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A Fuzzy Pay-Off Method for Real Option Valuation

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Abstract

Real option analysis offers interesting insights on the value of assets and on the profitability of investments, which has made real options a growing field of academic research and practical application. Real option valuation is, however, often found to be difficult to understand and to implement due to the quite complex mathematics involved. Recent advances in modeling and analysis methods have made real option valuation easier to understand and to implement. This paper presents a new method (fuzzy pay-off method) for real option valuation using fuzzy numbers that is based on findings from earlier real option valuation methods and from fuzzy real option valuation. The method is intuitive to understand and far less complicated than any previous real option valuation model to date. The paper also presents the use of number of different types of fuzzy numbers with the method and an application of the new method in an industry setting.

Keywords: Real Option Valuation, Fuzzy Real Options, Fuzzy Numbers

1 Introduction

Real option valuation (ROV) is based on the observation that the possibilities financial options give their holder resemble the possibilities to invest in real investments and possibilities found within real investments, i.e., managerial flexibility - "an irreversible investment opportunity is much like a financial call option" [18]. In other words, real option valuation is treating

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investment opportunities and the different types of managerial flexibility as options and valuing them with option valuation models. Real options are useful both, as a mental model for strategic and operational decision-making, and as a valuation and numerical analysis tool. This paper concentrates on the use of real options in numerical analysis, and particularly on the derivation of the real option value for a given investment opportunity, or identified managerial flexibility.

Real options are commonly valued with the same methods that have been used to value financial options, i.e., with Black-Scholes option pricing formula [2], with the binomial option valuation method [11], with Monte-Carlo based methods [3], and with a number of later methods based on these. Most of the methods are complex and demand a good understanding of the underlying mathematics, issues that make their use difficult in practice. In addition these models are based on the assumption that they can quite accurately mimic the underlying markets as a process, an assumption that may hold for some quite efficiently traded financial securities, but may not hold for real investments that do not have existing markets or have markets that can by no means be said to exhibit even weak market efficiency.

Recently, a novel approach to real option valuation was presented in [15], [16], and in [12], where the real option value is calculated from a pay-off distribution, derived from a probability distribution of the NPV for a project that is generated with a (Monte-Carlo) simulation. The authors show that the results from the method converge to the results from the analytical Black-Scholes method. The method presented greatly simplifies the calculation of the real option value, making it more transparent and brings real option valuation as a method a big leap closer to practitioners. The most positive issue in this method is that it does not suffer from the problems associated with the assumptions connected to the market processes connected to the Black-Scholes and the binomial option valuation methods. The method utilizes cash-flow scenario based estimation of the future outcomes to derive the future pay-off distribution this is highly compatible with the way cash-flow based profitability analysis is commonly done in companies.

All of the above mentioned models and methods use probability theory in their treatment of uncertainty, there are however, other ways than probability to treat uncertainty, or imprecision in future estimates, namely fuzzy logic and fuzzy sets. In classical set theory an element either (fully) belongs to a set or does not belong to a set at all. This type of bi-value, or true/false, logic is commonly used in financial applications (and is a basic assumption of probability theory). Bi-value logic, however, presents a problem, because financial decisions are generally made under uncertainty. Uncertainty in the financial investment context means that it is in practice impossible, ex-ante to give absolutely correct precise estimates of, e.g., future cash-flows. There may be a number of reasons for this, see, e.g., [14], however, the at the end of the day we our estimations are less than fully accurate.

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Fuzzy sets are sets that allow (have) gradation of belonging, such as "a future cash flow at year ten is about x euro". This means that fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain or imprecise knowledge, e.g., future cash-flow estimation, which human reasoning is especially adaptive to. "Fuzzy set-based methodologies blur the traditional line between qualitative and quantitative analysis, since the modeling may reflect more the type of information that is available rather than researchers' preferences" [20] and indeed in economics "The use of fuzzy subsets theory leads to results that could not be obtained by classical methods" [19]. The origins of fuzzy sets date back to an article by Lotfi Zadeh [23] where he developed an algebra for what he called fuzzy sets. This algebra was created to handle imprecise elements in our decision making processes, and is the formal body of theory that allows the treatment of practically all decisions in an uncertain environment. "Informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not" [1].

2 Fuzzy Sets and Fuzzy Numbers

A *fuzzy subset* A of a non-empty set X can be defined as a set of ordered pairs, each with the first element from X, and the second element from the interval $[0,1]$, with exactly one ordered pair present for each element of X. This defines a mapping,

$$\mu_A \colon X \to [0,1],$$

between elements of the set X and values in the interval $[0,1]$. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The set X is referred to as the universe of discourse for the fuzzy subset A. Frequently, the mapping μ_A is described as a function, the membership function of A. The degree to which the statement "x is in A" is true is determined by finding the ordered pair $(x, \mu_A(x))$. The degree of truth of the statement is the second element of the ordered pair. It is clear that A is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

It should be noted that the terms *membership function* and *fuzzy subset* get used interchangeably and frequently we will write simply $A(x)$ instead of $\mu_A(x)$. A γ-level set (or γ-cut) of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\gamma$ and defined by

$$[A]^\gamma = \{t \in X | A(t) \geq \gamma\},$$

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if $\gamma > 0$ and cl(supp$A$) if $\gamma = 0$, where cl(supp$A$) denotes the closure of the support of A. A fuzzy set A of X is called *convex* if $[A]^\gamma$ is a convex subset of X for all $\gamma \in [0,1]$. A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support [5]. Fuzzy numbers can be considered as possibility distributions.

Definition 2.1. *Let A be a fuzzy number. Then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\gamma \in [0,1]$. Let us introduce the notations*

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma$$

In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut, $\gamma \in [0,1]$.

Definition 2.2. *A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \le t \le a \\ 1 - \dfrac{t-a}{\beta} & \text{if } a \le t \le a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0,1].$$

The support of A is $(a - \alpha, b + \beta)$. A triangular fuzzy number with center a may be seen as a fuzzy quantity "x is approximately equal to a".

Definition 2.3. *The* possibilistic (or fuzzy) mean value *of fuzzy number A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ is defined in [5] by*

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \, d\gamma = \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma \, d\gamma.$$

Example 2.1. *If $A = (a, \alpha, \beta)$ is a triangular fuzzy number with center a, left-width $\alpha > 0$ and right-width $\beta > 0$ then a γ-level of A is computed by*

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0,1],$$

Then,

$$E(A) = \int_0^1 \gamma[a - (1-\gamma)\alpha + a + (1-\gamma)\beta] d\gamma = a + \frac{\beta - \alpha}{6}.$$

When $A = (a, \alpha)$ is a symmetric triangular fuzzy number we get $E(A) = a$.

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Definition 2.4. *A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a - \alpha \le t \le a \\ 1 & \text{if } a \le t \le b \\ 1 - \dfrac{t-b}{\beta} & \text{if } a \le t \le b + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation

$$A = (a, b, \alpha, \beta). \tag{1}$$

It can easily be shown that $[A]^\gamma = [a-(1-\gamma)\alpha, b+(1-\gamma)\beta]$ for all $\gamma \in [0, 1]$. The support of A is $(a - \alpha, b + \beta)$.

Example 2.2. *Let $A = (a, b, \alpha, \beta)$ be a fuzzy number of trapezoidal form with peak $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$. It is easy to compute that,*

$$E(A) = \frac{a+b}{2} + \frac{\beta - \alpha}{6}.$$

Fuzzy set theory uses fuzzy numbers to quantify subjective fuzzy observations or estimates. Such subjective observations or estimates can be, e.g., estimates of future cash flows from an investment. To estimate future cash flows and discount rates "One usually employs educated guesses, based on expected values or other statistical techniques" [4], which is consistent with the use of fuzzy numbers. In practical applications the most used fuzzy numbers are trapezoidal and triangular fuzzy numbers. They are used, because they make many operations possible and are intuitively understandable and interpretable.
When we replace non-fuzzy numbers (crisp, single) numbers that are commonly used in financial models with fuzzy numbers we can construct models that include the inaccuracy of human perception, or ability to forecast, within the (fuzzy) numbers. This makes these models more in line with reality, as they do not simplify uncertain distribution-like observations to a single point estimate that conveys the sensation of no-uncertainty. Replacing non-fuzzy numbers with fuzzy numbers means that the models that are built must also follow the rules of fuzzy arithmetic.

3 Fuzzy Numbers in Option Valuation

Fuzzy numbers (fuzzy logic) have been adopted to option valuation models in (binomial) pricing an option with a fuzzy payoff, e.g., in [17], and in

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Black-Scholes valuation of financial options in, e.g., [22]. There are also some option valuation models that present a combination of probability theory and fuzzy sets, e.g., [24]. Fuzzy numbers have also been applied to the valuation of real options in, e.g., [6, 9], and [7]. More recently there are a number of papers that present the application of fuzzy RO models in the industry setting, e.g., [8, 21]. There are also specific fuzzy models for the analysis of the value of optionality for very large industrial real investments, e.g., [10].

In the following section we will present a new method for valuation of real options from fuzzy numbers that is based on the previous literature on real option valuation, especially the findings presented in [15] and on fuzzy real option valuation methods, we continue by illustrating the use of the method with a selection of different types of fuzzy numbers and with a case application of the new method in an industry setting, and close with a discussion and conclusions.

4 New Fuzzy Pay-Off Method for Valuation of Real Options from Fuzzy Numbers

Two recent papers [16, 15] present a practical probability theory based method for the calculation of real option value (ROV) and show that the method and results from the method are mathematically equivalent to the Black-Sholes formula [2]. The method is based on simulation generated probability distributions for the NPV of future project outcomes. The method implies that: the real-option value can be understood simply as the average net profit appropriately discounted to Year 0, the date of the initial R & D investment decision, contingent on terminating the project if a loss is forecast at the future launch decision date. The project outcome probability distributions are used to generate a payoff distribution, where the negative outcomes (subject to terminating the project) are truncated into one chunk that will cause a zero payoff, and where the probability weighted average value of the resulting payoff distribution is the real option value.

We use fuzzy numbers in representing the expected future distribution of possible project costs and revenues, and hence also the profitability (NPV) outcomes. When using fuzzy numbers the fuzzy NPV itself is the payoff distribution from the project.

The method presented in [15] implies that the weighted average of the positive outcomes of the payoff distribution is the real option value; in the case with fuzzy numbers the weighted average is the fuzzy mean value of the positive NPV outcomes (which is nothing more than the possibility weighted average). Derivation of the fuzzy mean value is presented in [5].

This means that calculating the real option value (ROV) from a fuzzy NPV (distribution) is straightforward, it is the fuzzy mean of the possibility dis-

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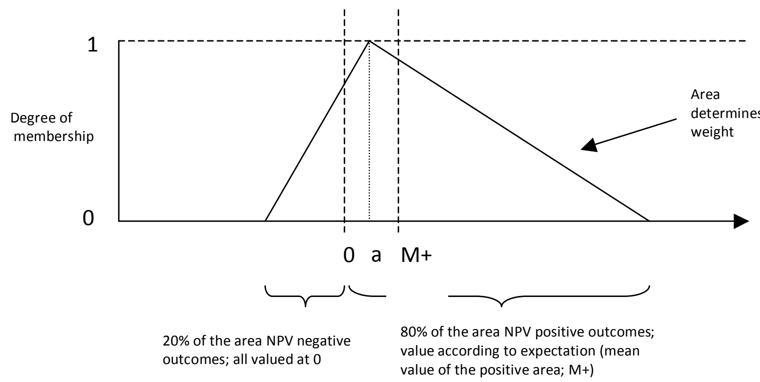

Figure 1: A triangular fuzzy number A, defined by three points $\{a, \alpha, \beta\}$ describing the NPV of a prospective project; (percentages 20% and 80% are for illustration purposes only).

tribution with values below zero counted as zero, i.e., the area weighted average of the fuzzy mean of the positive values of the distribution and zero (for negative values).

Definition 4.1. *We calculate the real option value from the fuzzy NPV as follows*

$$\mathrm{ROV} = \frac{\int_0^\infty A(x)\mathrm{d}x}{\int_{-\infty}^\infty A(x)\mathrm{d}x} \times E(A_+) \tag{2}$$

where A stands for the fuzzy NPV, $E(A_+)$ denotes the fuzzy mean value of the positive side of the NPV, and $\int_{-\infty}^\infty A(x)\mathrm{d}x$ computes the are below the whole fuzzy number A, and $\int_0^\infty A(x)\mathrm{d}x$ computes the area below the positive part of A.

It is easy to see that when the whole fuzzy number is above zero then ROV is the fuzzy mean of the fuzzy number, and when the whole fuzzy number is below zero the ROV is zero
The components of the new method are simply the observation that real option value is the probability weighted average of the positive values of a payoff distribution of a project, which is nothing more than the fuzzy NPV of the project, and that for fuzzy numbers the probability weighted average of the positive values of the payoff distribution is nothing more than the weighted fuzzy mean of the positive values of the fuzzy NPV, when we use fuzzy numbers.

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5 Calculating the ROV with the Fuzzy Pay-Off Method with a Selection of Different Types of Fuzzy Numbers

As the form of a fuzzy number may be variable the most used forms are the triangular and trapezoidal fuzzy numbers. These are very usable forms, as they are easy to understand and can be simply defined by three (triangular) and four (trapezoidal) values.
We shall calculate the positive area and the fuzzy mean of the positive area of a triangular fuzzy pay-off $A = (a, \alpha, \beta)$ in the case of $a - \alpha < 0 < a$. Variable z, where $0 \leq z \leq \alpha$, represents the distance of a general cut point from $a - \alpha$ at which we separate the triangular fuzzy number (distribution) into two parts - for our purposes the variable z gets the value $\alpha - a$ (we are interested in the positive part of A). Let us introduce the notation

$$(A|z)(t) = \begin{cases} 0 & \text{if } t \leq a - \alpha + z \\ A(t) & \text{otherwise} \end{cases}$$

for the membership function of the right-hand side of a triangular fuzzy number truncated at point $a - \alpha + z$, where $0 \leq z \leq \alpha$.
Then we can compute the expected value of this truncated triangular fuzzy number

$$\begin{aligned} E(A|z) = I_1 + I_2 = & \int_0^{z_1} \gamma(a - \alpha + z + a + (1-\gamma)\beta)d\gamma + \\ & \int_{z_1}^{1} \gamma(a - (1-\gamma)\alpha + a + (1-\gamma)\beta)d\gamma \end{aligned} \tag{3}$$

where

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}$$

and the integrals are computed by

$$I_1 = \int_0^{z_1} [(2a - \alpha + z + \beta)\gamma - \beta\gamma^2]d\gamma = (2a - \alpha + z + \beta)\frac{z^2}{2\alpha^2} - \beta\frac{z^3}{3\alpha^3}$$

and

$$I_2 = \int_{z_1}^{1} [(2a+\beta-\alpha)\gamma - \gamma^2(\beta-\alpha)]d\gamma = (2a+\beta-\alpha)\left(\frac{1}{2} - \frac{z^2}{2\alpha^2}\right) - g(\beta-\alpha)\left(\frac{1}{3} - \frac{z^3}{3\alpha^3}\right)$$

that is,

$$I_1 + I_2 = (2a - \alpha + z + \beta) \times \frac{z^2}{2\alpha^2} - \beta \times \frac{z^3}{3\alpha^3} + (2a + \beta - \alpha) \times \left(\frac{1}{2} - \frac{z^2}{2\alpha^2}\right) -$$

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$$(\beta - \alpha) \times \left(\frac{1}{3} - \frac{z^3}{3\alpha^3}\right) = \frac{z^3}{2\alpha^2} + \frac{2a - \alpha + \beta}{2} + \frac{\alpha - \beta}{3} - \alpha \times \frac{z^3}{3\alpha^3},$$

and we get,

$$E(A|z)) = \frac{z^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}.$$

If $z = \alpha - a$ then $A|z$ becomes A_+, the positive side of A, and therefore, we get

$$E(A_+) = \frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}.$$

To derive the real option value with the above formulas we must calculate the ratio between the positive area of the triangular fuzzy number and the total area of the same number and multiply this by $E(A_+)$, the fuzzy mean value of the positive part of the fuzzy number A, according to equation (2). Derivation of the fuzzy mean value for the positive part of trapezoidal fuzzy pay-off and the same for a special case fuzzy pay-off distribution are shown in Appendix 1.

6 A Simple Case: Using the New Method in Analyzing Acquisition Synergy as a Real Option

The problem at hand is to evaluate the value of uncertain synergies arising from a corporate acquisition that is estimated to last for seven years at maximum. The acquiring company has identified three possible scenarios, good, most likely, and bad, for the investments to realize the synergies and the synergy benefits.

The scenario values are given by managers as non-fuzzy numbers, they can have used any type of analysis tools, or models to reach these scenarios. From these cost & benefit scenarios three scenarios for the NPV are combined (PV benefits - PV investment costs), where the cost cash-flows (CF) are discounted at the risk free rate and the benefit CF discount rate is selected according to the risk (risk adjusted discount rate). The NPV is calculated for each of the three scenarios separately, for detailed calculation see Appendix 2.

The resulting fuzzy NPV is the payoff distribution for the synergies investment. The real option value for the investment can be calculated from the resulting fuzzy NPV, which is the pay-off distribution for the project, according to the formula presented in (2). In this case, as the whole distribution is above zero and the ROV is nothing else than the fuzzy mean value of the fuzzy NPV.

The company managers are accustomed to giving information in the form of scenarios (usually three) and they have a set of methods for building the scenarios - usually coming from past experience and based on looking at

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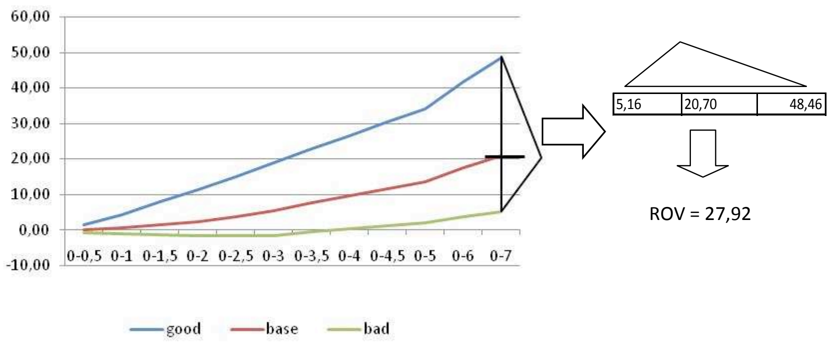

Figure 2: Three NPV scenarios for the duration of the synergies that are used to generate (triangular) fuzzy NPV.

issues like the most contributing single issues (or variables) and the markets, similar approaches are reported, e.g., in [13].
With the fuzzy pay-off method the scenario approach can also be fully omitted and the future forecast can be done, from the beginning of the process until the end, with fuzzy numbers - the end result will be a fuzzy NPV, which is the pay-off distribution for the project; this is the same result that we get if we use scenarios, however, does not require us to simplify the future to three alternative scenarios.

7 Discussion and Conclusions

There is reason to expect that the simplicity of the presented method is an advantage over more complex methods. Using triangular and trapezoidal fuzzy numbers make very easy implementations possible with the most commonly used spreadsheet software; this opens avenues for real option valuation to find its way to more practitioners. The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis. Fuzzy NPV is a distribution of the possible values that can take place for NPV; this means that it is by definition perceived as impossible at the time of the assessment that values outside of the number can happen this is in line with the situation that real option value is zero when all the values of the fuzzy NPV are lower than zero. If we compare this to the presented case, we can see that in practice it is often that managers are not interested to use the full distribution of possible outcomes, but rather want to limit their assessment to the most possible alternatives (and leaving out the tails of the distribution). We think that the tails should be included in the real option analysis, because even remote possibilities should be taken into consideration.

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The method brings forth an issue that has not gotten very much attention in academia, the dynamic nature of the assessment of investment profitability, i.e., the assessment changes when information changes. As cash flows taking place in the future come closer, information changes, and uncertainty is reduced this should be reflected in the fuzzy NPV, the more there is uncertainty the wider the distribution should be, and when uncertainty is reduced the width of the distribution should decrease. Only under full certainty should the distribution be represented by a single number, as the method uses fuzzy NPV there is a possibility to have the size of the distribution decrease with a lesser degree of uncertainty, this is an advantage vis-à-vis probability based methods.
The common decision rules for ROV analysis are applicable with the ROV derived with the presented method. We suggest that the single number NPV needed for comparison purposes is derived from the (same) fuzzy NPV by calculating the fuzzy mean value. This means that in cases when all the values of the fuzzy NPV are greater than zero the single number NPV equals ROV, which indicates immediate investment.
We feel that the presented new method opens possibilities for making simpler generic and modular real option valuation tools that will help construct real options analyses for systems of real options that are present in many types of investments.

APPENDIX 1.

Let us consider a trapezoidal fuzzy pay-off distribution A defined by

$$A(u) = \begin{cases} \dfrac{u}{\alpha} - \dfrac{a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \leq u \leq a_1 \\ 1 & \text{if } a_1 \leq u \leq a_2 \\ \dfrac{u}{-\beta} + \dfrac{a_2 + \beta}{\beta} & \text{if } a_2 \leq u \leq a_2 + \beta \\ 0 & \text{otherwise} \end{cases}$$

where the γ-level of A is defined by $[A]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$ and its expected value is caculated by

$$E(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}.$$

Then we have the following five cases,

Case 1: $z < a_1 - \alpha$. In this case we have $E(A|z) = E(A)$.

Case 2: $a_1 - \alpha < z < a_1$. Then introducing the notation,

$$\gamma_z = \frac{z}{\alpha} - \frac{a_1 - \alpha}{\alpha}$$

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we find

$$[A]^\gamma = \begin{cases} (z, -\gamma\beta + a_2 + \beta) & \text{if } \gamma \le \gamma_z \\ (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta) & \text{if } \gamma_z \le \gamma \le 1 \end{cases}$$

and,

$$E(A|z) = \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)d\gamma + \int_{\gamma_z}^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta)d\gamma$$

$$= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} + (z - a_1 + \alpha)\frac{\gamma_z^2}{2} - \alpha\frac{\gamma_z^3}{3}$$

Case 3: $a_1 < z < a_2$. In this case $\gamma_z = 1$ and

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta]$$

and we get,

$$E(A|z) = \int_0^1 \gamma(z - \gamma\beta + a_2 + \beta)d\gamma = \frac{z + a_2}{2} + \frac{\beta}{6}$$

Case 4: $a_2 < z < a_2 + \beta$. In this case we have

$$\gamma_z = \frac{z}{-\beta} + c\frac{a_2 + \beta}{\beta}$$

and,

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta],$$

if $\gamma < \gamma_z$ and we find,

$$E(A|z) = \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)d\gamma = (z + a_2 + \beta)\frac{\gamma_z^2}{2} - \beta\frac{\gamma_z^3}{3}.$$

Case 5: $a_2 + \beta < z$. Then it is easy to see that $E(A|z) = 0$

In the special case 1 we expect that the managers will have already performed a building of three scenarios and have assigned probabilities to each scenario (adding to 100%). We want to use all this information and hence will assign the same 'probabilities' to the scenarios resulting in a fuzzy number that has a graphical presentation of the following type (not in scale):

$$A(u) = \begin{cases} (\gamma_3 - \gamma_1)\frac{u}{\alpha} - (\gamma_3 - \gamma_1)\frac{a - \alpha}{\alpha} + \gamma_1 & \text{if } a - \alpha \le u \le a \\ \gamma_3 & \text{if } u = a \\ (\gamma_2 - \gamma_3)\frac{u}{\beta} - (\gamma_2 - \gamma_3)\frac{a}{\beta} + \gamma_3 & \text{if } a \le u \le a + \beta \\ 0 & \text{otherwise} \end{cases}$$

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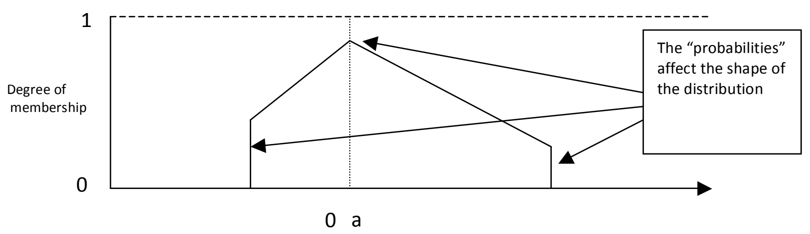

Figure 3: Calculation of the fuzzy mean for the positive part of a fuzzy pay-off distribution of the form of special case 1.

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \int_0^1 \gamma a_1(\gamma)d\gamma + \int_0^1 \gamma a_2(\gamma)d\gamma$$

$$\int_0^1 \gamma a_1(\gamma)d\gamma = \int_0^{\gamma_1} \gamma(a-\alpha)d\gamma + \int_{\gamma_1}^{\gamma_3} \gamma(\frac{\gamma-\gamma_1}{\gamma_3-\gamma_1}\alpha + a - \alpha)d\gamma$$

$$= (a-\alpha)\frac{\gamma_1^2}{2} + (a - \alpha - \frac{\alpha\gamma_1}{\gamma_3-\gamma_1})(\frac{\gamma_3^2}{2} - \frac{\gamma_1^2}{2}) + \frac{\alpha}{\gamma_3-\gamma_1}(\frac{\gamma_3^3}{3} - \frac{\gamma_1^3}{3})$$

$$\int_0^1 \gamma a_2(\gamma)d\gamma = \int_0^{\gamma_2} \gamma(a+\beta)d\gamma + \int_{\gamma_2}^{\gamma_3} \gamma(\frac{\gamma-\gamma_3}{\gamma_2-\gamma_3}\beta + a)d\gamma$$

$$= (a+\beta)\frac{\gamma_2^2}{2} + (a - \frac{\beta\gamma_3}{\gamma_2-\gamma_3})(\frac{\gamma_3^2}{2} - \frac{\gamma_2^2}{2}) + \frac{\beta}{\gamma_2-\gamma_3}(\frac{\gamma_3^3}{3} - \frac{\gamma_2^3}{3})$$

$$E(A) = \frac{\gamma_1^2}{2}\frac{\alpha\gamma_1}{\gamma_3-\gamma_1} + \frac{\gamma_2^2}{2}(\beta + \frac{\beta\gamma_3}{\gamma_2-\gamma_3}) + \frac{\gamma_3^2}{2}(2a - \alpha - \frac{\alpha\gamma_1}{\gamma_3-\gamma_1} - \frac{\beta\gamma_3}{\gamma_2-\gamma_3}) - \frac{\gamma_1^3}{3}\frac{\alpha}{\gamma_3-\gamma_1}$$

$$-\frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2-\gamma_3} + \frac{\gamma_3^3}{3}(\frac{\alpha}{\gamma_3-\gamma_1} + \frac{\beta}{\gamma_2-\gamma_3})$$

1. $z < a - \alpha : E(A|z) = E(A)$
2. $a - \alpha < z < a$:$\gamma_z = (\gamma_3 - \gamma_1)\frac{z}{\alpha} - (\gamma_3 - \gamma_1)\frac{a-\alpha}{\alpha} + \gamma_1$

$$E(A|z) = \frac{\gamma_z^2}{2}(z - a + \alpha + \frac{\alpha\gamma_1}{\gamma_3-\gamma_1}) + \frac{\gamma_2^2}{2}(\beta + \frac{\beta\gamma_3}{\gamma_2-\gamma_3}) +$$
$$\frac{\gamma_3^2}{2}(2a - \alpha - \frac{\alpha\gamma_1}{\gamma_3-\gamma_1} - \frac{\beta\gamma_3}{\gamma_2-\gamma_3}) - \frac{\gamma_z^3}{3}\frac{\alpha}{\gamma_3-\gamma_1}$$

$$-\frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2-\gamma_3} + \frac{\gamma_3^3}{3}(\frac{\alpha}{\gamma_3-\gamma_1} + \frac{\beta}{\gamma_2-\gamma_3})$$

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3. $a < z < a + \beta : \gamma_z = (\gamma_2 - \gamma_3)\dfrac{z}{\beta} - (\gamma_2 - \gamma_3)\dfrac{a}{\beta} + \gamma_3$

$$E(A|z) = \frac{\gamma_z^2}{2}\left(z + a - \frac{\beta}{\gamma_2 - \gamma_3}\right) + \frac{\gamma_2^2}{2}\left(\beta + \frac{\beta\gamma_3}{\gamma_2 - \gamma_3}\right) + \frac{\gamma_z^3}{3}\frac{\beta\gamma_3}{\gamma_2 - \gamma_3} - \frac{\gamma_2^3}{3}\frac{\beta}{\gamma_2 - \gamma_3}$$

4. $a + \beta < z : E(A|z) = 0$

APPENDIX 2.

Investment / cost cash-flows (PV@"risk free" level) Rf= 0,05

	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6
CF Good	1,00	1,00	1,00	1,00	1,00	1,00	0,00	0,00	0,00	0,00	0,00	0,00
CF Base	0,60	0,60	0,60	0,60	0,60	0,60	0,08	0,08	0,00	0,00	0,00	0,00
CF Bad	0,40	0,40	0,40	0,40	0,40	0,40	0,33	0,33	0,05	0,05	0,00	0,00
PV Good	1,00	0,98	0,95	0,93	0,91	0,89	0,00	0,00	0,00	0,00	0,00	0,00
PV Base	0,60	0,59	0,57	0,56	0,54	0,53	0,06	0,06	0,00	0,00	0,00	0,00
PV Good	0,40	0,39	0,38	0,37	0,36	0,35	0,28	0,27	0,04	0,04	0,00	0,00

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7
ΣPV Good	1,00	1,98	2,93	3,86	4,76	5,65	5,65	5,65	5,65	5,65	5,65	5,65
ΣPV Base	0,60	1,19	1,76	2,31	2,86	3,39	3,45	3,52	3,52	3,52	3,52	3,52
ΣPV Bad	0,40	0,79	1,17	1,54	1,91	2,26	2,54	2,81	2,86	2,90	2,90	2,90

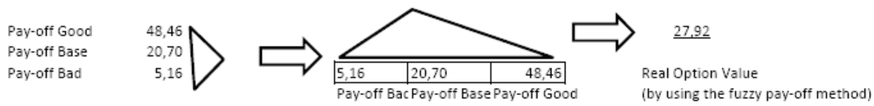
Revenue cash-flows (PV @ "risk adjusted" level) Ra = 0,15

	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	6
CF Good	1,80	3,50	4,50	4,80	5,40	6,10	6,30	6,50	6,80	6,90	15,50	15,50
CF Base	0,56	1,21	1,56	1,95	2,49	3,04	3,41	3,48	3,58	3,65	7,60	7,90
CF Bad	0,31	0,61	0,75	0,91	1,13	1,33	1,46	1,53	1,57	1,60	3,29	3,39
PV Good	1,80	3,26	3,91	3,89	4,08	4,30	4,14	3,99	3,89	3,68	7,71	6,70
PV Base	0,56	1,13	1,36	1,58	1,88	2,15	2,24	2,14	2,05	1,95	3,78	3,42
PV Good	0,31	0,57	0,65	0,74	0,85	0,94	0,96	0,94	0,90	0,85	1,64	1,46

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7
ΣPV Good	1,80	5,06	8,98	12,87	16,95	21,25	25,40	29,38	33,27	36,95	44,65	51,36
ΣPV Base	0,56	1,69	3,04	4,63	6,51	8,66	10,90	13,03	15,08	17,03	20,80	24,22
ΣPV Bad	0,31	0,88	1,53	2,27	3,12	4,06	5,02	5,96	6,86	7,71	9,35	10,81

	0-0,5	0-1	0-1,5	0-2	0-2,5	0-3	0-3,5	0-4	0-4,5	0-5	0-6	0-7
FNPV Good	1,40	4,27	7,81	11,33	15,05	18,99	22,85	26,57	30,41	34,05	41,76	48,46
FNPV Base	-0,04	0,50	1,29	2,31	3,65	5,27	7,44	9,51	11,56	13,51	17,29	20,70
FNPV Bad	-0,69	-1,10	-1,39	-1,59	-1,64	-1,59	-0,63	0,31	1,21	2,06	3,70	5,16

Fuzzy NPV (FNPV) is the pay-distribution from the project. In this case it is calculated by deducting the bad case cost from the good case revenue, the base case cost from the base case revenue, and the good case cost from the bad case revenue - this way the resulting distribution includes the extremes (this is in line with the standard operations with fuzzy numbers).



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