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Horii, Ryo and Ono, Yoshiyasu

Tohoku University, Osaka University

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Ryo Horii† and Yoshiyasu Ono‡

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Abstract

This paper presents a simple model in which the learning behavior of agents generates fluctuations in money demand and possibly causes a prolonged depression. We consider a stochastic Money-in-Utility model, where agents receive utility from holding money only when a liquidity shock (e.g., a bank run) occurs. Households update the subjective probability of the shock based on the observation and change their money demand accordingly. In this setting, we first derive a stationary cycles under perfect price adjustment, which is characterized by periods of gradual inflation and sudden sporadic falls of the price level. When the nominal stickiness is introduced, the liquidity shock is followed by a period of low output. We show that the adverse effects of the shocks are largest when they occur in succession in an economy which has enjoyed a long period of stability.

JEL Classifications: E32, E41, D83

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†Correspondence: Graduate School of Economics, Tohoku University, 27-1 Kawauchi, Aoba-ku, Sendai 980-8576, Japan; Tel/Fax: +81-(0)22-795-4272.

‡Institute of Social and Economic Research, Osaka University, 6-1 Mihogaoka, Ibaraki-shi, Osaka, 567-0047, JAPAN

1 Introduction

This paper presents a simple model in which the learning behavior of agents generates fluctuations in money demand and possibly causes a prolonged depression. It is commonly assumed in monetary macroeconomics models—both in money-in-utility models and cash-in-advance models—that the benefit from holding money is well known beforehand. However, we sometimes find ourselves not very sure about how much money will be needed but hold some just for precaution. For concreteness, suppose that we can do business using checks at normal times, but once some shock to the financial system (e.g., a bank run) occurs transactions cannot be settled without money. We do not know exactly when bank runs occur. Moreover, we do not know the precise probability that bank runs occur but have to learn from past history. In such a situation, the learning process will cause the money demand to fluctuate, which in turn affect other macroeconomic variables especially when prices are not fully flexible.

To examine this information-driven fluctuation, we introduce a stochastic version of Sidrauski's (1967) model: agents receive utility from holding money only when a liquidity shock occurs. In the model, the shock is generated by a Markov modulated Poisson process (MMPP), which means that shock follows a usual Poisson process, and the arrival rate changes unobservably between high (a dangerous state) and low (a safer state) according to a Markov Process. We show that if the shock does not occur for a while, agents gradually increase the belief of being in a safer state, reduce the shock probability, and lower money demand, causing inflation. Conversely, when they observe the shock, they strengthen the belief that they are in a dangerous state, increase their subjective probability for meeting with the shock again, and raise money demand, causing deflation.

An important finding is that the impact of a liquidity shock on the economy depends on the economy's history before encountering it. In a "turbulent" economy that is hit regularly by shocks, an additional shock actually has little impact on the economy because it has only minor effect on the belief about the fundamental state of the economy. By contrast, in a "stable" economy where shocks were rarely observed until recently, an occurrence of a shock has a significant effect on their belief (Intuitively, it is a surprise to agents). Still, if there are no more shocks to follow, the macroeconomic effect is limited since agents see the one-shot shock as a mare accident. However, if they observe a succession of shocks in a previously stable economy, they will completely change their belief about the fundamental state of the economy, as experienced in Japan in mid 1990s, and possibly as in the U.S. in late

2000s. As agents become quickly pessimistic, the aggregate money demand jumps up, which can a cause depression if the prices are not fully flexible. Moreover, as in the case of Japan, the recovery from the depression is shown to take a long time when the agents' pessimistic belief is so strong that it is not easily turned over by the gradually revealed information that no shock occurs.

There exist a number of earlier studies that analyzed the macroeconomic movements when an underlying state is only partially observable and information is revealed gradually (e.g., Caplin and Leahy 1993; Zeira 1994; Boldrin and Levine 2001; and Andolfatto and Gomme 2003). In particular, Chalkley and Lee (1998) considered unobservable changes in investment opportunities and showed that recovery from a recession is protracted when risk aversion of agents prevents them from acting promptly on receiving good news. Potter (2000) and Nieuwerburgh and Veldkamp (2006) explained slow recovery generated by an endogenous flow of information. If agents have a pessimistic belief, their activities are low, generating less public information, and therefore good news is only slowly revealed. These studies are complementary to this paper in providing alternative explanations of slow recovery,[1] but they do not show that negative shocks have the largest effect when the shocks hit an economy that was previously in good condition for a long time. This paper is also related to Farzin, Huisman and Kort (1998), Hassett and Metcalf (1999), Venegas-Martínes (2001), and Wälde (1999, 2005) in that the analysis includes a continuous-time stochastic optimization with discrete jumps in a state variable, although these studies consider the case in which agents know the true arrival rate of jumps.[2]

The organization of the paper is as follows. After introducing a stochastic Money-in-Utility model in Section 2, we describe the process of the liquidity shock and the evolution of the belief that is updated based on Bayes' law in section 3. Section 4 presents a benchmark result for the case where the price level is perfectly flexible,

[1] In our model, the recovery is slow not because information is scarce in depression but people's strong beliefs dwarf the significance of new, favorable information. In fact, the flow of information brought by no occurrence of the shock is largest when people are convinced of being in the dangerous state. However, it is also the time when their prior belief is strongest, and hence people only slowly change it.

[2] Technically speaking, the substantial differences are in that our model have multiple state variables and that the size of jump is not exogenously random but determined by the rational learning behavior. It is shown that, in our case, the Hamilton-Jacobi-Bellman (HJB) equations yields a system of partial delay differential equations (sometimes called difference-differential equations). We develop a numerical procedure to solve this problem via iteration without linearization. A set of Mathematica codes are available upon request.

and shows the pattern of price movements. The nominal stickiness is introduced in Section 5 to investigate how a depression is triggered and how economy recovers from it. Section 6 concludes the paper. Some mathematical proofs are collated in Appendix.

2 A Stochastic Money-in-Utility Model

This section sets up a stochastic Money-in-Utility model, where money holdings affect utility only at random discrete points in time. In the model, time is continuous,[3] and the economy is inhabited by a continuum of infinitely lived homogeneous households with measure one. At each date, they gain utility $u(c_t)$ from consumption c_t, where instantaneous felicity function $u(\cdot)$ is twice differentiable, $u'(\cdot) > 0$, and satisfies the Inada conditions. In addition, when a liquidity shock occurs, they experience utility loss $v(m_t) < 0$ according to their real money holding m_t. We assume $v'(m) > 0$, which means that the size of utility loss is small when their real money holdings are large. Function $v(\cdot)$ also satisfies $v''(m) < 0$, $\lim_{m\to 0} mv'(m) > 0$, and $\lim_{m\to\infty} v'(m) = 0$. Their expected utility EU_t is therefore given by

$$EU_t = E_t\left[\int_t^\infty u(c_\tau)e^{-\rho(\tau-t)}d\tau + \sum_{\tau\in S_{(t,\infty)}} v(m_\tau)e^{-\rho(\tau-t)}\right], \tag{1}$$

where ρ is the subjective discount rate and $S_{(t,\infty)}$ is the set of future dates at which the shock occurs. Note that the shock dates $S_{(t,\infty)}$ are stochastic and cannot be exactly anticipated in advance. Therefore, households are willing to hold money at all times for precaution.

We keep the remaining settings as simple as possible. Each household is endowed with one unit of labor at each point time, which is suppled to the labor market inelastically. A representative firm employs n_t units of labor and competitively produces yn_t units of goods, where $y > 0$ is a constant technology parameter and n_t is labor input. Note that, as long as prices are perfectly flexible, $n_t = 1$ holds and the output will be y. The monetary authority issues a constant amount of nominal money stock, the size of which is normalized to one.[4] Goods are perishable and thus cannot be stored. The households will not borrow or lend among themselves because

[3]We use a continuous time model in order to highlight the difference between the change in belief when bad news arrives and when there is no such news. This strategy is similar to Driffill and Miller (1993) and Zeira (1999), but in their models uncertainty eventually vanishes and the economy reaches a steady state since unobservable state is time invariant.

[4]This assumption is made only for the simplicity of the description of the model and notations.

they are identical. The firm has no value because of its linear production technology and perfect competition. Therefore, money is the only asset in this economy.

Let p_t denote the price of consumption good. Since firms are competitive, the nominal wage rate is given by the nominal marginal product $p_t y$. Then, the nominal money holding of the household evolves according to

$$\dot{M}_t = p_t y n_t - p_t c_t. \tag{2}$$

The objective of the representative household is to maximize expected utility (1) under budget constraint (2). To solve this problem, they need two sorts of additional information. One is the likelihood of encountering a shock in the future, because it determines the expected benefit of holding money. The next section explains how household estimate and update the likelihood through Bayesian learning. The other required information is the inflation rate, because it determines the real cost of holding money. We later investigate how the evolution pattern of the inflation rate is determined in the market, both for the case of perfectly flexible prices (Section 4) and for the case of sticky prices (Section 5).

3 Learning Process

There are two underlying states with different probabilities of the shock, called states H and L. In state $i \in \{H, L\}$, the shock occurs with probability θ^i per unit of time, where $\theta^H > \theta^L > 0$. The household cannot directly observe the current state but knows that the state evolves according to a Markov process: state H changes to state L with Poisson probability p^H per unit of time whereas state L changes to state H with probability p^L. We assume that the shock occurs much more frequently in state H than in state L and that the state change is a rare event when compared to the shock in state H. Formally,

Assumption 1 $\theta^H - \theta^L > p^H + p^L$.

By observing whether the shock occurs or not the household continuously revises its subjective shock probability in a Bayesian manner. Let $\theta_t \in \{\theta^H, \theta^L\}$ denote the true shock probability at time t, which is unknown to the household. Using information available up to time t, it forms a belief that current θ_t is θ^H with probability

The results to follow are essentially the same even when the nominal money growth rate is positive and constant. In that case the price level would not be stationary, and therefore we need to normalize the price level by by dividing by the nominal money supply.

λ_t^H and θ^L with probability λ_t^L. Obviously,

$$\lambda_t^L + \lambda_t^H = 1 \quad \text{for all } t. \tag{3}$$

In order to find how the household updates λ_t^i from t to $t+\Delta t$,[5] we first obtain the subjective probability that the shock does not occur between t and $t+\Delta t$ for given λ_t^i. It is denoted by $\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset\right]$, where $\text{Prob}_t[\cdot]$ is a probability operator based on information available at t, $S_{(a,b]}$ is the set of dates on which the shock actually occurs during $(a, b]$, and $\emptyset$ the empty set. Since the underlying state is either H or L at time $t + \Delta t$, this probability is divided into two components, $\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset \cap \theta_{t+\Delta t} = \theta^H\right]$ and $\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset \cap \theta_{t+\Delta t} = \theta^L\right]$.

Each of the two components is further divided into two probabilities. The former is the sum of the probability that 'the state is H at time t and neither the state change nor the shock occurs during the interval' and the probability that 'the present state is L and the state changes to H during the interval.' It is[6]

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset \cap \theta_{t+\Delta t} = \theta^H\right] = \left(1 - (\theta^H + p^H)\Delta t\right)\lambda_t^H + (p^L \Delta t)\lambda_t^L. \tag{4}$$

Similarly, the latter is

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset \cap \theta_{t+\Delta t} = \theta^L\right] = \left(1 - (\theta^L + p^L)\Delta t\right)\lambda_t^L + (p^H \Delta t)\lambda_t^H. \tag{5}$$

Summing up (4) and (5) yields

$$\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset\right] = 1 - \theta_t^e \Delta t, \tag{6}$$

where θ_t^e represents the expected (or subjective) probability of the shock per unit of time at time t,

$$\theta_t^e \equiv \theta^H \lambda_t^H + \theta^L \lambda_t^L. \tag{7}$$

Let us consider how the representative household updates its belief if it eventually finds that the shock did not occur during $(t, t+\Delta t]$. In this case the information that $S_{(t,t+\Delta t]} = \emptyset$ is added to its knowledge. Thus, using Bayes' law we find updated subjective probability $\lambda_{t+\Delta t}^i$ to be

$$\begin{aligned}\lambda_{t+\Delta t}^i &\equiv \text{Prob}_{t+\Delta t}\left[\theta_{t+\Delta t} = \theta^i\right] = \text{Prob}_t\left[\theta_{t+\Delta t} = \theta^i | S_{(t,t+\Delta t]} = \emptyset\right] \\ &= \frac{\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset \cap \theta_{t+\Delta t} = \theta^i\right]}{\text{Prob}_t\left[S_{(t,t+\Delta t]} = \emptyset\right]}.\end{aligned}$$

[5] Time interval Δt is taken to be so short that the probability that the liquidity shock and a state change coexist in the interval is negligible.

[6] Throughout the paper we ignore the second-order term of Δt and higher because $\Delta t \to 0$.

Since the numerator is given by (4) or (5) and the denominator by (6), $\lambda^H_{t+\Delta t}$ equals[7]

$$\lambda^H_{t+\Delta t} = \frac{\left(1-(\theta^H+p^H)\Delta t\right)\lambda^H_t + (p^L\Delta t)\lambda^L_t}{1-\theta^e_t\Delta t}.$$

From this equation we derive the time derivative of λ^H_t as

$$\dot{\lambda}^H_t = \lim_{\Delta t\to 0}\frac{\lambda^H_{t+\Delta t}-\lambda^H_t}{\Delta t} = (\theta^e_t-\theta^H-p^H)\lambda^H_t + p^L\lambda^L_t. \tag{8}$$

We next consider the case where the shock occurs during $(t, t+\Delta t]$. Since

$$\mathrm{Prob}_t\left[S_{(t,t+\Delta t]} \neq \emptyset \cap \theta_{t+\Delta t} = \theta^i\right] = \theta^i\lambda^i_t\Delta t \quad \text{for} \quad i \in \{L, H\}, \tag{9}$$

the probability that the shock occurs is

$$\mathrm{Prob}_t\left[S_{(t,t+\Delta t]} \neq \emptyset\right] = \left(\theta^H\lambda^H_t + \theta^L\lambda^L_t\right)\Delta t = \theta^e_t\Delta t, \tag{10}$$

which is consistent with (6). From Bayes' law dividing (9) by (10) gives the updated subjective probability that $\theta_{t+\Delta t} = \theta^i$ under the condition that the shock occurs during $(t, t+\Delta t]$. It is

$$\lambda^i_t = \lim_{t'\to t-}\frac{\theta^i\lambda^i_{t'}}{\theta^e_{t'}} \equiv \frac{\theta^i\lambda^i_{t-}}{\theta^e_{t-}}, \tag{11}$$

where subscript $t-$ represents the state just before t.[8] Finally, we obtain the dynamics of subjective probability θ^e_t. From (3) and (7),

$$\lambda^H_t = \frac{\theta^e_t-\theta^L}{\theta^H-\theta^L}, \quad \lambda^L_t = \frac{\theta^H-\theta^e_t}{\theta^H-\theta^L}. \tag{12}$$

Substituting (8) and (12) into the time derivative of (7) yields the time derivative of θ^e_t in the case where the shock does not occur at time t,

$$\dot{\theta}^e_t = (\theta^e_t-\theta^L-p^L)(\theta^e_t-\theta^H-p^H) - p^Lp^H \equiv g(\theta^e_t) \quad \text{for } t \notin S_{(0,\infty)}. \tag{13}$$

Under Assumption 1, function $g(\theta^e_t)$ has a U-shape, as illustrated in Figure 1(a). This function satisfies

$$\begin{aligned} &g(\theta) \lesseqgtr 0 \Longleftrightarrow \theta \gtreqless \theta^* \quad \text{for any } \theta \in \left[\theta^L, \theta^H\right], \text{ where} \\ &\theta^* \equiv \frac{\theta^L+\theta^H+p^L+p^H-\sqrt{(\theta^H+p^H-\theta^L-p^L)^2+4p^Lp^H}}{2} \in (\theta^L, \theta^H). \end{aligned} \tag{14}$$

[7] $\lambda^L_{t+\Delta t}$ is analogously obtained. From (3) it equals $1-\lambda^H_{t+\Delta t}$.

[8] Mathematically, θ_{t-} is the limit of θ_τ as τ approaches t from the left. θ_{t-} is different from θ_t when belief of the household changes discretely at time t. Similar notations are used, for example, in a textbook by Dockner *et al.* (2000, Chapter 8).

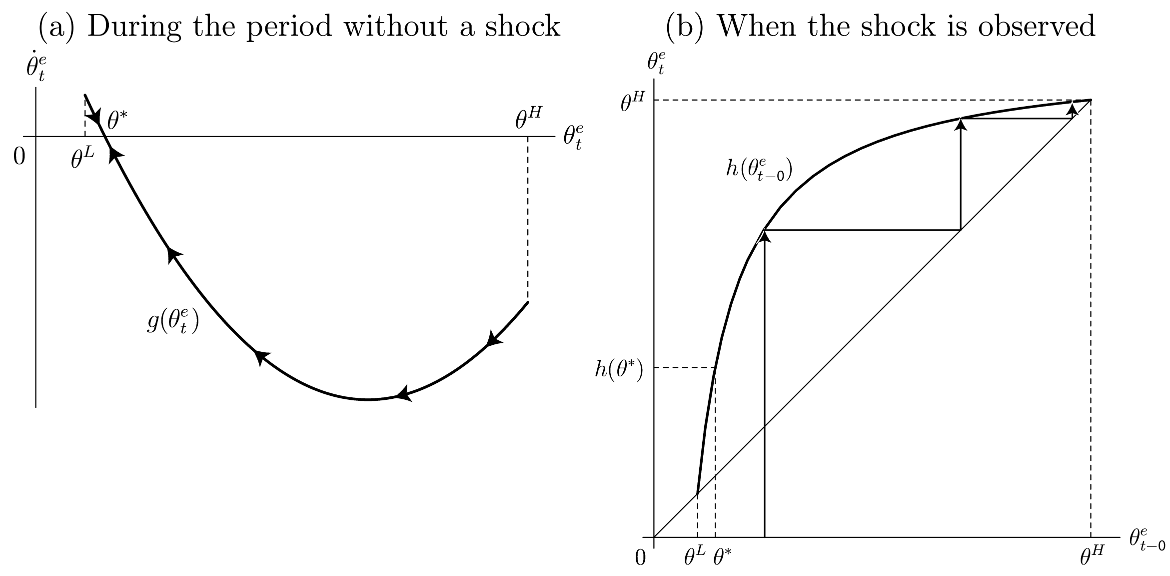

Figure 1: Movement of belief through Bayesian learning

Similarly, by substituting (11) and (12) into (7) we obtain the value of θ_t^e as a function of θ_{t-}^e in the case where a shock does occur at time t.

$$\theta_t^e = \theta^L + \theta^H - \frac{\theta^L \theta^H}{\theta_{t-}^e} \equiv h(\theta_{t-}^e) \quad \text{for } t \in S_{(0,\infty)}. \tag{15}$$

As shown in Figure 1(b), function $h(\theta)$ satisfies

$$h(\theta^H) = \theta^H, \text{ and } \theta^e < h(\theta^e) < \theta^H \text{ for all } \theta^e \in (\theta^L, \theta^H).$$

Equations (15) and (13) respectively describe the dynamics of θ_t^e with and without the shock. They jointly show that θ_t^e fluctuates within interval $\left(\theta^*, \theta^H\right]$. The liquidity shock is a rare event, and therefore causes a discrete change in people's expectation about the present state once it occurs. As function $h(\theta^e)$ is located above the 45-degree line in Figure 2, the more often people observe the shock, the more strongly they believe that they are in state H, and hence θ_t^e becomes closer to θ^H.

Conversely, in the absence of the shock people gradually become more and more optimistic and confident that the economy is in state L. Thus, their subjective probability of the shock gradually declines, converging to θ^*.[9] However, the U-shape of function $g(\theta_t^e)$ implies that the speed of adjusting belief is slow when θ_t^e is near θ^H. Note that $\theta_t^e \approx \theta^H$ is equivalent to $\lambda_t^H \approx 1$ from (12), which means that the precision

[9] θ_t^e never becomes lower than $\theta^*(> \theta^L)$ since people take into account the possibility that state L might have changed to state H even though the shock does not occur.

of the prior belief is quite high (i.e., people are quite sure that the current state is H). In that case any additional information has only a small impact on the posterior belief.

4 Information Cycles under Perfectly Flexible Prices

Let us examine how the market price evolves when households update their belief in the way explained in the previous section. In this section, we assume that price level p_t can be adjusted instantly so that $n_t = 1$ holds for all t. Note that there is no steady state in equilibrium at which the price level stays constant for all t because the decisions of household depend on θ^e_t, which is not constant. Thus, we instead search for a stationary relationship between θ^e_t and p_t. Specifically, we search for a function $p(\cdot)$ that satisfies[10]

$$p_t = p(\theta^e_t) \quad \text{for all } t. \tag{16}$$

Since we are interested in a monetary equilibrium path in which money has a positive value, we limit our attention to the path of equilibrium price that become neither zero or infinity:[11]

Assumption 2 *$p(\theta^e) \in (0, \infty)$ for all $\theta^e \in (\theta^*, \theta^H)$.*

If the price level is a function of θ^e_t holds, the inflation rate can also be written as a function of θ^e_t. From (16) and (13),

$$\pi_t \equiv \dot{p}_t / p_t = \frac{p'(\theta^e_t)}{p(\theta^e_t)} g(\theta^e_t) \equiv \pi(\theta^e_t) \quad \text{for } t \notin S_{(0,\infty)}. \tag{17}$$

However, recall that the belief θ^e_t jumps when a shock is observed. In that case, p_t may also jump. The following gives the ratio of the price level between before and after the shock.

$$\Pi_t \equiv p_t / p_{t-} = \frac{p(h(\theta^e_t))}{p(\theta^e_t)} \equiv \Pi(\theta^e_t) \quad \text{for } t \in S_{(0,\infty)}. \tag{18}$$

[10] This approach is similar to Lucas (1978).

[11] To see why this assumption is reasonable, suppose that $p_{t_0} = \infty$ for some date t_0, which means that money has no value at t_0. Then, it follows that $p_t = \infty$ for all $t \geq t_0$ since otherwise an arbitrage opportunity arises: consumers can obtain an arbitrary amount of money at date t_0 at no cost and then sell money (i.e., purchase goods) at a date in which p_t is finite to increase their expected utility. Since θ^e_t evolves within (θ^*, θ^H) recurrently, (16) implies that if $p(\theta^\infty) = \infty$ for some $\theta^\infty \in (\theta^*, \theta^H)$ then $p(\theta^e) = \infty$ for all $\theta^e \in (\theta^*, \theta^H)$. That is, if there is such θ^∞, then $p_t = \infty$ for all t and therefore money is never demanded. We also rule out the possibility that $p(\theta^e) = 0$ for some $\theta^e \in (\theta^*, \theta^H)$ because the value of consumption good never becomes zero from $u'(\cdot) > 0$.

Our task is to find a function $p(\theta_t^e)$ (and therefore also $\pi(\theta_t^e)$ and $\Pi(\theta_t^e)$) such that, given these, the household's optimization leads to the clearance of all markets. In the following, we will proceed in three steps: (i) We consider a dynamic programming problem and obtain a Hamilton-Jacobi-Bellman (HJB) equation, given $\pi(\theta_t^e)$ and $\Pi(\theta_t^e)$. (ii) We obtain the Euler equation from the first order and envelope conditions of the HJB equation. (iii) We substitute the equilibrium conditions to the Euler equation and examine the properties that must be satisfied by function $p(\theta_t^e)$.

When the inflation rate follows (17)-(18) and $n_t = 1$ holds, budget constraint (2) can be written as

$$\dot{m}_t = y - \pi(\theta_t^e)m_t - c_t \quad \text{for } t \notin S_{(0,\infty)}, \tag{19}$$

$$m_t = m_{t-}/\Pi(\theta_t^e) \quad \text{for } t \in S_{(0,\infty)}. \tag{20}$$

The household maximizes the expected utility (1) subject to (19) and (20), and also to the law of motion of thier belief (13) and (15). Let $U(\theta^e, m)$ denote the maximized value when the current belief and real money holding are θ^e and m. By considering a small time interval Δt, the Bellman equation for this problem can be written as

$$\begin{aligned} U(\theta^e, m) = \max_c \Big[& u(c)\Delta t + (\theta^e \Delta t)v(m'') \\ & + \frac{1}{1+\rho\Delta t}\left\{(1-\theta^e\Delta t)U(\theta^{e\prime}, m') + (\theta^e\Delta t)U(h(\theta), m'')\right\}\Big], \end{aligned} \tag{21}$$

where $\theta^{e\prime} = \theta^e + g(\theta^e)\Delta t$, $m' = m + (y - \pi(\theta^e)m - c)\Delta t$, and $m'' = m/\Pi(\theta^e)$. Observe that with probability $1 - \theta^e\Delta t$ there is no shock and the state changes from (θ^e, m) to $(\theta^{e\prime}, m')$, whereas with probability $\theta^e\Delta t$ there is a shock and the state changes to $(h(\theta^e), m'')$. Taking the limit $\Delta t \to 0$ in (21) yields the Hamilton-Jacobi-Bellman (HJB) equation for the problem:

$$\begin{aligned} \rho U(\theta^e, m) = \max_c \Big[& u(c) + \theta^e\big(v(m/\Pi(\theta^e)) + U(h(\theta^e), m/\Pi(\theta^e)) - U(\theta^e, m)\big) \\ & + g(\theta^e)U_\theta(\theta^e, m) + \big(y - \pi(\theta^e)m - c\big)U_m(\theta^e, m)\Big]. \end{aligned} \tag{22}$$

Differentiating the right hand side of (22) with respect to c gives the first order condition

$$u'(\widetilde{c}) = U_m(\theta^e, m), \tag{23}$$

where $\widetilde{c}$ denotes the optimal amount of consumption. Since θ_t^e and m_t evolves according to (13) and (19) during the period of no shock, equation (23) shows that the movement of consumption is characterized by

$$\frac{d}{dt}u'(\widetilde{c}_t) = g(\theta_t^e)U_{m\theta} + \big(y - \pi(\theta^e)m - \widetilde{c}\big)U_{mm} \quad \text{for } t \notin S_{(0,\infty)}, \tag{24}$$

abbreviating the arguments for $U(\cdot,\cdot)$ functions when they are (θ^e, m). From the envelope theorem, (22) can be differentiated with respect to m at $c = \widetilde{c}$ to give

$$\begin{aligned}(\rho + \pi(\theta^e) + \theta^e)U_m =& g(\theta^e)U_{\theta m} + \big(y - \pi(\theta^e)m - \widetilde{c}\big)U_{mm} \\ &+ \theta^e \Pi(\theta^e)^{-1}\big(v'(m/\Pi(\theta^e) + U_m(h(\theta^e), m/\Pi(\theta^e))\big).\end{aligned} \tag{25}$$

By substituting (23) and (24) for (25), we can eliminate the value function from it to obtain the Euler equation,

$$\frac{d}{dt}u'(\widetilde{c}_t) = (\rho + \pi(\theta^e) + \theta^e)u'(\widetilde{c}_t) - \theta^e \frac{v'(m_t/\Pi(\theta^e)) + u'(\widetilde{c}''_t)}{\Pi(\theta^e)} \quad \text{for } t \notin S_{(0,\infty)}, \tag{26}$$

where $\widetilde{c}''_t$ represents the optimal amount of consumption when a shock is observed and the state changes to $(h(\theta^e_t), m/\Pi(\theta^e_t))$.

Since all households are symmetric, the equilibrium of goods and money markets implies

$$\widetilde{c}_t (= \widetilde{c}''_t) = y, \;\; m_t = p(\theta^e_t)^{-1} \quad \text{for all } t. \tag{27}$$

Function $p(\cdot)$ is determined so that the household's demand for goods and money always satisfies (27). Substituting (27) into (26) yields a condition that must be satisfied for all possible values of θ^e,

$$\rho + \pi(\theta^e) = \theta^e \Pi(\theta^e)^{-1} v'(p(h(\theta^e))^{-1}) + \theta^e \left(\Pi(\theta^e)^{-1} - 1\right). \tag{28}$$

The left hand side represents the cost of holding money: the utility loss from postponing consumption plus the capital loss caused by inflation. In the other side are the expected benefits of holding money: the first term is the expected utility from holding money, whereas the second term represents the expected capital gain by the downward jump in the price level (the upward jump in the value of money) when the liquidity shock occurs. Thus, (28) shows that function $p(\cdot)$ is determined so that the cost and the benefit of holding money are equalized with each other.

From (17), (18) and (28), we obtain a (delay) differential equation for $p(\cdot)$:

$$\begin{aligned} p'(\theta^e) &= \frac{p(\theta^e)}{g(\theta^e)}\pi(\theta^e), \;\; \text{where} \\ \pi(\theta^e) &\equiv -(\rho + \theta^e) + \theta^e \frac{p(\theta^e)}{p(h(\theta^e))} \frac{v'(p(h(\theta^e))^{-1}) + u'(y)}{u'(y)}. \end{aligned} \tag{29}$$

Since functions u, v, g, h are already known, (29) is an autonomous differential equation with respect to function $p(\cdot)$. The following lemma gives a boundary condition with which function $p(\cdot)$ is pinned down.

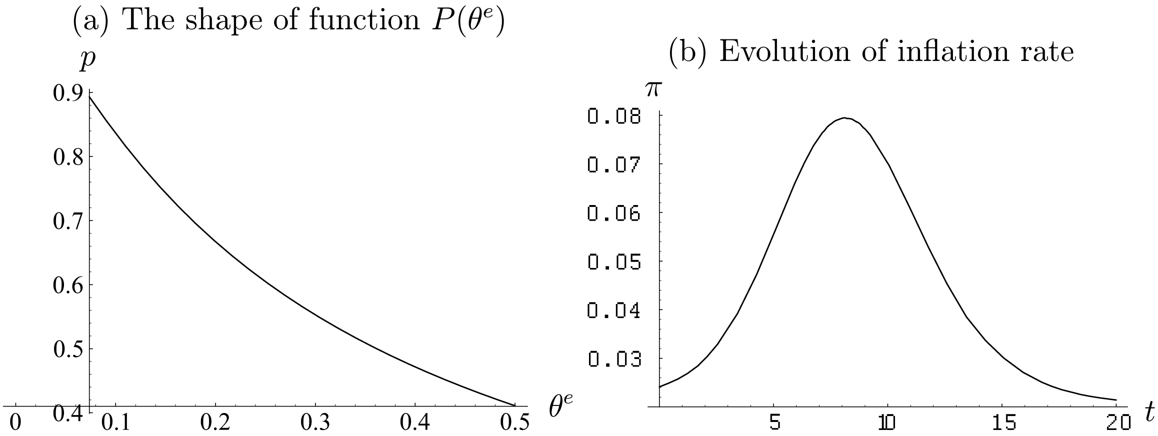

Figure 2: Inflation cycles without nominal frictions

Lemma 1 *Under Assumption 2 and transversality condition*[12]

$$\lim_{T\to\infty} E_t e^{-\rho(T-t)} u'(c_T) m_T = 0 \quad \textit{for all } t, \tag{30}$$

function $\pi(\cdot)$ *must satisfy* $\lim_{\theta^e \to \theta^*} \pi(\theta^e) = 0$.
proof: in appendix

Intuitively, the inflation rate at the limit ($\theta_t^e \to \theta^*$) must be must be equalized to the growth rate of nominal money supply, which is zero in our model. Note that there exists a non-zero possibility that the liquidity shock does not occur for an arbitrary long time. In that case, if $\lim_{\theta^e \to \theta^*} \pi(\theta^e) \neq 0$, the real money holding $m_t = p_t^{-1}$ diverges to either infinity (violating the transversality condition) or to zero (violating the assumption of the monetary equilibrium).

The stationary dynamics of a monetary equilibrium can be calculated from (29) and the boundary condition given by Lemma 1. Figure 2(a) shows the representative shape of function $p(\cdot)$ against θ^e, which is downward sloping.[13] A large value of θ_t^e means that people anticipates that the liquidity shock occurs with a high probability. In that situation, the marginal benefit of holding money is high. Thus, to clear the market for money, the value of money must be sufficiently high in relative to the value of good, which means a low price level.

During the period without the liquidity shock, θ_t^e gradually declines and p_t increases. Figure 2(b) shows the evolution of inflation rate against time as θ_t^e moves

[12] Operator E_t represents the expectation based on the information available to agents at date t.

[13] In all examples presented in this paper, we specify $u(c) = \ln c$, $v(m) = -m^{-1}$, $y = 1$, $\rho = .05$, $\theta^H = 1$, $\theta^L = .2$, $p^H = .05$ and $p^L = .02$. We have confirmed that our results are robust to changes in parameter values.

from θ^H to θ^*. Inflation accelerates temporarily when the households adjusts their belief responding to observing no shock for a certain time length, but it gradually falls to the rate of nominal money growth, which is zero in this case, as the economy converges to the most optimistic state. When the liquidity shock occurs, θ_t^e jumps up. Then p_t jumps down so that the (θ_t^e, p_t) pair is always on the curve depicted in panel (a). Thus, the dynamics of the economy is characterized by gradual inflation with sporadic and discrete falls in the price level.

At each event of the liquidity shock, price level must jump down in order to clear the increased liquidity demand induced by the change in people's belief. However, we rarely observe such a discrete fall in the price level in the aggregate economy; although we do sometimes observe a discrete fall in the prices of certain goods, the aggregated general price level tends to fall only slowly. One explanation for this is the existence of a (downward) nominal stickiness in the price level caused by staggered price adjustments, menu costs, labor unions, moral issues, and the all other factors discussed in the literature. If the price cannot jump downward, our model predicts that the demand for money exceeds the supply, and, by Warlas' law, a demand shortage occurs in the goods and labor market. The next section investigates this possibility.

5 Possibility of Depression under Sticky Prices

The discrete fall in the price level derived in the previous section implies that the instantaneous rate of inflation must be minus infinity. This section considers a more realistic setting where the price level cannot fall infinitely fast, or equivalently, where the rate of deflation is restricted to be within some finite bound.[14] Let us consider a model similar to the one analyzed in the previous section, with a only difference in

[14]This is equivalent to assuming that the nominal wage cannot fall infinitely fast because the production technology is linear and firms are competitive.

that the price level cannot fall faster than a certain rate,[15]

$$\dot{p}_t/p_t \geq -\delta, \quad \delta \in (0, \infty). \tag{31}$$

Note that condition (31) breaks the one-for-one relationship between the price level and the belief because p_t cannot jump while θ_t^e can. Thus, the state of the economy cannot be described solely by θ_t^e; but by the pair of (θ_t^e, p_t). This economy has two possibilities at each point in time. The first possibility is that constraint (31) is not binding and full employment obtains ($n_t = 1$ and $c_t = y$). The second possibility is that (31) is binding, i.e., $\dot{p}_t/p_t = -\delta$, and unemployment exists ($n_t < 1$ and $c_t < y$). Which one of these possibilities occurs depends on the state of the economy, summarized by (θ_t^e, p_t).

It is natural to guess that, for a given level of θ_t^e, there is a level of p_t at which the money market clears and full employment obtains. Let us denote this critical level by $\overline{p}(\theta_t^e)$. Price level p_t cannot be below the threshold $\overline{p}(\theta^e)$ since there is no upward stickiness in the price level and thus can be adjusted instantly if $p_t < \overline{p}(\theta^e)$. Similarly to the previous section, we limit our attention to the monetary equilibrium path by assuming that[16]

Assumption 3 $\overline{p}(\theta^e) \in (0, \infty)$ *for all* $\theta^e \in (\theta^*, \theta^H)$.

Unemployment occurs when constraint (31) is binding, i.e., when $p > \overline{p}(\theta^e)$. In this case, the economy experience deflation at the rate of δ. If (31) is not binding, full employment obtains and the price level evolves so that equilibrium condition $p = \overline{p}(\theta^e)$ is maintained. Let us denote by $C(\theta^e, p)$ aggregate demand for goods at state (θ^e, p). Then,

$$C(\theta^e, p) \begin{cases} = y & \text{if } p = \overline{p}(\theta^e), \\ < y & \text{if } p > \overline{p}(\theta^e). \end{cases} \tag{32}$$

[15] To keep the analysis to follow as tractable as possible, we employ a quite simple specification for the sticky price in condition (31). This assumption is motivated by the experience in Japan, where the rate of deflation remained at a few percentage points for nearly ten years after mid 1990s. When we explicitly model a staggered pricing behavior by monopolistically competing firms, the rate of deflation would differ depending on the state of economy. Nonetheless, the most of the main implications will not change because the most crucial assumption is that the price cannot jump. We could also assume a symmetric restriction such as $\dot{p}_t/p_t \in [-\delta, \delta]$. This would make the analysis a little complicated without changing the final results.

[16] We can show that if there is some $\theta^\infty \in (\theta^*, \theta^H)$ such that $\overline{p}(\theta^\infty) = \infty$ then $p_t = \infty$ for all t.

The inflation rate for a given state can be summarized as

$$\pi(\theta^e, p) = \begin{cases} \dfrac{\overline{p}'(\theta^e)}{\overline{p}(\theta^e)} g(\theta^e) & \text{if } p = \overline{p}(\theta^e), \\ -\delta & \text{if } p > \overline{p}(\theta^e). \end{cases} \tag{33}$$

The representative household maximize (1) under budget constraint (2). Since the demand for goods is $C(\theta^e_t, p_t)$ and the production function is yn_t, the amount of employment is determied as $n_t = C(\theta^e_t, p_t)/y$. The budget constraint can thus be written as

$$\dot{m}_t = C(\theta^e_t, p_t) - \pi(\theta^e_t, p_t)m_t + c_t \tag{34}$$

as long as p_t evolves continuously, and as (20) if p_t jumps. Note that, from (31), price level p_t never jumps down. At this point, however, we cannot rule out an upward jump in p_t, which may occur if current price level p_t is smaller than the new market clearing price level after the shock, $\overline{p}(h(\theta^e_t))$. Let us denote the value function of the household by $U(\theta^e, p, m)$, which now depends on the current value of p because it affects the aggregate demand and thus the household's income. The Bellman equation for this problem is

$$\begin{aligned} U(\theta^e, p, m) = \max_c \Big[& u(c)\Delta t + (\theta^e \Delta t) v(m'') \\ & + \frac{1}{1 + \rho\Delta t} \left\{ (1 - \theta^e \Delta t) U(\theta^{e\prime}, p', m') + (\theta^e \Delta t) U(h(\theta^e), p'', m'') \right\} \Big], \end{aligned} \tag{35}$$

where $\theta^{e\prime} = \theta^e + g(\theta^e)\Delta t$, $p' = p + \pi(\theta^e, p)p\Delta t$, $m' = m + (C(\theta^e, p) - \pi(\theta^e, p)m + c)\Delta t$, $p'' = \max\{\overline{p}(h(\theta)), p\}$, and $m'' = (p/p'')m$. Taking the limit of $\Delta t \to 0$ in (35) yields the HJB equation,

$$\begin{aligned} \rho U = \max_c \Big[& u(c) + \theta^e \big(v(m'') + U(h(\theta^e), p'', m'') - U \big) + g(\theta^e) U_\theta \\ & + \pi(\theta^e, p) p U_p + \big(C(\theta^e, p) - \pi(\theta^e, p)m - c \big) U_m \Big], \end{aligned} \tag{36}$$

where the arguments of function $U(\cdot, \cdot, \cdot)$ and its partial derivatives are abbreviated when they are (θ^e, p, m). The first order condition for (36) is $u'(\widetilde{c}) = U_m(\theta^e, p, m)$, where $\widetilde{c}$ is the optimal amount of consumption. Then, the envelope condition is

$$\begin{aligned} (\rho + \pi(\theta^e, p) + \theta^e) U_m = & \theta^e (p/p'') \big(v'(m'') + U_m(h(\theta^e), p'', m'') \big) + g(\theta^e) U_{\theta m} \\ & + \pi(\theta^e, p) p U_{pm} + \big(C(\theta, p) - \pi(\theta^e, p)m - \widetilde{c} \big) U_{mm}. \end{aligned} \tag{37}$$

Note that the RHS of (37) depends on whether p jumps or not in the event of the liquidity shock. As long as function $\overline{p}(\cdot)$ is weakly downward sloping, $p \geq \overline{p}(\theta^e) \geq$

$\overline{p}(h(\theta^e))$ and therefore $p'' = p$ and $m'' = m$. The following analysis focuses on this case and we leave for Appendix the analysis of the case of $p < \overline{p}(h(\theta^e))$.

Substituting the first order condition, its time derivative, and the conditions for the representative household, $\widetilde{c} = C(\theta^e, p)$ and $m = p^{-1}$, into (37) yields the Euler equation,

$$\left(\rho - \frac{\frac{d}{dt}u'(C(\theta^e,p))}{u'(C(\theta^e,p))}\right) + \pi(\theta^e,p) = \theta^e \frac{v'(p^{-1})}{u'(C(\theta^e,p))} + \theta^e \left(\frac{u'(C(h(\theta^e),p))}{u'(C(\theta^e,p))} - 1\right) \quad (38)$$

for all $t \notin S_{(0,\infty)}$. Equation (38) has an interpretation similar to (28). The cost of holding money, given by the LHS, is the sum of time preference and inflation. The benefit is the sum of the direct utility gain and the expected capital gain measured in terms of utility when a shock occurs and consumption jumps down.

Functions $\overline{p}(\cdot)$ and $C(\cdot,\cdot)$ are determined so that equation (38) holds for all possible pairs of (θ^e, p). Let us first consider the case in which current price p is at the market clearing level $\overline{p}(\theta^e)$. Recall that $p = \overline{p}(\theta^e)$ implies $C(\theta^e, p) = y$ and $\pi(\theta^e, p) = \overline{p}'(\theta^e)g(\theta^e)/\overline{p}(\theta^e)$ from (32) and (33). Substituting these for (38) gives a differential equation that determines the form of function $\overline{p}(\cdot)$:[17]

$$\begin{aligned} &\overline{p}'(\theta^e) = \frac{\overline{p}(\theta^e)}{g(\theta)}\gamma_{\overline{p}}(\theta^e), \quad \text{where} \\ &\gamma_{\overline{p}}(\theta^e) = -(\rho+\theta^e) + \theta^e \frac{v'\left(\overline{p}(\theta^e)^{-1}\right) + u'(C(h(\theta^e),\overline{p}(\theta^e)))}{u'(y)}. \end{aligned} \quad (39)$$

Function $\gamma_{\overline{p}}(\theta^e)$ in (39) represents the growth (inflation) rate of the market clearing price, $\dot{\overline{p}}_t/\overline{p}_t$. The difference between (29) and (39) lies in the fact that consumption is adjusted in the occurrence of the liquidity shock when nominal stickiness exists, while adjustment is done fully by the price level when price is completely flexible. A boundary condition for equation (39) is given by the following lemma.

Lemma 2 *Under Assumption 3 and transversality condition (30), function* $\gamma_{\overline{p}}(\cdot)$ *must satisfy* $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) = 0$
proof: in appendix

Next, consider the case in which current price p is above the market clearing level $\overline{p}(\theta^e)$. In this case, $p > \overline{p}(\theta^e)$ implies $C(\theta^e, p) < y$ and $\pi(\theta^e, p) = -\delta$. Substituting

[17] Equation (39) holds when $p \geq \overline{p}(h(\theta^e))$. The corresponding expression for $\gamma_{\overline{p}}(\theta^e)$ when $p < \overline{p}(h(\theta^e))$ is given by (44) in Appendix A.

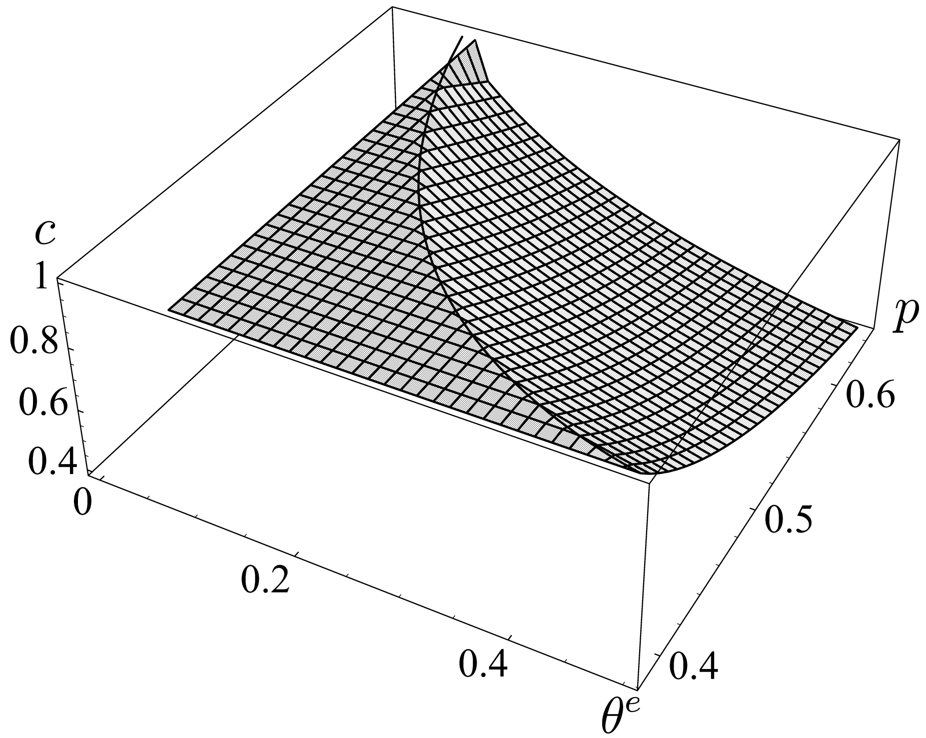

Figure 3: Representative shapes of function $C(\theta^e, p)$ and function $\overline{p}(\theta^e)$

this for (38) gives a partial (delay) differential equation for function $C(\cdot, \cdot)$,

$$\begin{aligned} g(\theta^e)C_\theta(\theta^e, p) - p\delta C_p(\theta^e, p) &= \frac{u'(C(\theta^e, p))}{u''(C(\theta^e, p))}\gamma_{u'}(\theta^e, p), \text{ where} \\ \gamma_{u'}(\theta^e, p) &= \rho - \delta + \theta^e - \theta^e \frac{v'(p^{-1}) + u'(C(h(\theta^e), p))}{u'(C(\theta^e, p))}. \end{aligned} \tag{40}$$

In (40), $\gamma_{u'}(\theta^e, p)$ represents the rate of change in marginal utility, $\dot{u}'/u'$. Combined with the boundary condition $C(\theta^e, \overline{p}(\theta^e)) = y$ for all θ^e, this partial (delay) differential equation determines the shape of function $C(\cdot, \cdot)$ for all $(\theta^e, p) \in \{(\theta^e, p)|p > \overline{p}(\theta^e))\}$.

5.1 Numerical Analysis

Since $\overline{p}(\cdot)$ and $C(\cdot, \cdot)$ are interrelated as described above, they are determined simultaneously so that they satisfy the system of partial differential equations, (39) and (40), along with two boundary conditions specified above. This problem can be solved numerically by combining a finite difference method and an appropraite iteration method.[18] Figure 3 shows a representative shape of function $C(\cdot, \cdot)$ in

[18] The details of the simulation procedure are availble upon request.

(a) phase diagram in (θ^e, p) space (b) phase diagram in (c, p) space

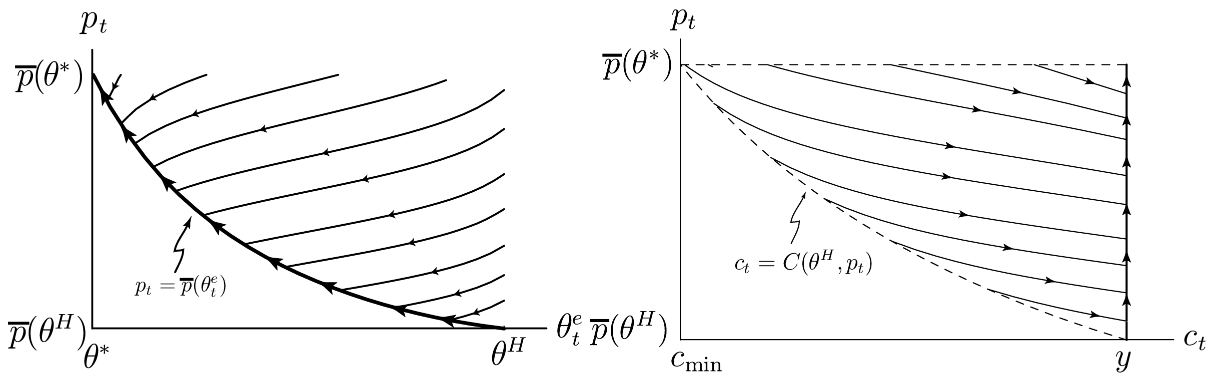

Figure 4: Evolution of the economy during the period of no shock.

(θ^e, p) space, where the solid curve on the edge represents function $\overline{p}(\cdot)$. Observe that function $\overline{p}(\cdot)$ is downward sloping in θ^e. The reason behind it is the same as the reason for the similar property of $p(\cdot)$ in the previous section: when θ^e is large, people's liquidity preference is high and thus a low price level (a high relative price of money to goods) is required to equalize the money demand to the money supply. The height of curved surface indicates the value of function $C(\theta^e, p)$ at each state in region $p > \overline{p}(\theta^e)$. $C(\theta^e, p)$ is equal to y on the curve of $p = \overline{p}(\theta^e)$ and gets smaller as the pair (θ^e, p) moves to the direction of north-east. That is, a pair of high θ_t^e and high p_t implies a combination of high liquidity preference, a low relative price of money to goods, and a small supply of real money stock. In that case, the excess demand for money is huge and therefore the aggregate demand for goods (and thus employment) is small.

Figure 4 illustrates the movement of the belief, price and consumption (output) during the period of no shock. For ease of visibility, we present two phase diagrams in (θ^e, p) spece and in (c, p) space. If the current price level p_t is higher than the market clearing price level $\overline{p}(\theta_t^e)$, the price gradually falls and consumption (output) grows according to

$$\frac{\dot{p}_t}{p_t} = -\delta, \quad \dot{c}_t = \frac{u'(c_t)}{u''(c_t)} \left[\rho - \delta + \theta^e - \theta_t^e \frac{v'\left(p_t^{-1}\right) + u'(c_t'')}{u'(c_t)} \right], \tag{41}$$

where $c_t'' \equiv C(h(\theta^e), p_t)$.[19] As long as no shock occurs, the pair (θ^e, p_t) follows (41) until it reaches the market clearing curve $p_t = \overline{p}(\theta_t^e)$ in a finite time. From that time

[19]The second equation in (41) is obtained by solving $\frac{d}{dt} u'(c_t)/u'(c_t) = \gamma_{u'}(\theta^e, p)$, where $\gamma_{u'}(\theta^e, p)$ is given by (40). Similarly, the first equation in (42) is given by $\gamma_{\overline{p}}(\theta^e, p)$ in (39).

on, consumption stays constant and the price level rises so that the pair traces the market clearing curve,

$$\frac{\dot{p}_t}{p_t} = -(\rho + \theta_t^e) + \theta_t^e \frac{v'(p_t^{-1}) + u'(c_t'')}{u'(y)}, \quad c_t = y. \tag{42}$$

As θ^e approaches θ^*, the price level converges to $p^* \equiv \overline{p}(\theta^*)$, and inflation rate converges to zero.

Now let us explain how the economy reacts to the liquidity shocks. In this model, the reaction is both qualitatively and quantitatively different depending on the history and the way the economy is hit by the shocks. To illustrate this point, the following consider three different examples. Figure 5(a) presents the result from a standard impulse-response exercise. In this exercise, it is assumed that there have been no shock for a long time and then the economy is hit by a one-shot liquidity shock. Before the shock, the household has the lowest possible belief $\theta^e = \theta^*$ and the price level is at the highest, $p_t = \overline{p}(\theta^*)$. At the time of the liquidity shock, the household updates its belief θ^e to $h(\theta^*)$, but p_t cannot jump immediately. Thus, as illustrated in the left panel, the pair (θ^e, p_t) jumps horizontally toward east. This means that the pair is now above the market clearing line ($p_t = \overline{p}(\theta_t^e)$), and therefore consumption and output must fall discretely (see the right panel). After the shock, both θ^e and p_t gradually falls through learning and deflation. This means that the pair (θ^e, p_t) approaches the market clearing curve, and accordingly consumption and output recover toward the initial level.

As a second example, Figure 5(b) illustrates a situation where the economy is hit regularly by shocks. Observe that, when compared to the case where the shock occurs only once, consumption falls only slightly each time the economy is hit by a shock, and that the recovery after the fall is fairly quick. There are two reasons behind this counter-intuitive result. When shocks occur regularly, the belief of the household is always near the highest level, θ^H. That is, the household believes almost surely that the economy is in the dangerous state (state H), and will not change its belief much when another shock is observed. In addition, the price level is already adjusted to this belief and near the lowest level $\overline{p}(\theta^H)$. Thus, even when θ_t^e jumps, the price need not fall significantly. As a result, the recovery process is quick.

Figure 5(c) displays the worst possibility as the third example. Similarly to example (a), we assume that there have been no shock for a long time before the economy is hit by a shock (this implies that the economy is initially at $\theta^e = \theta^*$ and $p_t = \overline{p}(\theta^*)$). However, in this example, the shock is not one-shot, but comes in a bunch for a short while. By observing the shocks, the belief θ_t^e jumps up again and

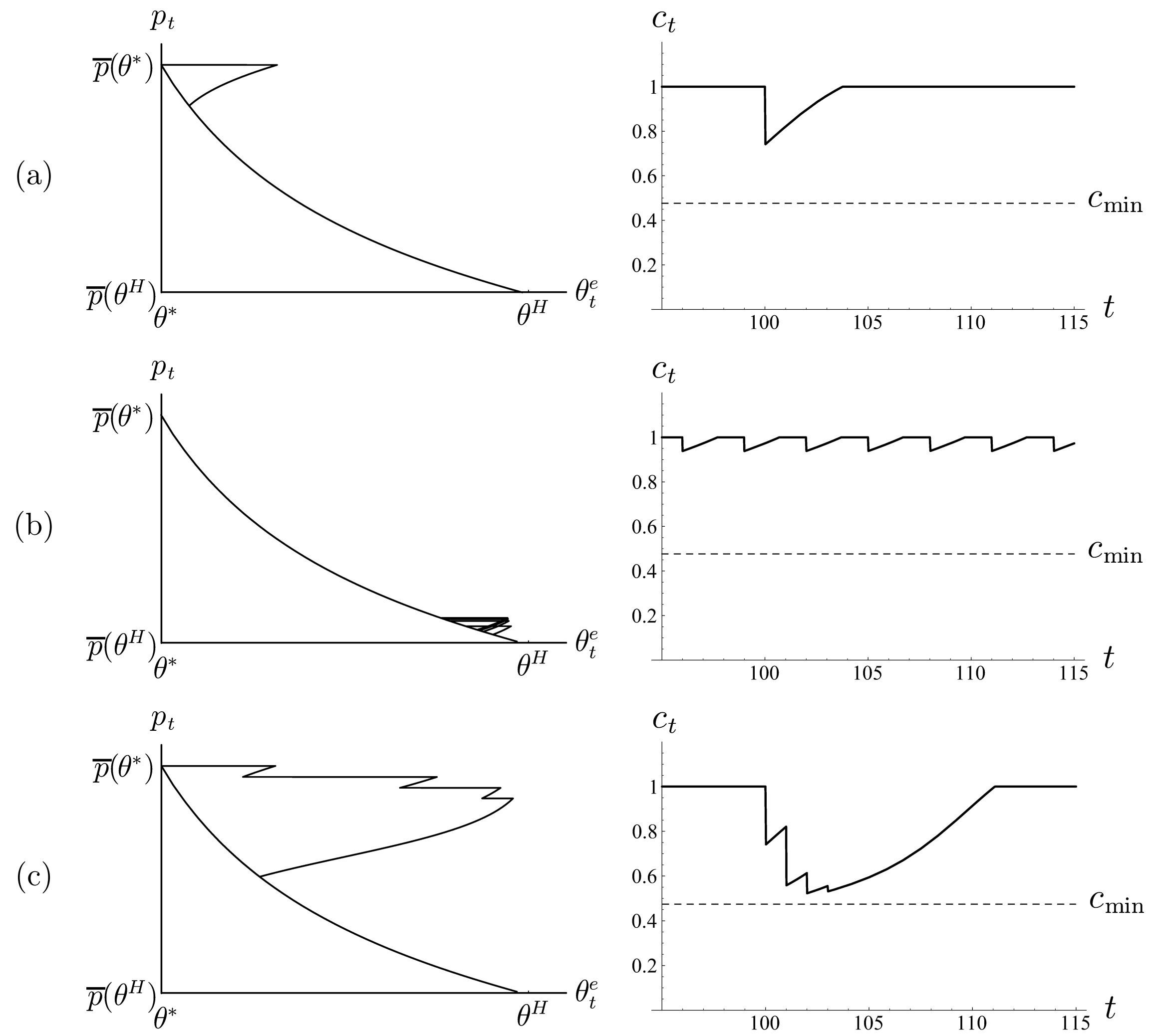

Figure 5: Reaction of the economy to liquidity shocks: three examples

again nearly to the highest level θ^H, while giving little time for the price level to adjust through deflation. As a result, the pair (θ^e, p_t) moves to the furthiest position from the market clearling curve, and consumption (outout) falls nearly to the lowest possible value ($c_{\min}$). In addition, even after the shock ceases, the recovery process is slow; it can be seen from the figure that the time path of c_t is convex in the phase of recovery. This is because once the household hold a strong belief that the underlying state is bad (i.e., $\theta_t^e \approx \theta^H$), the belief cannot be easily overturned by the additional information that no shock is observed for a while.[20] In other words, once agents become too pessimistic, it takes a long time until the recovery process accerelates.

[20]Recall the discussion in the end of section 3.

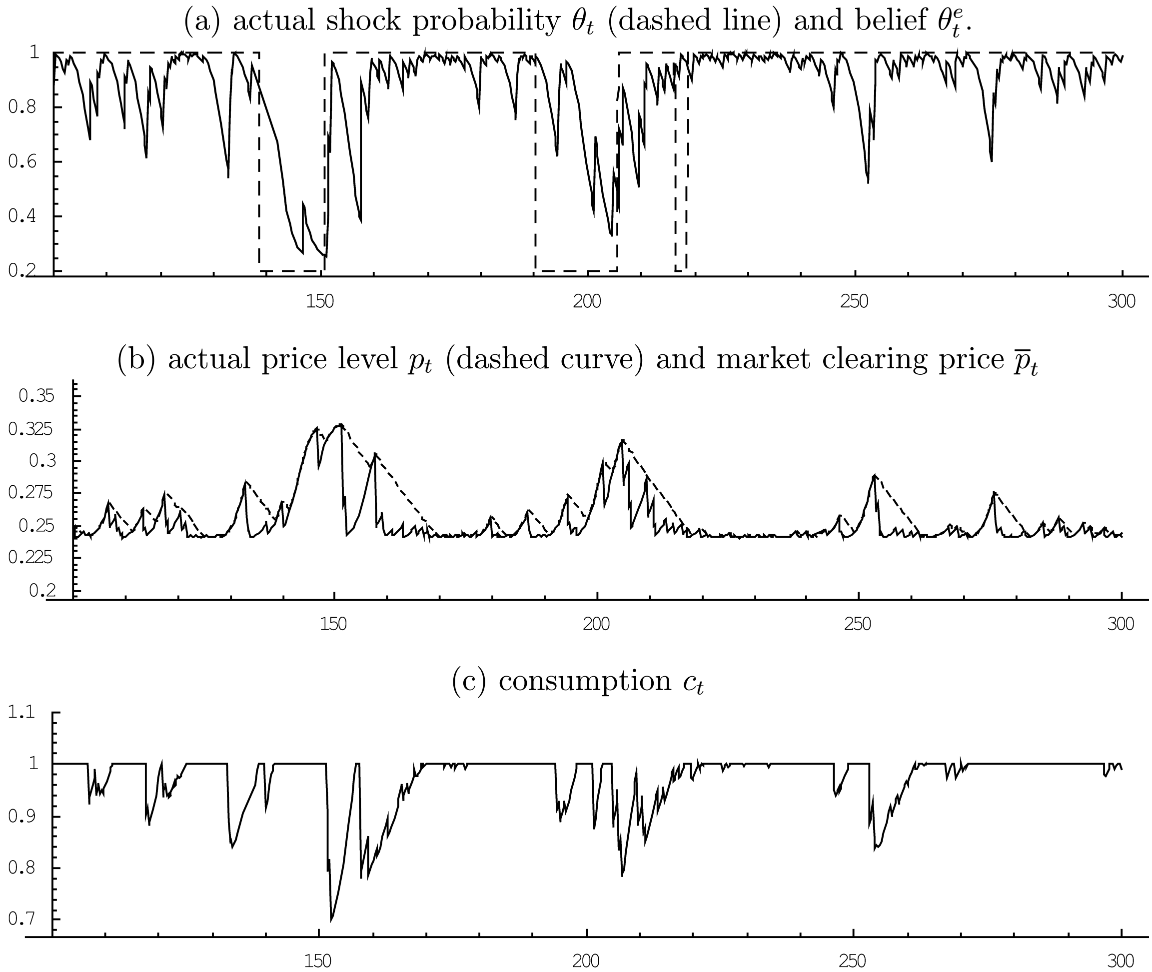

Figure 6: A simulated time path when shock are generated by Markov modulated Poisson process

This pattern of recovery is in contrast to example (a), where the shock is observed only once and may be viewed as a “mere accidient.” (Observe that the time path of c_t during the recovery process is actually concave in example (a)).

So far, we considered three examples where the shocks occur in specific patterns. However, in the actual model economy, the shocks are randomly generated by the Markov modulated Poisson process, as explained in Section 3. Figure 6 which depicts a simulated time path of the economy when shocks are randomly generated. Observe that the price level goes up when the economy stay in the safer fundamental state (state L) for some time. This is in fact dangerous because once the state switches to state H, where many shocks are likely to occur, the market clearing price ($\overline{p}_t$) jumps down to a lower level, but the actual price (p_t) can only fall gradually. This creates a large discrepancy between $\overline{p}_t$ and p_t, which translates into a period of stagnation where consumption c_t and thus output stay below the normal level. This mechanism provides one possible reasion why once a long-time (seemingly) stable economy experiences negative shocks it has to go through a long and deflationary period of depression.

6 Conclusion

This paper presents a theory of economic fluctuations and prolonged depression based on households' learning behavior. We consider a stochasic version of Money-in-Utility model, where agents recieve utility from money only when liquidity shocks occur, where the true shock probability unobservably changes between high and low. In this setting, we first exmained the way the households update thier belief through Bayesian learning. Second, using Hamilton-Jacobi-Bellman equations, we investigated the evolutions of consumption and money holding of the household who behaves rationally based on his belief about the state of the economy. Third, we derived a stationary cycle under perfect price adjustments in terms of a delay differential equation and demonstrated that the price level would experience sporadic downward jumps in such a setting. Forth, we extended the model to incorporate nominal stickiness. In this case, the belief and the slow-moving price cannot correspond one-to-one, and this discrepancy creates a period of stagnation. The stationary dynamics is given as a solution to a system of partial delay differential equations, which we solved numerically. It is shown that the reaction of the economy to negative shocks depends on the history and the pattern of the realization of the shocks. In particular, a successive occurrence of shocks may cause a depression if the economy has enjoyed a long period of stability before encountering the shocks.

Appendix

A Analysis of the case of $p < \overline{p}(h(\theta^e))$

If $p_t < \overline{p}(h(\theta^e_t))$, $p'' = \overline{p}(h(\theta^e_T))$ and $m'' = mp/\overline{p}(h(\theta^e_t))$ in (35), (36) and (37). Substituting the first order condition, its time derivative, and the equilibrium conditions,

$\widetilde{c} = C(\theta^e, p)$ and $m = p^{-1}$, into (37) yields the Euler equation,

$$\begin{aligned}
&\left(\rho - \frac{\frac{d}{dt}u'(C(\theta^e,p))}{u'(C(\theta^e,p))}\right) + \pi(\theta^e,p) \\
&\quad = \theta^e \frac{p}{\overline{p}(h(\theta^e))} \frac{v'(\overline{p}(h(\theta^e))^{-1})}{u'(C(\theta^e,p))} + \theta^e \left(\frac{p}{\overline{p}(h(\theta^e))} \frac{u'(y)}{u'(C(\theta^e,p))} - 1\right)
\end{aligned} \tag{43}$$

for all $t \notin S_{(0,\infty)}$. Substituting $p = \overline{p}(\theta^e)$, $C(\theta^e, p) = y$ and $\pi(\theta^e, p) = \overline{p}'(\theta^e)g(\theta^e)/\overline{p}(\theta^e)$, from (32) and (33), for (43) gives the growth rate of p_t during the period of full employment:

$$\gamma_{\overline{p}}(\theta^e) = -(\rho + \theta^e) + \theta^e \frac{\overline{p}(\theta^e)}{\overline{p}(h(\theta^e))} \frac{v'\left(\overline{p}(h(\theta^e))^{-1}\right) + u'(y)}{u'(y)}. \tag{44}$$

When unemployment exists (i.e., $p \geq \overline{p}(\theta^e)$), the rate of change in marginal utility is obtained by substituting $\pi(\theta^e, p) = -\delta$ for (43),

$$\gamma_{u'}(\theta^e, p) = \rho - \delta + \theta^e + \theta^e \frac{p}{\overline{p}(h(\theta^e))} \frac{v'\left(p^{-1}\right) + u'(y)}{u'(C(\theta^e,p))}. \tag{45}$$

B Proof of Lemmas

Let $\theta_{t,T}^{ns}$, $c_{t,T}^{ns}$, $p_{t,T}^{ns}$ and $m_{t,T}^{ns}$ denote respectively the values of θ_T, c_T, p_T and m_T conditional on that no shock occurs between t and T. Then, the probability that no shock occurs between t and T is given by $\exp\left(-\int_t^T \theta_t^{ns}\, d\tau\right)$.

The transversality condition (TVC) can be written as $\lim_{T\to\infty} E_t V_{t,T} = 0$, where $V_{t,T} \equiv e^{-\rho(T-t)}u'(c_T)m_T$ and E_t denotes the expectation taken upon the information available at t. Since $u'(c_T)m_T \geq 0$ for all T,

$$E_t V_{t,T} \geq \exp\left(-\int_t^T \theta_t^{ns}\, d\tau\right) e^{-\rho(T-t)} u'(c_{t,T}^{ns}) m_{t,T} \equiv V_{t,T}^{ns}. \tag{46}$$

Note that while $V_{t,T}$ is a random variable, $V_{t,T}^{ns}$ is a deterministic variable given the information available at t. From (46), a necessary condition for the TVC is

$$\lim_{T\to\infty} V_{t,T}^{ns} \leq 0. \tag{47}$$

Differentiating (46) with respect to T and using equilibrium condition $m_{t,T}^{ns} = 1/p_{t,T}^{ns}$ yield

$$\frac{dV_{t,T}^{ns}/dT}{V_{t,T}^{ns}} = -\theta_{t,T}^{ns} - \rho + \frac{du'(c_{t,T}^{ns})/dT}{u'(c_{t,T}^{ns})} - \frac{dp_{t,T}^{ns}/dT}{p_{t,T}^{ns}}. \tag{48}$$

Lemma 1

Without nominal stickiness, $c^{ns}_{t,T} = y$ for all T and thus $du'(c^{ns}_{t,T})/dT = 0$. From (29), $(dp^{ns}_{t,T}/dT)/p^{ns}_{t,T} = \pi(\theta^{ns}_T)$. Substituting these into (48) yields

$$\frac{dV^{ns}_{t,T}}{dT} = -\theta^{ns}_{t,T}\frac{p(\theta^{ns}_{t,T})}{p(h(\theta^{ns}_{t,T}))}\frac{v'(p(h(\theta^{ns}_{t,T}))^{-1}) + u'(y)}{u'(y)}V^{ns}_{t,T}. \tag{49}$$

Using $p(\theta^{ns}_{t,T}) = p^{ns}_{t,T} = 1/m^{ns}_{t,T}$ and the definition of $V^{ns}_{t,T}$ in (46), equation (49) reduces to $dV^{ns}_{t,T}/dT = -\exp(-\int_t^T \rho + \theta^{ns}_{t,v}\,dv)\theta^{ns}_{t,T}Z(h(\theta^{ns}_{t,T}))$, where $Z(\theta) \equiv (v'(p(\theta)^{-1}) + u'(y))\,/p(\theta)$. Integrating this differential equation with respect to T from t to ∞ and using the fact that $V^{ns}_{t,t} = u'(y)m_t$ give

$$\lim_{T\to\infty} V^{ns}_{t,T} = u'(y)m_t - \int_t^\infty \exp\left(-\int_t^T \rho + \theta^{ns}_{t,v}\,dv\right)\theta^{ns}_{t,T}Z(h(\theta^{ns}_{t,T}))\,dT. \tag{50}$$

Fix a small constant $a > 0$ and define a closed interval $A \equiv [h(\theta^*), h(\theta^* + a)] \in (\theta^*, \theta^H)$. Note that Assumption 2 implies that $Z(\theta) \in (0, \infty)$ for all $\theta \in (\theta^*, \theta^H)$. In addition, it is continuous in this interval because (29) implies that $p(\cdot)$ is differentiable. Thus, there exist finite constants $Z_{\min} \equiv \min_{\theta\in A} Z(\theta) \in (0, 1)$ and $Z_{\max} \equiv \max_{\theta\in A} Z(\theta) \in (0, 1)$. Whenever $\theta^e_t \in (\theta^*, \theta^* + a)$, $\theta^{ns}_{t,T} \in (\theta^*, \theta^* + a)$ for all $T \geq t$. and therefore there is upper and lower bounds for the second term in the RHS of (50), given by

$$\left(\frac{\theta^* Z_{\min}}{\rho + \theta^* + a}, \frac{(\theta^* + a)Z_{\max}}{\rho + \theta^*}\right) \equiv (I_{\min}, I_{\max}) \subset (0, \infty). \tag{51}$$

Now suppose that $\lim_{\theta^e\to\theta^*} \pi(\theta^e) < 0$. Then as θ^e_t converges to θ^*, $p_t \to 0$ and therefore $m_t \to \infty$. However, this violates the TVC since conditions (47), (50) and (51) imply that the TVC requires $m_t \leq I_{\max}/u'(y)$ whenever $\theta^e_t \in (\theta^*, \theta^* + a)$.

Suppose conversely that $\lim_{\theta^e\to\theta^*} \pi(\theta^e) < 0$. Then as θ^e_t converges to θ^*, $p_t \to \infty$ and therefore $m_t \to 0$. For sufficiently small m_t, (50) and (51) imply $\lim_{T\to\infty} V^{ns}_{t,T} < 0$. Since $V^{ns}_{t,t} = u'(c_t)m_t > 0$ and $V^{ns}_{t,T}$ is continuous in T, there should be a value of $T \geq t$ such that $V^{ns}_{t,T} = 0$. From the definition of $V^{ns}_{t,T}$ in (46) this implies that $m^{ns}_{t,T} = 0$ and therefore $p(\theta^{ns}_{t,T}) = p^{ns}_{t,T} = \infty$, violating Assumption 2.

Lemma 2

We first derive a contradiction under assumption $\lim_{\theta^e\to\theta^*} \gamma_{\overline{p}}(\theta^e) < 0$. Fix $a > 0$ and define $A = [h(\theta^*), h(\theta^* + a)]$. Then, from $\overline{p}(\theta) \in (0, \infty)$ and its continuity, there exists $\overline{p}_{\min} \equiv \min_{\theta\in A} \overline{p}(\theta) \in (0, \infty)$. The assumption $\lim_{\theta^e\to\theta^*} \gamma_{\overline{p}}(\theta^e) < 0$ implies that $\overline{p}(\theta) \to 0$ as $\theta \to \theta^*$. Recall, in addition, that p_t falls at the rate of δ whenever

$p_t > \overline{p}(\theta^*)$. Thus, there is a positive probability that $(\theta_t^e,\ p_t)$ pair satisfies $\theta_t^e < \theta^* + a$ and $p_t \leq \overline{p}_{\min}$ when the liquidity shock does not occur for a sufficiently long while.

Suppose that the current $(\theta_t^e,\ p_t)$ pair satisfies the above inequalities. Then $p_{t,T}^{ns} < \overline{p}(h(\theta_{t,T}^{ns}))$ for all $T \geq t$, which means that the analysis in Appendix A applies. Substituting the results obtained in Appendix A into (48) yields

$$\frac{dV_{t,T}^{ns}/dT}{V_{t,T}^{ns}} = \begin{cases} -\theta_{t,T}^{ns} - \rho + \gamma_{u'}(\theta_{t,T}^{ns}) + \delta & \text{if } p_{t,T}^{ns} > \overline{p}(\theta_{t,T}^{ns}), \\ -\theta_{t,T}^{ns} - \rho - \gamma_{\overline{p}}(\theta_{t,T}^{ns}) & \text{if } p_{t,T}^{ns} = \overline{p}(\theta_{t,T}^{ns}). \end{cases} \tag{52}$$

Substituting (44) and (45) into (52) and using $p_{t,T}^{ns} = 1/m_{t,T}^{ns}$ and the definition of $V_{t,T}^{ns}$ in it, equation (52) reduces to $dV_{t,T}^{ns}/dT = -\exp(-\int_t^T \rho + \theta_{t,v}^{ns}\, dv)\theta_{t,T}^{ns}\overline{Z}(h(\theta_{t,T}^{ns}))$, where $\overline{Z}(\theta) \equiv \left(v'(\overline{p}(\theta)^{-1}) + u'(y)\right)/\overline{p}(\theta)$. Integrating this differential equation with respect to T from t to ∞ and using the fact that $V_{t,t}^{ns} = u'(c_{t,T}^{ns})m_t$ give

$$\lim_{T\to\infty} V_{t,T}^{ns} = u'(c_{t,T}^{ns})m_t - \int_t^\infty \exp\left(-\int_t^T \rho + \theta_{t,v}^{ns}\, dv\right)\theta_{t,T}^{ns}\overline{Z}(h(\theta_{t,T}^{ns}))\, dT. \tag{53}$$

Note that $h(\theta_{t,T}^{ns}) \in A$ for all $T \geq t$ and that there exists a finite constant $\overline{Z}_{\max} \equiv \max_{\theta \in A} \overline{Z}(\theta)$. From $c_{t,T}^{ns} \leq y$, $u'(c_{t,T}^{ns}) \geq u'(y)$ for all T. Thus (47) and (53) jointly imply that

$$m_t \leq \frac{(\theta^* + a)\overline{Z}_{\max}}{(\rho + \theta^*)u'(y)}. \tag{54}$$

While assumption $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) < 0$ implies that an arbitrarily large $m_t = 1/p_t$ realizes with a positive probability, the RHS of (54) is constant. Thus (54) and hence the TVC will be violated with a positive probability.

Next, assume conversely that $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) > 0$, which means that $\overline{p}(\theta^e)$ become arbitrarily large as $\theta^e \to \theta^*$. Then, $\theta_{t,T}^{ns} \in (\theta^*, \theta^* + a)$ and $p_{t,T}^{ns} = \overline{p}(\theta_{t,T}^{ns}) > \overline{p}_{\max} \equiv \max_{\theta \in A} \overline{p}(\theta)$ for sufficiently large T. In this case, Analysis in Section 4 applies and full employment obtains. From $m_{t,T}^{ns} = 1/p_{t,T}^{ns}$ and (39),

$$\frac{dm_{t,T}^{ns}}{dT} = (\rho + \theta_{t,T}^{ns})m_{t,T}^{ns} - \theta_{t,T}^{ns}\frac{v'(m_{t,T}^{ns})m_{t,T}^{ns} + u'(C(h(\theta_{t,T}^{ns}), 1/m_{t,T}^{ns}))m_{t,T}^{ns}}{u'(y)} \tag{55}$$

for sufficiently large T. As $T \to \infty$, $\overline{p}(\theta_{t,T}^{ns}) \to \infty$ and therefore $m_{t,T}^{ns} \to 0$. In this case, (55) implies $\lim_{T\to\infty} dm_{t,T}^{ns}/dT < \theta_T^{ns}u'(y)^{-1}\lim_{m\to 0} v'(m)m < 0$, where the latter inequality follows from the definition of $v(\cdot)$. These properties jointly imply that there is a finite T such that $m_{t,T}^{ns} = 0$ and therefore $\overline{p}(\theta_{t,T}^{ns}) = p_{t,T}^{ns} = \infty$, violating Assumption 3.

References

[1] Andolfatto, David and Paul Gomme (2003), “Monetary Policy Regimes and Beliefs,” *International Economic Review*, **44**(1), 1-30.

[2] Boldrin, Michele and David K. Levine. (2001). “Growth Cycles and Market Crashes.” *Journal of Economic Theory* **96**, 13–39.

[3] Caplin, Andrew and John Leahy. (1993). “Sectoral Shocks, Learning, and Aggregate Fluctuations.” *Review of Economic Studies* **60**, 777–794.

[4] Chalkley, Martin and In Ho Lee (1997), “Learning and Asymmetric Business Cycles,” *Review of Economic Dynamics*, **1**, 623-645.

[5] Driffill, John and Marcus Miller (1993), “Learning and Inflation Convergence in the ERM,”*Economic Journal*, **103**, 369-378.

[6] Dockner, Engelbert, Steffen Jørgensen, Ngo Van Long and Gerhard Sorger. (2000). *Differential Games in Economics and Management Science.*: Cambridge University Press.

[7] Farzin, Y. H.,Huisman, K. J. M., Kort, P. M. (1998) ”Optimal timing of technology adoption,” Journal of Economic Dynamics and Control, vol. 22(5), pages 779-799.

[8] Hassett, Kevin A, Metcalf, Gilbert E (1999) ”Investment with Uncertain Tax Policy: Does Random Tax Policy Discourage Investment?,” Economic Journal, vol. 109(457), pages 372-93.

[9] Lucas, Robert E. Jr. (1978), “Asset Prices in an Exchange Economy,” *Econometrica*, **46**(6), 1429-1445.

[10] Nieuwerburgh, Stijn Van and Laura L. Veldkamp. (2006). “Learning Asymmetries in Real Business Cycles.” *Journal of Monetary Economics* **53**(4), 753–772.

[11] Potter, Simon M. (2000), “A Nonlinear Model of Business Cycle,” *Studies in Nonlinear Dynamics and Econometrics*, **4**(2), 85-93.

[12] Sidrauski, Miguel (1967), “Rational Choice and Pattern of Growth in a Monetary Economy,” *American Economic Review*, **57**(2), 534-544.

[13] Venegas-Martinez, Francisco (2001) "Temporary stabilization: A stochastic analysis," Journal of Economic Dynamics and Control, vol. 25(9), pages 1429-1449.

[14] Wälde, Klaus, (1999) "Optimal Saving under Poisson Uncertainty," Journal of Economic Theory, vol. 87(1), pages 194-217.

[15] Wälde, Klaus (2005) "Endogenous Growth Cycles," International Economic Review, vol. 46(3), pages 867-894.

[16] Zeira, Joseph. (1994). "Informational Cycles." *Review of Economic Studies* **61**, 31–44.

[17] Zeira, Joseph (1999), "Informational Overshooting, Booms, and Crashes," *Journal of Monetary Economics*, **43**, 237-257.