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Abstract

The tomahawk bifurcation is used by Fujita et al. (1999) in a model with two regions to explain the formation of a core-periphery urban pattern from an initial uniform distribution. Baldwin et al. (2003) show that the tomahawk bifurcation disappears when the two regions have an uneven population of immobile agricultural workers. Thus, the appearance of this type of bifurcation is the result of assumed exogenous model symmetry. We provide a general analysis in a regional model of the class of bifurcations that have crossing equilibrium loci, including the tomahawk bifurcation, by examining arbitrary smooth parameter paths in a higher dimensional parameter space. We find that, in a parameter space satisfying a mild rank condition, generically in all parameter

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paths this class of bifurcations does not appear. In other words, conclusions drawn from the use of this bifurcation to generate a core-periphery pattern are not robust. Generically, this class of bifurcations is a myth, an urban legend.

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1. Introduction

Economic activities are not distributed uniformly in space. Manufacturing often concentrates in a few regions, resulting in a core-periphery pattern. How does one region come to dominate others and become a manufacturing core? The literature often considers a two region system. Beginning with a uniform distribution of immobile agricultural workers or farmers, Fujita et al. (1999) explain the emergence of the core-periphery urban pattern using the dynamics of a tomahawk bifurcation when transportation cost varies and other parameters are fixed (see also Fujita and Mori, 1997). When transportation cost is high, the symmetric equilibrium, where both regions have the same mobile manufacturing population, is the only equilibrium and it is stable. When transportation cost is moderate, two other stable equilibria emerge; when this happens, one of the two regions attracts all of the manufacturing, resulting in a core-periphery pattern. When transportation cost is low, the symmetric equilibrium becomes unstable and the only stable equilibria are the two core-periphery equilibria.

Is this type of bifurcation robust? Baldwin et al. (2003) examine the case where one region has slightly more immobile agricultural workers than the other. The model still preserves the feature of catastrophic agglomeration but the tomahawk bifurcation disappears. This means that the tomahawk bifurcation results from exogenous model symmetry. In addition, they show that, in the footloose entrepreneur model, the tomahawk bifurcation appears in the case of symmetric immobile population in regions but disappears with asymmetric populations (see also Forslid and Ottaviano, 2003).

To illustrate how the underlying exogenous parameters, such as the location of immobile population, affect bifurcation patterns, let's consider the following one-dimensional dynamical system with two parameters:

$$\dot{x} = a + bx - x^3$$

where $x \in \Re$ and parameters $(a, b) \in \Re^2$. This system exhibits the standard pitchfork bifurcation when $a = 0$ (see Figure 1; the solid and dashed lines indicate stable and unstable equilibria respectively). To show that this bifurcation is not robust, we perturb a to 0.005 and obtain Figure 2 instead. The general contour is still the same and the stable and unstable regions change slightly, but the equilibrium loci do not cross each other. This is a saddle-node bifurcation. The same pattern appears when we perturb a to -0.005 in Figure 3. The full equilibrium diagram against the two-dimensional parameter space (a, b) is plotted in Figure 4. Consider all one dimensional paths in (a, b) space and the equilibrium diagram generated by taking a slice of the three-dimensional picture along any path. We can see that only in some paths passing through $(0, 0)$, there is a pitchfork bifurcation.

[Figure 1 Here]
[Figure 2 Here]
[Figure 3 Here]
[Figure 4 Here]

This paper provides a general analysis of the class of bifurcations having crossing equilibrium loci in a two region model. This class includes, for example, the tomahawk, the pitchfork, and the transcritical bifurcations. It is well-known that such bifurcations are not stable: "all bifurcations of one-parameter families at an equilibrium with a zero eigenvalue can be perturbed into saddle-node bifurcations" (Guckenheimer and Holmes, 1997, p. 149). Baldwin et al. (2003) demonstrate exactly this by adding a slight population asymmetry while letting the dynamical system change along the transportation cost axis. We examine equilibrium dynamics along arbitrary smooth (C^r) parameter paths in a higher dimensional parameter space. We show that in a parameter space satisfying a mild rank condition, generically[1] in all parameter paths this class of bifurcations does not appear. Thus, these kinds of

[1] Here, a generic property means a property satisfied by parameter paths in an open and dense set.

bifurcations are not robust, and their appearance relies on the strategic choice of very specific parameter values. The rank condition just mentioned requires that the Jacobian matrix of the dynamical system with respect to endogenous variables and parameters has full rank at every equilibrium for all parameter values, and is standard in the general equilibrium literature on smooth economies. We show that it is easy to find such a parameter space.

Section 2 introduces the benchmark model and extends it to more exogenous parameters. Section 3 discusses migration dynamics and presents the main result. Section 4 concludes.

2. The Model

The core-periphery model features a two-region economy with the same resources in both regions. The same populations of immobile farmers in both regions produce a homogeneous agricultural good under constant returns to scale. A population of mobile manufacturing workers can migrate between regions. These manufacturing workers move to the region where they enjoy a higher utility level. The transportation of manufactured goods across regions bears a cost while transport of the agricultural good does not. Manufacturing firms produce differentiated products under increasing returns to scale technologies, competing monopolistically. There are two types of pecuniary externalities that generate forces causing agglomeration. These forces imply positive feedback that comes from firms locating near each other. First, manufacturing production will concentrate where there is a large market with many workers consuming manufactured goods. Second, workers will move to the region where production concentrates because the manufactured goods are cheaper there. The benchmark model is introduced formally next. We then expand the model by adding three more exogenous parameters to conduct genericity analysis in a higher dimensional parameter space.

The Benchmark Model

There are two regions in the economy indexed by $i \in \{1, 2\}$. There are two types of commodities: a homogeneous agricultural good and horizontally differentiated manufactured goods. There is a continuum of manufactured goods of size $n \in \Re_+$, determined endogenously. Each manufactured good is denoted by $j \in [0, n]$.

Let $p_i^A \in \Re_{++}$ denote the local price of the agricultural good, and let $p_i(j)$, where $p_i : [0, n] \to \Re_{++}$ is a measurable function, denote the local price of each manufactured good j in region i. There are two types of consumers: immobile farmers of population L_i^A in region $i \in \{1, 2\}$, and mobile workers of total population L^M who migrate between regions. Each worker is endowed with one unit of labor, supplied inelastically.

Let $A \in \Re_+$ denote the quantity of the agricultural good, and let $m(j)$, where $m : [0, n] \to \Re_+$ is a measurable function, denote the quantity of manufactured good j. All consumers have the same utility function

$$u(m, A) = M^{\mu} A^{1-\mu},$$

where $M = \left[\int_0^n m(j)^{\rho} dj\right]^{\frac{1}{\rho}}$ and $0 < \mu, \rho < 1$. A consumer in region i with income Y solves the following problem.

$$\begin{aligned} \underset{A,\, m(j) \in \Re_+}{Max} \quad & u(m, A), \\ s.t.\ p_i^A A + \int_0^n & p_i(j)\, m(j)\, dj = Y. \end{aligned} \tag{1}$$

The demand functions are

$$\begin{aligned} \hat{A}_i(Y) &= (1-\mu)\, Y / p_i^A, \\ \hat{m}_i(j, Y) &= \mu Y G_i^{\frac{\rho}{1-\rho}} / p_i(j)^{\frac{1}{1-\rho}}, \end{aligned}$$

where $G_i = \left[\int_0^n p_i(j)^{\frac{\rho}{\rho-1}} dj\right]^{\frac{\rho-1}{\rho}}$ is the manufacturing price index.

There are two types of workers, skilled workers who work in the manufacturing sector and unskilled workers or farmers who work in the agricultural sector. Skilled workers can move between regions, whereas unskilled workers cannot move between regions. Neither type of worker can change type to move to the other sector. Each worker is also a consumer, and supplies one unit of labor inelastically to the sector in which they are employed.

The agricultural good is produced by farmers with a one-to-one (labor input)-output ratio. For simplicity, the transportation of the agricultural good is assumed to bear no cost. Thus, the equilibrium agricultural commodity price is the same in both regions by no arbitrage; let $p_1^A = p_2^A = p^A$. Farmers retain all the revenue and they have income p^A.

Manufactured goods are produced by firms that employ mobile workers. Labor is the only input required. All firms have the same inverse production function

$$l = F + cq,$$

where $F, c > 0$ are the fixed and the marginal input requirements in terms of labor, whereas l units of labor are required for q units of output. The production technology exhibits increasing returns to scale due to fixed costs. There is free entry into the market that is subject to the fixed cost. Because of increasing returns to scale, each j-good is produced by and is the only product of an operating firm. Operating firms choose locations and engage in Chamberlinian monopolistic competition. Each firm chooses a location and charges a uniform free-on-board price for its product. Firms make decisions simultaneously. Let $w_i \in \Re_{++}$ denote the wage rate in region i. Suppose a firm locates in region i, charges price p, pays wage w_i, and sells output $q(p)$, where $q : \Re_{++} \to \Re$ is the demand of consumers. Its profit is

$$\pi_i(p) = pq(p) - w_i[F + cq(p)].$$

A firm in region i solves the following problem.

$$\underset{p \in \Re_{++}}{Max} \; \pi_i(p). \tag{2}$$

It is well-known that because of the assumed constant elasticity utility function and the iceberg transportation cost (to be defined shortly), the elasticity of demand faced by a firm is independent of the locations of its consumers. A monopolistically competitive firm charges a price marked up from the marginal cost. The profit-maximizing price for a firm in region i is $p_i = cw_i/\rho$. Its maximized profit is

$$\pi_i^* = \frac{1-\rho}{\rho} c w_i \left[q - \frac{F}{(1-\rho)c} \right].$$

The transportation cost of manufactured goods takes the Samuelson iceberg form. If one unit of good is shipped across regions, the fraction $1/T$ of the unit arrives ($T > 1$). Since firms are identical and their behavior differs only in location, we label firms and their products with their locations. This simplifies the notation to $j \in \{1, 2\}$. We replace $p_i(j)$ with p_i^j, which denotes the price of region j products in region i, and replace $\hat{m}_i(j, Y)$ with $\hat{m}_i^j(Y)$, which denotes the demand for region j

products by region i consumers (with prices an implicit argument). We denote the utility function by $u(m_i^1, m_i^2, A_i)$, replacing the function m with scalars m_i^1 and m_i^2 representing the quantity of manufactured goods consumed by a region i worker, and letting A_i represent the agricultural commodity consumption of a region i worker. Let $m_{A_i}^1$, $m_{A_i}^1$, and A_{A_i} denote the analogous quantities for region i farmers. The superscript denotes the region in which commodities are produced. Let n_i denote the number of firms in region i. The total number of operating firms equals the total variety of products; $n_1 + n_2 = n$. Note that $G_i = \left[n_1 (p_i^1)^{\frac{\rho}{\rho-1}} + n_2 (p_i^2)^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}}$.

A region i firm charges a free-on-board price $p_i = cw_i/\rho$. Thus, $p_i^i = p_i$ and $p_i^j = p_j T$ for $j \neq i$ by no arbitrage. Substituting Y with w_i, we have region i manufacturing workers' indirect utility:

$$v_i = \mu^\mu (1-\mu)^{1-\mu} w_i G_i^{-\mu} \text{ for } i \in \{1, 2\}.$$

Manufacturing workers are freely mobile. They choose a region that offers the highest utility level.

Extended Parameters

Above is the standard model of the new economic geography. The model is usually studied with varying transportation cost. In order to facilitate analysis in a higher dimension, we augment the system with three more exogenous parameters. These parameters do not change the model significantly, but they do accommodate asymmetric parameterizations. Let Θ be an open subset of $\Re^3$; its elements are denoted by $\theta = (\nu_1, \nu_2, \gamma)$, where $\nu_i \in (-F, \infty)$ (for $i = 1, 2$) and $\gamma \in (-1, 1)$. These parameters enter the model in the following way:

(i) ν_i parameterizes "regional fixed inputs": The fixed labor input of a region i firm is $F + \nu_i$. Note that although firms' profit function is changed to

$$\pi_i(p) = pq(p) - w_i [F + \nu_i + cq(p)],$$

their chosen price cw_i/ρ is not affected.

(ii) γ parameterizes "regional amenity": Workers have preferences over regions as follows. If a worker lives in region 2, her utility function is unchanged. If she lives in region 1, her utility is factored up by $(1+\gamma)$. The new utility function of region 1 workers is

$$(1+\gamma) u\left(m_1^1, m_1^2, A_1\right).$$

This captures regional differences such as the weather and the landscape. Note that region 1 workers' indirect utility is changed to

$$v_1 = \mu^{\mu} (1-\mu)^{1-\mu} (1+\gamma) w_1 G_1^{-\mu}.$$

An *economy* is specified by a vector of exogenous parameters $\theta \in \Theta$. The standard model is parameterized at $\theta = (0,0,0)$. The basic structure of the extended model and its equilibrium remain the same as those of the standard model, but there are many other interesting ways to extend the standard model to more parameters; we view this set of extended parameters as a natural example.

Equilibrium

To facilitate the analysis, we present the definition of equilibrium in a general equilibrium format. Let L_i^M denote the worker population in region i, and let A_i, (m_i^1, m_i^2) denote the consumption of agricultural and manufactured goods, respectively, in region i. Let A_{Ai}, m_{Ai}^1, m_{Ai}^2 denote the consumption of farmers in region i. Let q^i denote the output level of each region i firm. An *allocation* in the economy is described by the following list of variables: $\left\{ L_i^M, A_i, A_{Ai}, \left\{ m_i^j, m_{Ai}^j \right\}_{j=1}^2, n_i, q^i \right\}_{i=1}^2$. A *feasible* allocation satisfies the following constraints:

$$L_1^M + L_2^M = L^M. \tag{3}$$

$$L_1^M m_1^1 + L_1^A m_{A1}^1 + L_2^M m_2^1 T + L_2^A m_{A2}^1 T - q^1 = 0. \tag{4}$$

$$L_1^M m_1^2 T + L_1^A m_{A1}^2 T + L_2^M m_2^2 + L_2^A m_{A2}^2 - q^2 = 0. \tag{5}$$

$$L_1^M A_1 + L_1^A A_{A1} + L_2^M A_2 + L_2^A A_{A2} - L^A = 0. \tag{6}$$

Equation (3) balances the total manufacturing worker population, each providing one unit of labor inelastically, and the total labor used. Equations (4) and (5) balance the consumption of each manufactured good and the amount produced. Equation (6) balances consumption of agricultural commodity and the amount produced.

Facing prices p^A, p_1, p_2, w_1, and w_2, the following conditions are satisfied in equilibrium. (Note that we have already imposed no-arbitrage on the transportation of goods.) The entry of new firms drives the profit of operating firms down to zero.

$$\pi_1^* = \pi_2^* = 0. \tag{7}$$

Workers in the manufacturing sector are identical and freely mobile; they migrate to the region with a higher utility level. Let $\lambda = L_1^M / L^M$ denote region 1's share of manufacturing worker population. In equilibrium, manufacturing workers' utility levels must be the same in both regions if there are manufacturing workers in both regions. Thus, the migration equilibrium condition is

$$v_1 = v_2, \text{ if } 0 < \lambda < 1. \tag{8}$$

Note that manufacturing workers' utility v_i is not defined if there are no manufacturing workers in region i. For completeness, we define the potential manufacturing wage of a region as the limit of the equilibrium manufacturing wage when worker population approaches zero. Then, the potential utility is derived accordingly. Having all manufacturing workers in one region constitutes a (boundary) equilibrium if the potential utility in the other region is not higher. However, since the crossing part of the bifurcation is interior, we focus on $\lambda \in (0, 1)$.

An *equilibrium* consists of a list of prices and a feasible allocation such that conditions (1), (2), (7), and (8) are satisfied. We simplify the system as follows. First, by (1), the demand by workers in region i for the agricultural good and manufactured goods are $A_i = (1-\mu)\, w_i / p^A$ and $m_i^j = \mu w_i G_i^{\frac{\rho}{1-\rho}} \left(p_i^j\right)^{\frac{-1}{1-\rho}}$, respectively, and the demand by farmers in region i for the two types of goods are $A_{Ai} = (1-\mu)$ and $m_{Ai}^j = \mu p^A G_i^{\frac{\rho}{1-\rho}} / \left(p_i^j\right)^{\frac{1}{1-\rho}}$. By (2), $p_i = c w_i / \rho$. Then by (7),

$$\begin{aligned} q^i &= \frac{\rho\,(F+\nu_1)}{c\,(1-\rho)}, \\ n_i &= \frac{L_i^M}{(F+\nu_i) + c\frac{\rho(F+\nu_i)}{c(1-\rho)}} = \frac{L_i^M\,(1-\rho)}{F+\nu_i}. \end{aligned}$$

Plugging the results above into equations (4) and (5), we have

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho}T\right)^{\frac{1}{1-\rho}}} - \frac{\rho\,(F+\nu_1)}{c\,(1-\rho)} = 0, \tag{9}$$

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_2}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_2}{\rho}\right)^{\frac{1}{1-\rho}}} - \frac{\rho\,(F+\nu_2)}{c\,(1-\rho)} = 0, \tag{10}$$

Equation (8) can be replaced with

$$(1+\gamma)\,w_1 G_1^{-\mu} - w_2 G_2^{-\mu} = 0. \tag{11}$$

Finally, normalizing the agricultural price to $p^A = 1$, we have a system of three variables and three equations. The three equations are the last three above, and the three variables are w_1, w_2, and λ. Let $w = (w_1, w_2)$ and let f_1, f_2, g denote the left-hand side functions of (9), (10), and (11), respectively. Let $F(w_1, w_2, \lambda; \theta) = (f_1, f_2, g)$, $F : \Re^3_{++} \times \Theta \rightarrow \Re^3$. We will focus on the parameter space Θ; $F(w, \lambda, \theta) = 0$ defines the reduced form static equilibrium concept for a parameterized economy.

Since the focus is on migration dynamics, the adjustment of market prices is assumed to take no time. Once all workers choose a region to live in, commodity markets reach an equilibrium instantaneously given the population distribution. For fixed parameters, let $w(\lambda, \theta)$ denote the equilibrium price under population λ, which is derived from $\{f_i(w, \lambda, \theta) = 0\}_{i=1,2}$. In Proposition 1, we will show that this solution is unique. With this structure, the migration balance condition (11), after solving for $w(\lambda, \theta)$ but with λ as the remaining endogenous variable, is

$$g(w(\lambda, \theta), \lambda, \theta) = 0.$$

Note that $F(w, \lambda, \theta) = 0$ if and only if $g(w(\lambda, \theta), \lambda, \theta) = 0$. Let $f = (f_1, f_2)$. This approach is valid if there exists a unique solution $w(\lambda, \theta)$ to $\{f_i(w, \lambda, \theta) = 0\}_{i=1,2}$ for any fixed λ and any fixed admissible parameters $\theta \in \Theta$. We use the following sufficient condition for existence and uniqueness of equilibrium: the system $f(w, \lambda, \theta)$ satisfies the *index condition* if $|-D_w f(w, \lambda, \theta)| > 0$ at every equilibrium for all $\lambda \in (0,1)$ and for all $\theta \in \Theta$ (as in Mas-Colell, 1995, Definition 17.D.2). This index condition implies that the equilibrium of $F(w, \lambda, \theta) = 0$ is unique by the Index Theorem (see Mas-Colell, 1995; and Kehoe, 1998).

To explain further, the index condition is a standard condition from the smooth economies literature that implies existence and uniqueness of equilibrium. In simple terms, it uses the mathematical theory for an index of a fixed point to force uniqueness of equilibrium. For example, in the classical exchange economy with two commodities, the condition tells us that the slope of the derivative of aggregate excess demand has the same sign at all equilibria, namely whenever aggregate excess demand is zero, so by continuity aggregate excess demand crosses zero at most once. Existence of equilibrium also follows from the index theorem.

Proposition 1. *$f(w,\lambda,\theta)$ satisfies the index condition $|-D_w f(w,\lambda,\theta)| > 0$.*

Proof.

$$\begin{aligned}
\frac{\partial f_1}{\partial w_1} &= \left(\frac{c}{\rho}\right)^{\frac{-1}{1-\rho}} (w_1)^{\frac{-2+\rho}{1-\rho}} B_1 + \left(\frac{cw_1}{\rho}\right)^{\frac{-1}{1-\rho}} \lambda L^M \mu G_1^{\frac{\rho}{1-\rho}} \\
\frac{\partial f_1}{\partial w_2} &= \left(\frac{cw_1}{\rho}\right)^{\frac{-1}{1-\rho}} (1-\lambda) L^M \mu G_2^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} \\
\frac{\partial f_2}{\partial w_1} &= \left(\frac{cw_2}{\rho}\right)^{\frac{-1}{1-\rho}} \lambda L^M \mu G_1^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} \\
\frac{\partial f_2}{\partial w_2} &= \left(\frac{c}{\rho}\right)^{\frac{-1}{1-\rho}} (w_2)^{\frac{-2+\rho}{1-\rho}} B_2 + \left(\frac{cw_2}{\rho}\right)^{\frac{-1}{1-\rho}} (1-\lambda) L^M \mu G_2^{\frac{\rho}{1-\rho}}
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} + (1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} + L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} + L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} > 0 \\
B_2 &= \lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} + (1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} + L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T^{\frac{-\rho}{1-\rho}} + L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} > 0.
\end{aligned}$$

So,

$$\begin{aligned}
|-D_w f(w,\lambda,\theta)| &= \frac{\partial f_1}{\partial w_1}\frac{\partial f_2}{\partial w_2} - \frac{\partial f_1}{\partial w_2}\frac{\partial f_2}{\partial w_1} \\
&= \left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_1 w_2)^{\frac{-2+\rho}{1-\rho}} B_1 B_2 + \left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_1)^{\frac{-2+\rho}{1-\rho}} (w_2)^{\frac{-1}{1-\rho}} B_1 (1-\lambda) L^M \mu G_2^{\frac{\rho}{1-\rho}} \\
&+ \left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_2)^{\frac{-2+\rho}{1-\rho}} (w_1)^{\frac{-1}{1-\rho}} B_2 \lambda L^M \mu G_1^{\frac{\rho}{1-\rho}} \\
&+ \left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_1 w_2)^{\frac{-1}{1-\rho}} \lambda (1-\lambda) \left(L^M \mu\right)^2 G_1^{\frac{\rho}{1-\rho}} G_2^{\frac{\rho}{1-\rho}} \\
&- \left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_1 w_2)^{\frac{-1}{1-\rho}} \lambda (1-\lambda) \left(L^M \mu\right)^2 G_1^{\frac{\rho}{1-\rho}} G_2^{\frac{\rho}{1-\rho}} T^{\frac{-2\rho}{1-\rho}}.
\end{aligned}$$

The first three terms are all positive, and the fourth and fifth terms become

$$\left(\frac{c}{\rho}\right)^{\frac{-2}{1-\rho}} (w_1 w_2)^{\frac{-1}{1-\rho}} \lambda (1-\lambda) \left(L^M \mu\right)^2 G_1^{\frac{\rho}{1-\rho}} G_2^{\frac{\rho}{1-\rho}} \left(1 - T^{\frac{-2\rho}{1-\rho}}\right) > 0.$$

Since $T > 1$ and $\frac{-2\rho}{1-\rho} < 0$, we have $T^{\frac{-2\rho}{1-\rho}} < 1$. Therefore, $|-D_w f(w, \lambda, \theta)| > 0$.

This property holds for all values of endogenous variables, not just equilibrium values. ∎

Corollary 1. $\forall \theta \in \Theta$, $\forall \lambda \in (0, 1)$, *equilibrium in commodity markets defined by equations (9) and (10) exists and is unique.*

The index condition implies a unique equilibrium for a system of excess demand functions (see, for example, Mas-Colell, 1995; and Kehoe, 1998). It can easily be verified that f satisfies the properties of excess demand functions such as: Walras' Law holds; f is bounded from below; and if there is a sequence of prices with a component approaching zero, then the excess demand approaches infinity.

Notice that since the Index Condition for our model relies on derivatives with respect to endogenous variables, it is verified for the parameter space that is a product of ours and transport cost T, for example.

3. Migration Dynamics

The free migration condition requires that at an interior equilibrium ($0 < \lambda < 1$), skilled workers receive the same utility level in both regions. Various migration dynamics can be added, in a consistent manner, on top of this migration equilibrium condition. Given some parameters $\theta \in \Theta$, a C^2 vector field $h(\lambda, \theta)$, $h : (0, 1) \times \Theta \to \mathbb{R}$ describes the dynamics of λ after solving for $w(\lambda, \theta)$:

$$\dot{\lambda} = h(\lambda, \theta) .$$

The dynamics are consistent with the migration condition if the following properties are satisfied for all $(\lambda, \theta) \in (0, 1) \times \Theta$:

(D1) If $h(\lambda, \theta) = 0$, then $g(w(\lambda, \theta), \lambda, \theta) = 0$.

(D2) If $D_\theta g(w(\lambda, \theta), \lambda, \theta)$ has full rank (equal to 1), then $D_\theta h(\lambda, \theta)$ has full rank (equal to 1).

Condition D1 says that stationary points of h select from solutions to the migration equilibrium condition $g(w(\lambda, \theta), \lambda, \theta) = 0$. Moreover, condition D2 says that the dynamics of h preserve the rank of the Jacobian matrix of g in the parameter space. The function g is the difference in indirect utility for the two regions. A stronger condition on the derivatives, which is not needed in our analysis, would be: when an exogenous change in parameters (keeping endogenous variables fixed) makes utility higher in a region, population wants to move there. Conditions D1 and D2 rule out strange dynamics that alter the nature of the economy. Our genericity analysis in fact applies to all C^2 dynamics that satisfy conditions D1 and D2.

A common example of dynamics satisfying our assumptions is replicator dynamics (Weibull, 1995; Fujita et al., 1999; and Baldwin et al., 2003).[2] The population change in a region is proportional to the difference between the local utility level and the average utility level:

$$h(\lambda, \theta) = \lambda \left[v_1(\lambda, \theta) - (\lambda v_1(\lambda, \theta) + (1-\lambda) v_2(\lambda, \theta)) \right].$$

Definition 1. A *dynamic equilibrium* of an economy $\theta \in \Theta$ is a population ratio $\lambda \in (0,1)$ such that $h(\lambda, \theta) = 0$.

Under D1, implicit in this definition is the fact that commodity markets clear, since $w(\lambda, \theta)$ is an argument of g.

Parameter Paths

The vector field $h(\lambda, \theta)$ for dynamics is defined over the whole parameter space Θ. Previous literature has examined dynamics when the transportation cost is changed, keeping other parameters fixed. This is a very special parameter path that follows along the transportation cost axis. The general case is when many parameters change simultaneously, resulting in a one dimensional smooth path through the multi-dimensional parameter space Θ. Therefore, we proceed to examine the dynamics along arbitrary "parameter paths" in the parameter space.

A *parameter path* is a C^r map $\eta : [0,1] \to \Theta$ where $r \geq 2$. In other words $\eta \in C^r([0,1], \Theta)$, where we impose the standard C^r topology on this space of parameter paths. The path η defines a one-parameter family of vector fields $h(\lambda, \eta(t))$, where

[2] The replicator dynamics satisfy conditions D1 and D2.

$t \in [0,1]$ is used to index the parameter path. Let $E(\eta) = \{(\lambda, t) \in (0,1) \times [0,1] \mid h(\lambda, \eta(t)) = 0\}$ denote the set of dynamic equilibrium points.

Given this structure, we can define bifurcations. An *equilibrium locus* from an equilibrium point $(\lambda, t) \in E(\eta)$ is the image of a continuous map $e : [0,1] \to (0,1) \times [0,1]$ such that $e(0) = (\lambda, t)$ and $e(z) \in E(\eta)$ for $z \in [0,1]$. The equilibrium locus takes as its domain the unit interval purely for convenience. A parameter path η has a *bifurcation with crossing equilibrium loci* at $\left(\hat{\lambda}, \hat{t}\right) \in E(\eta)$ if for any neighborhood around $\left(\hat{\lambda}, \hat{t}\right)$ there are more than two distinct equilibrium loci from $\left(\hat{\lambda}, \hat{t}\right)$. This type of bifurcation includes the tomahawk, the pitchfork, and the transcritical bifurcations.

Next, we claim that a necessary condition for having crossing equilibrium loci at $\left(\hat{\lambda}, \hat{t}\right)$ is that $D_{(\lambda,t)} h\left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$ does not have full rank. It is easy to see this as follows.

$D_{(\lambda,t)} h\left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$, a vector with two components, has full rank if and only if it is not zero. Say $D_t h\left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$ is nonzero. By the implicit function theorem, $h(\lambda, \eta(t)) = 0$ can be locally solved as a C^1 function of λ. This means that $E(\eta)$ is a C^1 curve in a neighborhood of $\left(\hat{\lambda}, \hat{t}\right)$. Therefore, in a small neighborhood, there can be only two distinct equilibrium loci from $\left(\hat{\lambda}, \hat{t}\right)$. An analogous argument applies if $D_\lambda h\left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$ is nonzero.

Therefore, if a path η has $D_{(\lambda,t)} h(\lambda, \eta(t))$ with full rank at all of its equilibria (namely where $h(\lambda, \eta(t)) = 0$), it does not have bifurcations with crossing equilibrium loci. The next proposition says that if the parameter space is chosen properly, generically in all paths there is no such kind of bifurcation. More precisely, the set of paths without such bifurcations is open and dense.

We say that parameter space $\widehat{\Theta}$ satisfies the *rank condition* for h if $D_{(\lambda,\theta)} h(\lambda, \theta)$ has full rank whenever $h(\lambda, \theta) = 0$ (such parameter spaces are used in Debreu, 1970; Dierker, 1974; and Mas-Colell, 1985). This condition is standard in the smooth economies literature of general equilibrium theory, and is satisfied by an open set of economies. In the language of that literature, it is called a *regular parameterization.*

Proposition 2. *For any h satisfying D1 and D2, parameter space Θ satisfies the rank condition for h.*

Note that parameter spaces with more exogenous variables but containing Θ as a subspace also satisfy the rank condition. Thus, it is easy to find such parameter spaces as long as they contain a minimum set of parameters that have a full rank Jacobian matrix with respect to endogenous variables and exogenous parameters at equilibrium (see also Berliant and Zenou, 2002; and Berliant and Kung, 2006).

Proof of Proposition 2. Using the definition of $F = (f_1, f_2, g)$ and $\theta = (\upsilon_1, \upsilon_2, \gamma)$, it is straightforward to calculate

$$D_\theta F(w, \lambda, \theta) = \begin{pmatrix} -\frac{\rho}{c(1-\rho)} & 0 & 0 \\ 0 & -\frac{\rho}{c(1-\rho)} & 0 \\ 0 & 0 & w_1 G_1^{-\mu} \end{pmatrix}.$$

By the index condition and the Implicit Function Theorem,

$$D_\theta w(\lambda, \theta) = -D_\theta f(w, \lambda, \theta) [D_w f(w, \lambda, \theta)]^{-1}.$$

Then,

$$\begin{aligned} D_\theta g(w(\lambda, \theta), \lambda, \theta) &= D_\theta w(\lambda, \theta) D_w g(w, \lambda, \theta) + D_\theta g(w, \lambda, \theta) \\ &= -D_\theta f(w, \lambda, \theta) [D_w f(w, \lambda, \theta)]^{-1} D_w g(w, \lambda, \theta) + D_\theta g(w, \lambda, \theta) \\ &= -\begin{pmatrix} D_\theta f_1(w, \lambda, \theta) & D_\theta f_2(w, \lambda, \theta) \end{pmatrix} [D_w f(w, \lambda, \theta)]^{-1} D_w g(w, \lambda, \theta) + D_\theta g(w, \lambda, \theta). \end{aligned}$$

This expression is a linear combination of three vectors $D_\theta f_1$, $D_\theta f_2$ and $D_\theta g$, that are linearly independent whenever $F(w, \lambda, \theta) = 0$ since $D_\theta F$ has full rank. Thus, we can conclude that $D_\theta g(w(\lambda, \theta), \lambda, \theta) \neq 0$ and has full rank whenever $g(w(\lambda, \theta), \lambda, \theta) = 0$. By conditions D1 and D2, we know that $D_\theta h(\lambda, \theta)$ has full rank whenever $h(\lambda, \theta) = 0$.
∎

Note that $D_{(\lambda,\theta)} h(\lambda, \theta)$ having full rank does not imply that $D_{(\lambda,t)} h(\lambda, \eta(t))$ has full rank, but rather implies a generic property of the parameter paths η:

Proposition 3. *For dynamics h satisfying D1, the set of parameter paths η that do not have bifurcations with crossing equilibrium loci is open and dense for any open parameter space $\widehat{\Theta}$ satisfying the rank condition for h, for example any $\widehat{\Theta}$ with Θ a lower dimensional subspace of $\widehat{\Theta}$.*

We will use the following Theorem in the proof of Proposition 3. For a C^r map $\Phi : A \to B$ between manifolds A and B, we say that $b \in B$ is a *regular value* of Φ if $D_a\Phi(a)$ has full rank whenever $\Phi(a) = b$. We cite the following theorem (see Guillemin and Pollack, 1974, p. 68; and Mas-Colell, 1985, p. 320):

Transversality Theorem. *Suppose that* $\Phi : X \times S \to \Re^n$ *is a* C^r *map where* X and S *are* C^r *boundariless manifolds with* $r > \max\{0, \dim(X) - n\}$, *and let* $\Phi_s(x) = f(x, s)$, $\Phi_s : X \to \Re^n$. *If* $c \in \Re^n$ *is a regular value for* Φ, *then except for* s *in a set of measure zero in* S, c *is a regular value for* Φ_s.

The proof of Proposition 3 follows closely the proof of Mas-Colell (1985, Proposition 8.8.2, p. 345).

Proof of Proposition 3. The set of paths η such that $D_{(\lambda,t)}h(\lambda, \eta(t))$ has full rank whenever $h(\lambda, \eta(t)) = 0$ is open because of continuity.[3] To show that this set is also dense, for any path η, we construct a path η' that is arbitrarily close to η and does not have bifurcations with crossing equilibrium loci.

For any path η, define a map $\phi : (0, 1) \times [0, 1] \times \Re^3 \to \Re$,

$$\phi(\lambda, t, a) = h(\lambda, \eta(t) + a)$$

where $a \in \mathbb{R}^3$ and $\eta(t) + a \in \widehat{\Theta}$. Then, $D_a\phi(\lambda, t, a) = D_\theta h(\lambda, \theta)$; the latter has full rank whenever $\phi(\lambda, t, a) = h(\lambda, \eta(t) + a) = 0$ (using the rank condition). By the Transversality Theorem, $D_{(\lambda,t)}\phi(\lambda, t, a)$ has full rank whenever $\phi(\lambda, t, a) = 0$ for almost all a. So, we can pick any a' with this property arbitrarily close to zero and set $\eta'(t) = \eta(t) + a'$. Therefore, $D_{(\lambda,t)}h(\lambda, \eta'(t)) = D_{(\lambda,t)}\phi(\lambda, t, a')$ has full rank whenever $h(\lambda, \eta'(t)) = \phi(\lambda, t, a') = 0$. Then, $\eta'(t)$ is the path we want. ∎

Evidently, Proposition 3 holds for any open parameter space $\widehat{\Theta}$ satisfying the rank condition for h, including all $\widehat{\Theta}$ that contain Θ as a subspace. *For example,* $\widehat{\Theta}$ *could include all of the parameters in* Θ *but also the transport cost parameter* T. It is simply easier to exposit our analysis with fewer parameters.

[3] A simple proof by contradiction works well here.

4. Conclusion

The study of bifurcations provides interesting insights into the complex dynamic behavior of a system. It is important to study an economic system in a one dimensional parameter space when the chosen parameter is the main force changing the economy. In the case of the new economic geography, that parameter is transportation cost. However, the real world has many parameters, and the choice of parameters affects the equilibrium (bifurcation) diagram of a system. This raises the following question: Given enough parameters, what kind of dynamic behavior is typical? We characterize the generic pattern of dynamic regional systems along general smooth paths of parameter change in a higher (for example, 3) dimensional parameter space. We show that, in a parameter space satisfying the rank condition, there is a generic (open and dense) set of parameter paths that do *not* have bifurcations with crossing equilibrium loci. Thus, the use of such bifurcations, for example the tomahawk bifurcation, to generate core-periphery urban patterns from an initial uniform pattern is suspect because it relies on the strategic choice[4] of very specific parameter values and paths of parameter change. This has led to an urban legend.

It is easy, but notationally burdensome, to extend our results to more general models. For example, the two region framework can easily be replaced with n regions. The arguments are basically unchanged.

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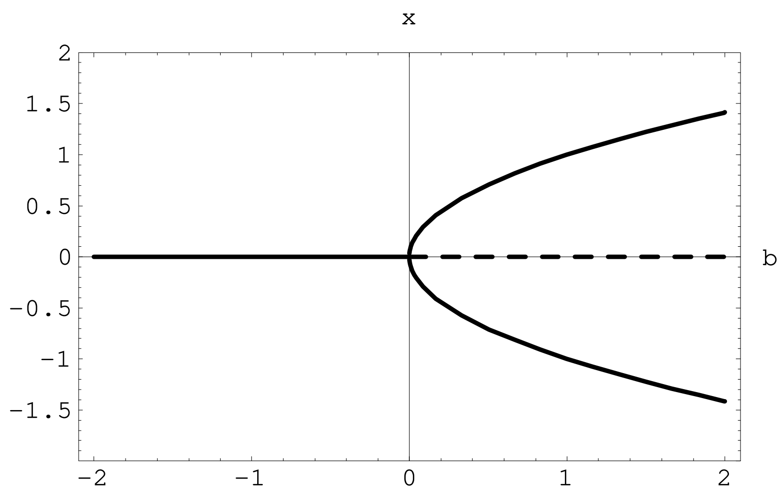

Figure 1: $a = 0$, $-2 < b < 2$.

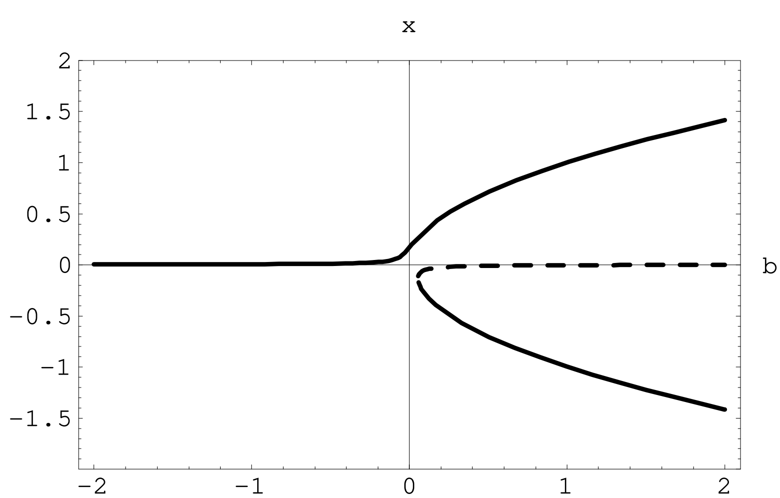

Figure 2: $a = 0.005$, $-2 < b < 2$.

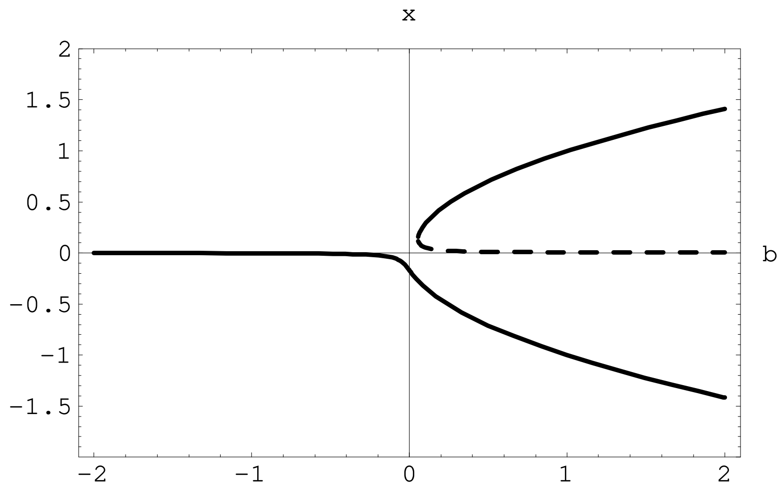

Figure 3: $a = -0.005$, $-2 < b < 2$.

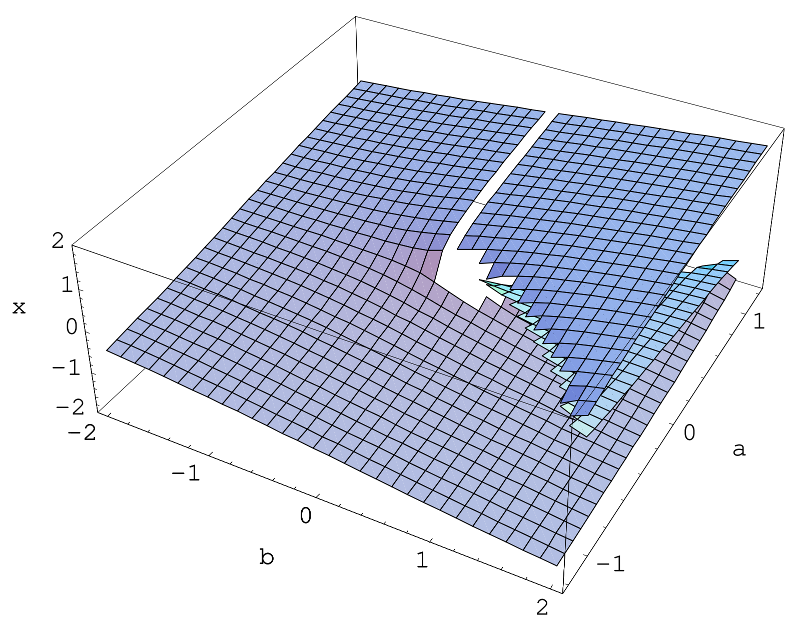

Figure 4: $-1.2 < a < 1.2$, $-2 < b < 2$.