Most Stringent Test for Location Parameter of a Random Number from Cauchy Density

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Abstract:

We study the test for location parameter of a random number from Cauchy density, focusing on point optimal tests. We develop analytical technique to compute critical values and power curve of a point optimal test. We study the power properties of various point optimal tests. The problem turned out to be different in its nature, in that, the critical value of a test determines the power properties of test. We found that if for given size α and any point θ_m in alternative space, if the critical value of a point optimal test is 1, the test optimal for that point is the most stringent test.

Keywords: Cauchy density, Power Envelop, Location Parameter, Stringent Test

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1. Introduction

Cauchy Distribution (or Lorentz Distribution in terminology of Physicists) has its applications in Physics, Spectroscopy and in Statistics. It is used to measure the sensitivity of an estimator\test statistics to normality assumptions due to its heavier tails which are extremely unlikely under normality assumptions.

Since the moments are not defined for Cauchy distribution, the tests/estimators based on asymptotic properties are not appropriate while studying the properties of Cauchy distributions. Statistician had discussed some statistical properties of the distribution. The principal focus of studies is on goodness of fit to the Cauchy distribution. The studies of Rublyk (1997), Rublyk (1999) Gurtler & Henze (2000), Rublyk (2001), Rublyk (2003) and Matsui & Takemura (2005) are the examples. Other studies explore parameter estimation, & behavior of Likelihood function etc. See Copas (1975), Besbeas & Morgan (2001) and Lawless (1972) for example.

For an observation X from Cauchy distribution, we explore the tests for location parameter of the distribution with focus on point optimal (Neyman-Pearson) tests & develop method to find out the most stringent test. Lehman (1986) discussed that for location parameter of an observation X from Cauchy density, UMP test does not exist. Obviously the Stringent test is the feasible choice in absence of UMP test if we can find it. We study the power properties of a large number of NP[1] tests and develop technique to find out power envelop and power curve of a point optimal test of given size, and hence to find shortcoming of test. By luck of draw, the problem turned to be different in its nature, in that, we are able to construct the power envelope analytically. Similarly we can trace power curves of a point optimal test as well. This made possible the computation of shortcoming of a large number of point optimal tests. The techniques are discussed below.

[1] The terms 'point optimal test', 'Neyman-Pearson test' and abbreviation 'NP test' are used interchangeably in the paper. Furthermore T^{θ} is mathematical shorthand for NP test optimal for point θ.

2. The problem

Let Ca(θ) denotes the Cauchy distribution, with location parameter θ and unit scale parameter & X ~ Ca(θ) i.e.

$$f(X|\theta) \quad = \quad \frac{1}{\pi\left[1 + (X-\theta)^2\right]}$$

We are interested in testing the Hypothesis H_0: θ = 0 versus H_1: θ > 0. The Null space Θ_0 = (0) and the alternative space $\Theta_1 = R^+$.

3. Point Optimal Test & Power Envelope

3.1 Test Statistics for Point Optimal Test & Critical Values

Given the problem, Neyman Pearson Lemma allows us to construct test optimal for a point $\theta \in \Theta_1$. Let L(X,θ) denote density at X given the location parameter θ than the test statistics is:

$$L(X,0,\theta) = \frac{L(X,\theta)}{L(X,0)}$$

For test of size α the critical values can be computed by assuming H_0 is true & finding $C_\alpha(\theta)$ such that

$$P\left(\frac{L(X,\theta)}{L(X,0)} \geq C_\alpha(\theta)\right) = \alpha \qquad [1]$$

Where P(.) denotes probability & the subscript α refers to size of the test. Now For Cauchy density,

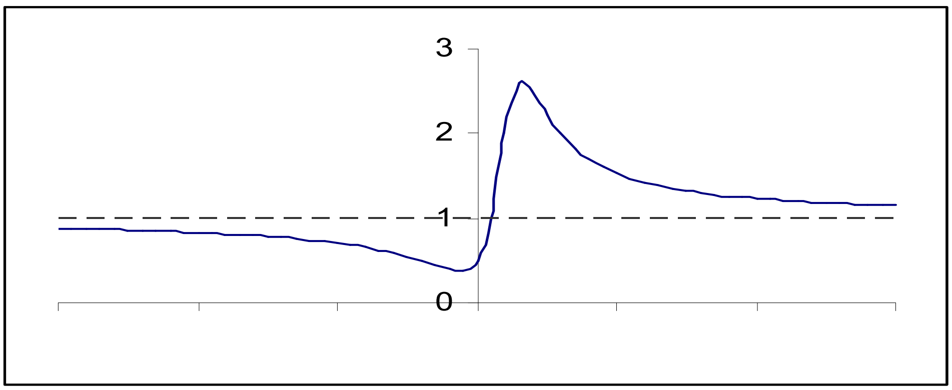
$$\frac{L(X,\theta)}{L(X,0)} = \frac{\left[\pi\left[1 + (X-\theta)^2\right]\right]^{-1}}{\left[\pi\left(1 + X^2\right)\right]^{-1}} = \frac{1 + X^2}{1 + (X-\theta)^2}$$

Given an observation X, we reject the null if $L(X,0,\theta) > C_\alpha(\theta)$ and accept otherwise. This procedure maximizes power at $\theta \in \Theta_1$. By changing θ, we get different test statistics optimizing power for that new point.

Now, for $\theta \in \Theta_1$, we plot $L(X,0,\theta)$ for different values of X. A Typical graph is shown in fig 1.

Remark: For the figures labeled 'typical graph', there is no significance of scale especially if it is not visible. Just general shape of objective function is described by the graph.

Fig 1: A Typical Plot of $L(X,0,\theta)$



The solid line denotes the values of $L(X, 0,\theta)$. The graph cuts the line y =1 (the dotted line) at one point and converges to 1 as X converges to $\pm\infty$

The graph of $L(X,0,\theta)$ is increasing in a finite interval and decreasing otherwise. $L(X,0,\theta) = 1$ only at $X= \theta/2$. Little mathematical formulation verifies that for $X>\theta/2$, $L(X,0,\theta)$ is greater than 1 & smaller than 1 otherwise. For finite θ, the graph of $L(X,0,\theta)$ converges to 1 if $X \to \pm\infty$.

Below we discuss how to find out the critical value $C_\alpha(\theta)$ analytically.

Theorem 1:

Given an NP test of size α , optimal for alternative θ ,

$$C_\alpha(\theta) = \frac{8 + 8k + 4\theta^2 + 4\theta^2 k + \sqrt{\left(8 + 8k + 4\theta^2 + 4\theta^2 k\right)^2 - 4\left(4 + 4k + k\theta^4 + 4k\theta^2\right)(4 + 4k)}}{2\left(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2\right)}$$

Where $C_\alpha(\theta)$ be defined in [1] above and

$$k = \tan^2(\pi\theta)$$

Proof:

Using definition of $C_\alpha(\theta)$,given size of test = α & assuming H_0 is true, $C_\alpha(\theta)$ is the value such that

$$P\left(\frac{L(X,\theta)}{L(X,0)} \geq C_\alpha(\theta)\right) = \alpha \qquad [2]$$

To find $C_\alpha(\theta)$, solving the equation,

$$\frac{1 + X^2}{1 + (X - \theta)^2} = C_\alpha(\theta) \qquad [3a]$$

For simplicity, denote $C_\alpha(\theta)$ by C, we can rewrite [3a] as:

$$X^2(1 - C) + 2CX\theta + 1 - C - C\theta^2 = 0. \qquad [3c]$$

Supposing l, m being the roots of equation [3b], since l, m specify the range where we reject the Null. So assuming the Null is true, integration of the density function over range (l, m) must be equal to size of test α i.e.

$$\int_l^m \frac{1}{\pi\left(1 + X^2\right)}\,dx = \alpha$$

$$\Rightarrow \quad \frac{1}{\pi}\left[\tan^{-1}(m) - \tan^{-1}(l)\right] = \alpha$$

$$\Rightarrow \quad \frac{m - l}{1 + lm} = \tan(\pi\alpha) \qquad [3c]$$

Now l, m being roots of equation [3b] quadratic in X, therefore:

$$l + m = -2\frac{C\theta}{1 - C} \qquad [3d]$$

$$lm = \frac{1 - C - C\theta^2}{1 - C} \qquad [3e]$$

Solving [3c], [3d], and [3e] for C yield:

$$\frac{8 \cdot C + 4 \cdot C^2 \cdot \theta^2 - 4 - 4 \cdot C}{\left(2 - 2 \cdot C - C\theta^2\right)^2} = tan^2(\pi\alpha) \qquad [4a]$$

Assuming $k = tan^2(\pi\theta)$ [4b]

And solving [4] for C gives:

$$C = \frac{8 + 8k + 4\theta^2 + 4\theta^2 k \pm \sqrt{\left(8 + 8k + 4\theta^2 + 4\theta^2 k\right)^2 + 4(4 + 4K + k\theta^4 + 4k\theta^2)(4 + 4k)}}{2\left(4 + 4K + k\theta^4 + 4k\theta^2\right)}$$

This equation gives two values of C corresponding to

P[L(X,0,θ) >C] = α and P[L(X,0, θ)<C] = α

Obviously we are interested in first expression, for that we have to choose the larger root of C. i.e.

$$C = \frac{8 + 8k + 4\theta^2 + 4\theta^2 k + \sqrt{\left(8 + 8k + 4\theta^2 + 4\theta^2 k\right)^2 + 4(4 + 4K + k\theta^4 + 4k\theta^2)(4 + 4k)}}{2\left(4 + 4K + k\theta^4 + 4k\theta^2\right)} \qquad [5]$$

Replacing C by $C_\alpha(\theta)$, yield proof of theorem.

We completely specify our test statistics and the critical values to be used throughout in the discussion of Neyman-Pearson tests as:

Test Statistics: $L(X,0,\theta) = (1+X^2)/(1+(X-\theta)^2)$

Critical Value: $C_\alpha(\theta)$ (defined & derived in theorem 1)

The test statistics and the critical values vary with the variation of θ.

3.2 Critical value & the power properties:

It was observed that power of an NP test depends crucially on $C_\alpha(\theta)$. The relationship of power of a test and the critical value $C_\alpha(\theta)$ is discussed in the following theorem.

Theorem 2:

For convenience, let θ denote the point at which we are maximizing power & β be the point in R^+ for which we want to compute the power of a test. Further T^θ denote NP test optimal for $\theta \in R^+$. Than for an NP test;

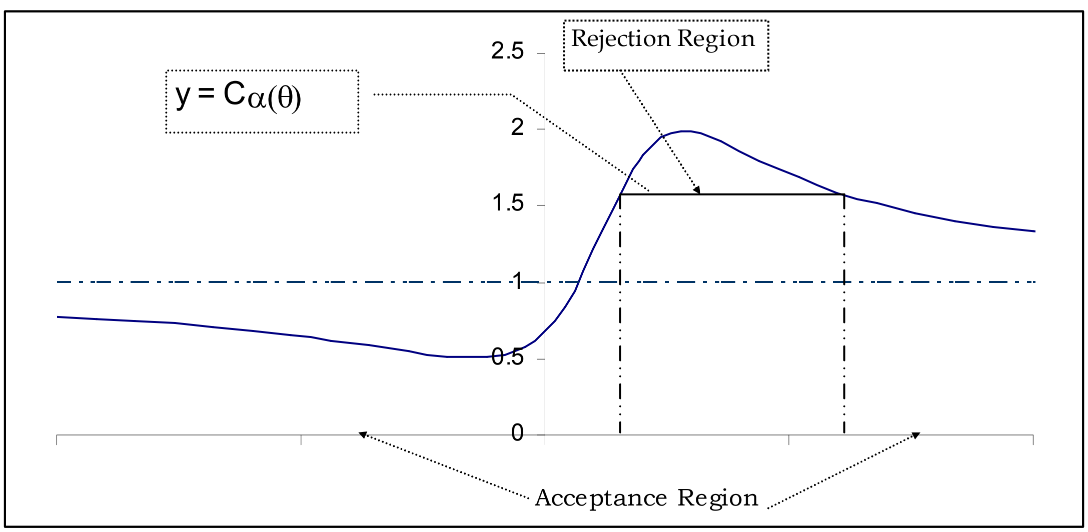
1. If $C_\alpha(\theta) > 1$, power of T^θ converges to zero for the large β.
2. If $C_\alpha(\theta) \le 1$, power of T^θ converges to 1 for the large β.

Proof:

Remember we reject the null if $L(X,0,\theta) > C_\alpha(\theta)$

a) If $C_\alpha(\theta) > 1$, than roots of equation [3b] lies on RHS of point $X=\theta/2$. Let *l, m* be the roots of quadratic equation [3b], than the roots determine boundaries of rejection region. It can be shown unique maxima of $L(X,0,\theta)$ lies inside interval (*l, m*). Therefore the rejection region is bounded by roots *l, m*. Than for some $\theta \in R^+$, power of test is just integration of density function $f(X|\beta)$ on interval (*l, m*).

Fig 2: Rejection region when $C_\alpha(\theta) > 1$



Value of $L(X,0,\theta)$ is large than $C_\alpha(\theta)$ for points between root of $L(X,0,\theta) = C_\alpha(\theta)$ which is the rejection region

Now Let $R(T, \beta)$ denote power of test T at β & test T^θ is such that $C_\alpha(\theta) > 1$, than

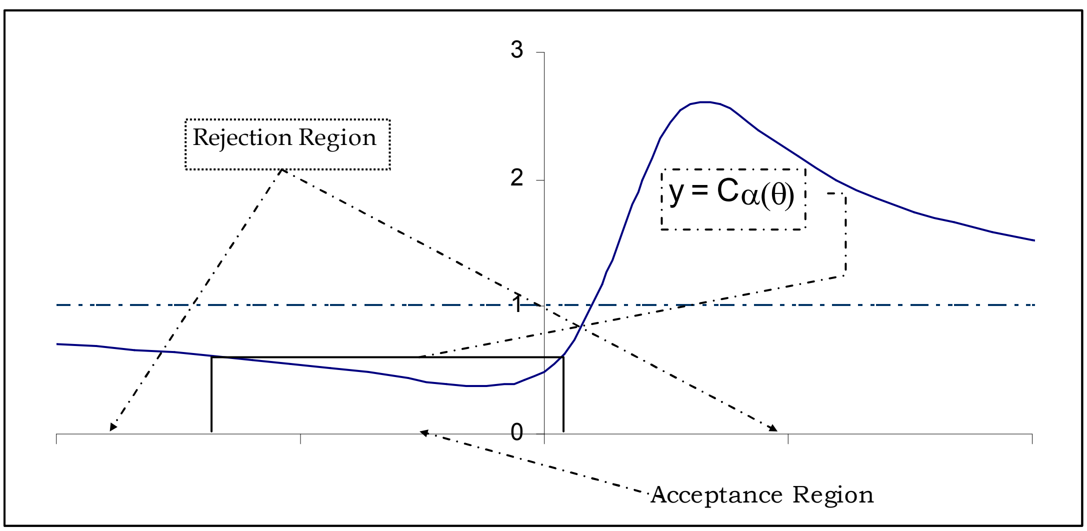
$$\lim_{\beta\to\infty} R\left(T^{\theta},\beta\right) = \lim_{\beta\to\infty} \int_{l}^{m} \frac{1}{\pi\left[1+\left(x-\beta\right)\right]}\,dx$$

$$= \lim_{\beta\to\infty} \frac{1}{\pi}\left[\left(tan^{-1}\right)\left(m-\beta\right) - tan^{-1}\left(l-\beta\right)\right]. \qquad = 0$$

b) We divide proof of [b] in to two parts

i) If $C_{\alpha}(\theta)$ <1, than roots of equation [3b] lies on left of point X=θ/2. Again if *l, m* be the roots of quadratic equation [3b], they determine boundaries of rejection region. It can be shown unique minima of L(X,0,θ) lies inside interval (*l, m*). Therefore the rejection region is R- (*l, m*). Therefore for some $\beta \in R^{+}$, power of test is integration of density function f(X|β) on real line minus integration of density function on interval (*l, m*).

Fig 3: Rejection region when $C_{\alpha}(\theta)$ <1



Value of L(X,0,θ) is smaller than $C_{\alpha}(\theta)$ for points between root of L(X,0,θ) = $C_{\alpha}(\theta)$ which is the acceptance region

Now to get power, we have to integrate on overall range except the interval (*l, m*), and probability of overall range is 1, so in this case, power of an NP test, is 1- integration of density function on interval (*l, m*) therefore

$$\lim_{\beta\to\infty} R\left(T^{\theta},\beta\right) = \lim_{\beta\to\infty}\left[1-\int_{l}^{m}\frac{1}{\pi\left[1+(X-\beta)^{2}\right]}\,dX\right]$$

$$= 1-\lim_{\beta\to\infty}\frac{1}{\pi}\left[\left(tan^{-1}\right)(m-\beta)-tan^{-1}(l-\beta)\right]. \qquad = 1$$

ii) If $C_{\alpha}(\theta)$ =1, than we are just at X=θ/2and as we had discussed, for all points on right side of θ/2, L(X,0,θ) >1. The two roots of quadratic equation [3b] are θ/2 and ∞. So power of such test for a $\beta \in R^{+}$ is integration of density function f(X|β) on range (θ/2, ∞). Finding probability of accepting Null for large β;

$$\lim_{\beta\to\infty} 1-R(T,\beta) = \lim_{r,\beta\to\infty}\int_{-\gamma}^{\frac{\theta}{2}}\frac{1}{\pi\left[1+(x-\beta)\right]}\,dx$$

$$= \lim_{r,\beta\to\infty}\frac{1}{\pi}\left[tan^{-1}\left(\frac{\theta}{2}-\beta\right)-tan^{-1}(-\gamma-\beta)\right] \qquad = 0$$

Since probability of accepting Null for large β is 0, the power of test will be 1.

Theorem 2 realizes the important role of the critical value in determining the power properties of a test. If the critical value is larger than 1, the test is sure to have zero power for the large alternatives. Whereas the test with critical value smaller than 1 has 100% power for the same alternatives. Therefore the former tests with critical value larger than 1 are sure to have 100% shortcoming & should never be used in absence of precise prior information of the alternative.

3.3 Power Envelope

For a test optimal at point $\theta \in \Theta_1$ we discussed how we can find power of the test at any arbitrary point. Let T^{θ} denote test optimal for θ, the locus of $R(T^{\theta}, \theta)$, $\theta \in \Theta_1$

forms the power envelope. The algorithm to trace power envelope is discussed in greater detail in appendix.

4. Performance of Conventional NP tests

Up to best of our knowledge, none of existing studies had addressed the problem we are discussing. However in general hypothesis testing problems, when there is no UMP test, different strategies had been recommended in the Literature to design a feasible NP test. Below we discuss the some of the strategies & their performance in the present problem. We discuss three types of conventional hypothesis testing strategies:

A) Maximizing power in neighborhood of Null (the Locally Most Powerful or LMP tests)

B) Maximizing power for extremely large alternatives (Berenblutt & Webb Type tests)

C) Maximizing power for some intermediate choice of alternatives

4.1 The Locally Most Powerful test

The rational of a Locally Most Powerful test (LMP) is to maximize power for an alternative in the neighborhood of Null. Choice of an LMP test for testing location parameter in the problem discussed turned out to be most unsuitable as the LMP test possess 100% shortcoming. It turns out that for an LMP, $C_\alpha(\theta) > 1$, therefore according to theorem 2, the test has zero power for the large alternatives. Whereas, as we will show later, there is a certain class of test for which $C_\alpha(\theta) < 1$ & hence possess 100% power for large alternatives. Therefore the LMP test has 100% shortcoming. The following Lemma proves the claim.

Lemma 1:

If θ is so small that $\theta^2 \cong 0$ than $C_\alpha(\theta) > 1$.

Proof:

We can rewrite expression for $C_\alpha(\theta)$ derived in theorem 1 as:

$$C_\alpha(\theta) = \frac{8 + 8k + 4\theta^2 + 4\theta^2 k + 4\theta\sqrt{(1+k)(2+\theta^2)}}{2(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2)} \tag{6}$$

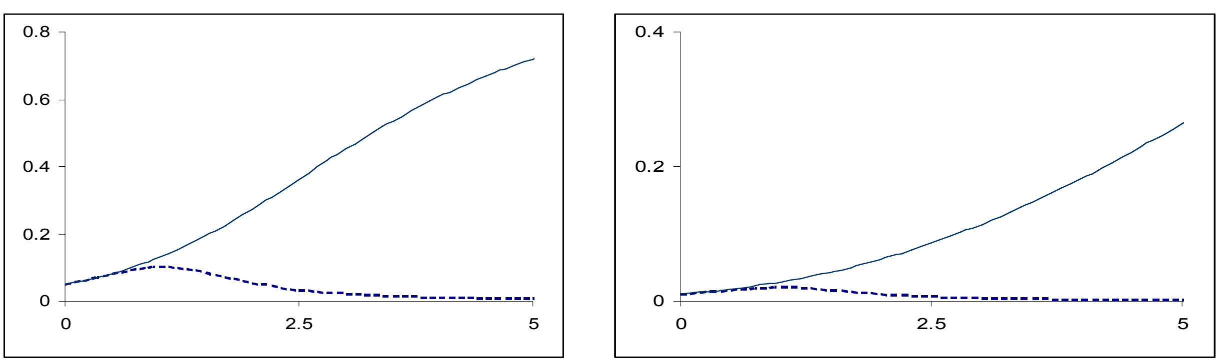
Let θ be so small that it higher powers are negligible than the expression reduces to:

$$C_\alpha(\theta) = \frac{8 + 8k + 4\theta\sqrt{2(1+k)}}{2(4 + 4\cdot k)} \tag{6b}$$

Obviously, for $\theta > 0$ the numerator of expression on right hand side of [6b] is greater than denominator. Hence $C_\alpha(\theta)>1$.

The Lemma gives crucial information about the power curve of LMP test, in that, the critical value of LMP is larger than 1 & thus according to theorem 2, have zero power for large alternatives. Therefore, LMP has 100% shortcoming. So LMP should never be used if we don't have precise information of θ. Empirical results shows that LMP is not good even outside a small neighborhood of the point for which it is optimal. A test of size 5% optimal for $\theta = 0.5$ has only 1% power for $\theta = 4$ and for $\theta = 15$, its power is zero up to 3 decimal places.

Fig 4: Performance of LMP test



The solid lines in the two graphs represent the power envelope whereas the dotted lines represent power curves of test optimal for $\theta = 0.1$. Size of test is 5% for the left panel and 1% for the right panel. Immediate decline in power of LMP is obvious in the two panels.

So having no prior information of the true parameter, the choice of LMP test is hazardous.

4.2 Berenblutt & Webb type test

In their discussions of tests for autocorrelation, Berenblutt & Webb (1973) recommend to use another extreme strategy, to maximize power for largest possible alternative. The alternative space we have is R^+ ranging from 0 to ∞, therefore we have maximize power for ∞ . One of the result of following Lemma is the proof that; $\beta \to \infty \Rightarrow C_\alpha(\theta) \to 0 < 1$, and thus (by theorem 2) the test has 100% power for large alternatives.

Lemma 2:

For $C_\alpha(\theta)$ defined in [2], following results hold.

R1 $$\lim_{\beta\to\infty} C_\alpha(\theta) = 0$$

R2 $$\lim_{\beta\to\infty} C_\alpha(\theta)\,\theta = 0$$

R3 $$\lim_{\beta\to\infty} C_\alpha(\theta)\cdot\theta^2 = \frac{2 + 2\cdot k + 2\cdot\sqrt{(1+k)\cdot\left(2+\theta^2\right)}}{k}$$

Where ‘k’ is defined in [4b] above.

Proof:

Again for simplicity, denote $C_\alpha(\theta)$ by C than for R1

$$\lim_{\beta\to\infty} C_\alpha(\theta) = \lim_{\theta\to\infty} \frac{8 + 8k + 4\theta^2 + 4\theta^2 k + 4\theta\sqrt{(1+k)\left(2+\theta^2\right)}}{2\left(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2\right)}$$

$$= \lim_{\theta\to\infty} \frac{\dfrac{8 + 8\cdot k + 4\cdot\theta^2 + 4\cdot\theta^2\cdot k}{\theta^4} + \dfrac{4\cdot\theta\cdot\sqrt{(1+k)\cdot\left(2+\theta^2\right)}}{\theta^4}}{2\cdot\dfrac{\left(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2\right)}{\theta^4}}$$

$$= 0$$

.

Similarly for R2

$$\lim_{\beta\to\infty} C_\alpha(\theta)\,\theta = \lim_{\theta\to\infty} \frac{\dfrac{8\theta + 8\cdot k\theta + 4\cdot\theta^3 + 4\cdot\theta^3\cdot k}{\theta^4} + \dfrac{4\cdot\theta^2\cdot\sqrt{(1+k)\cdot\left(2+\theta^2\right)}}{\theta^4}}{2\cdot\dfrac{\left(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2\right)}{\theta^4}}$$

$$= 0$$

And for R3:

$$\lim_{\beta\to\infty} C_\alpha(\theta)\,\theta^2 = \lim_{\theta\to\infty} \frac{\dfrac{8\theta^2 + 8\cdot k\theta^2 + 4\cdot\theta^4 + 4\cdot\theta^4\cdot k}{\theta^4} + \dfrac{4\cdot\theta^3\cdot\sqrt{(1+k)\cdot\left(2+\theta^2\right)}}{\theta^4}}{2\cdot\dfrac{\left(4 + 4\cdot k + k\theta^4 + 4\cdot k\theta^2\right)}{\theta^4}}$$

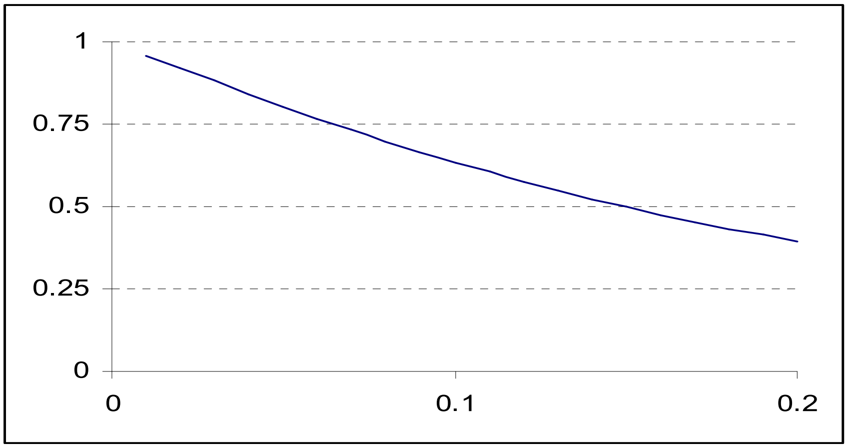
$$= \frac{2 + 2k + 2\sqrt{1+k}}{k}$$

Hence the results

Solving equation [3] for X & using results of Lemma 2, one can find out the range for which B&W type test rejects Null. It is observed that power curve of B&W type test is increasing in β. So, obviously the tests of this type are preferable to LMP in that they have smaller shortcoming than LMP.

Practically, if we design test maximizing power for a very large alternative, it has negligible discrimination between Null and an alternative in neighborhood of Null. Therefore this type of test is expected to have smaller power for alternatives close to null. Therefore such test possess heavy shortcoming (But not larger than that of LMP). For different sizes, we computed the shortcoming of B&W type. Figure below shows the plot shortcoming of B&W type test versus its size.

Fig 5: Shortcoming of B&W type tests



Size of test represented on X-axis & shortcoming of B&W on Y-axis. Shortcoming of B&W type test is less than 1, but still massive

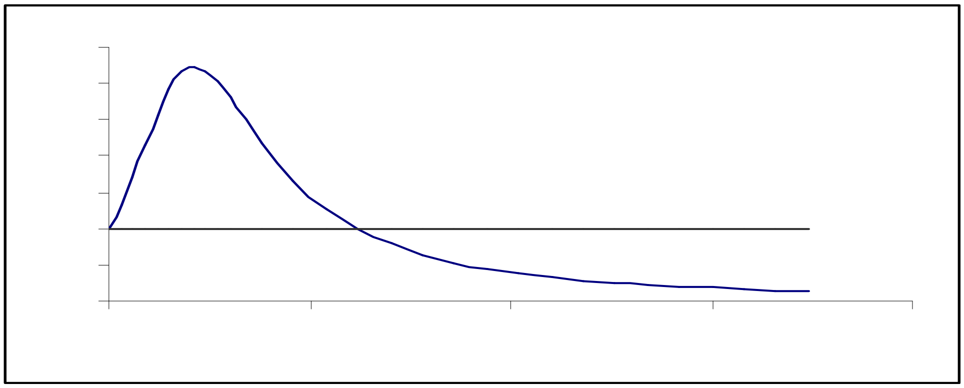
For size 1%, the test has 96% shortcoming, which is obviously disappointing, and for size 20%, the shortcoming is 40%. That is, we still have a lot of loss in using this type of test. As we will soon show, there is an intermediate class of tests other than B&W type tests, all of which has 100% power for infinitely large alternatives. Is it possible to choose a test in that class which has smaller shortcoming? Next we investigate the same question.

4.3 Intermediate choice of Alternative

Cases where both LMP and B&W type tests perform poorly, it is natural to choose an intermediate alternative to optimize power for. Several strategies are recommended by different writers to choose a suitable intermediate. See Efron (1975), Davies (1969) Fraser et al (1976) & King (1985) for example.

As we discussed, critical value of a test is 'critical' in determining the power properties, choosing an intermediate value, we must have to consider the behavior of critical value. The typical plot of $C_\alpha(\theta)$ follow the pattern given in fig below.

Fig 6: A typical plot of $C_\alpha(\theta)$



$C_\alpha(\theta)$ is large than 1 for a finite range and than decreases continuously

We see that for test of fixed size, there exist a $\theta_m \in R^+$ with $C_\alpha(\theta) = 1$ and

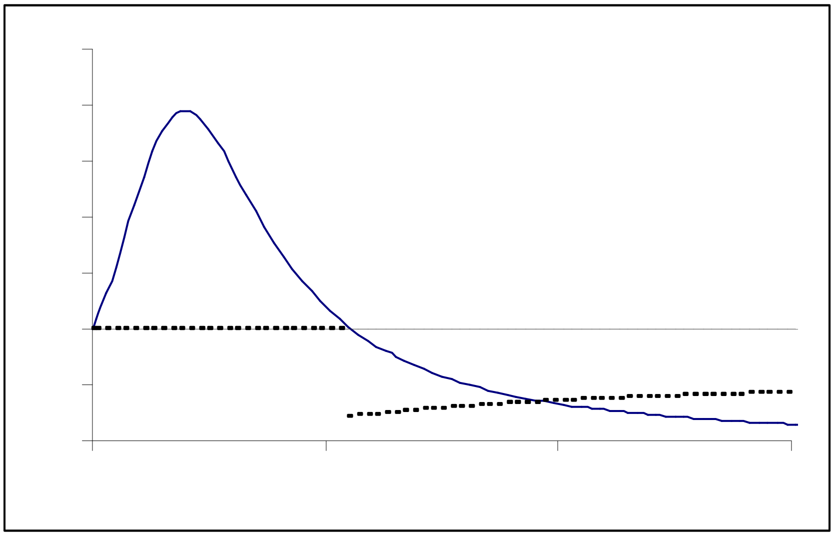
$$C_\alpha(\theta) > 1 \qquad \theta \in (0, \theta_m)$$

$$C_\alpha(\theta) < 1 \qquad \theta \in (\theta_m, \infty)$$

Now if we choose an alternative in $(0, \theta_m)$, $C_\alpha(\theta) > 1$, an NP test optimal in this interval is the test with maximum shortcoming. Whereas for choice of $\theta \in (\theta m, \infty)$ the test is certain to have 100% and power for very large alternative. Now first step in an analysis is to search θ_m so that we can avoid choosing a test with 100% shortcoming.

Luckily, we are able to compute critical value of an NP test analytically as well as we can analytically compute power of an of NP test at any point. This makes computation of exact power envelope, power curve & shortcoming of a test possible. We compute the shortcoming of a large number of NP tests. E.g. for size of test > 5% we compute shortcoming of test optimizing power at each point of the vector (0, 0.5…10000) to choose a feasible alternative which minimizes the shortcoming (The algorithm given appendix). We found that for the test optimal for θ_m (θ_m defined above), shortcoming is minimum. Let T^θ denote test optimal for θ and $S(T^\theta)$ denote the shortcoming of T^θ . For fixed size the typical plot of T^θ versus $S(T^\theta)$ is given in fig below:

Fig 7: Shortcoming of NP Tests of fixed size



Dotted line indicates shortcoming of the test, whereas the solid line is for value of $C_\alpha(\theta)$. Shortcoming suddenly drops down for $C_\alpha(\theta) = 1$ and then increases monotonically with θ

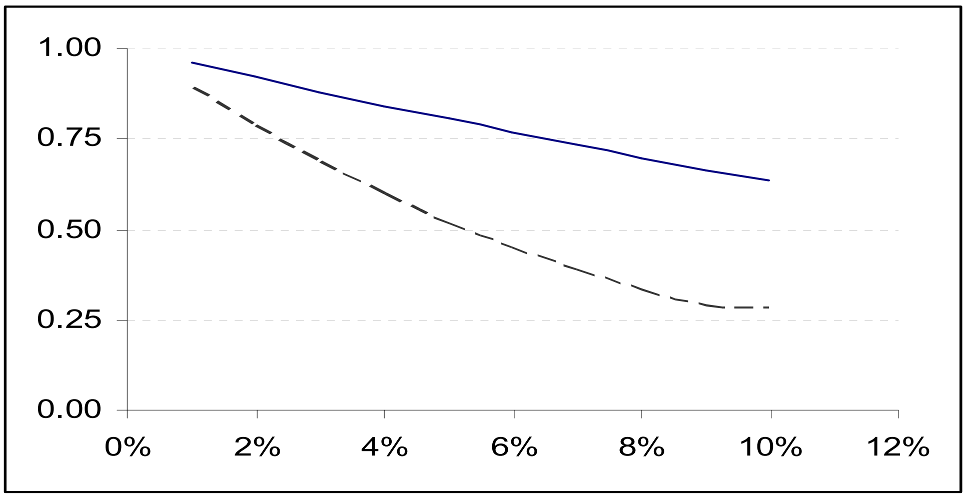
Tests optimal for any point in (0, θ_m) have 100% shortcoming, which is according to our expectation. Than shortcoming suddenly drops down for $T^{\theta m}$ and than increases monotonically with θ and converges to shortcoming of B& W type tests. The pattern of plot remains same for different size of test.

There are some other strategies to choose a feasible alternative, e.g. using the information matrix etc. But in our case, any strategy for choosing a feasible intermediate point optimal test falls into one of the two categories discussed & therefore there is no need to study them separately.

5. Recommended test

Studying power properties of a large number of point optimal tests, we reach the conclusion that $T^{\theta m}$ possesses smallest shortcoming, & hence is the most stringent test. Therefore, given the size of test α, the problem of searching for a stringent test reduces to search for a point θ_m in R^+ such that $C_\alpha(\theta_m) = 1$. Having an analytical formula to compute $C_\alpha(\theta)$, this is worth spending few moments on computer. In table 1, we tabulate θ_m for different sizes. It was observed that $T^{\theta m}$ has significantly smaller shortcoming than that of B&W. Below we plot the shortcoming of the two types ($T^{\theta m}$ and B&W) of test for size 1-10%.

Fig 8: Comparison of $T^{\theta m}$ and B&W test



Dotted line represent shortcoming of $T^{\theta m}$ which is much smaller than shortcoming of B&W type test represented by solid line.

5.1 Other Characteristic of Recommended Test

5.1.1 Effect of size of test on shortcoming of $T^{\theta m}$

It can be seen from fig 8 that shortcoming of $T^{\theta m}$ increases by a large factor if we reduce size of test by 1%. Reducing size of test from 5% to 4% increases the shortcoming from 51% to 59%. Hence the arbitrary choice of size of test is risky. It's better to choose the larger size since it cuts down shortcoming by multiple factor.

5.1.2 Abuse of $T^{\theta m}$, consequences

In absence of any prior belief about alternative, $T^{\theta m}$ is the only feasible option, since it possess minimum shortcoming. But suppose we have solid reasons to believe that the alternative is some $\beta \neq \theta_m$. So, one may think to use an NP test optimal for β to get a larger power. There are two possibilities

If $\beta < \theta_m$ than using T^{β} may be beneficial by a large amount in the neighborhood of β but as we had discussed, the test may have zero power if the true parameter is larger than our belief. We are at risk of 100% shortcoming.

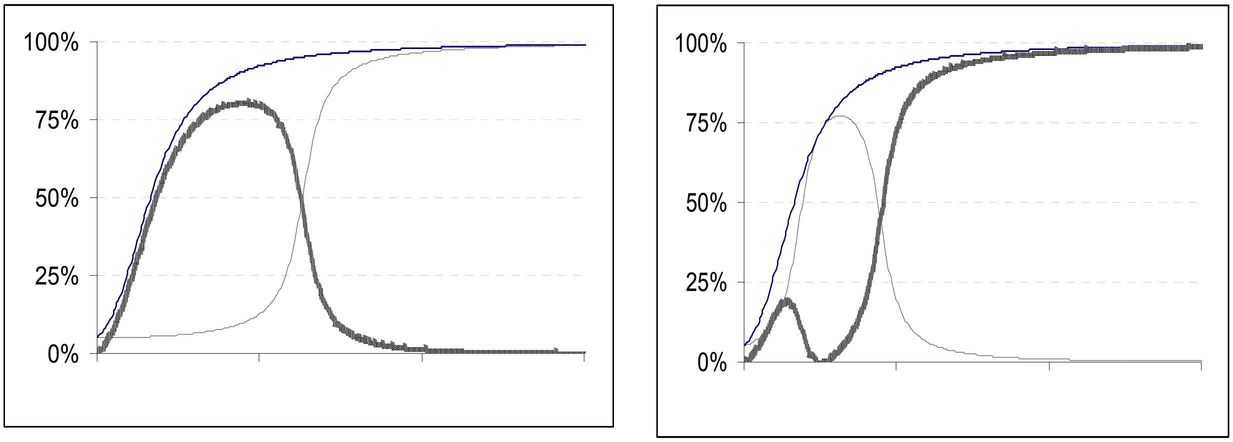
If $\beta > \theta_m$ than surprisingly, T^{β} is not much beneficial even in the neighborhood of β. We computed the maximum shortcoming of $T^{\theta m}$ of different sizes on the range (θ_m, ∞) which is tabulated below. It turns out that the maximum difference between power envelope & power curve of $T^{\theta m}$ is negligible in interval (θ_m, ∞).

Table 1:

Size of test	θ_m	SC of $T^{\theta m}$ on (θ_m, ∞)
1%	63.642	0.0009
2%	31.790	0.0018
3%	21.158	0.0027
4%	15.832	0.0036
5%	12.628	0.0045
6%	10.485	0.0054
7%	8.948	0.0063
8%	7.790	0.0072
9%	6.885	0.0081
10%	6.156	0.0091

For test of any size, the maximum advantage we can have to maximize power beyond θ_m is less than 1%.Hence maximizing for a larger alternative does not have any practical benefit even at the point for which we maximize power.

Figure 9: Abuse of $T^{\theta m}$



Solid lines in the two graphs represent power envelope and lines with dashed lines represent power curves of $T^{\theta m}$. The Dotted line in left panel represent power curve of an NP test optimal in $(0, \theta_m)$. The test has significant gain over $T^{\theta m}$ for a small range, but zero power for large alternatives. In right panel, dotted line denote point wise shortcoming of $T^{\theta m}$ which is negligible for range (θ_m, ∞) Whereas, optimizing beyond θ_m will always increase shortcoming. Optimal choice in this case is again $T^{\theta m}$.

Solid lines in the two graphs represent power envelope and lines with cross marks represent power curves of $T^{\theta m}$. The Dotted line in left panel represent power curve of an NP test optimal in $(0, \theta_m)$. The test has significant gain over $T^{\theta m}$ for a small rang, but zero power for large alternatives. In right panel, dotted line denote point wise shortcoming of $T^{\theta m}$ which is negligible for range (θ_m, ∞).

Whereas, optimizing beyond θ_m will always increase shortcoming. Optimal choice in this case is again $T^{\theta m}$.

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6. Appendix

6.1 Computation of power curve of a point optimal test & power envelope & shortcoming of a test.

Suppose we want to trace power curve of a test of size α which optimizes power at $\theta \in R^+$. A lot of computational burden is released by the analytic formula in our hands. Even without computing the actual test statistics, we can compute the critical value by the formula derived in theorem1. Putting the critical value in [3b] and solving for X, yield the roots *l, m*. Theorem 2 allows us to specify the range for which value of actual test statistics will be larger than critical value. That is if critical value '$C_\alpha(\theta)$ ' is larger than 1, the rejection region is bounded by the roots i.e.

$L(X,0,\theta) > C_\alpha(\theta)$ if $l < X < m$

Now to compute power of test for some $\theta \in R^+$, the power of test is probability that random variable X lies in specified range.

$$\lim_{\beta\to\infty} R\left(T^{\theta},\beta\right) = P\,[\,l \le X \le m\,]$$

$$= \int_{l}^{m} \frac{1}{\pi\left[1+\left(x-\beta\right)\right]}\,dx$$

Now once we have specified range for which null should be rejected for T^θ, The locus of $R(T, \beta)$, gives the power curve. The power curves discussed in paper were traced by computing power at $\beta = 0, 0.5 \ldots 10000$ for size of test > 5% and at $\beta = 0, 1 \ldots 50000$ for size of test < 5%. Computing $R(T^\beta , \beta)$ yield power envelop for $\beta = 0, 0.5 \ldots 10000$. For any test T, $R(T^\beta , \beta) - R(T^\theta, \beta)$, $\beta = 0, 0.5 \ldots 10000$ yield vector of point wise shortcoming, and maximum of this vector is the shortcoming of the test.

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