Pricing the implicit contracts in the Paris Club debt buybacks

Manas, Arnaud and Daniel, Laurent

Banque de France

December 2007

Pricing the implicit contracts in the Paris Club debt buybacks

Arnaud Manas* Laurent Daniel†

July 4, 2007

Abstract — *In 2005, more than 20 billion dollars were bought back by Paris Club debtors: Russia USD 15 billion Poland USD 5.4 billion and Peru USD 1.5 billion. During the first half of 2006, more than USD 30 billion in buybacks was announced: Russia USD 22 billion, Algeria USD 8 billion dollars, Brazil USD 1.5 billion. The buybacks consisted of the prepayment of debts at par with no penalties. These transactions were carried out at a discount of more than 20% compared to their net present value. The total loss incurred by creditors in the three buybacks is estimated at more than USD 10 billion. This raises the question as to why the Paris Club creditors agreed to the buybacks voluntarily. It appears that these buybacks are the result of the exercise of specific contracts previously agreed with the debtors in the 1990s, without receiving any compensation for this and without assessing the consequences. These implicit contracts make it possible to formalise the respective interests for creditors and debtors. Their pricing requires the use of financial mathematics tools (derivatives) and stochastic models for interest rates (Vasicek), but applied in the Paris Club framework.*

Keywords — *buyback, Paris Club, par value, Vasicek model, creditor cartel*

JEL Classification — *F34, C72*

*Banque de France – Research and Foreign Relations Department – Debt Unit

†Banque de France – Research and Foreign Relations Department – International Monetary Relations Unit

Part I
Introduction

In 2005, Russia [1] ,Poland [2] and Peru [3] bought back more than USD 20 billion in debt from Paris Club members. The buybacks consisted of prepayment of debts at par with no penalties. These transactions were carried out at a discount of more than 20% compared to their net present value. The total loss incurred by creditors in the three buybacks is estimated at more than USD 5 billion.

This raises the question as to why the Paris Club creditors agreed to the buybacks voluntarily. It appears that these buybacks are similar to the exercise of specific contracts previously agreed with the debtors in the 1990s, without receiving any compensation for this and without assessing the consequences. These implicit contracts make it possible to formalise the respective interests for creditors and debtors.

The Paris Club was created by the main sovereign creditors in 1956 to maximise their bargaining power vis-à-vis debtor countries. Club members act together to reschedule the debt of countries facing liquidity problems. The key feature of debt in the Paris Club is illiquidity. Unlike bond issues, the claims are not transferable. Therefore, debtors hold a buyback monopoly with regard to their creditors. However, creditors can securitise their claims in the market at a discount. The price of a buyback is theoretically the result of a negotiation between the debtor and the cartel within an Edgeworth box.

To avoid these negotiations, the Paris Club has put in place rules based on fair treatment between debtors and creditors and on reconciliation of diverging interests. On the one hand, debtors were eager to have the right to buy back their debt at a discount as had been made possible for some countries in the case of commercial credits (Brady bonds). On the other hand, some creditors, in the event of early repayment, wished to obtain penalty fees in compensation for the breakage costs in accordance with the standards recommended by the OECD:

"In the event of an early voluntary repayment of all or part of a loan, the debtor compensates the governmental institution, which gives its financial support, for all the costs and losses arising from this early repayment and, in particular, for the cost stemming from replacing the fixed-rate flows interrupted by the early repayment."

These two positions were difficult to reconcile. Indeed, for some creditors, the principle of a discount was unacceptable because they considered discounts

[1] For a face value of USD 15 billion, see www.clubdeparis.org

[2] For a face value of EUR 12.6 billion.

[3] Peru's finance minister, Pedro Pablo Kuczynski, expects the Paris Club to accept the country's offer to buy back some US $1.5 billion of debt, according to The Financial Times. The country hopes to save USD 300 million in debt servicing charges annually in retiring the debt early. [. . .] The government will finance the buy-back by selling longer-term sovereign debt both on the local and international capital markets. Peru's total debt with the USD 8.5 billion, according to the paper.

to be concessional treatment. Their position was not to add concessions to concessions as some initial contracts already provided preferential treatment to debtors (e.g. decrease of debt stock or interest rate charges) and considered discounted buybacks to be a "second discount". This accounting position did not take into account financial market developments. The opponents of the discount principle refused for the value of the debt be modified in any way relative to its historical value. Reciprocally, the payment of breakage costs through penalty fees was unanimously rejected by debtors.

For these reasons, Paris Club members chose a median way and a compromise "neither discount nor penalty fees", allowing, by the late 1990s, early buybacks of debt at par value (in addition to debt investment swaps with discounts of nearly 50%). In fine, the combination of improved refinancing conditions, compared to those prevailing in the 1980s and the neither-nor rule of the 1990s led to buybacks in 2005.

During the negotiations, a pool of united creditors and one single debtor gathered under the auspices of the Paris club (there is no such link between debtors in spite of unsuccessful attempts in the past to federate debtors in a "counter-cartel"). Processed claims include both loans for commercial or aid purposes and debts that have already been rescheduled. Only the debts due by a sovereign country or those benefiting from a sovereign guarantee are likely to be processed by the Paris Club. The rescheduling negotiation between a debtor and the Paris Club results in a new homogeneous scheduling, aiming to lengthen and harmonise the maturity of the initial loans (interest rates are rarely modified).

The 2005 and 2006 buybacks can partly be attributed to the improving economic situation in the 2000s, i.e. low interest rates and good economic prospects for the five debtors. In the early 1980s, when debtors took out their initial loan, the world long term interest rate was above 10% due to the high level of inflation,. These loans were restructured and rescheduled in the Paris Club during the 1990s. The disinflation in the 1990s led to a sharp decrease in interest rates to less than 5% as of 2003. Consequently, the net present values of these debts increased, creating a growing interest in refinancing.

From a practical viewpoint, Paris Club loans can be seen as standard loans with fixed rates in most cases. Interest is generally paid on an annual or half-yearly basis and the repayment of the principal is made in fine. These loans are like bullet bonds and their pricing is quite easy because traditional pricing techniques are available.

It is important to mention a number of academic papers on buybacks. Sovereign credit buybacks were studied in the 1980s. The aim of these studies was mainly to determine whether these buybacks were positive or negative for a country. These papers were based on a macroeconomic analysis for the debtor and a model for the buyback in terms of social surplus. To sum up, these papers resulted in two opposing positions: for Krugman [19], buybacks were in general positive for the debtor, whereas for Rogoff and Bulow [6], buybacks were a "boondoggle" for the debtor. The main limitations of these studies were that the buyback price was an exogenous factor. The specific case of "secret" buybacks was studied by Cohen [10].

Buybacks were also studied from a financial viewpoint. Brennan and Swartz [5] analysed the characteristics of saving bonds and callable bonds using an interest rate model based on a Gauss-Wiener process. Büttler [8] sums up the three main approaches used to price bonds: direct pricing of the underlying asset without an interest rate model, indirect pricing through an interest rate model with discrete time periods and indirect pricing through an interest rate model with continuous time periods. Other studies focused on the numerical solving of these models, e.g. Barone-Adosi and Whaley [2] or Büttler and Waldvogel [9]. For interest rate models with continuous time periods, Jamshidian [18] provided an explicit solution for European calls on zero coupon bonds based on the Vasicek model [26].

To the best of our knowledge, no studies have been conducted on debt buybacks for the Paris Club. In our study, we apply the results of Jamshidian to bonds with a payment *in fine* (bullet bonds). For cases with weak mean-reverting tendencies in the Vasicek model, we obtain a simple solution. This paper prices implicit options for four different types of behaviour and different interpretations of the rules. Finally, we rank the contracts and the underlying incentives from "virtuous" to "hypocritical".

In a first part, Paris Club loans are presented and a model of the profit of creditors and debtors is proposed. In the second part, European implicit contracts are priced (compulsory or optional, and individual or collective buybacks).

Part II

The debt buyback scheme in the Paris Club

In this paper, we use an indirect approach with continuous rates. First, we present our hypothesis to model Paris Club buybacks and then we explain the interest rate model used. Second, we determine the profits of the creditors and debtors when buybacks occur.

We assume that the debtor's debts for a value of K are n bullet bonds indexed from 1 to n. To simplify, we liken a claim to a creditor, assuming that each creditor owns a single claim on the debtor. This claims previously rescheduled by Paris Club has the following characteristics: these fixed-rate bonds all have the same residual maturity D because of past restructurings. They have a repayment type similar to a bullet bond with repayment of the principal at the end and annual interest payments. There are m payments, the first $m-1$ are only composed of interest and the last is composed of interest and capital. The debts have the same face value K/n.

These assumptions are not restrictive. In the case of a mixed portfolio containing fixed-rate and variable-rate debt, the portfolio can be treated as a single portfolio containing only fixed-rate debt (see equivalence in the appendix). The assumption of identical residual maturities stems from the rationale behind the rescheduling agreements, which align repayments. In addition, the largest debts can be broken up to make debts of similar amounts.

Moreover, we use a continuous rate and we assume that the initial (continuous) rate of each credit follows a normal distribution. It is represented by a random real variable (RRV) Q_i. The RRV Q_i are identically distributed following the same law $Q_i \sim N(\mu_Q, \sigma_Q)$ where μ_Q is the expectation and σ_Q the standard deviation. We use $\mu_Q = \bar{q}$.

Thus, the i$^{\text{th}}$ credit can be represented by a portfolio of $m-1$ zero coupon bonds that are the interest payments (nominal $(K/n)(e^{Q_i D/m}-1)$ and the payment at jD/m (for $1 \leq j \leq m-1$) and a zero coupon bond corresponding to interest and capital (nominal $(K/n)e^{Q_i D/m}$ maturity D.

Date	Interest	Capital
D/m	$(K/n)(e^{Q_i D/m}-1)$	
$2D/m$	$(K/n)(e^{Q_i D/m}-1)$	
...	...	...
$(m-1)D/m$	$(K/n)(e^{Q_i D/m}-1)$	
D	$(K/n)(e^{Q_i D/m}-1)$	(K/n)

We consider that the interest rate follows a Vasicek model. This model has advantages such as return to the average and symmetrical distribution. Moreover, the use of a normal distribution enables us to obtain analytical distributions. In this model, spot and forward rates follow a normal process and not a log-normal process as in the Black-76 model. These properties are highly suitable for pricing contracts over a long period (maturity of more than 15 years). However, it has the drawback of not excluding negative interest rates and positive rates that are too high (see infra).

Spot rate r_t follows the stochastic process of Ornstein-Uhlenbeck:

$$dr_t = \alpha(\gamma - r_t)dt + \rho dW_t \tag{1}$$

The spot rate and the forward rate have the following distributions:

$$E(R_{s,t}|r_t) = r_t e^{-\alpha(s-t)} + \gamma(1 - e^{-\alpha(s-t)}) \tag{2}$$

$$E(F_{s,t,r_t}|r_t) = r_t e^{-\alpha(s-t)} + \gamma(1 - e^{-\alpha(s-t)}) - \frac{\rho^2}{2\alpha}(1 - e^{-\alpha(s-t)})^2 \tag{3}$$

$$Var(R_{s,t}|r_t) = \frac{\rho^2}{2\alpha}(1 - e^{-2\alpha(s-t)}) \tag{4}$$

$$Var(F_{s,t,r_t}|r_t) = \frac{\rho^2}{2\alpha}(1 - e^{-2\alpha(s-t)}) \tag{5}$$

In this model, t value of a zero coupon bond with a capital of 1 unit and a maturity s is expressed in exponential-affine form:

$$P(t,s,r_t) = E\left(e^{\int_t^s R_u du}|r_t\right) = e^{-A(t,s)r_t + B(t,s)} \tag{6}$$

with

$$A(t,s) = \frac{1}{\alpha}(1-e^{-\alpha(s-t)}) \tag{7}$$

$$\begin{aligned} B(t,s) &= \frac{1}{\alpha}(1-e^{-\alpha(s-t)})\gamma-(s-t)\gamma+\frac{\rho^2}{4\alpha^3}(2\alpha(s-t) \\ &\quad -3+4e^{-\alpha(s-t)}-e^{-2\alpha(s-t)}) \end{aligned} \tag{8}$$

Moreover, rates used by creditors and debtors have a risk premium on top of the rate described above. This premium[4] is considered constant overtime.

For the debtor, the premium corresponds to the spread demanded by international markets, which is defined exogenously by markets. A single spread is defined for the debtor. The forward rate for the debtor is:

$$F^{debtor}_{s,t,r_t} = F_{s,t,r_t} + y \tag{9}$$

For creditors, the risk premium is explained by the individual perception of default risk as well as the preference for liquidity[5]. Because the rate is exogenous, these risk factors are entirely explained by spreads. We assume that creditors' spreads are normally distributed Z_i RVV.) with $Z_i \sim N(\mu_Z, \sigma_Z)$ average μ_Z (written $\bar{z}$ and standard deviation σ_Z :

$$F^{creditor}_{s,t,r_t} = F_{s,t,r_t} + Z_i \tag{10}$$

We will now consider the profit of each player.

The debtor assesses the value of its credit at t with the net present value for the remaining payments at the prepayment time (T) and calculates its potential profit. This is tantamount to calculating the net present value of the corresponding zero coupon bond at the debtor's refinancing rate. The debtor's profit is the difference between the net present value and the price of the buyback (the nominal value). Given that the repayment is assessed in the future ($T > t$), the forward [6] rate of the debtor and creditors and must be used.

The value at t must take into account the discount factor. Symmetrically, the creditor assesses its credit at t. Its profit is the difference between the buyback price (nominal value) and the future payments that will be not received.

The debtor's profit is a RRV calculated at time T of prepayment via the equivalent portfolio of zero coupon bonds:

---

[4]A brownian process of Y_t and Z_t^i with $Y_t \sim N(y_0, \sigma_{Y_0}\sqrt{t})$ and $Z_t^i \sim N(z_0^i, \sigma_{Z_0^i}\sqrt{t})$ is probably closer to the real value but has two drawbacks: it is impossible to have explicit expression for calls III and IV and for calls I and II, results are close to those with a fixed spread. Rate volatility $\sigma_{F_{t,T}}$ is replaced by $\sqrt{\sigma^2_{F_{t,T}} + \sigma^2_{Y_T}}$.

[5] The Stability and Growth Pact constraint may increase the preference for liquidity for EU countries with excessive deficits and/or debt. Indeed, the Stability and Growth Pact creates an asymmetry because only gross debt is considered. It favours more "liquid" forms of external credit to reduce debt.

[6]As Vasicek notes, the use of the forward rate at T and not the expectated spot rate enables us to take account of the liquidity (or term)premium.

$$\Pi_i^{debtor}(F_{t,T,r_t}^{debtor}) = \sum_{j=1}^{\frac{m(D-T)}{D}} (K/n)(e^{Q_i \frac{D}{m}} - 1)P(T, T + \frac{jD}{m}, F_{t,T,r_t}^{debtor}) + (K/n)P(T, D, F_{t,T,r_t}^{debtor}) - K/n \quad (11)$$

This analytical formula is specific to Jamshidian's work [18]. The forward rate for which the profit is zero (strike rate) can be determined algebraically in a general case and must be calculated using a numerical process.

However, if α is small, an exact solution exists for bullet bonds. This particular case is tantamount to a situation in which the mean reversion of the interest rate towards the long term rate is weak compared to the stochastic term. This reflects the reality of the Paris Club because its use of discounting with the non-continuous rate yields the same expression.

With α small, (7) and (8) imply $A(t, t + DJ/m) \approx jD/m$ and $B(t, t + Dj/m) \approx 0$. So

$$P(T, T + \frac{jD}{m}, F_{t,T,r_t}^{debtor}) = e^{-A(t,t+Dj/m)F_{t,T,r_t}^{debtor} + B(t,t+Dj/m)} \approx e^{-F_{t,T,r_t}^{debtor} jD/m} \quad (12)$$

Thus, because (11) and (12)

$$\Pi_i^{debtor}(F_{t,T,r_t}^{debtor}) = (K/n)(\frac{e^{(Q_i - F_{t,T,r_t}^{debtor})D/m} - 1}{e^{-F_{t,T,r_t}^{debtor} D/m} - 1})(e^{-F_{t,T,r_t}^{debtor}(D-T)} - 1) \quad (13)$$

The debtor's profit is positive when the forward rate is inferior to the credit rate:

$$\Pi_i^{debtor}(F_{t,T,r_t}^{debtor}) > 0 \Leftrightarrow F_{t,T,r_t}^{debtor} < Q_i \quad (14)$$

In this case, the debtor's strike rate F^* is known: it is equal to the initial rate $F^* = Q_i$.

Moreover, it is possible, when the forward and the initial rate are low, to use an approximation at the first order for the debtor's profit from (13):

$$\Pi_i^{debtor}(F_{t,T,r_t}^{debtor}) = \frac{K}{n}(D - T)(Q_i - F_{t,T,r_t}^{debtor}) \quad (15)$$

The creditor's profit is:

$$\Pi_i^{creditor}(F_{t,T,r_t}^{creditor}) = -\sum_{j=1}^{\frac{m(D-T)}{D}} (K/n)(e^{\frac{Q_i D}{m}} - 1)P(T, T + \frac{jD}{m}, F_{t,T,r_t}^{creditor}) - (K/n)P(T, D, F_{t,T,r_t}^{creditor}) + K/n \quad (16)$$

The creditor's profit is positive when the forward rate is above the initial credit rate:

$$\Pi_i^{creditor}(F_{t,T,r_t}^{creditor}) > 0 \Leftrightarrow Q_i < F_{t,T,r_t}^{creditor} \quad (17)$$

First approximation:

$$\Pi_i^{creditor}(F_{t,T,r_t}^{creditor}) = (K/n)(D-T)(F_{t,T,r_t}^{creditor} - Q_i) \quad (18)$$

After using the debtor's and the creditor's spreads (with (9) and (10)):

$$\Pi_i^{debtor}(F_{t,T,r_t}) = (K/n)(D-T)(Q_i - y - F_{t,T,r_t}) \quad (19)$$

$$\Pi_i^{creditor}(F_{t,T,r_t}) = (K/n)(D-T)(F_{t,T,r_t} + Z_i - Q_i) \quad (20)$$

The sum of creditors and debtor profits is the paretian profit:

$$\Pi^{paretian} = \Pi_i^{creditor} + \Pi^{debtor} = \frac{K}{n}(D-T)(Z_i - y)$$

Part III
Pricing the contracts

The Club informally specifies the type of option, depending on the clout and the superior interests of the creditors and the clauses in the multilateral and bilateral agreements. Depending on their interpretation of the "de minimis" clause, the creditors may make the buyback optional or mandatory. Depending on the wording and interpretation of bilateral agreements, creditors may grant the debtor the right to choose which debts it wants to buy back. Therefore:

1. The buyback may be mandatory or optional for creditors. In other words, the creditors may or may not have the right to reject the debtor's buyback offer.

2. The buyback offer may be collective or selective. The debtor may be bound to make a buyback offer to all of its creditors (i.e. the Paris Club members) or, it may select only some of its creditors.

3. The buyback offer may be revocable or irrevocable. Either the debtor may retract its offer, if it deems it helpful to do so, or else it may be required to go ahead with the offer once it is announced, even if it turns out to be unfavourable for the debtor.

This means that there are four possible types of contracts:

	Mandatory (M) offers that creditors cannot reject	Optional (O) offers that creditors can reject
Collective buybacks (C) that the debtor must offer to all creditors	Type I CM contract "standard form"	Type II CO contract
Selective buybacks (S) where the debtor selectively offers to buy back the debts held by individual creditors	Type III SM contract	Type IV SO contract

The collective-mandatory or type I contract (CM) is the standard form for prepayment buybacks. It is akin to a conventional call option (see above).

Furthermore, the attractiveness and the value of each option contract depend on the specific rights included in each option. It is already clear that the options that let the debtor select which debts it wants to buy back (SM and SO) are more advantageous for the debtor. On the other hand, the CM and CO options are more advantageous for creditors because they can use the cartel power of the Paris Club. In the same vein, mandatory buybacks (CM and SM) are advantageous for the debtor, while the CO and SO options are the least disadvantageous for creditors. All in all, the SM contract is the most advantageous for debtors and the SO contract is the least disadvantageous for creditors.

1 Pricing of type I contracts: collective and mandatory buybacks (CM)

The debtor proposes to the cartel to buy all his credits and the cartel cannot refuse the offer. Of course, the debtor proposes this buyback only if it considers that its profit is positive which depends on the market spot rate. This profit is the sum of the profits of each credit. The portfolio of n credits is like a single credit.

The debtor exercises its contract if it considers that the value of this composite asset is greater than the nominal value (debtor payoff expectation is positive). This condition exists when the refinancing spot rate (market rate plus spread [7]) is inferior to the credit rate.

Because the debtor only considers the level of the forward rate when making the decision, the option is only exercised if the profit expectation for the whole operation is positive:

$$E\left(\sum_{1\leq i\leq n}\Pi_i^{debtor}(F_{t,T,r_t})|F_{t,T,r_t}\right) > 0 \quad \Leftrightarrow \quad E(Q_i) - F_{t,T,r_t} - y > 0$$

[7] We can assume that the debtor does not consider the possibility of defaulting, otherwise , it would have already chosen to to do so.

$$\Leftrightarrow \quad F_{t,T,r_t} < \bar{q} - y \tag{21}$$

The strike rate f^* is defined and is $q - \bar{y}$. If condition 21 is met, the portfolio payoff is the sum of individual profits. Otherwise it is zero. Therefore:

$$\begin{aligned}
F_{t,T,r_t} < \bar{q} - y \Rightarrow \mathit{Payoff}_T(F_{t,T,r_t}) &= \sum_{1 \le i \le n} (K/n)(D-T)(Q_i - y - F_{t,T,r_t}) \\
F_{t,T,r_t} > \bar{q} - y \Rightarrow \mathit{Payoff}_T(F_{t,T,r_t}) &= 0
\end{aligned}$$

Contract I corresponds to the total payoff expectation at exercise date (T) in a discounted risk-neutral universe. This approach uses Jamshidian's work, which provides the price for European call options on zero coupon bonds through the discounted payoff expectation when the call is exercised. This payoff is calculated at T as the difference between the value of the bond at maturity D (discounted in T with the forward rate for T at t) and the exercise price. The payoff expectation is then discounted. The value of the contract is then:

$$\begin{aligned}
C_T^I &= P(t,T,r_t) E\left(\mathit{Payoff}_T(F_{t,T,r_t})\right) \\
&= P(t,T,r_t) K(D-T) E\left(\sum_{1 \le i \le n} Q_i/n - y - F_{t,T,r_t} | F_{t,T,r_t} < \bar{q} - y \right) \\
&= P(t,T,r_t) K(D-T) E\left(\sum_{1 \le i \le n} Q_i/n - y | F_{t,T,r_t} < \bar{q} - y \right) \\
&\quad - P(t,T,r_t) K(D-T) E(F_{t,T,r_t} | F_{t,T,r_t} < \bar{q} - y)
\end{aligned}$$

As the initial rate Q_i and the forward rate F_{t,T,r_t} are independent

$$E\left(\sum_{1 \le i \le n} Q_i/n - y | F_{t,T,r_t} < \bar{q} - y \right) = E\left(\sum_{1 \le i \le n} Q_i/n - y \right) P\left(F_{t,T,r_t} < \bar{q} - y\right)$$

The price of the contract is then:

$$\begin{aligned}
C_T^I &= P(t,T,r_t) K(D-T) \left((\bar{q} - y) P\left(F_{t,T,r_t} < \bar{q} - y\right) - E(F_{t,T,r_t} | F_{t,T,r_t} < \bar{q} - y) \right) \\
&= P(t,T,r_t) K(D-T) \left((\bar{q} - y) N\left(\frac{\bar{q} - y - f_{t,T}}{\sigma_{F_{t,T}}} \right) + \frac{\sigma_{F_{t,T}}}{\sqrt{2\pi}} \int_{-\infty}^{\bar{q}-y} \frac{-f}{\sigma_{F_{t,T}}^2} e^{-\left(\frac{f - f_{t,T}}{\sqrt{2}\sigma_{F_{t,T}}} \right)^2} df \right) \\
&= e^{-r_t(T-t)} K(D-T) \left((\bar{q} - y - f_{t,T}) N\left(\frac{\bar{q} - y - f_{t,T}}{\sigma_{F_{t,T}}} \right) + \frac{\sigma_{F_{t,T}}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{q} - y - f_{t,T}}{\sigma_{F_{t,T}}} \right)^2} \right)
\end{aligned} \tag{22}$$

$f_{t,T}$ and $\sigma_{F_{t,T}}$ are respectively the expectation and the standard deviation of the forward rate (see 4 and 3).

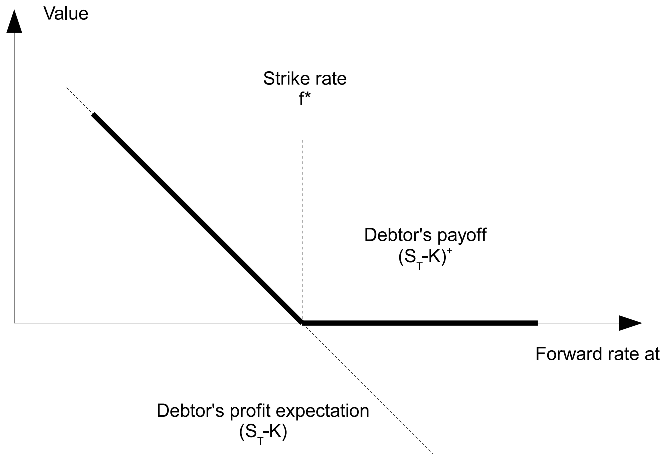

Figure 1: Payoff of the mandatory collective contract

In this case, the debtor's contract is like a standard call option on an asset with a nominal K and a value $S_T = K(D-T)f^* - F_{t,T,r_t}) + K$. The debtor's average profit is the conditional expectation of the payoff:

$$E(Gain(F_{t,T,r_t})|F_{t,T,r_t}) = (\bar{q} - y - F_{t,T,r_t})^+ \; = (S_T - K)^+$$

In this case, the buyback contract is like a call option incorporated in a Paris Club loan embedded call option. These kinds of options exist in some bond contracts such as saving bonds and parity bonds.

By analysing the value of the contract relative to the forward rate $f_{t,T}$: it appears that it is zero when $f_{t,T} \gg \bar{q} - y$ and when $f_{t,T} \ll -y$. Between these limits the linear term is dominant.

There is a maximum for $\frac{\partial C_T}{\partial T} = 0$ and $T = T_{opt}$. If $t \leq T \leq D$ then $C_T \leq C_{T_{opt}}$

Numerical simulations[8], we get $T_{opt} = \frac{1}{3(r_t + 1/D)}$ show that this extremum is reached after 1 to 3 years for classical parameters (such as maturity of 10 to 20 years). Indeed, the time value is zero for an immediate buyback ($T = t$) and rises proportionally to the square root of time. Moreover, the contract value decreases quasi linearly and is zero at maturity ($T = D$).

At the first order 21, the contract value could be expressed as the sum of the intrinsic value and the time value.

$$C_T^I = \frac{1}{2} K(D-T) e^{-r_T(T-t)} \big(\underbrace{(\bar{q} - y - f_{t,T})}_{intrinsic value} + \underbrace{\sigma_{F_{t,T}} \sqrt{\frac{2}{\pi}}}_{time value} \big) \qquad (23)$$

[8]With the approximation $C_T^I \approx \frac{KD}{2} \left((\bar{q} - y - f_{t,T}) + \sigma_{F_{t,T}} \sqrt{\frac{2T}{\pi}} (1 - (f_{t,T} + \frac{1}{D})T) \right)$

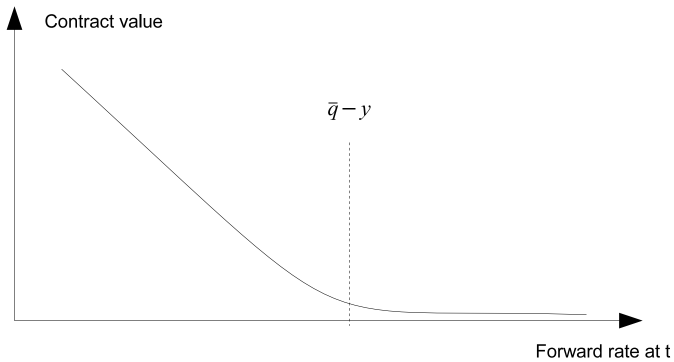

Figure 2: Value of the mandatory collective contract

The intrinsic value depends on the strike rate. The only endogenous factor is the spread as the rate of the composite asset depends on the initial contracts. The spreads depend on market perception of default. As the debtor wants the lowest possible price, this mechanism is virtuous as it is an incentive to reduce the spread.

The pricing for the American call is more difficult. The upper bound for European Calls is a minimum value for the American call but there is no exact solution. Therefore only an approximation is possible for callable bonds.

During the initial negotiation, the call had a value $C_0 = 0$. Indeed, the liquidity crisis that triggered the intervention of the Paris Club temporarily increased the country spread to a level of y' ($r_0 + y' > \bar{q}$). The buyback could occur only after the crisis and when the spread returned to $y < y'$.

2 Pricing of type II contracts: optional collective buybacks

For this contract, creditors could refuse the buyback if their profit is negative. This condition is the following:

$$\begin{aligned} E\left(\sum_{1\leq i\leq n} \Pi_i^{creditor}(F_{t,T,r_t})|F_{t,T,r_t}\right) > 0 \quad &\Leftrightarrow \quad F_{t,T,r_t} + E(Z_i) - E(Q_i) > 0 \\ &\Leftrightarrow \quad F_{t,T,r_t} > \bar{q} - \bar{z} \qquad (24) \end{aligned}$$

The debtor payoff depends on the cartel's profit. At maturity it is positive and equal to the debtor's profit expectation ($\sum_{1\leq i\leq n} \Pi_i^{debtor}(F_{t,T,r_t})$) when its expectation and that of the cartel are positive (conditions (21) and (24)). It is

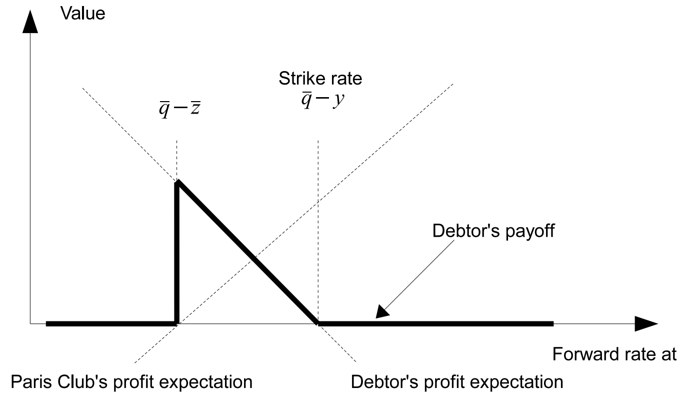

Figure 3: Payoff for the optional collective buyback

zero in other cases, that is to say when the profit expectation of the debtor or the Club is negative. Therefore:

$$(F_{t,T,r_t} > \bar{q} - \bar{z}) \quad \wedge \quad (F_{t,T,r_t} < \bar{q} - y)$$
$$\Rightarrow \mathit{Payoff}_T(F_{t,T,r_t}) = \sum_{1 \le i \le n} \frac{K}{n}(D-T)(Q_i - y - F_{t,T,r_t})$$

$$(F_{t,T,r_t} < \bar{q} - \bar{z}) \quad \vee \quad (F_{t,T,r_t} > \bar{q} - y)$$
$$\Rightarrow \mathit{Payoff}_T(F_{t,T,r_t}) = 0$$

The condition can also be expressed as $\bar{q} - \bar{z} < F_{t,T,r_t} < \bar{q} - y$, possible only if $\bar{z} > y$ that is to say when there is a paretian payoff (positive sum game); when creditors consider the debtor risk to be at a higher level than the market.

This contract combines the plain-vanilla call option under the mandatory collective contract and an "asset-or-nothing put option[9]. This asset-or-nothing put option is broken down into a plain-vanilla put option and a cash-or-nothing put option for an amount equal to the Pareto payoff.

[9]The asset or nothing put option is a binary put option with a price equal to the value of the asset or zero: A binary option is a type of option where the payoff is either some fixed amount of some asset or nothing at all. The two main types of binary options are the cash-or-nothing binary option and the asset-or-nothing binary option. The cash-or-nothing binary option pays some fixed amount of cash if the option expires in-the-money while the asset-or-nothing pays the value of the underlying security. Thus, the options are binary in nature because their are only two possible outcomes. They are also called all or nothing options or digital options. For example, suppose I buy a binary cash-or-nothing call option on XYZ Corp's stock struck at \$100 with a binary payoff of \$1000. Then if at the future maturity date, the stock is trading at or above \$100, I receive \$1000. If it stock is trading below \$100, I receive nothing.(source: Wikipedia)

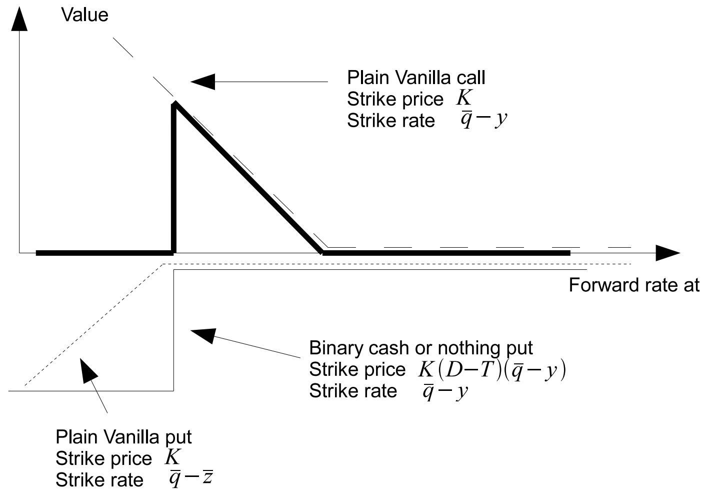

Figure 4: Breakdown into a standard call, a standard put and a binary put

By discounting the expectation of the payoff we obtain, as above, the contract value

$$\begin{aligned} C_T^{II} &= P(t,T,r_t)E\left(Payoff_T(F_{t,T,r_t})\right) \\ &= e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})\left(N\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right)-N\left(\frac{\bar{q}-\bar{z}-f_{t,T}}{\sigma_{F_{t,T}}}\right)\right) \\ &\quad +e^{-r_t(T-t)}K(D-T)\frac{\sigma_{F_{t,T}}}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right)^2}-e^{-\frac{1}{2}\left(\frac{\bar{q}-\bar{z}-f_{t,T}}{\sigma_{F_{t,T}}}\right)^2}\right) \end{aligned} \quad (25)$$

The contract value relative to the forward rate $f_{t,T}$ is zero $f_{t,T} \gg \bar{q}-y$ or $f_{t,T} \ll \bar{q}-\bar{z}$.

An approximation of (25) gives:

$$C_T^{II} = e^{-r_t(T-t)}K(D-T)\frac{1}{2\sigma_{F_{t,T}}\sqrt{2\pi}}(\bar{z}-y)^2 \quad (26)$$

This differential breaks down into the value of the plain-vanilla put option and the cash-or-nothing put option on the Pareto gain.

It should be noted that the approximated value of the contract shows no first order dependence on the level of future interest rates.

Furthermore, this contract encourages virtuous behaviour in appearance only. The debtor's gain-maximising strategy does indeed consist of reducing the market spread y which constitutes virtuous behaviour. But it is also in the debtor's interest to increase his creditors' spread $\bar{z}$, meaning their perception of the debtor's specific risk. This type of behaviour may give rise to contradictory

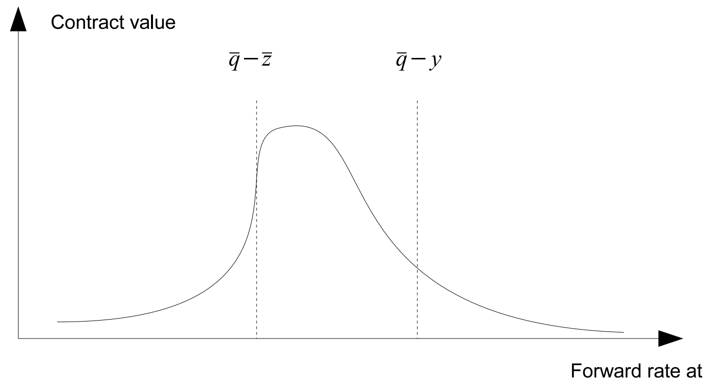

Figure 5: Price of the mandatory collective call

specific signals, or even putting on a virtuous front for the markets, while trying to make creditors wary at the same time. Therefore, this form of contract is likely to give rise to specific default behaviour.

3 Pricing of type III contracts: mandatory selective buybacks

Under a mandatory selective buyback contract, the debtor may choose to prepay only those debts where its profit will be positive, meaning debts where the initial rate is higher than the current rate, plus the spread. $\Pi_i^{debtor}(F_{t,T,r_t}) > 0$

As a consequence, the condition to exercise each option (see (9)) is

$$\Pi_i^{debtor}(F_{t,T,r_t}) > 0 \Leftrightarrow F_{t,T,r_t} < Q_i - y \tag{27}$$

The payoff on each option is equal to the debtor's profit on each credit when the exercise condition is checked (27); it is zero in other cases:

$$\begin{aligned} F_{t,T,r_t} < Q_i - y \Rightarrow \mathit{Payoff}_T^i(F_{t,T,r_t}) &= \Pi_i^{debtor}(F_{t,T,r_t}) \\ &= \frac{K}{n}(D-T)(Q_i - y - F_{t,T,r_t}) \\ F_{t,T,r_t} > Q_i - y \Rightarrow \mathit{Payoff}_T^i(F_{t,T,r_t}) &= 0 \end{aligned}$$

The contract value is the discounted expectation of the sum of payoffs on the whole credit. It is also the sum of the value of contracts (the same) on each credit:

$$
\begin{aligned}
C_T^{III} &= P(t,T,r_t)\sum_{1\le i\le n} E\left(Payoff_T^i(F_{t,T,r_t})\right)\\
&= P(t,T,r_t)\frac{K}{n}(D-T)\sum_{1\le i\le n} E(Q_i-y-F_{t,T,r_t}|-y<F_{t,T,r_t}<Q_i-y)
\end{aligned}
$$

The conditional expectation could be expressed by neglecting the second order terms:

$$
\begin{aligned}
E(Q_i-y-F_{t,T,r_t}|F_{t,T,r_t}<Q_i-y) \;\approx\; & E(Q_i-y)-E(Q_i-y|Q_i-y<F_{t,T,r_t})\\
& -E(F_{t,T,r_t})+E(F_{t,T,r_t}|Q_i-y<F_{t,T,r_t})
\end{aligned}
$$

The conditional expectations $E(Q_i-y|Q_i-y<F_{t,T,r_t})$ and $E(F_{t,T,r_t}|Q_i-y<F_{t,T,r_t})$ could be calculated by using the lemma presented in Appendix I.

Hence:

$$
\begin{aligned}
& E(Q_i-y-F_{t,T,r_t}|-y<F_{t,T,r_t}<Q_i-y)=\\
& (\bar{q}-y-f_{t,T})N\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)+\frac{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)^2} \qquad (28)
\end{aligned}
$$

So

$$
\begin{aligned}
C_T^{III} &= e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})N\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)\\
&\quad +e^{-r_t(T-t)}K(D-T)\frac{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)^2} \qquad (29)
\end{aligned}
$$

It appears that when $\sigma_Q=0$ (that is to say creditors have the same characteristics and adopt the same behaviour, they behave in the same way as the cartel), contract value is equal to the value of the first type of contract $C_T^I=C_T^{III}$.

At the first order, the contract value is (see (29))

$$
C_T^{III}=\frac{1}{2}K(D-T)e^{-r_T(T-t)}\left(\underbrace{(\bar{q}-y-f_{t,T})}_{intrinsic\,value}+\underbrace{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}\sqrt{\frac{2}{\pi}}}_{time\,value}\right) \qquad (30)
$$

This result shows that the behaviour of contract III is identical to type I and does not depend on the strike rate $f^*=\bar{q}-y$. These contract incentives are virtuous in the case of contract I.

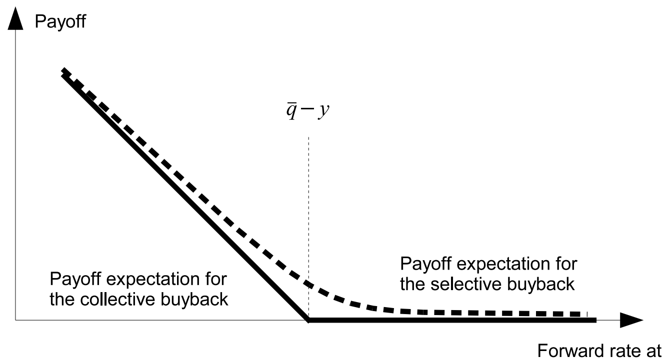

Figure 6: Payoffs of the mandatory collective contract and the mandatory selective contract

Moreover, the average payoff is:

$$E\left(\mathit{Payoff}\left(F_{t,T,r_t}\right)|F_{t,T,r_t}\right) = (\bar{q}-y-F_{t,T,r_t})N\left(\frac{\bar{q}-y-F_{t,T,r_t}}{\sigma_Q}\right) - \frac{\sigma_Q}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{q}-y-F_{t,T,r_t}}{\sigma_Q}\right)^2}$$

A comparison with the collective contract shows that the "intrinsic" terms are identical. The differential between these two contracts is positive (the selective contract is favourable to the debtor).

4 Pricing of type IV contracts: optional selective buybacks

Under an optional selective buyback, the debtor offers to buy back debts when both the debtor's profit and the creditor's profit are positive. Only creditors whose payoff is positive accept the buyback. The debtor's total profit expectation is therefore the conditional expectation on the creditor's profits when the latter are positive.

As a consequence, the condition to exercise each option (see 20 and 27) is

$$(\Pi_i^{creditor}(F_{t,T,r_t}) > 0) \wedge (\Pi_i^{debtor}(F_{t,T,r_t}) > 0) \Leftrightarrow Q_i - Z_i < F_{t,T,r_t} < Q_i - y$$

The payoff is then

$$\begin{aligned} F_{t,T,r_t} > Q_i - Z_i) \quad \wedge \quad & (F_{t,T,r_t} < Q_i - y) \\ & \Rightarrow \mathit{Payoff}_T^i(F_{t,T,r_t}) = \Pi_i^{debtor}(F_{t,T,r_t}) = \frac{K}{n}(D-T)(Q_i - y - F_{t,T,r_t}) \end{aligned}$$

$$(F_{t,T,r_t} < Q_i - Z_i) \quad \vee \quad (F_{t,T,r_t} > Q_i - y)$$
$$\Rightarrow Payoff^{\,i}_{T}(F_{t,T,r_t}) = 0$$

The contract value is

$$\begin{aligned}
C_T^{IV} &= P(t,T,r_t) \sum_{1\le i\le n} E\left(Payoff^{\,i}_{T}(F_{t,T,r_t})\right) \\
&= P(t,T,r_t)(K/n)(D-T) \sum_{1\le i\le n} E(Q_i - F_{t,T,r_t} - y | Q_i - Z_i < F_{t,T,r_t} < Q_i - y)
\end{aligned}$$

By neglecting the second order terms conditions:

$$\begin{aligned}
E(Q_i - F_{t,T,r_t} - y | Q_i - Z_i < F_{t,T,r_t} < Q_i - y) \;=\; & E(Q_i - y) \\
& - E(Q_i - y | Q_i - y < F_{t,T,r_t}) \\
& - E(Q_i - y | F_{t,T,r_t} + Z_i - y < Q_i - y) \\
& - E(F_{t,T,r_t}) \\
& + E(F_{t,T,r_t} | Q_i - y < F_{t,T,r_t}) \\
& + E(F_{t,T,r_t} | F_{t,T,r_t} < y - Z_i)
\end{aligned}$$

With the lemma (see Appendix I) applied to $E(Q_i - y | Q_i - y < F_{t,T,r_t})$, $E(Q_i - y | F_{t,T,r_t} + Z_i - y < Q_i - y)$, $E(F_{t,T,r_t} | Q_i - y < F_{t,T,r_t})$ and $E(F_{t,T,r_t} | F_{t,T,r_t} < y - Z_i)$ and with the sum on all the credits, we obtain:

$$\begin{aligned}
C_T^{IV} \approx\; & e^{-r_t(T-t)} K(D-T)(\bar{q} - y - f_{t,T}) \left(N\left(\frac{\bar{z} - \bar{q} + f_{t,T}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2 + \sigma_Z^2}} \right) - N\left(\frac{y - \bar{q} + f_{t,T}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2}} \right) \right) \\
& + e^{-r_t(T-t)} K(D-T) \frac{\sigma_Q^2 + \sigma_{F_{t,T}}^2}{\sqrt{2\pi}} \left(\frac{e^{-\frac{1}{2}\left(\frac{y - \bar{q} + f_{t,T}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2}} \right)^2}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2}} - \frac{e^{-\frac{1}{2}\left(\frac{\bar{z} - \bar{q} + f_{t,T}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2 + \sigma_Z^2}} \right)^2}}{\sqrt{\sigma_Q^2 + \sigma_{F_{t,T}}^2 + \sigma_Z^2}} \right)
\end{aligned}$$

Moreover the average payoff is:

$$\begin{aligned}
E\left(Payoff\,(F_{t,T,r_t}) | F_{t,T,r_t}\right) \;=\; & K(D-T) \\
& (\bar{q} - y - F_{t,T,r_t}) \left(N\left(\frac{\bar{q} - y - F_{t,T,r_t}}{\sigma_Q} \right) - N\left(\frac{\bar{q} - \bar{z} - F_{t,T,r_t}}{\sqrt{\sigma_Q^2 + \sigma_Z^2}} \right) \right)
\end{aligned}$$

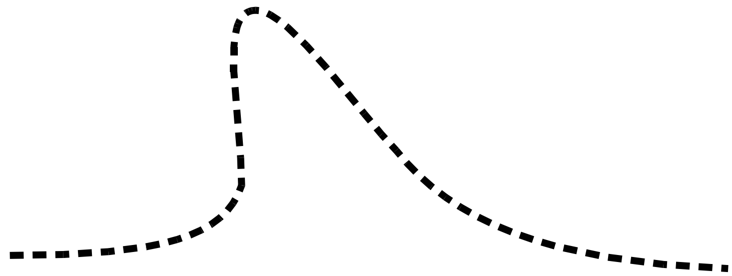

Figure 7: Comparison of the payoffs for the optional collective contract and optional selective contract

$$+K(D-T)\frac{\sigma_Q^2}{\sqrt{2\pi}}\left(\frac{e^{-\frac{1}{2}\left(\frac{\bar{q}-y-F_{t,T,r_t}}{\sigma_Q}\right)^2}}{\sigma_Q}-\frac{e^{-\frac{1}{2}\left(\frac{\bar{q}-\bar{z}-F_{t,T,r_t}}{\sqrt{\sigma_Q^2+\sigma_Z^2}}\right)^2}}{\sqrt{\sigma_Q^2+\sigma_Z^2}}\right)$$

With $\epsilon=\frac{\sigma_Z}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}$, we obtain at the first order

$$C_T^{IV}\approx e^{-r_t(T-t)}K(D-T)\frac{1}{2\sqrt{2\pi}\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\left((\bar{z}-y)^2+\frac{\epsilon^2}{2}(\bar{q}-y-f_{t,T})\frac{5}{2}(\bar{q}-f_{t,T}-\bar{z})\right) \tag{31}$$

This approximated value, as in the case of contract II, does not depend on the interest rate but on the paretian payoff.

For extreme rates, this payoff is zero. Indeed, as in the optional collective contract, a buyback is possible only when the profits of the debtor and the creditors are positive. The incentive is similar to that produced by the mandatory selective contract. This type of option may give rise to "hypocritically" virtuous behaviour.

The value for the contracts can be summarised as follows:

$$\begin{aligned} C_T^I &= e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})N\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right) \\ &\quad +e^{-r_t(T-t)}K(D-T)\frac{\sigma_{F_{t,T}}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right)^2} \end{aligned}$$

$$\begin{aligned} C_T^{II} &= e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})\left(N\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right)-N\left(\frac{\bar{q}-\bar{z}-f_{t,T}}{\sigma_{F_{t,T}}}\right)\right) \\ &\quad +e^{-r_t(T-t)}K(D-T)\frac{\sigma_{F_{t,T}}}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sigma_{F_{t,T}}}\right)^2}-e^{-\frac{1}{2}\left(\frac{\bar{q}-\bar{z}-f_{t,T}}{\sigma_{F_{t,T}}}\right)^2}\right) \end{aligned}$$

$$C_T^{III} = e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})N\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)$$
$$+e^{-r_t(T-t)}K(D-T)\frac{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{q}-y-f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)^2}$$

$$C_T^{IV} \approx e^{-r_t(T-t)}K(D-T)(\bar{q}-y-f_{t,T})\left(N\left(\frac{\bar{z}-\bar{q}+f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2+\sigma_Z^2}}\right)-N\left(\frac{y-\bar{q}+f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)\right)$$
$$+e^{-r_t(T-t)}K(D-T)\frac{\sigma_Q^2+\sigma_{F_{t,T}}^2}{\sqrt{2\pi}}\left(\frac{e^{-\frac{1}{2}\left(\frac{y-\bar{q}+f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}\right)^2}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2}}-\frac{e^{-\frac{1}{2}\left(\frac{\bar{z}-\bar{q}+f_{t,T}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2+\sigma_Z^2}}\right)^2}}{\sqrt{\sigma_Q^2+\sigma_{F_{t,T}}^2+\sigma_Z^2}}\right)$$

As regards irrevocable vs. revocable buyback offers, the following point must be stressed. Logically, the debtor will only exercise the option when his payoff is positive. Consequently, the debtor is not concerned about sustaining losses (except if exogenous variables were to change between the time the buyback is announced and the time it is completed). Therefore, the debtor enjoys an implicit right to retract the buyback offer. This reasoning holds for mandatory contracts where all of the information is known.

Part IV

Numerical simulations

The numerical simulations with the following parameters confirm that the maximal value for the European type I and III contracts occur after approximately two years. The computed value for the type IV contract is effectively a lower bound value as it should be higher than the value of the type II contract.

Type I : 4,7%	Type II : 2,1%
Type III : 5,2%	Type IV : 2.1%

The average initial rates of the debt contracted in the 1990s was at approximately 8% ($\bar{q}$) with a standard deviation of 2% (σ_Q). The market spread could amount to 300 basis points y. For such parameters, the strike rate for a type I contract was 5% (8%-3%). When taking a long-term interest rate of 5% equal to the initial spot rate –implying a zero intrinsic value-with a standard deviation of 2% and a residual maturity of 10 years, the value of the type I contract amounts to at least 11.7% of the face value.

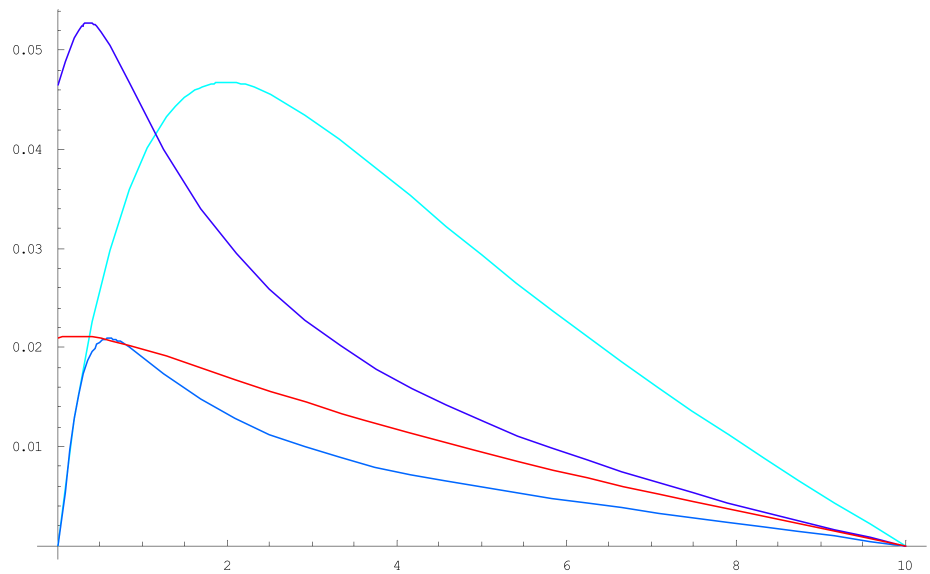

Figure 8: Numerical simulations of the contracts

From top to bottom: type III, I, II and IV. As abscissa: time to exercise (in years). As ordinate; the value of the contract as a fraction of the principal.

Part V
Conclusion

Mandatory buyback contracts carry a relatively high implicit cost. However, optional buybacks have a fairly low value, but they may have a perverse effect in that the debtor may be tempted to put on a reassuring front for the markets and another, more troubling, front for its main creditors. This is why the arrangement for buybacks at par does not seem appropriate for optimal management of the liquidity of the Club's claims. The discounted buyback arrangement conceived by the Club Secretariat seems clearly preferable, since it may be carried out at any time and it limits risks and moral hazard.

Several further developments may be considered. More specifically, it might be a good idea to price American-style call options for these four types of contracts. Furthermore, we could investigate the dynamics between the debtor and the creditors from the point of view of the debtor's strategy for influencing the market and the creditors' perception of its risk. We could also look at pricing the supplementary option resulting from the revocability of buyback offers.

Appendix I

Demonstration of lemma

If X and Y are normal $X \sim N(\mu_X, \sigma_X)$ and $Y \sim N(\mu_Y, \sigma_Y)$ then

$$E(X|X>Y) = \mu_X N\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) + \frac{\sigma_X^2}{\sqrt{2\pi}\sqrt{\sigma_X^2 + \sigma_Y^2}} e^{-\frac{1}{2}\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)^2}$$

$$E(X|X>Y) = \mu_X N\left(\frac{-\mu_X + \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) - \frac{\sigma_X^2}{\sqrt{2\pi}\sqrt{\sigma_X^2 + \sigma_Y^2}} e^{-\frac{1}{2}\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)^2}$$

By definition $E(X|X>Y) = \int\int_{-\infty<y<x<+\infty} x f_X(x) f_Y(y) dxdy$

After changing variables:

$$\begin{cases} x = u + v \\ y = u - v \end{cases}$$

$$\int\int_{-\infty<y<x<+\infty} x f_X(x) f_Y(y) dxdy = \int\int_{v>0 and -\infty<u<+\infty} (u+v) f_X(u+v) f_Y(u-v) \left|J(u,v)\right| dudv$$

with Jacobian $|J(u,v)| = 2$

$$\begin{aligned} E(X|X>Y) &= \int\int_{v>0} 2(u+v) f_X(u+v) f_Y(u-v) \left|J(u,v)\right| dudv \\ &= \int\int_{v>0 and -\infty<u<+\infty} \frac{u+v}{\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left(\frac{u+v-\mu_X}{\sigma_X}\right)^2} e^{-\frac{1}{2}\left(\frac{u-v-\mu_Y}{\sigma_Y}\right)^2} dudv \\ &= \frac{1}{\pi\sigma_X\sigma_Y} \int\int_{v>0 and -\infty<u<+\infty} (u+v) e^{-\frac{1}{2}(au^2 - 2u(b+cv))^2} e^{-\frac{1}{2}(av^2 - 2dv + e)^2} dudv \\ &= \frac{1}{\pi\sigma_X\sigma_Y} \int_0^{+\infty} e^{-\frac{1}{2}(av^2 - 2dv + e)^2} \left(\underbrace{\int_{-\infty}^{+\infty} (u+v) e^{-\frac{1}{2}(au^2 - 2u(b+cv))^2} du}_{I} \right) dv \end{aligned}$$

with

$$\begin{aligned} a &= \frac{\sigma_X^2 + \sigma_Y^2}{(\sigma_X\sigma_Y)^2} \\ b &= \frac{\sigma_X^2\mu_Y + \sigma_Y^2\mu_X}{(\sigma_X\sigma_Y)^2} \\ c &= \frac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X\sigma_Y)^2} \\ d &= \frac{-\sigma_X^2\mu_Y + \sigma_Y^2\mu_X}{(\sigma_X\sigma_Y)^2} \\ e &= \frac{\sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}{(\sigma_X\sigma_Y)^2} \end{aligned}$$

The first integral is

$$
\begin{aligned}
I &= \int_{-\infty}^{+\infty} (u+v)e^{-\frac{1}{2}(au^2-2u(b+cv))^2}\,du \\
&= \int_{-\infty}^{+\infty} ue^{-\frac{1}{2}(au^2-2u(b+cv))^2}\,du + \int_{-\infty}^{+\infty} ve^{-\frac{1}{2}(au^2-2u(b+cv))^2}\,du
\end{aligned}
$$

$$
\begin{aligned}
I &= e^{\frac{(cv+b)^2}{2}a}\left(\int_{-\infty}^{+\infty}\left(\sqrt{\frac{a}{2}}u-\frac{cv+b}{\sqrt{2}}a\right)e^{-\left(\sqrt{\frac{a}{2}}u-\frac{cv+b}{\sqrt{2}}a\right)^2}du+\frac{cv+b}{\sqrt{2}}a\int_{-\infty}^{+\infty}e^{-\left(\sqrt{\frac{a}{2}}u-\frac{cv+b}{\sqrt{2}}a\right)^2}du\right) \\
&\quad +ve^{\frac{(cv+b)^2}{2}a}\int_{-\infty}^{+\infty}e^{-\left(\sqrt{\frac{a}{2}}u-\frac{cv+b}{\sqrt{2}}a\right)^2}du \\
&= e^{\frac{(cv+b)^2}{2}a}\left(\frac{cv+b}{a\sqrt{a}}\sqrt{2}\pi\right)+ve^{\frac{(cv+b)^2}{2}a}\frac{1}{\sqrt{a}}\sqrt{2}\pi \\
&= e^{\frac{(cv+b)^2}{2}a}\frac{\sqrt{2\pi}}{a\sqrt{a}}(cv+b+av)
\end{aligned}
$$

Then

$$
\begin{aligned}
E(X|X>Y) &= \frac{1}{\pi\sigma_X\sigma_Y}\int_0^{+\infty}e^{-\frac{1}{2}(av^2-2dv+e)^2}I\,dv \\
&= \frac{1}{\pi\sigma_X\sigma_Y}\int_0^{+\infty}e^{-\frac{1}{2}(av^2-2dv+e)^2}e^{\frac{(cv+b)^2}{2}a}\frac{\sqrt{2\pi}}{a\sqrt{a}}(cv+b+av)dv \\
&= \frac{1}{\pi\sigma_X\sigma_Y}\frac{\sqrt{2\pi}}{a\sqrt{a}}\int_0^{+\infty}e^{-\frac{1}{2}a\left((a^2-c^2)v^2-2v(ad+cb)+ae-b^2\right)}dv \\
&= \frac{1}{\pi\sigma_X\sigma_Y}\frac{\sqrt{2\pi}}{a\sqrt{a}}e^{\frac{ab^2+ad^2-a^2e+c^2e+2bcd}{2(a^2-c^2)}}\underbrace{\int_0^{+\infty}((c+a)v+b)\,e^{-\frac{1}{2}a\left(\frac{(a^2-c^2)v-(ad+cb)}{\sqrt{a(a^2-c^2)}}\right)^2}dv}_{J}
\end{aligned}
$$

With 2 variable changes

$$
\begin{aligned}
J &= \int_0^{+\infty}((c+a)v+b)\,e^{-\frac{1}{2}a\left(\frac{(a^2-c^2)v-(ad+cb)}{\sqrt{a(a^2-c^2)}}\right)^2}dv \\
&= a\frac{a+c}{a^2-c^2}\int_0^{+\infty}\frac{v(a^2-c^2)-(ad+bc)}{a}e^{-\frac{1}{2}a\left(\frac{(a^2-c^2)v-(ad+cb)}{\sqrt{a^2-c^2}}\right)^2}dv \\
&\quad +\left(b+\frac{ad+bc}{a-c}\right)\int_0^{+\infty}e^{-\frac{1}{2}a\left(\frac{(a^2-c^2)v-(ad+cb)}{\sqrt{a^2-c^2}}\right)^2}dv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{a-c}\int_{\left(-\frac{ad+cb}{\sqrt{2a(a^2-c^2)}}\right)^2}^{+\infty} e^{-z}dz + \left(b+\frac{ad+bc}{a-c}\right)\sqrt{2\frac{a}{a^2-c^2}}\int_{-\frac{ad+cb}{\sqrt{2a(a^2-c^2)}}}^{+\infty} e^{-z^2}dz \\
&= \frac{a}{a-c}e^{-\frac{(ad+cb)^2}{2a(a^2-c^2)}} + a\left(\frac{b+d}{a-c}\right)\sqrt{\frac{2\pi a}{a^2-c^2}}N\left(\frac{ad+cb}{\sqrt{a(a^2-c^2)}}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
E(X|X>Y) &= \frac{1}{\pi\sigma_X\sigma_Y}\frac{\sqrt{2\pi}}{a\sqrt{a}}e^{\frac{ab^2+ad^2-a^2e+c^2e+2bcd}{2(a^2-c^2)}} \\
&\quad \left(\frac{a}{a-c}e^{-\frac{(ad+cb)^2}{2a(a^2-c^2)}} + a\left(\frac{b+d}{a-c}\right)\sqrt{\frac{2\pi a}{a^2-c^2}}N\left(\frac{ad+cb}{\sqrt{a(a^2-c^2)}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
E(X|X>Y) = \ & \frac{1}{\sigma_X\sigma_Y}\sqrt{\frac{2}{\pi}}\frac{e^{\frac{b^2-ae}{2a}}}{\sqrt{a}(a-c)} + \frac{1}{\sigma_X\sigma_Y}\frac{\sqrt{2}}{a\sqrt{a}}e^{\frac{ab^2+ad^2-a^2e+c^2e+2bcd}{2(a^2-c^2)}} \\
& a\left(\frac{b+d}{a-c}\right)\sqrt{\frac{a}{a^2-c^2}}N\left(\frac{ad+cb}{\sqrt{a(a^2-c^2)}}\right)
\end{aligned}
$$

We could replace now a, b, c, d and e:

$$
\frac{1}{\sigma_X\sigma_Y}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{a}(a-c)} = \frac{1}{\sqrt{2\pi}}\frac{\sigma_X^2}{\sqrt{\sigma_X^2+\sigma_Y^2}}
$$

$$
\frac{b^2-ae}{2a} = -\frac{(\mu_X-\mu_Y)^2}{2(\sigma_X^2+\sigma_Y^2)}
$$

$$
ab^2+ad^2-a^2e+c^2e+2bcd = 0
$$

$$
\frac{1}{\sigma_X\sigma_Y}\left(\frac{b+d}{a-c}\right)\sqrt{\frac{1}{a^2-c^2}} = \frac{1}{2}\mu_X
$$

$$
\frac{ad+bc}{\sqrt{a(a^2-c^2)}} = \frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}
$$

Finally

$$
E(X|X>Y) = \mu_X N\left(\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right) + \frac{\sigma_X^2}{\sqrt{2\pi}\sqrt{\sigma_X^2+\sigma_Y^2}}e^{-\frac{1}{2}\left(\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)^2}
$$

As $E(X|X>Y)+E(X|X<Y)=E(X)$, we obtain

$$
E(X|X>Y) = \mu_X N\left(\frac{-\mu_X+\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right) - \frac{\sigma_X^2}{\sqrt{2\pi}\sqrt{\sigma_X^2+\sigma_Y^2}}e^{-\frac{1}{2}\left(\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)^2}
$$

References

[1] Ball, C.A.; Torous, W.N.: (1983): *Bond Price Dynamics and Options* Journal of Financial and Quantitative Analysis 18, 517-531.

[2] G.Barone-Adesi, R.Whaley *Efficient Analytic Approximation of American Option Values*, Journal of Finance, XLII, n. 2, June 1987.

[3] Black, F.; Scholes, M.: (1973):*The Pricing of Options and Corporate Liabilities*; Journal of Political Economy 81, 637-654.

[4] Black (1976) *The pricing of commodity contracts*, Journal of Financial Economics, 3, 167-179

[5] Brennan, M. J. & E. S. Schwartz (1977). *Savings bonds, retractable bonds and callable bonds.* Journal of Financial Economics 5, 67-88;

[6] Bulow Jeremy & Rogoff Kenneth (1988) *The Buyback Boondoggle*,Brookings Papers on Economic Activity, 2: 1988

[7] Bulow Jeremy & Rogoff Kenneth (1988) *Sovereign debt repurchases: no cure for overhang*, Quarterly Journal of Economics 1219-1235

[8] Büttler, H.-J. (1995), *Evaluation of callable bonds: Finite difference methods, stability and accuracy.* Economic Journal 105, 374-384;

[9] Büttler, H.-J. & J. Waldvogel (1996), *Pricing callable bonds by means of Green's function*, Mathematical Finance 6, 53-88;

[10] Cohen Daniel & Verdier Thierry (1995), *"Secret" buyback of LDC debt*, Journal of International Economics 39 (1995) 317-334

[11] d'Halluin Y., Forsyth, Vetzal & Labahn , *A numerical PDE Approach for Pricing Callable Bonds*, http://www.scicom.uwaterloo.ca/ paforsyt/numcall.pdf(2001)

[12] Duffie D, Kan R, *A yield-Factor Model of Interest rates*, Mathematical finance, 64: 379-406

[13] Eichengreen Barry, Portes Richard (1985) *Debt and Default in the 1930s: causes and consequences*, NBER Working Paper n. 1772

[14] Elliott RJ, van der Hoek J, *Stochastic flows and the forward measure*, Finance and stochastics, 5: 511-525, 2001

[15] Froot Kenneth (1988),*Buybacks, Exit Bonds and the Optimality of debt and liquidity relief*, NBER Working Paper n. 2675, August 1988

[16] Heath D, Jarrow R, Morton A, *Bond pricing and the term structure of interest rates: a new methodology*, Econometrica, 60: 77105, 1992

[17] Ho T, Lee SB (1986) *Term Structure Movements and pricing interest rate contingent claims*, the Journal of Finance, XLI: 1011-1029

[18] Jamshidian, Farid (1989); *An Exact Bond Option Formula*, Journal of Finance, Vol. 44 No 1 – March 1989, 205-209

[19] Krugman Paul (1988), *Market Based Debt Reduction Schemes*, NBER Working Paper n. 2587, May 1988

[20] Mallier R. ; Deakin S. (2002), *A Green's Function for a Convertible Bond Using the Vasicek Model*, Journal of applied mathematics 2:5 (2002) 219-232

[21] Mamon Rogemar (2004), *Three ways to solve for bond prices in the Vasicek Model*, Journal of applied mathematics and decision sciences

[22] Merton, R.C.: (1973): *Theory of Rational Option Pricing*; Bell Journal Of Economics and Management 4, 141-183.

[23] Schaefer, S.M.; Schwartz, E.S.: (1987), *Time–Dependent Variance and the Pricing of Bond Options*; Journal of Finance 5, 1113-1128.

[24] Stoll, Hans (1969), *The Relationship Between Put and Call Option Prices*, Journal of Finance 24, 801-824

[25] Thomas J. (1996), *Default Costs, Willingness to Pay and Sovereign Debt Buybacks*, University of St Andrews (1996)

[26] Vasicek O. (1977), *An equilibrium characterization of the term structure*, Journal of Financial Economics 5, 177-188