Bartlett's formula for a general class of non linear processes

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Abstract

A Bartlett-type formula is proposed for the asymptotic distribution of the sample autocorrelations of nonlinear processes. The asymptotic covariances between sample autocorrelations are expressed as the sum of two terms. The first term corresponds to the standard Bartlett's formula for linear processes, involving only the autocorrelation function of the observed process. The second term, which is specific to nonlinear processes, involves the autocorrelation function of the observed process, the kurtosis of the linear innovation process and the autocorrelation function of its square. This formula is obtained under a symmetry assumption on the linear innovation process. An application to GARCH models is proposed.

Keywords : Bartlett's formula, nonlinear time series model, sample autocorrelation, GARCH model, weak white noise.

1 Introduction

In time series analysis, the estimation of the autocorrelation function plays a crucial role, in particular for identification problems (see *e.g.* Brockwell and Davis (1991)). Bartlett (1946) derived an explicit formula for the asymptotic covariance between sample autocorrelations. This formula is given in most time series textbooks, and most time series software packages plot the sample autocorrelation function with significance limits obtained from this formula[1]. Bartlett's formula was obtained for linear processes and it is well known (see *e.g.* Berlinet and Francq (1997), Diebold (1986), Romano and Thombs (1996)) that Bartlett's formula may be completely wrong for series exhibiting conditional heteroscedasticity or any other form of nonlinearity. In particular, Kokoszka and Politis (2008) show that the use of Bartlett's formula is unwarranted for ARCH or stochastic volatility processes. The aim of this paper is to generalize Bartlett's formula to a wide class of nonlinear processes.

In order to give a precise definition of a linear process, first recall that the Wold decomposition (see Brockwell and Davis (1991), Section 5.7) states that any purely non deterministic stationary process can be written in the form

$$X_t = \sum_{\ell=-\infty}^{\infty} \phi_\ell \epsilon_{t-\ell}, \qquad (\epsilon_t) \sim \mathrm{WN}(0, \sigma^2) \tag{1.1}$$

where $\sum_\ell \phi_\ell^2 < \infty$. The process (ϵ_t) is called the linear innovation process of the process $X = (X_t)$, and the notation $(\epsilon_t) \sim \mathrm{WN}(0, \sigma^2)$ signifies that (ϵ_t) is a *weak white noise*, that is a stationary sequence of centered and uncorrelated random variables with common variance σ^2. An independent and identically distributed (iid) sequence of random variables with mean 0 and common variance σ^2 is sometimes called a *strong white noise*, and will be denoted by $\mathrm{IID}(0, \sigma^2)$. Obviously a strong white noise is also a weak white noise, because independence entails uncorrelatedness, but the reverse is not true. The process X is said to be *linear* when $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$, and is said to be *nonlinear* in the opposite case. The autoregressive moving average (ARMA) model with iid noise is the leading example of linear process (see *e.g.* Brockwell and Davis, 1991). Examples of nonlinear models include, among many others, the self-exciting threshold autoregressive (SETAR) model (see Tong, 1990), the smooth transition autoregression (STAR) model (see Teräsvirta (2004) and the references therein), the exponential autoregressive (EXPAR) model introduced by Haggan and Ozaki (1981), the bilinear model (see Granger and Andersen, 1978) and the generalized autoregressive conditional heteroscedastic (GARCH) model introduced by Engle (1982) and Bollerslev (1986). Because numerous real time-series, in particular stock market returns, exhibit dynamics which can not be well mimicked by ARMA models with iid noises, nonlinear models are becoming more and more employed (see Tong (1990) and Fan and Yao (2003) for reference books on nonlinear time series analysis).

Before fitting any time series model to real data, it is common practice to draw the empirical autocovariances and analyze their significance. Because the standard Bartlett's formula can be

[1] See *e.g.* the function `acf()` of the statistical software R, with its argument `ci.type = c("white", "ma")`.

unreliable when the underlying series is non linear, it is important to have an appropriate tool which could be used in very general settings. A question is therefore whether the standard Bartlett formula can be extended. More precisely, our aim in this paper is to derive a formula giving the asymptotic covariances between empirical autocovariances, in function of characteristics of the underlying processes. As we will see, the theoretical autocorrelations of the observed process will not suffice to characterize those asymptotic covariances, as is the case in the linear framework. It will also be of interest to know whether the standard Bartlett's formula can provide good approximations of the asymptotic autocovariances when the underlying process is non linear.

The plan of the paper is as follows. In Section 2 we begin by recalling the standard Bartlett's formula. Section 3 states a generalized Bartlett's formula which can be applied to both linear and nonlinear processes. Section 4 illustrates the generalized Bartlett's formula with GARCH models. Proofs are relegated to Section 5.

2 Notation and Bartlett's formula for linear processes

The autocorrelation function of a real-valued stationary process $X = (X_t)$ is defined by

$$\rho_X(\cdot) = \frac{\gamma_X(\cdot)}{\gamma_X(0)}, \qquad \gamma_X(i) = \mathrm{Cov}(X_t, X_{t+i}) \quad \text{for all integers } t, i.$$

Assume that X is centered and that the observations are $X_1, \dots, X_n$. The autocorrelation $\rho_X(i)$ and autocovariance $\gamma_X(i)$, for $0 \le i < n$, are generally estimated by their sample versions

$$\hat\rho_X(i) = \hat\rho_X(-i) = \frac{\hat\gamma_X(i)}{\hat\gamma_X(0)}, \quad \hat\gamma_X(i) = \hat\gamma_X(-i) = \frac{1}{n}\sum_{t=1}^{n-i} X_t X_{t+i}.$$

For fixed $m \ge 1$, let us consider the following vectors of sample and theoretical autocovariances and autocorrelations

$$\gamma_m = (\gamma_X(0), \dots, \gamma_X(m)), \qquad \hat\gamma_m = (\hat\gamma_X(0), \dots, \hat\gamma_X(m)),$$

$$\rho_m = (\rho_X(1), \dots, \rho_X(m)) \quad \text{and} \quad \hat\rho_m = (\hat\rho_X(1), \dots, \hat\rho_X(m)).$$

The following theorem is standard (see Brockwell and Davis (1991), Chapter 7) and gives the asymptotic distribution of $\sqrt{n}\,(\hat\gamma_m - \gamma_m)$ and $\sqrt{n}\,(\hat\rho_m - \rho_m)$ in the case where X is a linear process.

Theorem 2.1 *Let $X = (X_t)$ be a linear process, that is a process satisfying (1.1) with $(\epsilon_t) \sim$ IID$(0, \sigma^2)$, $\sigma^2 > 0$. Assume also that $E(\epsilon_t^4) = \kappa\sigma^4 < \infty$ and $\sum_{\ell=-\infty}^{\infty} |\phi_\ell| < \infty$. The vectors $\sqrt{n}\,(\hat\gamma_m - \gamma_m)$ and $\sqrt{n}\,(\hat\rho_m - \rho_m)$ are asymptotically normally distributed with mean zero and variance given by Bartlett's formulas*

$$\lim_{n\to\infty} n Cov\{\hat\gamma_X(i), \hat\gamma_X(j)\} = v_{i,j}, \qquad \lim_{n\to\infty} n Cov\{\hat\rho_X(i), \hat\rho_X(j)\} = w_{i,j},$$

where for $i, j > 0$

$$v_{i,j} = (\kappa-3)\gamma_X(i)\gamma_X(j) + \sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}, \tag{2.1}$$

$$\begin{aligned} w_{i,j} = & \sum_{\ell=-\infty}^{\infty}\rho_X(\ell)\left\{2\rho_X(i)\rho_X(j)\rho_X(\ell)-2\rho_X(i)\rho_X(\ell+j)\right. \\ & \left.-2\rho_X(j)\rho_X(\ell+i)+\rho_X(\ell+j-i)+\rho_X(\ell-j-i)\right\}. \end{aligned} \tag{2.2}$$

It is important to note that the iid assumption on (ϵ_t) is very restrictive. Only linear models, essentially the ARMA models with iid noises, are covered by Theorem 2.1. In view of Wold's decomposition, if one can replace the assumption $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$ by the assumption $(\epsilon_t) \sim \mathrm{WN}(0, \sigma^2)$, then one can cover almost all the stationary nonlinear processes.

3 Bartlett's formula for non linear processes

Standard Bartlett's formula (2.2) only depends on the autocorrelation function of the process $X = (X_t)$, but is restricted to linear processes. The following theorem provides an extension of Bartlett's formula to nonlinear processes which, under a symmetry assumption, involves in addition the Kurtosis of the linear innovations ϵ_t of X and the autocorrelation function ρ_{ϵ^2} of (ϵ_t^2).

Theorem 3.1 *We consider the framework and assumptions of Theorem 2.1, but we relax the linearity assumption* $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$ *and we make the following symmetry assumption*

$$E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = 0 \qquad \text{when} \qquad t_1 \neq t_2,\ t_1 \neq t_3 \text{ and } t_1 \neq t_4. \tag{3.1}$$

Then $\rho_{\epsilon^2} = \sum_{h=-\infty}^{+\infty}\rho_{\epsilon^2}(h)$ *exists, and we have the generalized Bartlett's formula for autocovariances*

$$\lim_{n\to\infty} n\,Cov\left\{\hat\gamma_X(i), \hat\gamma_X(j)\right\} = v_{i,j} + v_{i,j}^*, \tag{3.2}$$

where $v_{i,j}$ *is defined by (2.1) and*

$$\begin{aligned} v_{i,j}^* = & (\kappa-1)\Big\{(\rho_{\epsilon^2}-3)\gamma_X(i)\gamma_X(j) \\ & + \sum_{\ell=-\infty}^{\infty}\gamma_X(\ell-i)\left\{\gamma_X(\ell-j)+\gamma_X(\ell+j)\right\}\rho_{\epsilon^2}(\ell)\Big\}. \end{aligned} \tag{3.3}$$

If

$$\sqrt{n}\left(\hat\gamma_m-\gamma_m\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{\hat\gamma_m}\right) \qquad \text{when } n\to\infty, \tag{3.4}$$

where the elements of $\Sigma_{\hat\gamma_m}$ *are given by (3.2), then*

$$\sqrt{n}\left(\hat\rho_m-\rho_m\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{\hat\rho_m}\right), \tag{3.5}$$

where the elements of $\Sigma_{\hat{\rho}_m}$, for $i,j>0$, are given by the generalized Bartlett's formula for autocorrelations

$$\lim_{n\to\infty} nCov\{\hat{\rho}_X(i),\hat{\rho}_X(j)\} = w_{i,j} + w^*_{i,j}, \tag{3.6}$$

where $w_{i,j}$ is defined by (2.2) and

$$\begin{aligned} w^*_{i,j} &= (\kappa-1)\sum_{\ell=-\infty}^{\infty} \rho_{\epsilon^2}(\ell)\left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - 2\rho_X(j)\rho_X(\ell)\rho_X(\ell+i)\right.\\ &\quad \left.-2\rho_X(i)\rho_X(\ell)\rho_X(\ell+j) + \rho_X(\ell+i)\left\{\rho_X(\ell+j)+\rho_X(\ell-j)\right\}\right]. \end{aligned} \tag{3.7}$$

We now give a series of remarks.

Remark 3.1 Following Remark 1 of Theorem 7.2.2 in Brockwell and Davis (1991), $w_{i,j}$ can also be written as

$$w_{i,j} = \sum_{\ell=1}^{\infty} w_i(\ell)w_j(\ell), \qquad \text{where} \qquad w_i(\ell) = \{2\rho_X(i)\rho_X(\ell) - \rho_X(\ell+i) - \rho_X(\ell-i)\}.$$

Similarly we have

$$w^*_{i,j} = (\kappa-1)\sum_{\ell=1}^{\infty} \rho_{\epsilon^2}(\ell)w_i(\ell)w_j(\ell),$$

which shows that, whenever $\rho_X(\cdot)$, κ and $\rho_{\epsilon^2}(\cdot)$ are available, the standard and generalized Bartlett's formulas are computed very similarly.

Remark 3.2 Even for non linear processes, standard Bartlett's coefficients $v_{i,j}$ and $w_{i,j}$ provide good approximations of $\sqrt{n}\mathrm{Cov}(\hat{\gamma}_X(i),\hat{\gamma}_X(j))$ and $\sqrt{n}\mathrm{Cov}(\hat{\rho}_X(i),\hat{\rho}_X(j))$ when i or j is very large, because

$$v^*_{i,j}\to 0 \quad \text{and} \quad w^*_{i,j}\to 0 \qquad \text{when} \quad i\to\infty \quad \text{or} \quad j\to\infty.$$

Note however that, for fixed (i,j), it is easy to find examples of nonlinear processes such that $v^*_{i,j}/v_{i,j}$ and $w^*_{i,j}/w_{i,j}$ are arbitrarily large.

The following remark concerns the technical assumptions of the theorem.

Remark 3.3 The proof of the theorem reveals that the symmetry assumption (3.1) is only needed to obtain a tractable form for the asymptotic covariances, but is not required for their existence. Note also that (3.4) is not entailed by the assumptions made in Theorem 3.1, but general assumptions, such as mixing assumptions, are available in the literature in order to obtain a central limit theorem implying (3.4) and (3.5) (see *e.g.* Berlinet and Francq (1997) or Romano and Thombs (1996)).

The following remark shows that the validity of the standard Bartlett's formulas is actually not limited to the case where ϵ_t is a strong noise.

Remark 3.4 When the ϵ_t^2's are uncorrelated the standard Bartlett formulas apply because

$$v_{i,j}^* = -2(\kappa-1)\gamma_X(i)\gamma_X(j) + (\kappa-1)\gamma_X(i)\{\gamma_X(j)+\gamma_X(-j)\} = 0$$

and $w_{i,j}^* = 0$.

We now consider the particular case where X is a weak white noise.

Corollary 3.1 (Weak white noise) *If $X = (\epsilon_t)$, where (ϵ_t) satisfies the assumptions of Theorem 3.1, then for $i, j \geq 0$, the generalized Bartlett's formula for autocovariances (3.2) holds with*

$$\begin{cases} v_{i,j} = v_{i,j}^* = 0 & \textit{if} \quad i \neq j \\ v_{i,i} = \gamma_\epsilon^2(0) \quad \textit{and} \quad v_{i,i}^* = \rho_{\epsilon^2}(i)\gamma_{\epsilon^2}(0) & \textit{if} \quad i > 0 \\ v_{0,0} = \gamma_{\epsilon^2}(0) \quad \textit{and} \quad v_{0,0}^* = (\rho_{\epsilon^2} - 1)\gamma_{\epsilon^2}(0). \end{cases}$$

Under the addition assumption (3.4), then for $i, j > 0$, the generalized Bartlett's formula for autocorrelations (3.6) holds with

$$\begin{cases} w_{i,j} = w_{i,j}^* = 0 & \textit{if} \quad i \neq j \\ w_{i,i} = 1 \quad \textit{and} \quad w_{i,i}^* = \frac{\gamma_{\epsilon^2}(i)}{\gamma_\epsilon^2(0)} & \textit{if} \quad i > 0. \end{cases} \tag{3.8}$$

Remark 3.5 In the case of GARCH processes, Kokoszka and Politis (2008) established the limiting distribution of $\hat{\rho}_m$. Their formula for the asymptotic variance coincide with (3.8).

It should be noted that the additional term $w_{i,i}^*$ can be arbitrarily large, as the next example shows.

Example 3.1 Romano and Thombs (1996) considered weak white noises of the form $\epsilon_t = \eta_t\eta_{t-1}\cdots\eta_{t-k+1}$ where $(\eta_t) \sim \mathrm{IID}(0, \sigma^2)$, with $\sigma^2 > 0$, $E\eta_1^4 = \mu_4 < \infty$ and $k \geq 1$. It is clear that (3.1) is satisfied for such noises. We have $\gamma_\epsilon(0) = \sigma^{2k}$ and

$$\gamma_{\epsilon^2}(i) = \begin{cases} \sigma^{4i}(\mu_4 - \sigma^4)^{k-i} & \text{for } i = 0, \ldots, k-1 \\ 0 & \text{for } i \geq k \end{cases}$$

Moreover, it can be seen that (3.4) is verified, using the central limit theorem for m-dependent processes (see e.g. Theorem 6.4.2, Brockwell and Davis, 1991). It follows that the conclusion of Corollary 3.1 holds with

$$w_{i,i}^* = \frac{\gamma_{\epsilon^2}(i)}{\gamma_\epsilon^2(0)} = \left(\frac{\mu_4}{\sigma^4} - 1\right)^{k-i}$$

when $i < k$ and $w_{i,i}^* = 0$ when $i \geq k$. Note that $w_{i,i}^* \geq 0$, showing larger variances for the sample autocorrelations than would be expected from the use of the standard Barlett formula. The next example shows that this is not always the case.

Example 3.2 Romano and Thombs (1996) also considered weak white noises of the form $\epsilon_t = \eta_t/\eta_{t-1}$ where $(\eta_t) \sim \mathrm{IID}(0, \sigma^2)$ and$E\eta_1^{-4} < \infty$. It is interesting to note that (3.1) may not hold because

$$E\epsilon_t^2\epsilon_{t-1}\epsilon_{t-2} = E\frac{\eta_t^2}{\eta_{t-1}^2}\frac{\eta_{t-1}}{\eta_{t-2}}\frac{\eta_{t-2}}{\eta_{t-3}} = \left\{E\left(\frac{1}{\eta_1}\right)\right\}^2.$$

When the marginal distribution of η_1 is symmetric (3.1) is however satisfied. In this case we have

$$\gamma_{\epsilon^2}(i) \quad = \quad \begin{cases} \mu_4 E\left(\frac{1}{\eta_1^4}\right) - \left\{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)\right\}^2 & \text{for } i = 0 \\ \sigma^2 E\left(\frac{1}{\eta_1^2}\right) - \left\{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)\right\}^2 & \text{for } i = 1 \\ 0 & \text{for } i \geq 2. \end{cases}$$

Note that (3.4) is satisfied, for the reasons given in the previous example, and that Corollary 3.1 holds with

$$w_{1,1}^* = \frac{1}{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)} - 1 < 0,$$

by Jensen's inequality. In this case, application of the standard Bartlett formula would lead to an over-evaluation of the asymptotic variance of $\sqrt{n}\hat{\rho}_\epsilon(1)$.

The next result shows that Bartlett's formula is also particularly simple for the autocorrelations of MA(q) at lags $i > q$.

Corollary 3.2 (Moving average with non independent linear innovations) *If $X_t = \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q}$, where (ϵ_t) satisfies the assumptions of Theorem 3.1, then the asymptotic covariances $w_{i,j} + w_{i,j}^*$ defined in Theorem 3.1 are such that*

$$w_{i,i} = \sum_{\ell=-q}^{q} \rho_X^2(\ell) \quad \textit{and} \quad w_{i,i}^* = \frac{1}{\gamma_\epsilon^2(0)} \sum_{\ell=-q}^{q} \gamma_{\epsilon^2}(i-\ell)\rho_X^2(\ell)$$

for all $i > q$.

4 Application to ARMA-GARCH models

The following lemma shows that the symmetry assumption (3.1) is satisfied for GARCH models with a symmetric innovation process.

Lemma 4.1 *Let (ϵ_t) be a GARCH(p, q) process defined by*

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t \\ h_t = \omega + \sum_{i=1}^q \alpha_i\epsilon_{t-i}^2 + \sum_{j=1}^p \beta_j h_{t-j}, \end{cases} \tag{4.1}$$

where $\omega > 0$, $\alpha_i \geq 0$ $(i = 1, \ldots, q)$, $\beta_j \geq 0$ $(j = 1, \ldots, p)$, and where $(\eta_t) \sim \mathrm{IID}(0, 1)$, $E\eta_t^4 < \infty$, with η_t independent of $\{\epsilon_u, u < t\}$. Assume also that $E\epsilon_t^4 < \infty$. If the distribution of η_1 is symmetric then (3.1) holds true.

From Ling and McAleer (2002), there exists a solution to (4.1) such that $E\epsilon_t^4 < \infty$ if $\rho(A^{(2)}) < 1$, where $\rho(A^{(2)})$ denotes the spectral radius of $A^{(2)} = EA_t \otimes A_t$, the symbol $\otimes$ standing for the Kronecker product, and

$$A_t = \begin{pmatrix} \eta_t^2 \boldsymbol{\alpha}'_{1:q-1} & \eta_t^2 \alpha_q & \eta_t^2 \boldsymbol{\beta}'_{1:p-1} & \eta_t^2 \beta_p \\ I_{q-1} & 0_{q-1} & 0_{(q-1)\times(p-1)} & 0_{q-1} \\ \boldsymbol{\alpha}'_{1:q-1} & \alpha_q & \boldsymbol{\beta}'_{1:p-1} & \beta_p \\ 0_{(p-1)\times(q-1)} & 0_{p-1} & I_{p-1} & 0_{p-1} \end{pmatrix},$$

with $\boldsymbol{\alpha}_{1:q-1} = (\alpha_1, \dots, \alpha_{q-1})'$, and $\boldsymbol{\beta}_{1:p-1} = (\beta_1, \dots, \beta_{p-1})'$. Note that A_t is written for $p \geq 2$ and $q \geq 2$, but can be straightforwardly modified when $p < 2$ or $q < 2$. It is well known that the square of a GARCH process admits an ARMA representation of the form

$$\epsilon_t^2 - \sum_{i=1}^{p\wedge q} (\alpha_i + \beta_i)\epsilon_{t-i}^2 = \omega + \nu_t - \sum_{i=1}^{p} \beta_i \nu_{t-i},$$

where $\nu_t = \epsilon_t^2 - h_t = (\eta_t^2 - 1)h_t$ is a weak white noise. From this ARMA equation, the autocorrelation function $\rho_{\epsilon^2}(\cdot)$ can be easily computed (see *e.g.* Section 3.3 in Brockwell and Davis, 1991). It can be shown that $\rho_{\epsilon^2}(h) \geq 0$ for all h. Thus, in view of the form of $w_{i,j}^*$ given in Remark 3.1, the presence of GARCH effects makes the autocorrelations more difficult to estimate. More precisely, we have the following result.

Proposition 4.1 *Under the assumptions of Theorem 3.1, if the linear innovation process* (ϵ_t) *is a GARCH process satisfying the assumptions of Lemma 4.1 then*

$$w_{i,i}^* \geq 0 \quad \textit{for all} \quad i > 0.$$

Moreover, if $\alpha_1 > 0$*, if* $Var(\eta_t^2) \neq 0$ *and* $\sum_{h=-\infty}^{+\infty} \rho_X(h) \neq 0$ *we have*

$$w_{i,i}^* > 0 \quad \textit{for all} \quad i > 0.$$

To compute the generalized Bartlett's formula, we also need $\kappa - 1 = \gamma_{\epsilon^2}(0)/\gamma_\epsilon^2(0)$, where $\gamma_\epsilon(0) = \omega \left\{1 - \sum_{i=1}^{p\wedge q} (\alpha_i + \beta_i)\right\}^{-1}$ and $\gamma_{\epsilon^2}(0) = E\epsilon_t^4 - \gamma_\epsilon^2(0)$. It can be shown that

$$E\epsilon_t^4 = \mathbf{e}_1 \left(I_{(p+q)^2} - A^{(2)}\right)^{-1} \left\{\underline{b}^{(2)} + \gamma_\epsilon(0) \left(EA_t \otimes \underline{b}_t + E\underline{b}_t \otimes A_t\right) \mathbf{1}_{p+q}\right\}$$

where $\mathbf{e}_1 = (1, 0'_{p+q-1})'$, $\underline{b}_t = (\omega\eta_t, 0'_{q-1}, \omega, 0'_{p-1})'$, $\underline{b}^{(2)} = E\underline{b}_t \otimes \underline{b}_t$ and $\mathbf{1}_{p+q} = (1, \dots, 1)' \in \mathbb{R}^{p+q}$.

It is then easy to compute Bartlett's coefficients $v_{i,j} + v_{i,j}^*$ and $w_{i,j} + w_{i,j}^*$. An approximate of the standard deviation of $\hat\rho_X(i)$ is then given by $\sigma_{\hat\rho_X(i)} = \sqrt{(w_{i,i} + w_{i,i}^*)/n}$. Using the delta method (see *e.g.* Proposition 6.4.3 in Brockwell and Davis, 1991), one can also obtain asymptotic standard deviations for the sample partial autocorrelations $\hat r_X(i)$, or for any other statistic depending on a finite number of sample autocovariances/autocorrelations. Statistical issues are not considered in the present paper, but it is clear that $\sigma_{\hat{\rho}_X(i)}$ and all the other theoretical moments must be replaced by estimates in statistical applications.

As an illustration, consider the following ARMA(2,1)-GARCH(1,1) model

$$\left\{\begin{array}{l} X_t - 0.8X_{t-1} + 0.8X_{t-2} = \epsilon_t - 0.8\epsilon_{t-1} \\ \epsilon_t = \sigma_t\eta_t, \quad \eta_t \text{ iid } \mathcal{N}(0,1) \\ \sigma_t^2 = 1 + 0.2\epsilon_{t-1}^2 + 0.6\sigma_{t-1}^2. \end{array}\right. \tag{4.2}$$

Figure 1 displays the autocorrelation and partial autocorrelation functions, as well as bands in full lines, in which the sample autocorrelations and sample partial autocorrelations should be included with a probability approximately equal to 95%, when $n = 1,000$. The bands in dotted lines are obtained by the standard Bartlett formula. It is seen that the variability of the autocorrelations is strongly underestimated when this formula is used.

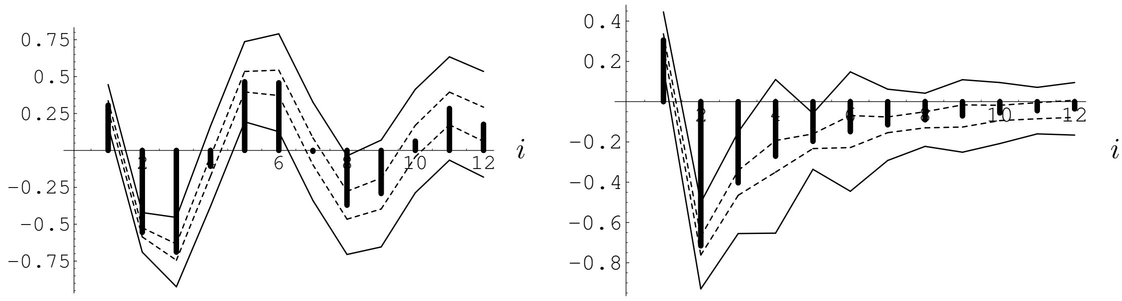

Figure 1: The left panel displays the autocorrelations $\rho_X(i)$ of Model (4.2) and the band $\rho_X(i) \pm 1.96\sigma_{\hat{\rho}_X(i)}$ in full lines, for $n = 1,000$. The right panel is similar for the partial autocorrelations $r_X(i)$. The bands in dotted lines are obtained by the standard Bartlett formula.

5 Proofs

Proof of Theorem 3.1. Using (3.1) and setting $\phi_{\ell_1,\ell_2,\ell_3,\ell_4} = \phi_{\ell_1}\phi_{\ell_2}\phi_{\ell_3}\phi_{\ell_4}$, we obtain

$$\begin{aligned} EX_tX_{t+i}X_{t+h}X_{t+j+h} &= \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \phi_{\ell_1,\ell_2,\ell_3,\ell_4} E\epsilon_{t-\ell_1}\epsilon_{t+i-\ell_2}\epsilon_{t+h-\ell_3}\epsilon_{t+j+h-\ell_4} \\ &= \sum_{\ell_1,\ell_3} \phi_{\ell_1,\ell_1+i,\ell_3,\ell_3+j} E\epsilon_{t-\ell_1}^2\epsilon_{t+h-\ell_3}^2 + \sum_{\ell_1,\ell_2} \phi_{\ell_1,\ell_1+h,\ell_2,\ell_2+h+j-i} E\epsilon_{t-\ell_1}^2\epsilon_{t+i-\ell_2}^2 \\ &+ \sum_{\ell_1,\ell_2} \phi_{\ell_1,\ell_1+h+j,\ell_2,\ell_2+h-i} E\epsilon_{t-\ell_1}^2\epsilon_{t+i-\ell_2}^2 - 2E\epsilon_t^4 \sum_{\ell_1} \phi_{\ell_1,\ell_1+i,\ell_1+h,\ell_1+h+j}. \end{aligned} \tag{5.1}$$

The last equality is obtained by summing over $\ell_1,\ell_2,\ell_3,\ell_4$ such that the indices of $\{\epsilon_{t-\ell_1},\epsilon_{t+i-\ell_2},\epsilon_{t+h-\ell_3},\epsilon_{t+j+h-\ell_4}\}$ are equal two-by-two, which corresponds to the first three sums, and then removing two times the sum in which the four indices are equal. We have also

$$\gamma_X(i) = \sum_{\ell_1,\ell_2} \phi_{\ell_1}\phi_{\ell_2} E\epsilon_{t-\ell_1}\epsilon_{t+i-\ell_2} = \gamma_\epsilon(0)\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}. \tag{5.2}$$

By stationarity,

$$\lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\gamma_X(i),\hat\gamma_X(j)\right\} = \sum_{h=-\infty}^{\infty} \mathrm{Cov}\left\{X_tX_{t+i}, X_{t+h}X_{t+j+h}\right\}.$$

In view of (5.1) and (5.2), the existence of the last sum is guaranteed by the conditions $\sum|\phi_{\ell_1}|<\infty$ and $\sum|\rho_{\epsilon^2}(h)|<\infty$, and this sum is equal to

$$\begin{aligned}
&\sum_{\ell_1,\ell_3}\phi_{\ell_1,\ell_1+i,\ell_3,\ell_3+j}\sum_h\left\{E\epsilon^2_{t-\ell_1}\epsilon^2_{t+h-\ell_3}-\gamma^2_\epsilon(0)\right\}+\sum_{h,\ell_1,\ell_2}\phi_{\ell_1,\ell_1+h,\ell_2,\ell_2+h+j-i}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\\
&+\sum_{h,\ell_1,\ell_2}\phi_{\ell_1,\ell_1+h+j,\ell_2,\ell_2+h-i}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}-2E\epsilon_t^4\sum_{h,\ell_1}\phi_{\ell_1,\ell_1+i,\ell_1+h,\ell_1+h+j}\\
=\;&\gamma_{\epsilon^2}(0)\rho_{\epsilon^2}\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}\sum_{\ell_3}\phi_{\ell_3}\phi_{\ell_3+j}+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\sum_h\phi_{\ell_1+h}\phi_{\ell_2+h+j-i}\\
&+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\sum_h\phi_{\ell_1+h+j}\phi_{\ell_2+h-i}-2E\epsilon_t^4\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}\sum_h\phi_{\ell_1+h}\phi_{\ell_1+h+j},
\end{aligned}$$

using Fubini's theorem for the permutation of summation symbols. Using again (5.2) and $\gamma_{\epsilon^2}(0)=(\kappa-1)\gamma^2_\epsilon(0)$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\gamma_X(i),\hat\gamma_X(j)\right\}\\
=\;&\gamma_{\epsilon^2}(0)\rho_{\epsilon^2}\gamma_\epsilon^{-2}(0)\gamma_X(i)\gamma_X(j)+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\gamma_\epsilon^{-1}(0)\gamma_X(\ell_2+j-i-\ell_1)\\
&+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\gamma_\epsilon^{-1}(0)\gamma_X(\ell_2-j-i-\ell_1)-2E\epsilon_t^4\gamma_\epsilon^{-2}(0)\gamma_X(i)\gamma_X(j)\\
=\;&\left\{(\kappa-1)\rho_{\epsilon^2}-2\kappa\right\}\gamma_X(i)\gamma_X(j)\\
&+\gamma_\epsilon^{-1}(0)\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}\left\{\gamma_X(\ell_2+j-i-\ell_1)+\gamma_X(\ell_2-j-i-\ell_1)\right\}\left\{\gamma_{\epsilon^2}(i-\ell_2+\ell_1)+\gamma^2_\epsilon(0)\right\}.
\end{aligned}$$

Setting $\ell=\ell_2-\ell_1$, we finally obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\gamma_X(i),\hat\gamma_X(j)\right\}=\left\{(\kappa-1)\rho_{\epsilon^2}-2\kappa\right\}\gamma_X(i)\gamma_X(j)\\
&+\gamma_\epsilon^{-2}(0)\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell)+\gamma^2_\epsilon(0)\right\}\\
&=(\rho_{\epsilon^2}-3)(\kappa-1)\gamma_X(i)\gamma_X(j)+(\kappa-3)\gamma_X(i)\gamma_X(j)\\
&+\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\\
&+(\kappa-1)\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\rho_{\epsilon^2}(i-\ell).
\end{aligned}$$

Setting $h=i-\ell$ and using the parity of the autocorrelation functions, the last sum can be written as

$$\sum_{h=-\infty}^{\infty}\gamma_X(-h+i)\left\{\gamma_X(-h+j)+\gamma_X(-h-j)\right\}\rho_{\epsilon^2}(h),$$

which gives (3.2).

The vector $(\hat{\rho}_X(i), \hat{\rho}_X(j))$ is a function of $(\hat{\gamma}_X(0), \hat{\gamma}_X(i), \hat{\gamma}_X(j))$. The Jacobian of this transformation is

$$J = \begin{pmatrix} -\frac{\gamma_X(i)}{\gamma_X^2(0)} & \frac{1}{\gamma_X(0)} & 0 \\ -\frac{\gamma_X(j)}{\gamma_X^2(0)} & 0 & \frac{1}{\gamma_X(0)} \end{pmatrix}.$$

Let Σ be the variance matrix of $(\hat{\gamma}_X(0), \hat{\gamma}_X(i), \hat{\gamma}_X(j))$. By the delta method, we obtain

$$\begin{aligned} & \lim_{n\to\infty} n\text{Cov}\left\{\hat{\rho}(i), \hat{\rho}(j)\right\} = J\Sigma J'(1,2) \\ = \quad & \frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)}\Sigma(1,1) - \frac{\gamma_X(i)}{\gamma_X^3(0)}\Sigma(1,3) - \frac{\gamma_X(j)}{\gamma_X^3(0)}\Sigma(2,1) + \frac{1}{\gamma_X^2(0)}\Sigma(2,3). \end{aligned}$$

Using (3.2) to determine the elements of Σ, this asymptotic covariance is

$$\begin{aligned} & \{(\kappa-1)\rho_{\epsilon^2} - 2\kappa\} \left\{ \frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)}\gamma_X^2(0) - \frac{\gamma_X(i)}{\gamma_X^3(0)}\gamma_X(0)\gamma_X(j) \right. \\ & \left. - \frac{\gamma_X(j)}{\gamma_X^3(0)}\gamma_X(i)\gamma_X(0) + \frac{1}{\gamma_X^2(0)}\gamma_X(i)\gamma_X(j) \right\} \\ & + \gamma_\epsilon^{-2}(0) \sum_{\ell=-\infty}^{\infty} \left[\frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)} 2\gamma_X^2(\ell)\left\{\gamma_{\epsilon^2}(-\ell) + \gamma_\epsilon^2(0)\right\} \right. \\ & - \frac{\gamma_X(i)}{\gamma_X^3(0)}\gamma_X(\ell)\left\{\gamma_X(\ell+j) + \gamma_X(\ell-j)\right\}\left\{\gamma_{\epsilon^2}(-\ell) + \gamma_\epsilon^2(0)\right\} \\ & - \frac{\gamma_X(j)}{\gamma_X^3(0)}\gamma_X(\ell)\left\{\gamma_X(\ell-i) + \gamma_X(\ell-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell) + \gamma_\epsilon^2(0)\right\} \\ & \left. + \frac{1}{\gamma_X^2(0)}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i) + \gamma_X(\ell-j-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell) + \gamma_\epsilon^2(0)\right\} \right]. \end{aligned}$$

As function of the autocorrelations, the previous quantity is written as

$$\begin{aligned} & \sum_{\ell=-\infty}^{\infty} \left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - \rho_X(i)\rho_X(\ell)\left\{\rho_X(\ell+j) + \rho_X(\ell-j)\right\}\right. \\ & \left. - \rho_X(j)\rho_X(\ell)\left\{\rho_X(\ell-i) + \rho_X(\ell-i)\right\} + \rho_X(\ell)\left\{\rho_X(\ell+j-i) + \rho_X(\ell-j-i)\right\}\right] \\ & + (\kappa-1) \sum_{\ell=-\infty}^{\infty} \rho_{\epsilon^2}(\ell) \left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - \rho_X(i)\rho_X(\ell)\left\{\rho_X(\ell+j) + \rho_X(\ell-j)\right\}\right. \\ & \left. - \rho_X(j)\rho_X(\ell-i)\left\{\rho_X(\ell) + \rho_X(\ell)\right\} + \rho_X(i-\ell)\left\{\rho_X(-\ell+j) + \rho_X(-\ell-j)\right\}\right]. \end{aligned}$$

Noting that

$$\sum_\ell \rho_X(\ell)\rho_X(\ell+j) \quad = \quad \sum_\ell \rho_X(\ell)\rho_X(\ell-j),$$

we obtain, with $w_{i,j}$ given by (2.2) and $w_{i,j}^*$ given by (3.7),

$$\lim_{n\to\infty} n\text{Cov}\left\{\hat{\rho}(i), \hat{\rho}(j)\right\} \quad = \quad w_{i,j} + w_{i,j}^*.$$

□

Proof of Corollary 3.1. When $X = (\epsilon_t)$,

$$\begin{aligned}
v_{i,j} &= (\kappa-3)\gamma_\epsilon(i)\gamma_\epsilon(j) + \gamma_\epsilon(0)\left\{\gamma_\epsilon(j-i) + \gamma_\epsilon(-j-i)\right\} \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ \gamma_\epsilon^2(0) & \text{if} \quad i = j > 0 \\ (\kappa-1)\gamma_\epsilon^2(0) & \text{if} \quad i = j = 0, \end{cases} \\
w_{i,j} &= -2\rho_\epsilon(j)\rho_\epsilon(i) + \rho_\epsilon(j-i) + \rho_\epsilon(-j-i) \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j > 0, \end{cases} \\
v^*_{i,j} &= (\rho_{\epsilon^2} - 3)(\kappa-1)\gamma_\epsilon(i)\gamma_\epsilon(j) + (\kappa-1)\gamma_\epsilon(0)\left\{\gamma_\epsilon(i-j) + \gamma_\epsilon(i+j)\right\}\rho_{\epsilon^2}(i) \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ (\kappa-1)\gamma_\epsilon^2(0)\rho_{\epsilon^2}(i) & \text{if} \quad i = j > 0 \\ (\rho_{\epsilon^2} - 1)(\kappa-1)\gamma_\epsilon^2(0) & \text{if} \quad i = j = 0, \end{cases} \\
w^*_{i,j} &= (\kappa-1)\left[-2\rho_\epsilon(i)\rho_\epsilon(j) + \rho_{\epsilon^2}(i)\left\{\rho_\epsilon(i+j) + \rho_\epsilon(i-j)\right\}\right] \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ (\kappa-1)\rho_{\epsilon^2}(i) & \text{if} \quad i = j > 0. \end{cases}
\end{aligned}$$

The conclusion then follows from $(\kappa - 1) = \gamma_{\epsilon^2}(0)/\gamma_\epsilon^2(0)$.

□

Proof of Corollary 3.2. Because $\rho_X(\ell) = 0$ for $|\ell| > q$, we have

$$\begin{aligned}
w_{i,j} &= \sum_{\ell=-q}^{q} \rho_X(\ell)\left[2\rho_X(i)\rho_X(j)\rho_X(\ell) - 2\rho_X(i)\rho_X(\ell+j)\right. \\
&\quad \left. -2\rho_X(j)\rho_X(\ell+i) + \rho_X(\ell+j-i) + \rho_X(\ell-j-i)\right]
\end{aligned}$$

and for $i, j > q$

$$w_{i,j} = \sum_{\ell=-q}^{q} \rho_X(\ell)\rho_X(\ell+j-i).$$

The expression of $w_{i,i}$ follows. Similarly, for $i > q$

$$\begin{aligned}
w^*_{i,i} &= (\kappa-1)\sum_{\ell=-\infty}^{\infty} \rho_{\epsilon^2}(\ell)\rho_X(\ell+i)\left\{\rho_X(\ell+i) + \rho_X(\ell-i)\right\} \\
&= (\kappa-1)\sum_{\ell=-q}^{q} \rho_{\epsilon^2}(i-\ell)\rho_X^2(\ell).
\end{aligned}$$

□

Proof of Lemma 4.1. It is shown in Francq and Zakoïan (2004) that, if the distribution of η_t is symmetric then

$$\forall j, \quad E\left\{g(\epsilon_t^2, \epsilon_{t-1}^2, \dots)\epsilon_{t-j} f(\epsilon_{t-j-1}, \epsilon_{t-j-2}, \dots)\right\} = 0, \tag{5.3}$$

for any functions f and g such that the expectation exists. Let four indices t_i, $i = 1, \dots, 4$, such that $t_1 \le t_2 \le t_3 \le t_4$. We will show that $E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = 0$ when one of the indices is different from the three others.

If $t_3 < t_4$, then

$$E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} \quad = \quad E\left[E\left(\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} \mid \{\epsilon_u, u < t_4\}\right)\right] = E\left[\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}h_{t_4}E\left(\eta_{t_4} \mid \{\epsilon_u, u < t_4\}\right)\right] = 0,$$

because h_{t_4} is measurable with respect to the σ-field generated by $\{\epsilon_u, u < t_4\}$ and because η_{t_4} is centered and independent of $\{\epsilon_u, u < t_4\}$. The result can also be obtained from (5.3) with $g = 1$, $t - j = t_4$ and $f(\epsilon_{t_4-1}, \epsilon_{t_4-2}, \dots) = \epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}$.

Assume therefore that $t_1 < t_2 \le t_3 = t_4$. Applying (5.3) with $g(x) = f(x) = x$, we have

$$E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = E\left\{g(\epsilon_{t_3}^2)\epsilon_{t_2}f(\epsilon_{t_1})\right\} = 0$$

and the conclusion follows.

$\square$

Proof of Proposition 4.1. Because

$$w_{i,i}^* = (\kappa - 1)\sum_{\ell=1}^{\infty}\rho_{\epsilon^2}(\ell)w_i^2(\ell),$$

the first result follows from Lemma 5.1 below. To prove the second part, note that $\mathrm{Var}\epsilon_t^2 = Eh_t^2\mathrm{Var}(\eta_t^2) + \mathrm{Var}h_t \ge \omega^2\mathrm{Var}(\eta_t^2) > 0$. Thus $\kappa - 1 > 0$. Note also that the condition $\sum_\ell |\phi_\ell| < \infty$ implies $\sum_{-\infty}^{\infty} |\rho_X(i)| < \infty$. The conclusion thus follows from Lemmas 5.1-5.2 below.

$\square$

Lemma 5.1 *If (ϵ_t) is a GARCH process and $E\epsilon_t^4 < \infty$ then*

$$\gamma_{\epsilon^2}(h) = Cov(\epsilon_t^2, \epsilon_{t-h}^2) \ge 0 \qquad \forall h,$$

with strict equality if $\alpha_1 > 0$.

Proof. It suffices to show that we have a MA(∞) of the form

$$\epsilon_t^2 = c + \nu_t + \sum_{\ell=1}^{\infty}\phi_\ell\nu_{t-\ell}, \quad \text{with} \quad \phi_\ell \ge 0 \quad \forall \ell.$$

Indeed, $\nu_t := \epsilon_t^2 - h_t = (\eta_t^2 - 1)h_t$ being a weak white noise, we have

$$\gamma_{\epsilon^2}(h) = E\nu_1^2\sum_{\ell=0}^{\infty}\phi_\ell\phi_{\ell+|h|}, \qquad \text{with the notation } \phi_0 = 1.$$

Denoting by B the backshift operator, and introducing the notation $\alpha(z) = \sum_{i=1}^q \alpha_i z^i$, $\beta(z) = \sum_{j=1}^p \beta_j z^j$ and $\phi(z) = \sum_{\ell=1}^{\infty}\phi_\ell z^\ell$, we obtain

$$\epsilon_t^2 = \{1 - (\alpha + \beta)(1)\}^{-1}\omega + \{1 - (\alpha + \beta)(B)\}^{-1}(1 - \beta(B))\nu_t = c + \phi(B)\nu_t.$$

Since $1-\beta(B)=1-(\alpha+\beta)(B)+\alpha(B)$, we obtain ϕ_ℓ as the coefficient of z^ℓ in the division of $\alpha(z)$ by $1-(\alpha+\beta)(z)$ according to the increasing powers of z. By recurrence on ℓ, it is easy to see that these coefficients are positive because the polynomials $\alpha(z)$ and $(\alpha+\beta)(z)$ have positive coefficients. It is sufficient to show the positivity of the coefficients c_i in the expansion

$$\frac{\alpha(B)}{1-(\alpha+\beta)(B)}=\sum_{i=1}^{\infty}c_iB^i.$$

By induction we prove that

$$c_i\geq\alpha_1(\alpha_1+\beta_1)^{i-1},\quad i\geq 1. \tag{5.4}$$

We have $c_1=\alpha_1$. Moreover, with by convention $\alpha_i=0$ if $i>q$ and $\beta_j=0$ if $j>p$,

$$c_{i+1}=c_1(\alpha_i+\beta_i)+\ldots+c_i(\alpha_1+\beta_1)+\alpha_{i+1}.$$

Thus if (5.4) holds up to the order i, using the positivity of the GARCH coefficients, we have $c_{i+1}\geq\alpha_1(\alpha_1+\beta_1)^i$. The conclusion follows.

□

Lemma 5.2 *Let $\rho(\cdot)$ be an autocorrelation function. If $\sum_{h=-\infty}^{+\infty}\rho(h)$ exists and is not equal to zero, for all $i>0$ we have*

$$w_i(\ell):=2\rho(i)\rho(\ell)-\rho(\ell+i)-\rho(\ell-i)\neq 0\quad \textit{for some}\quad \ell>0.$$

Proof. Suppose that for some $i>0$,

$$2\rho(i)\rho(\ell)=\rho(\ell+i)+\rho(\ell-i),\quad\forall\ell\geq 1.$$

Then, the equality also holds for any $\ell\in\mathbb{Z}$. Moreover, summing over ℓ yields

$$2\rho(i)\sum_{-\infty}^{\infty}\rho(\ell)=\sum_{-\infty}^{\infty}\rho(\ell+i)+\rho(\ell-i)=2\sum_{-\infty}^{\infty}\rho(\ell).$$

It follows that $\rho(i)=1$. But taking $\ell=i$ in the relation then yields $\rho(2i)=1$. By induction $\rho(ni)=1$. Letting $n\to\infty$ we have a contradiction.

□

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