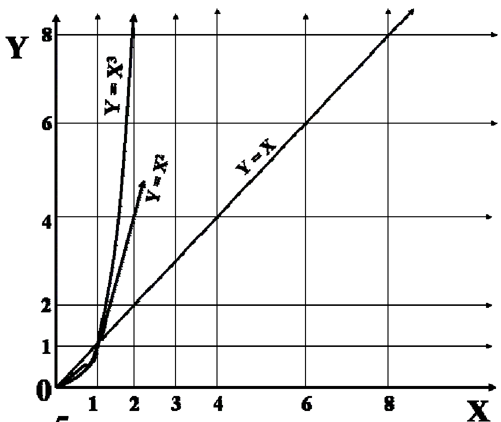

Figure 5

Thus far we can only see that it is in a linear function where the derivative is actually the rate of change.

3. Concern No. 2

The derivative in many instances can be illustrated as a tangent, the tangent supposedly showing the rate of change at that particular moment of a function, a non linear function obviously as a linear function can not have a tangent, it is its own tangent so to speak.

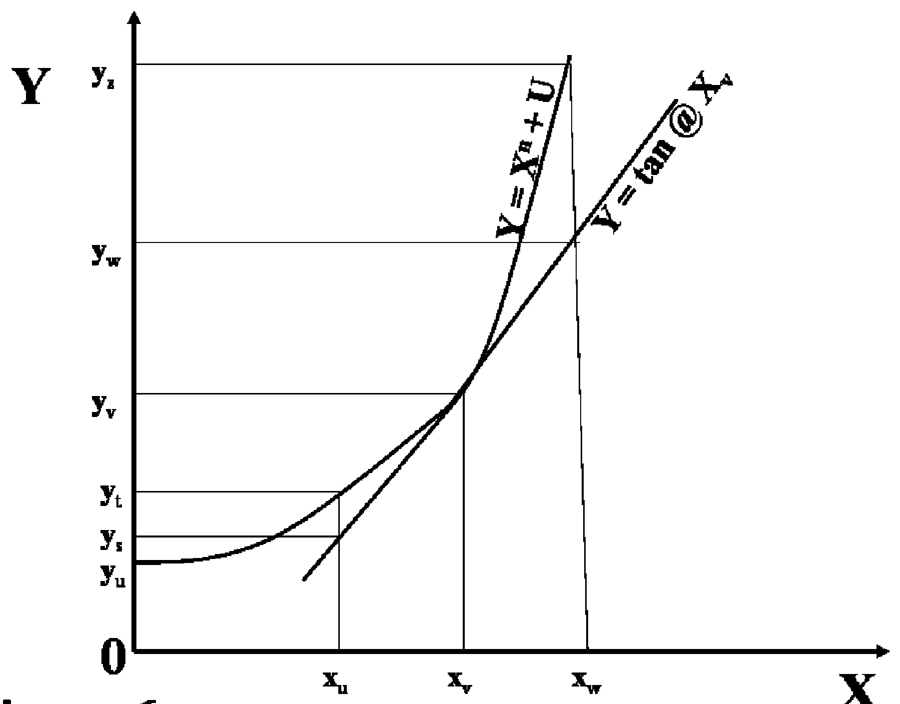

Figure 6

Figure 6 shows a function and a tangent at $X = x_v$. The reality of a tangent, in the case of figure 6, the tangent at x_v is that it is only equal to the function $Y = X^n + U$ only at xv and the corresponding Y value is y_v. Before xv, the tangent approaches the main function, $Y = X^n + U$, however it is not the main function, that must be understood. Looking at the tangent in figure 6 assume that xv and x_u are integers that follow each other on a normal number line, by normal it is meant, 1,2,3,4,5… 87,88,89…. Therefore if xv is 11, then x_u would be 10, and if x_v is 67, x_u would be 66. The same however can not hold for Y as being the independent variable they are determined by the function.

When we add a unit we would like to see the corresponding change in the dependent variable, that is the idea of the derivative. At the present to make the argument clear, let us say we are at xu and desire to add one more unit to arrive at $X = x_v$. What would be the expected corresponding change in Y. From figure 6 we see that to add one more unit to x_u we get x_v. Figure 6 illustrates a lot about the process, what is happening to the tangent as well as to the original function.

At xu, the corresponding value of Y on the real function is y_t, the real function of course being $Y = X_n + U$. However in regards to the tangent, the corresponding y value at x_u is y_s and as can be seen from figure 6 $y_t > y_s$. This is a very important concern because it means that when we arrive at x_v, there is a tangent, meaning that the real function and the tangent have equal y values, in our example that is y_v. To arrive at y_v, it means that the tangent has actually gained more than the actual function. That is partly why the derivative of a polynomial is always higher than the actual real difference. If the derivative was to be accurate, the function would take the shape of figure 7.1. 7.2 is how it would be accurate in the future that is how the function would be, but both figure 7.1 and 7.2 can not exist in reality, a function will never be both linear and non linear at the same time. However for the derivative to equal the real change it would be as figure 7.1, the darker line illustrating the hypothetical function.

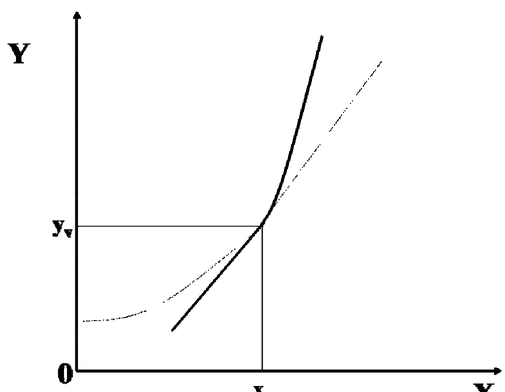

Figure 7.1

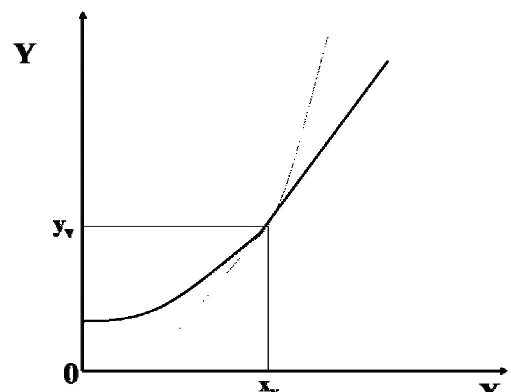

Figure 7.2

The tangent therefore does not show the rate of change, though it must be clear it approximates the rate o change and the larger the x value the closer is the approximation as shall be seen.

4. Further Concerns with the derivative and the Tangent

The derivative function of a polynomial in accepted theory is there to show the rate of change. It is usually easier to explain with illustrations. Figure 8 shows a polynomial function $Y = X^n$ and its derivative $Y = nX^{n-1}$. At x_v the real value of the function is y_v and the value of the derivative is y_u. yu is supposedly the corresponding rate of change of $Y = X_n$ at x_v. Therefore at any point that is X, the corresponding Y value for the derivative is actually the rate of change of the main function. This is more clearly shown in figure 9.

Figure 9 shows the main function, Y = Xn, the corresponding derivative, $f'(X)$ = nXn-1 as well as the corresponding tangent at xv. The corresponding tangent of course must have a slope that is equal to y value of the derivative at xv, therefore the tangent of Y = Xn at X = xv is defined by the function Y = yuX + A. yu being the slope or gradient and A being the intercept of Y, A can be either positive or negative, but there must be a Y intercept because the tangent can never go through the origin if the original function itself goes through the origin.

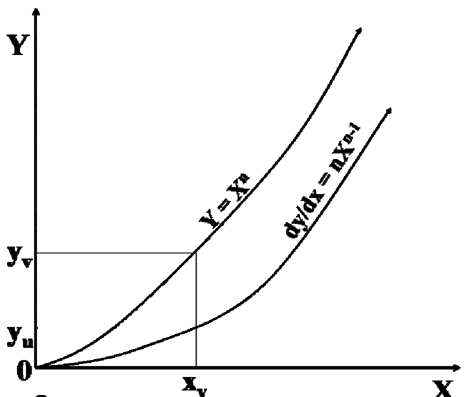

Figure 8

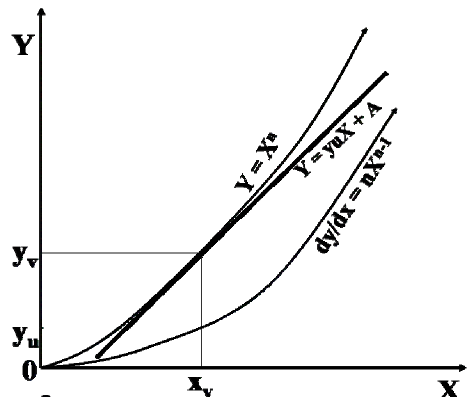

Figure 9

By looking at figure 9 and recalling the explanations about the tangent above, it should now be obvious that the derivative of a function in this case a polynomial function will never be accurate unless it is a linear function. One can be very positive that when Newton derived the derivative he was dealing mostly with linear functions and then merely generalized for all polynomials. A linear function can be said to be a polynomial of degree 1, therefore the generalization was easier, but who knows, I have never had the privilege to see Newton's papers. But because the derivative shows a corresponding value of a slope, a slope of a tangent, it will over estimate the rate of change, particularly in polynomials.

5. The Greater X, the More Accurate the Derivative

Though the rules of differentiation as we know it is not accurate in measuring the rate of change, it does become more accurate the larger the X value becomes. Oddly enough it is the behavior of the tangent that illustrates this in a simplified matter. We can look at figure 10.

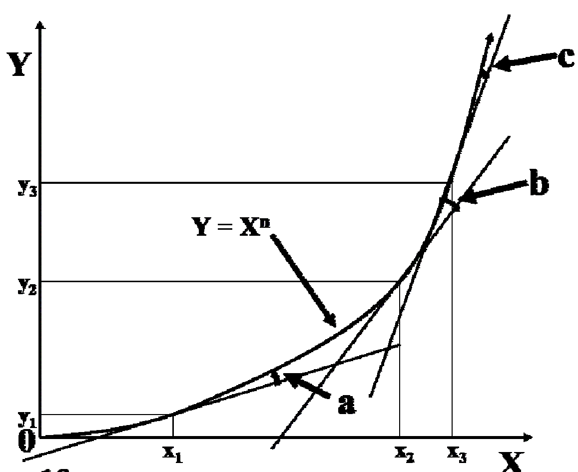

Figure 10

Figure 10 illustrates a polynomial function $Y = X^n$. There are three tangents to the function at x_1, x_2 and x_3, for simplicities sake the corresponding Y values are y_1, y_2, and y_3. Each tangent has a corresponding angle to the function, again this is an illustration, and curves will usually not have easily defined angles with straight lines. What one will see is that the corresponding angles will not all be the same size. One will find that angle a is the greatest and angle c the smallest. This is because as X gets larger the tangent remains closer to the function than when X is smaller. Therefore $a > b > c$. As the angle is getting smaller it means the corresponding derivative is in ratio terms, meaning ***derivative / real difference***, is getting smaller, approaching 1.

6. Getting to the Exact Rate of Change

From the theoretical explanations above it must be understood that the derivative does not measure exact change except for a linear function, for any polynomial with a degree 2 or higher the derivative is not accurate. Table 1 shows Y = X and the difference, *diff,* between each x accumulation as well as the derivative. The last column shows the difference in percentage terms of $f'(X)$/ difference, as can be seen from table 1 $f'(X)$/ difference in percentage terms is 100%, they are equal, therefore the derivative is equal.

Difference Between 'real difference' and differentiation of Y = X				
X	X = Y	diff	$f'(X) = 1$	%
0	0	n/a	0	N/A
1	1	1	1	100.00
2	2	1	1	100.00
3	3	1	1	100.00
4	4	1	1	100.00
5	5	1	1	100.00
6	6	1	1	100.00
7	7	1	1	100.00
8	8	1	1	100.00
9	9	1	1	100.00
10	10	1	1	100.00
11	11	1	1	100.00
12	12	1	1	100.00
13	13	1	1	100.00
14	14	1	1	100.00
15	15	1	1	100.00

Table 1

Table 1 is as expected from theory, however let us look at table 2 that shows the difference between the derivative and the 'real difference' for $Y = X^2$.

Difference Between 'real difference' and differentiation of $Y = X^2$				
X	$X^2 = Y$	diff	$f'(X) = 2X$	%
0	0	n/a	0	N/A
1	1	1	2	200.00
2	4	3	4	133.33
3	9	5	6	120.00
4	16	7	8	114.29
5	25	9	10	111.11
6	36	11	12	109.09
7	49	13	14	107.69
8	64	15	16	106.67
9	81	17	18	105.88
10	100	19	20	105.26
11	121	21	22	104.76
12	144	23	24	104.35
13	169	25	26	104.00
14	196	27	28	103.70
15	225	29	30	103.45
20	400	39	40	102.56
30	900	59	60	101.69
50	2,500	99	100	101.01

Table 2

Table 2 establishes and confirms the theory that has been built in the preceding pages, that the derivative will always be higher than the 'real difference', and it is higher for the reasons discussed above, the tangent in particular. As well as the fact that between X = 0, and X = 1, the derivative can greatly exaggerate what the rate of change is and must continuously recover. Therefore as the independent variable gets larger, as expected in ratio terms the derivative gets closer to the 'real difference', though in the case of $Y = X^2$, the derivative is always 1 greater than the 'real difference'.

Tables 3, 4, and 5 illustrate the real difference and the derivative for $Y = X^3$, $Y = X^4$, and $Y = X^5$ respectively. As can be seen from tables, 3, 4, and 5, the derivative always higher than the real difference, however the ratio of the difference falls as X, or the independent variable increases. This would be a real problem for example if one is using the derivative to illustrate the marginal revenue.

Difference Between 'real difference' and differentiation of $Y = X^3$				
X	$X^3 = Y$	diff	$f'(X) = 3X^2$	%
0	0	n/a	0	N/A
1	1	1	3	300.00
2	8	7	12	171.43
3	27	19	27	142.11
4	64	37	48	129.73
5	125	61	75	122.95
6	216	91	108	118.68
7	343	127	147	115.75
8	512	169	192	113.61
9	729	217	243	111.98
10	1,000	271	300	110.70
11	1,331	331	363	109.67
12	1,728	397	432	108.82
13	2,197	469	507	108.10
14	2,744	547	588	107.50
15	3,375	631	675	106.97
20	8,000	1,141	1,200	105.17
30	27,000	2,611	2,700	103.41
50	125,000	7,351	7,500	102.03
Table 3				

Difference Between 'real difference' and differentiation of $Y = X^4$				
X	$X^4 = Y$	diff	$f'(X) = 4X^3$	%
0	0	n/a	0	N/A
1	1	1	4	400.00
2	16	15	32	213.33
3	81	65	108	166.15
4	256	175	256	146.29
5	625	369	500	135.50
6	1,296	671	864	128.76
7	2,401	1,105	1,372	124.16
8	4,096	1,695	2,048	120.83
9	6,561	2,465	2,916	118.30
10	10,000	3,439	4,000	116.31
11	14,641	4,641	5,324	114.72
12	20,736	6,095	6,912	113.40
13	28,561	7,825	8,788	112.31
14	38,416	9,855	10,976	111.37
15	50,625	12,209	13,500	110.57
20	160,000	29,679	32,000	107.82
30	810,000	102,719	108,000	105.14
50	6,250,000	485,199	500,000	103.05
Table 4				

Difference Between 'real difference' and differentiation of $Y = X^5$				
X	$X^5 = Y$	diff	$f'(X) = 5X^4$	%
0	0	n/a	0	N/A
1	1	1	5	500.00
2	32	31	80	258.06
3	243	211	405	191.94
4	1,024	781	1,280	163.89
5	3,125	2,101	3,125	148.74
6	7,776	4,651	6,480	139.32
7	16,807	9,031	12,005	132.93
8	32,768	15,961	20,480	128.31
9	59,049	26,281	32,805	124.82
10	100,000	40,951	50,000	122.10
11	161,051	61,051	73,205	119.91
12	248,832	87,781	103,680	118.11
13	371,293	122,461	142,805	116.61
14	537,824	166,531	192,080	115.34
15	759,375	221,551	253,125	114.25
20	3,200,000	723,901	800,000	110.51
30	24,300,000	3,788,851	4,050,000	106.89
50	312,500,000	30,024,751	31,250,000	104.08
Table 5				

Now that a theory has been established as to why the derivative is always higher, a theory and real prove, the prove being in tables 1 – 5, it is time to correct the mistake and look for a new derivative function that will suite our needs for precision and accuracy.

6.2 Finding the Accurate Function for Rate of Change

To find the accurate differential we must find the function that will define $f'(X)$ – real difference (*diff)*

$f'_k = f'(X) - \text{diff}$ (3)

We can build a general formulae step by step, but first we need to understand why there is a difference between the derivative and the real change, a more compelling reason than merely the tangent, because that will not allow us to extract a real derivative. Take a polynomial say of degree 3, say the function $Y = X^3$. $X^3 = X \cdot X \cdot X$. Therefore $Y = X^3$ has properties of $Y = X^2$ and $Y = X^2$ in turn has properties that influence it from X. These properties need to be taken out because they are included in the derivative. The larger the polynomial function the more amplified are these properties that need to be taken out. We need to sort of distill the derivative in order to arrive at the real difference, to distill implies purify. In the case of $Y = X^3$, we need to get rid of the properties from the derivative that defines $Y = X^2$ and $Y = X$.

One can see this effect if one looks at tables 1 – 5 and by how much percentage points, by how high the ratio is that defines derivative over the real difference. When X = 1, Y = 1, and the derivative is 100% in table 1, 200% in table 2, 300% in table 3, 400% in table 4, and 500% in table 5. Table 1, is when X^1, table 2, X^2, table 3, X^3, table 4, X^4 and table 5, X^5. One can see the influence gets larger and larger, it is this influence that must be removed, this residual influence of $Y = X^n$, and the residual influence is equal to $Y = X^{n-1}$ as well as the influence of $\sum(X^{n-2} \rightarrow X^1)$. Obviously the greatest influence would be the preceding polynomial. $\sum(X^{n-2} \rightarrow X^1)$ is the residual effect of all the other preceding polynomials without including the immediate preceding polynomial, $X^1 = X$.

Knowing what causes the difference between the derivative and the real derivative we can then attempt to build a general formula for the rate of change, in polynomials, but the conclusions will be universal, for all derivatives. Let us look at $Y = X^2$.

Correcting the derivative of Y = X2 should be a fairly simple affair as the only effect that has to be taken out will be the effect of Y = X. Looking at table 1, we look for the pattern that defines the difference between the derivative and the real difference, we see that it is 1, the derivative of Y = X, that also happens to be the real difference. Therefore the rate of change for $Y = X^2$ the real derivative, f'_k is:

$f'_k(X) = f'(X) - f'_k(X^{n-1}) - R$ (4)
where R is defined as the residual created by $\sum(X^{n-2} \rightarrow X)$
in this case $X^{n-2} = 0$ and X is already included. Therefore equation (4) becomes
$f'_k(X) = 2X - 1$ (5)

Equation (4) can be the general formula, but it is better to make it look more simple. Not that for $Y = X^2$, the real difference is just the derivative minus 1. That 1 is the real change of Y = X.

It becomes more complex when we move up to the real change of $Y = X^3$. How do we get the real change, the real derivative. We have a guide in equation (4) Therefore we know that:

$f'k(X) = f'(X) - f'k(X^{n-1}) - R$ that leads to
$f'k(X) = 3X^2 - 2X - R$ where
$3X^2 = f'(X)$ and
$2X = f'k(X^{n-1})$ from equation (5)
Now we need to find R the function that defines the residual effects of all the other residual effects.

Table 6 is an extension of table 3, it gives us a step by step solution on finding R.

				1	2	3	4	
X	$X^3 = Y$	diff	$f'(X) = 3X^2$	2X	$3X^2$ - 2X	residual	add 1	**$f'_k(X)$**
0	0	0	0	0	0	0		
1	1	1	3	2	1	0	1	1
2	8	7	12	4	8	1	2	7
3	27	19	27	6	21	2	3	19
4	64	37	48	8	40	3	4	37
5	125	61	75	10	65	4	5	61
6	216	91	108	12	96	5	6	91
7	343	127	147	14	133	6	7	127
8	512	169	192	16	176	7	8	169
9	729	217	243	18	225	8	9	217
10	1,000	271	300	20	280	9	10	271
11	1,331	331	363	22	341	10	11	331
12	1,728	397	432	24	408	11	12	397
13	2,197	469	507	26	481	12	13	469
14	2,744	547	588	28	560	13	14	547
15	3,375	631	675	30	645	14	15	631

table 6

As mentioned above we first must subtract 2X from $3X^2$ and we arrive at what is the residual. This is under 3 in table 6. Trying to make sense of the residual we see from table 6 that the residual is 1 less than X. To get a function of the residual one will find that to create a function divisible by X one must always add 1 or subtract 1, in all polynomial functions by taking this action one will find the residual will then be divided by X. Adding 1 to the residual we get X. Therefore when $Y = X^3$, the residual is:

$R = (X – 1)$

Therefore taking our guide from equation (4) we get

$f'_k(X) = f'(X) – f'_k(X^{n-1}) – R$ (4)

$f'_k(X) = 3X^2 -2X – (X – 1)$ (6)

we have to subtract 1 from X because we added 1 in table six in order to get the X in the first place. From equation 6 we get the real derivative for Y = X3:

$f'_k(X) = 3X^2 – 2X – X + 1$ and by simplification we get

$f'_k(X) = 3X^2 – 3X + 1$ (7)

We can test it out to be sure, we can take X = 7 and we get

$3(7^2) – 3(7) + 1$

= 147 – 21 +1 = 127 and it is dead on. At one it equals 1.

To clarify our understanding let us take one more example, let us take the polynomial $Y = X^4$.We take guidance from equation (4).

$f'_k(X) = f'(X) – f'_k(X^{n-1}) – R$ (4)

Therefore when $Y = X^4$ the real rate of change would be

$f'_k(X) = 4X^3 - 3X^2 – 3X – R$. where

$4X^3 = f'(X)$

$3X^2 – 3X = f'_k(X^{n-1})$

We now will need to solve for R.

Solving for R, that is to say to see the function that defines the residual we shall take the same route as above, the same route as was used to solve for R to find the real rate of change for $Y = X^3$, the $f'_k(X^3)$. The process is made easy to understand by including a table to show each step by step process. We include table 7.

In table 7, the first step is to take out the known and be left with R, we therefore find $4X^3 – 3X^2 – 3X$. This value is clearly shown in table 7. When X = 13 from table 7 we see that $4X^3 – 3X^2 -3X$ is 8242 and when X = 4, $4X^3 – 3X^2 – 3X$ is 196.

X	$X^4 = Y$	diff	$f'(X) = 4X^3$	take out of $3X^2$ $3X^2$	take out of 3X 3X	**1st Residual $4X^3$- $3X^2$-3X** **$4X^3$-$3X^2$-3X**	Ist Residual - diff	take out 1	take out X = R
0	0	0	0	0	0	0	0	-1	#DIV/0!
1	1	1	4	3	3	-2	-3	-4	-4
2	16	15	32	12	6	14	-1	-2	-1
3	81	65	108	27	9	72	7	6	2
4	256	175	256	48	12	196	21	20	5
5	625	369	500	75	15	410	41	40	8
6	1,296	671	864	108	18	738	67	66	11
7	2,401	1,105	1,372	147	21	1,204	99	98	14
8	4,096	1,695	2,048	192	24	1,832	137	136	17
9	6,561	2,465	2,916	243	27	2,646	181	180	20
10	10,000	3,439	4,000	300	30	3,670	231	230	23
11	14,641	4,641	5,324	363	33	4,928	287	286	26
12	20,736	6,095	6,912	432	36	6,444	349	348	29
13	28,561	7,825	8,788	507	39	8,242	417	416	32
14	38,416	9,855	10,976	588	42	10,346	491	490	35
15	50,625	12,209	13,500	675	45	12,780	571	570	38

Table 7

Having arrived at the known solution of $4X^3 – 3X^2 – 3X$ we subtract the difference from the original table 4 and we arrive at what is called the 1st residual - diff. The diff being the real difference. Arriving at the solution we subtract 1. As mentioned above at this stage we must always add or subtract 1 in order to have a function that is related to X. At this stage after adding one we find that R at X = 15 is 570 and at say 3 is 6. After taking out 1 we find we can divide by X. The remaining solution follows the function 3X – 7, therefore R for $Y = X^4$ is:
(3X – 7)X + 1. This can be simplified to:
$3X^2 – 7X + 1$

Taking guidance from equation (4)

$f'_k(X) = f'(X) – f'_k(X^{n-1}) – R$ (4) we get
$f'_k(X) = 4X^3 – 3X^2 – 3X – (3X^2 – 7X + 1)$, this can be simplified to:
$f'_k(X) = 4X^3 – 3X^2 – 3X – 3X^2 + 7X – 1$ further simplification:
$f'_k(X) = 4X^3 – 6X^2 + 4X – 1$ (8)

To get real change for $Y = X^5$ the $f'_k(X)$ taking guidance from above we get
$f'_k(X) = 5X^4 – 4X^3 – 6X^2 + 4X – R$
and we solve for R, not forgetting to add or subtract 1 before solving for R.

6.3 Conclusion

The derivative is not an accurate measure of the rate of change, it over estimates. Take a Revenue function of the form $Y = X^4$. When we increase from 8 units to 9 units the derivative $f'(X) = 4X^3$ would suggest that additional revenue will equal 243, the reality is that it would be 217, and the f'_k (X) would be right and the derivative would have over estimated.

But it is a simple matter:
$f'_k = X_i^n – X^n_{i-1}$ (9)
for example for a function $Y = X^7$ at 4 the real change =
$f'_k = 4^7 – 3^7 = 14\ 197$

In many instances one would however have to solve for R as set out in equation (4), it's the age of computers it is easier than Newton's time.

But it is most important to remember equation 4, that is the Khumalo derivative symbolized as f'_k.

Reference:

Khumalo, B. (2009). The Variable Time: Crucial to Understanding Knowledge Economics. The Icfai University Journal of Knowledge Management, Volume 7, No 1, pp 34 -67.