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Abstract. In this paper we investigate the impact of delayed tax revenues on the fiscal policy out-comes. Choosing the delay as a bifurcation parameter we study the direction and the stability of the bifurcating periodic solutions. With respect to the delay we show when the system is stable. Some numerical examples are finally given for justifying the theoretical results.

Keywords: delay differential equation, stability, Hopf bifurcation, IS-LM model.
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1. Introduction

The differential equations with time delay play an important role for economy, engineering, biology and social sciences, because a lot of phenomena are described with their help. In this paper, we consider a model from economy, of the IS-LM type with time delay and we study how the delay affects the

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macroeconomic stability. The Hopf bifurcation and normal form theories are tools for establishing the existence and the stability of the periodic solutions. Similar ideas can be found in [5], [9], [10].

In [2] Cesare and Sportelli taking into account the papers [6], [7], [8], study a dynamic IS-LM model of the following type:

$$\begin{aligned}
&\dot{Y}(t)=\alpha[I(Y(t),r(t))+g-S(Y(t)-T(Y(t),Y(t-\tau))))-T(Y(t),Y(t-\tau)))]\\
&\dot{r}(t)=\beta[L(Y(t),r(t))-M(t)]\\
&\dot{M}(t)=g-T(Y(t),Y(t-\tau))),
\end{aligned}$$

with Y as income, I as investment, g as government expenditure (constant), S as savings, T as tax revenues, r as rate of interest, L as liquidity, M as real money supply and α, β as positive constants. The time delay τ appears in function T:

$$T(Y(t),Y(t-\tau)))=d(1-\varepsilon)Y(t)+d\varepsilon Y(t-\tau), \tag{1}$$

where $d \in (0,1)$ is a common average tax rate and $\varepsilon \in (0,1)$ is the income tax share.

Based on the papers [2], [1], [11], we consider the following IS-LM model:

$$\begin{aligned}
&\dot{Y}(t)=\alpha[I(Y(t),r(t))+g-S(Y(t)-T(Y(t),Y(t-\tau)))-T(Y(t),Y(t-\tau))]\\
&\dot{r}(t)=\beta[L(Y(t),r(t))-M(t)]\\
&\dot{K}(t)=I(Y(t-\tau),r(t))-\delta K(t)\\
&\dot{M}(t)=g-T(Y(t),Y(t-\tau)),
\end{aligned} \tag{2}$$

with $\delta > 0$ and with the initial conditions:

$$Y(t)=\varphi(t), t\in[-\tau,0], r(0)=r_1, M(0)=M_1, K(0)=K_1.$$

In the following analysis we will consider the function T given by (1), the investment, the saving and the liquidity of the form:

$$\begin{aligned}
&I(Y(t),r(t))=aY(t)^{\alpha_1}r(t)^{-\alpha_2}, \quad a>0, \alpha_1>0, \alpha_2>0,\\
&S(Y(t)-T(Y(t),Y(t-\tau)))=s(Y(t)-T(Y(t),Y(t-\tau))), \quad s\in(0,1),\\
&L(r(t))=mY(t)+\frac{\gamma_0}{r(t)-r_2}, \quad m>0, \gamma_0>0, r_2>0.
\end{aligned} \tag{3}$$

The paper is organized as follows. In section 2 we investigate the local stability of the equilibrium point associated to system (2). Choosing the delay as a bifurcation parameter some sufficient conditions for the existence of Hopf bifurcation are found. In section 3 there is the main aim of the paper, namely the direction, the stability and the period of a limit cycle solution. Section 4 gives some numerical simulations which show the existence and the nature of the periodic solutions. Finally, some conclusions are given.

2. The qualitative analysis of system (2).

Using functions (1) and (3), system (2) becomes:

$$\begin{aligned}
&\dot{Y}(t)=\alpha[((s-1)(1-\varepsilon)d-s)Y(t)+d\varepsilon(s-1)Y(t-\tau)+aY(t)^{\alpha_1}r(t)^{-\alpha_2}+g]\\
&\dot{r}(t)=\beta[mY(t)+\frac{\gamma_0}{r(t)-r_2}-M(t)]\\
&\dot{K}(t)=aY(t-\tau)^{\alpha_1}r(t)^{-\alpha_2}-\delta K(t)\\
&\dot{M}(t)=g-d(1-\varepsilon)Y(t)-d\varepsilon Y(t-\tau),
\end{aligned} \tag{4}$$

$\alpha,\beta>0$, $\alpha_1,\alpha_2>0$, $m>0$, $\gamma_0>0$, $r_2>0$, $\delta>0$, $s\in(0,1)$, $d\in(0,1)$, $\varepsilon\in(0,1)$.

System (4) is a system of equations with time delay. The qualitative analysis is done using the methods from [3].

The equilibrium point of system (4) has the coordinates Y_0, r_0, K_0, M_0, where:

$$Y_0=\frac{g}{d}, r_0=[\frac{s(1-d)}{a}Y_0^{1-\alpha_1}]^{-\frac{1}{\alpha_2}}, K_0=\frac{s(1-d)Y_0}{\delta}, M_0=mY_0+\frac{\gamma_0}{r_0-r_2}. \tag{5}$$

Using the translation

$$Y=x_1+Y_0, r=x_2+r_0, K=x_3+K_0, M=x_4+M_0$$

in (4) and considering the Taylor expansion of the right members from (4) until the third order, we have:

$$\dot{x}(t)=Ax(t)+Bx(t-\tau)+F(x(t),x(t-\tau)) \tag{6}$$

where

$$A=\begin{pmatrix} \alpha a_1 & a\alpha\rho_{01} & 0 & 0 \\ \beta m & \beta\gamma_0\gamma_1 & 0 & -\beta \\ 0 & a\rho_{01} & -\delta & 0 \\ a_4 & 0 & 0 & 0 \end{pmatrix}, B=\begin{pmatrix} \alpha b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\rho_{10} & 0 & 0 & 0 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$, and

$$a_1 = (s-1)d(1-\varepsilon) - s + a\rho_{10}, b_1 = (s-1)d\varepsilon, a_4 = -d(1-\varepsilon), b_4 = -d\varepsilon$$
$$\rho_{01} = -\alpha_2 Y_0^{\alpha_1} r_0^{-(\alpha_2+1)}, \rho_{10} = \alpha_1 Y_0^{\alpha_1-1} r_0^{-\alpha_2}$$

and

$$F(x(t), x(t-\tau)) = (F^1(x(t), x(t-\tau)), F^2(x(t), x(t-\tau)), F^3(x(t), x(t-\tau)), F^4(x(t), x(t-\tau)))^T, \tag{8}$$

$$F^1(x(t), x(t-\tau)) = \frac{\alpha a}{2}\rho_{20}x_1^2(t) + \alpha a\rho_{11}x_1(t)x_2(t) + \frac{\alpha a}{2}\rho_{02}x_2^2(t) + \frac{\alpha a}{6}(\rho_{30}x_1^3(t) + 3\rho_{21}x_1(t)^2x_2(t) + 3\rho_{12}x_1(t)x_2(t)^2 + \rho_{03}x_2(t)^3)$$

$$F^2(x(t), x(t-\tau)) = \frac{\gamma_0}{2}\gamma_2 x_2(t)^2 + \frac{\gamma_0}{6}\gamma_3 x_2(t)^3$$

$$F^3(x(t), x(t-\tau)) = \frac{a}{2}\rho_{20}x_1(t-\tau)^2 + a\rho_{11}x_1(t-\tau)x_2(t) + \frac{a}{2}\rho_{02}x_2(t)^2 + \frac{a}{6}(\rho_{30}x_1^3(t-\tau) + 3\rho_{21}x_1(t-\tau)^2x_2(t) + 3\rho_{12}x_1(t-\tau)x_2(t)^2 + \rho_{03}x_2(t)^3)$$

$$F^4(x(t), x(t-\tau)) = 0$$

and

$$\rho_{20} = \alpha_1(\alpha_1-1)Y_0^{\alpha_1-2}r_0^{-\alpha_2}, \quad \rho_{11} = -\alpha_1\alpha_2 Y_0^{\alpha_1-1}r_0^{-(\alpha_2+1)}$$
$$\rho_{02} = \alpha_2(\alpha_2+1)Y_0^{\alpha_1}r_0^{-(\alpha_2+2)}, \quad \rho_{30} = \alpha_1(\alpha_1-1)(\alpha_1-2)Y_0^{\alpha_1-3}r_0^{-\alpha_2}$$
$$\rho_{21} = -\alpha_1\alpha_2(\alpha_1-1)Y_0^{\alpha_1-2}r_0^{-(\alpha_2+1)}, \quad \rho_{12} = \alpha_1\alpha_2(\alpha_2+1)Y_0^{\alpha_1-1}r_0^{-(\alpha_2+2)}$$
$$\rho_{03} = -\alpha_2(\alpha_2+1)(\alpha_2+2)Y_0^{\alpha_1}r_0^{-(\alpha_2+3)},$$
$$\gamma_1 = -\frac{1}{(r_0-r_2)^2}, \gamma_2 = \frac{2}{(r_0-r_2)^3}, \gamma_3 = -\frac{6}{(r_0-r_2)^4}.$$

The characteristic equation of linear part from (6) is:

$$det(\lambda I - A - Be^{-\lambda\tau}) = (\lambda+\delta)\Delta(\lambda,\tau) = 0 \tag{9}$$

where

$$\Delta(\lambda,\tau) = P(\lambda) + e^{-\lambda\tau}Q(\lambda) \tag{10}$$

$$P(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \quad Q(\lambda) = q_2\lambda^2 + q_1\lambda + q_0$$

and

$$\begin{array}{l} p_2 = -(\alpha a_1 + \beta\gamma_0\gamma_1), \quad p_1 = \alpha\beta(\gamma_0 a_1\gamma_1 + ma\rho_{01}), \quad p_0 = \alpha a\beta a_4\rho_{01} \\ q_2 = -\alpha b_1, \quad q_1 = \alpha\beta\gamma_0 b_1\gamma_1, \quad q_0 = \alpha\beta b_4\rho_{01}. \end{array}$$

To investigate the local stability of the equilibrium point, we begin by considering, as usual, the case without delay ($\tau = 0$). In this case the characteristic polynomial is:

$$(\lambda+\delta)(P(\lambda)+Q(\lambda)) = 0$$

hence, according to the Hurwitz criterion, the equilibrium point is stable if and only if:

$$p_2 + q_2 > 0, \quad (p_1+q_1)(p_2+q_2) > p_0 + q_0.$$

When $\tau > 0$, standard results on stability of systems of delay differential equations postulate that a equilibrium point is asymptotically stable if an only if all roots of equation (10) have a negative real part. It is well known that equation (10) is a transcendental equation which has an infinite number of complex roots and the some possible roots with positive real part are finite in number.

We want to obtain the values τ_0 such that the equilibrium point (5) changes from local asymptotic stability to instability or vice versa. We need the imaginary solutions of equation $\Delta(\lambda,\tau) = 0$. Let $\lambda = \pm i\omega$ be these solutions and without loss of generality we assume $\omega > 0$. We suppose that $P(i\omega) + Q(i\omega) \neq 0$, for all $\omega \in \mathbb{R}$. The previous conditions are equivalent to $(p_1+q_1)(p_2+q_2) \neq p_0 + q_0$.

A necessary condition to have ω as a solution of $\Delta(i\omega,\tau) = 0$ is that ω must be a root of the following equation:

$$f(\omega) = \omega^6 + a_F\omega^4 + b_F\omega^2 + c_F = 0 \tag{11}$$

where $a_F = p_2^2 - q_2^2 - 2p_1$, $b_F = 2q_0q_2 - 2p_0p_2 - q_1^2 + p_1^2$, $c_F = p_0^2 - q_0^2$.

Let $k = -\frac{a_F}{3}$ and $f_D = \frac{1}{4}[f(k)]^4 + \frac{1}{9}[f^{'}(k)]^3$.

Using the results from [2], it results:

Proposition 1. *1. Let $\varepsilon > \frac{1}{2}$. Then the following cases can be discerned:*

(i) If $a_F \geq 0$ or $b_F \leq 0$, then equation (11) has only one real positive root;

(ii) If $a_F < 0$, $b_F > 0$ and $f_D \leq 0$, then equation (11) has only one real positive root.

2. If $\varepsilon < \frac{1}{2}$, $a_F < 0$, $b_F < 0$ and $f_D \leq 0$ then equation (11) has two real positive roots which are distinct if $f_D \neq 0$.

3. If $\varepsilon = \frac{1}{2}$ and:

(i) $a_F < 0$, $b_F = 0$ then equation (11) has only one real positive root;

(ii)$a_F < 0$, $b_F > 0$, then equation (11) has two real positive roots;

(iii)$b_F < 0$, then equation (11) has only one real positive root.

Also, we have:

Theorem 1. *If we suppose that the equilibrium point (Y_0, r_0, K_0, M_0) is locally asymptotically stable without time delay, then in conditions of Proposition 1 there exists only one stability switch.*

Theorem 2. *If τ_0 is a stability switch and $f_D \neq 0$, then a Hopf bifurcation occurs at τ_0, where*

$$\tau_0 = \frac{1}{\omega_0} arctg \left(\frac{\omega_0(\omega_0^4 q_2 - \omega^2(q_0 - q_1p_2 + q_2p_1) + q_0p_1 - p_0q_1)}{\omega_0^4(q_1 - q_2p_2) + \omega_0^2(q_0p_2 - q_1p_1 + p_0q_2) - p_0q_0} \right)$$

and ω_0 is a root of (11).

3. The normal form for system (4). Cyclical behavior.

In this section we describe the direction, stability and the period of the bifurcating periodic solutions of system (4). The method we use is based on the normal form theory and the center manifold theorem introduced by Hassard [4]. Taking into account the previous section, if $\tau = \tau_0$ then all roots of equation (9) other than $\pm i\omega_0$ have negative real parts, and any root of equation (9) of the form $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ satisfies $\alpha(\tau_0) = 0$, $\omega(\tau_0) = \omega_0$

and $\dfrac{d\alpha(\tau_0)}{d\tau} \neq 0$. For notational convenience let $\tau = \tau_0 + \mu, \mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value for equations (4).

Define the space of continuous real-valued functions as $C = C([-\tau_0, 0], \mathbb{R}^4)$.

In $\tau = \tau_0 + \mu, \mu \in \mathbb{R}$, we regard μ as the bifurcation parameter. For $\Phi \in C$ we define a linear operator:

$$L(\mu)\Phi = A\Phi(0) + B\Phi(-\tau)$$

where A and B are given by (7) and a nonlinear operator $F(\mu, \Phi) = F(\Phi(0), \Phi(-\tau))$, where $F(\Phi(0), \Phi(-\tau))$ is given by (8). By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions, $\eta(\theta, \mu)$ with $\theta \in [-\tau_0, 0]$ such that:

$$L(\mu)\Phi = \int_{-\tau_0}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \theta \in [-\tau_0, 0].$$

For $\Phi \in C^1([-\tau_0, 0], \mathbb{R}^4)$ we define:

$$\mathcal{A}(\mu)\Phi(\theta) = \begin{cases} \dfrac{d\Phi(\theta)}{d\theta}, & \theta \in [-\tau_0, 0) \\ \int_{-\tau_0}^{0} d\eta(t, \mu)\phi(t), & \theta = 0, \end{cases}$$

$$R(\mu)\Phi(\theta) = \begin{cases} 0, & \theta \in [-\tau_0, 0) \\ F(\mu, \Phi), & \theta = 0. \end{cases}$$

We can rewrite (6) in the following vector form:

$$\dot{u}_t = \mathcal{A}(\mu)u_t + R(\mu)u_t \tag{12}$$

where $u = (u_1, u_2, u_3, u_4)^T$, $u_t = u(t + \theta)$ for $\theta \in [-\tau_0, 0]$.

For $\Psi \in C^1([0, \tau_0], \mathbb{R}^{*4})$, we define the adjunct operator $\mathcal{A}^*$ of $\mathcal{A}$ by:

$$\mathcal{A}^*\Psi(s) = \begin{cases} -\dfrac{d\Psi(s)}{ds}, & s \in (0, \tau_0] \\ \int_{-\tau_0}^{0} d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases}$$

We define the following bilinear form:

$$< \Psi(\theta), \Phi(\theta) >= \bar{\Psi}^T(0)\Phi(0) - \int_{-\tau_0}^{0}\int_{\xi=0}^{0} \bar{\Psi}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$.

We assume that $\pm i\omega_0$ are eigenvalues of $\mathcal{A}(0)$. Thus, they are also eigenvalues of $\mathcal{A}^*$. We can easily obtain:

$$\Phi(\theta) = ve^{\lambda_1\theta}, \quad \theta \in [-\tau_0, 0] \tag{13}$$

where $v = (v_1, v_2, v_3, v_4)^T$,

$$v_1 = 1, v_2 = -\frac{\beta(a_4 + b_4e^{-\lambda_1\tau_0} - m\lambda_1)}{\lambda_1(\lambda_1 - \beta\gamma_0\gamma_1)}, v_4 = \frac{a_4 + b_4e^{-\lambda_1\tau_0}}{\lambda_1},$$

$$v_3 = \frac{a}{\lambda_1 + \delta}[\rho_{10}e^{-\lambda_1\tau_0} + \rho_{01}\frac{\beta(a_4 + b_4e^{-\lambda_1\tau_0} - m\lambda_1)}{\lambda_1(\beta\gamma_0\gamma_1 - \lambda_1)}]$$

is the eigenvector of $\mathcal{A}(0)$ corresponding to $\lambda_1 = i\omega_0$ and

$$\Psi(s) = we^{\lambda_2 s}, \quad s \in [0, \infty)$$

where $w = (w_1, w_2, w_3, w_4)$,

$$w_1 = \frac{\lambda_2 - \beta\gamma_0\gamma_1}{\alpha a\rho_{01}}\frac{1}{\bar{\eta}}, w_2 = \frac{1}{\bar{\eta}}, w_3 = 0, w_4 = -\frac{\beta}{\lambda_2}\frac{1}{\bar{\eta}}$$

$$\eta = \frac{\lambda_1 - \beta\gamma_0\gamma_1}{a\alpha\rho_{01}}(1+\alpha b_1\frac{\lambda_1\tau_0e^{-\lambda_1\tau_0} - e^{-\lambda_1\tau_0} + 1}{\lambda_1^2}) + v_2 - \frac{\beta}{\lambda_1}(v_4 + b_4\frac{\lambda_1\tau_0e^{-\lambda_1\tau_0} - e^{-\lambda_1\tau_0} + 1}{\lambda_1^2})$$

is the eigenvector of $\mathcal{A}(0)$ corresponding to $\lambda_2 = -i\omega_0$.

We can verify that: $< \Psi(s), \Phi(s) >= 1, < \Psi(s), \bar{\Phi}(s) >=< \bar{\Psi}(s), \Phi(s) >= 0, < \bar{\Psi}(s), \bar{\Phi}(s) >= 1$.

Using the approach of Hassard [4], we next compute the coordinates to describe the center manifold Ω_0 at $\mu = 0$. Let $u_t = u_t(t + \theta), \theta \in [-\tau_0, 0)$ be the solution of equation (12) when $\mu = 0$ and

$$z(t) =< \Psi, u_t >, \quad w(t, \theta) = u_t(\theta) - 2Re\{z(t)\Phi(\theta)\}.$$

On the center manifold Ω_0, we have:

$$w(t, \theta) = w(z(t), \bar{z}(t), \theta)$$

where

$$w(z,\bar z,\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar z + w_{02}(\theta)\frac{\bar z^2}{2} + w_{30}(\theta)\frac{z^3}{6} + \dots$$

in which z and $\bar z$ are local coordinates for the center manifold Ω_0 in the direction of Ψ and $\bar\Psi$ and $w_{02}(\theta) = \bar w_{20}(\theta)$. Note that w and u_t are real.

For solution $u_t \in \Omega_0$ of equation (12), since $\mu = 0$, we have:

$$\dot z(t) = \lambda_1 z(t) + g(z,\bar z) \tag{14}$$

where

$$\begin{aligned} g(z,\bar z) &= \bar\Psi(0)F(w(z(t),\bar z(t),0) + 2Re(z(t)\Phi(0))) = \\ &= g_{20}\frac{z(t)^2}{2} + g_{11}z(t)\bar z(t) + g_{02}\frac{\bar z(t)^2}{2} + g_{21}\frac{z(t)^2\bar z(t)}{2} + \dots \end{aligned}$$

where

$$g_{20} = F^1_{20}\bar w_1 + F^2_{20}\bar w_2, g_{11} = F^1_{11}\bar w_1 + F^2_{11}\bar w_2, g_{02} = F^1_{02}\bar w_1 + F^2_{02}\bar w_2, \tag{15}$$

with

$$\begin{aligned} &F^1_{20} = \alpha a(\rho_{20} + 2\rho_{11}v_2 + \rho_{02}v_2^2), F^1_{11} = \alpha a(\rho_{20} + \rho_{11}(\bar v_2 + v_2) + \rho_{02}v_2\bar v_2), \\ &F^2_{20} = \gamma_0\gamma_2 v_2^2, F^2_{11} = \gamma_0\gamma_2 v_2\bar v_2, F^1_{02} = \bar F^1_{20}, F^2_{02} = \bar F^2_{20}, \end{aligned}$$

and

$$g_{21} = F^1_{21}\bar w_1 + F^2_{21}\bar w_2 \tag{16}$$

where

$$\begin{aligned} &F^1_{21} = \alpha a\rho_{20}(2w^1_{11}(0) + w^1_{20}(0)) + \alpha a\rho_{11}(2w^2_{11}(0) + w^2_{20}(0) + 2w^1_{11}(0)\bar v_2 + \\ &w^1_{20}(0)v_2) + \alpha a\rho_{02}(2w^2_{11}(0)v_2 + w^2_{20}(0)\bar v_2) + \alpha a(\rho_{30} + 2\rho_{21}v_2 + 2\rho_{12}v_2^2 + \\ &+\rho_{03}v_2^2\bar v_2 + \rho_{12}v_2^2 + \rho_{21}\bar v_2) \end{aligned}$$

$$F^2_{21} = \gamma_0\gamma_2(2w^2_{11}(0)v_2 + w^2_{20}(0)\bar v_2) + \gamma_0\gamma_3 v_2^2\bar v_2.$$

The vectors $w_{20}(\theta)$, $w_{11}(\theta)$ with $\theta \in [-\tau, 0]$ are given by:

$$\begin{aligned} w_{20}(\theta) &= -\frac{g_{20}}{\lambda_1}ve^{\lambda_1\theta} - \frac{\bar g_{02}}{3\lambda_1}\bar v_e^{\lambda_2\theta} + E_1e^{2\lambda_1\theta} \\ w_{11}(\theta) &= \frac{g_{11}}{\lambda_1}ve^{\lambda_1\theta} - \frac{\bar g_{11}}{\lambda_1}\bar v e^{\lambda_2\theta} + E_2 \end{aligned} \tag{17}$$

where

$$E_1 = -(A + e^{-\lambda_1\tau_0}B - 2\lambda_1 I)^{-1}F_{20}, \quad E_2 = -(A + B)^{-1}F_{11},$$

where $F_{20} = (F_{20}^1, F_{20}^2, F_{20}^3, 0)^T$, $F_{11} = (F_{11}^1, F_{11}^2, F_{11}^3, 0)^T$.

Based on the above analysis and calculation, we can see that each g_{ij} in (15), (16) are determined by the parameters and delay from system (4). Thus, we can explicitly compute the following quantities:

$$\begin{aligned} &C_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2} \\ &\mu_2 = -\frac{Re(C_1(0))}{Re\lambda'(\tau_0)}, T_2 = -\frac{Im(C_1(0)) + \mu_2 Im\lambda'(\tau_0)}{\omega_0}, \beta_2 = 2Re(C_1(0)). \end{aligned} \tag{18}$$

and

$$\lambda' = \frac{\lambda(q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau}}{3\lambda^2 + 2p_2\lambda + p_1 + e^{-\lambda\tau}(2q_2\lambda + q_1 - \tau(q_2\lambda^2 + q_1\lambda + q_0))}$$

In summary, this leads to the following result:

Theorem 3. *In formulas (18), μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exit for $\tau > \tau_0(< \tau_0)$; β_2 determines the stability of the bifurcating periodic solutions: the solutions are orbitally stable (unstable) if $\beta_2 < 0(> 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(< 0)$.*

4. Numerical example.

In this section we find the waveform plots through the formula:

$$X(t+\theta) = z(t)\Phi(\theta) + \bar{z}(t)\bar{\Phi}(\theta) + \frac{1}{2}w_{20}(\theta)z^2(t) + w_{11}(\theta)z(t)\bar{z}(t) + \frac{1}{2}w_{02}(\theta)\bar{z}(t)^2 + X_0,$$

where $z(t)$ is the solution of (14), $\Phi(\theta)$ is given by (13), $w_{20}(\theta), w_{11}(\theta), w_{02}(\theta) = \bar{w}_{20}(\theta)$ are given by (17) and $X_0 = (Y_0, r_0, K_0, M_0)^T$ is the equilibrium state.

For the numerical simulations we use Maple 9.5. We consider system (4) with $a = 0.38$, $\alpha = 0.96$, $\beta = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.83$, $\gamma_0 = 1$, $d = 0.1$, $s = 0.3$,

$r_2 = 0.003$, $\delta = 0.2$, $m = 0.005$, $g = 50$. The equilibrium point is: $Y_0 = 500$, $r_0 = 0.03572181612$, $K_0 = 675$, $M_0 = 33.06065092$. In what follows we consider two different shares ϵ of delay tax revenues: $\varepsilon = 0.3$ and $\varepsilon = 0.8$.

For $\varepsilon = 0.3$ we obtain: $\mu_2 = 1.654628706 \cdot 10^{-8}$, $\beta_2 = 2.224294680 \cdot 10^{-9}$, $T_2 = 2.092652051 \cdot 10^{-9}$, $\omega_0 = 0.6685954740$, $\tau_0 = 4.965007916$. Then the Hopf bifurcation is supercritical, the solutions are orbitally unstable and the period of the solution is increasing. The wave plots are given in the following figures:

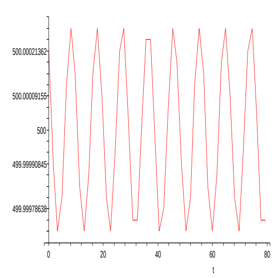
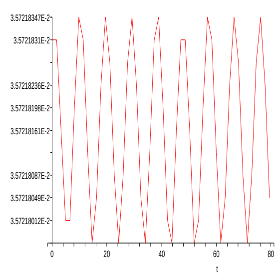
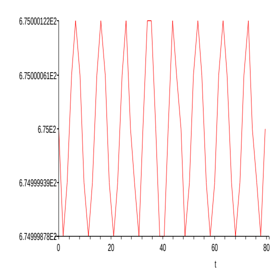
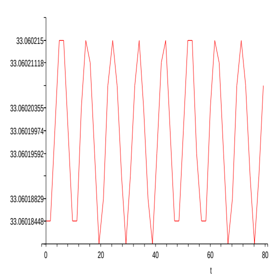
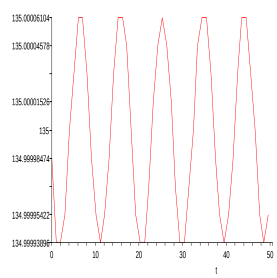
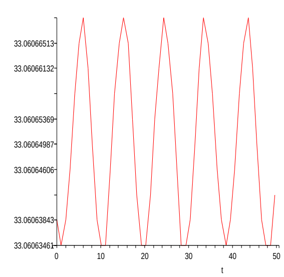
Fig.1.Waveplot $(t, Y(t))$	Fig.2.Waveplot $(t, r(t))$	Fig.3.Waveplot $(t, K(t))$
		

Fig.4.Waveplot $(t, M(t))$	Fig.5.Waveplot $(t, I(t))$	Fig.6.Waveplot $(t, L(t))$
		

For $\varepsilon = 0.8$ we obtain: $\mu_2 = -9.160756314 \cdot 10^{-8}$, $\beta_2 = -1.968119398 \cdot 10^{-8}$, $T_2 = -1.608068638 \cdot 10^{-8}$, $\omega_0 = 0.8553440397$, $\tau_0 = 3.918246696$. Then the Hopf bifurcation is subcritical, the solutions are orbitally stable and the period of the solution is decreasing. The wave plots are given in the following figures:

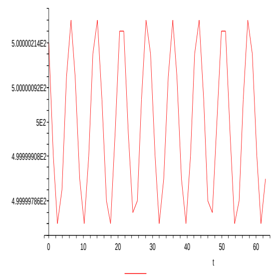
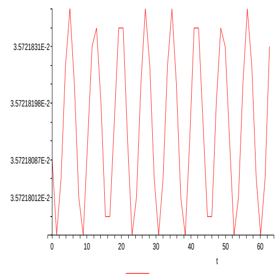
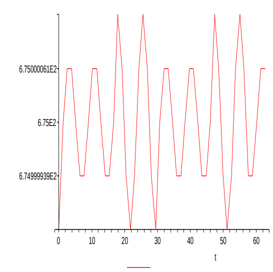
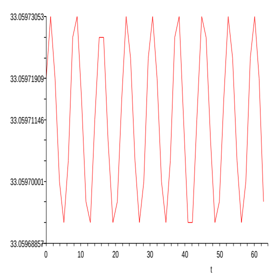
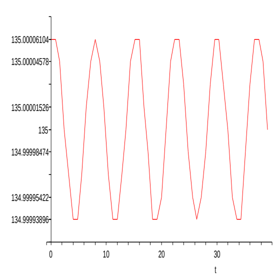
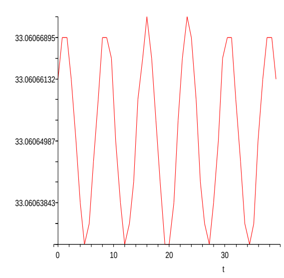
Fig.7.Waveplot $(t, Y(t))$	Fig.8.Waveplot $(t, r(t))$	Fig.9.Waveplot $(t, K(t))$
		

Fig.10.Waveplot $(t, M(t))$	Fig.11.Waveplot $(t, I(t))$	Fig.12.Waveplot $(t, L(t))$
		

5. Conclusions.

From the analysis of the model with continuous time, it results that the model accepts a limit cycle. The nature of the limit cycle is given by the coefficients (18) which include the parameters of the model. We establish the nature of the limit cycle. Because the expressions of the parameter and the coefficients (18) are difficult to analyze directly, the use of Maple 9.5 was essential. The paper's results confirm that a series of economical processes, where a variable with time delay intervenes, have a limit cycle, thus, allowing a prediction concerning the evolution of the model.

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