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27 February 2009

Online at https://mpra.ub.uni-muenchen.de/13624/
MPRA Paper No. 13624, posted 26 Feb 2009 04:55 UTC

COULD MARKETS' EQUILIBRIUM SETS BE FRACTAL ATTRACTORS?

C-René Dominique*

"[A]n estimate of probability after the fact is impossible unless
we can assume that the past is a reliable guide to the future."
Jacques Bernoulli (1654 – 1705)

"No one will be considered literate tomorrow who is not familiar with fractals."
John Archibald Wheeler (1911 -)

ABSTRACT: The assumption that markets are positive linear structures moving toward stable fixed-point equilibia is not supported by empirical investigations. This note reformulates the purest and the simplest of all Walrasian models, i. e., a pure exchange economy, and shows that event such a simple market moves toward a compact time-invariant set of prices due to the constant destruction and creation of excess demands under the impulsion of self-interested agents with strong monotone preferences. Fractal attractors better explain continuous market fluctuations, 'black swans', and the reason behind the flawed risk assessments of market risks of the financial engineers of Wall Street.

Key Words: Market Equilibria, Market Fluctuations, 'Black Swans', Fractal Attractors, Risk Assessment.

I- INTRODUCTION

As both quantities and prices are always positively valued, neo-classical economists adhere to the Walrasian practice of representing the economy including its individual markets as a positive linear system moving inexorably toward an asymptotically stable fixed-point equilibrium price vector **p*** from any initial price vector **p**(0) > **0** (see, Arrow and Debreu, 1954; Arrow, Brock and Hurwicz, 1958; Arrow and Hurwicz, 1959, among others). Therefore, any observed deviations from **p*** are viewed as temporary disturbances that are normally distributed about **p*** with zero mean μ and constant variance σ^2. It then follows that σ, the standard deviation, may be used to assess market risks.

Succinctly put, this is in essence the neo-classical economic paradigm. Its logic underpins the Bayesian viewpoint which holds that randomness can be quantified. Accordingly, an outcome is risky whenever its randomness can be quantified with numerical probabilities; otherwise, the outcome is termed uncertain. As is well-known, the returns on any asset are risky. To determine the price of a security, for example, it is necessary to have an idea of the risk associated with its future returns at the time of the pricing. Belief in the above paradigm naturally led adherents to claim that there is a 'y' percent probability that the returns will fall within ± 'x'σ about **p***.

There are a number of unresolved issues relative to that set of priors. In the first place, market prices are easily observable, but there is no way of knowing whether or not observed prices are transient or equilibrium prices, and there does not seem to be a particular **p*** to which the system returns when disturbances die-out. In fact, disturbances are recurrent, and their origin is unexplained. Second, attempts, no matter how sophisticated, to forecast **p*** are always plagued with errors. And, risk assessments, based on Abraham de Moivre's bell-curve (a. k. a. the Gaussian distribution) systematically lead assessors astray. Indeed, recent risk assessments on Wall Street are now recognized as an important factor behind the collapse of the capital market in October 2008.

These incongruities and systematic failures can not be simply brushed aside. Instead, they cast a serious doubt on both the existence of an asymptotically stable fixed-point equilibrium price vector and the notion that price fluctuations are random. Indeed, I have argued elsewhere (Dominique, 2008b) that randomness does not seem to exist in nature. When outcomes appear random, it is because the deterministic mechanism generating the outcome is hidden from observation. I believe that that was the position of Albert Einstein too, and it is supported by countless thought experiments. Moreover, statistical analyses of price movements, whether the calculation of correlation dimension (Grassberger and Proccacia, 1983), positive Lyapunov exponent, power spectral analysis, or rescaled range analysis (see Hurst et al., 1951), etc., systematically reveal the presence on non-linearity. Yet, non-linear modeling still does not figure prominently in economists' tool kit.

The purpose of this note is to propose a return to basics. It will examine the simplest of all general equilibrium models, i. e., the Walrasian pure exchange economy, for the presence of a fractal attractor. Success will prove, by extension at least, the presence of non-linearity in more realistic economic models with feedbacks, time-to-build, delays, and institutional rigidities.

II- THE WALRASIAN SYSTEM

A Walrasian pure exchange model supposes that there exist $i \varepsilon m$ consumers and $j \varepsilon n$ goods. The column vector $\mathbf{x}$ represents the n goods while the row vector $\mathbf{p}$ represents their prices. For any non-negative set of prices, the demand of i for good j is $x^i_j(\mathbf{p})$, and i's initial endowments are $\Sigma_j \omega^i_j$. The market demand of all i for good j is $\Sigma_i x^i_j(\mathbf{p})$. Similarly, the market supply function for any non-negative set of prices is accordingly $\Sigma_i \omega^i_j(\mathbf{p})$. The difference between the market demand and supply functions is the excess demand $\zeta_j(\mathbf{p})$. If the nth good is taken as the numéraire, the n – 1 excess demands are represented as a column vector $\zeta(\mathbf{p}) = (\zeta_j(p_1), \zeta_j(p_2), \ldots, \zeta_j(p_{n-1}))^T$, where the T represents the transpose operation. The market equilibrium is then defined as the vector of non-negative price vector $\mathbf{p}^*$ such that excess demands $\zeta_j(p_j)$, $\forall j \varepsilon (n\text{-}1)$ are non positive.

The problem that the market[1] must solve is:

$$(1) \qquad d\,\mathbf{p}/dt = \zeta(\mathbf{p}) = [\,\mathrm{diag}\,(1/p_j)\,(\mathbf{B} - \mathrm{diag}\,(\Sigma_j\,\omega^i_j)]\,\mathbf{p}, \quad \text{subject to } \mathbf{p}(0) > \mathbf{0} \text{ and } \textstyle\sum_j \omega^i_j > 0.$$

The bracketed term is the (n-1) x (n-1) matrix $\mathbf{M}_k$, and[2]

$$
\mathbf{B} = \begin{bmatrix}
\Sigma_{i\varepsilon m}\, (\Pi\, \alpha^{i}_{1}\omega^{i}_{1}) \ldots\ldots\ \Sigma_{i\varepsilon m}\, (\Pi\, \alpha^{i}_{1}\omega^{i}_{n-1}) \\
\Sigma_{i\varepsilon m}\, (\Pi\, \alpha^{i}_{2}\, \omega^{i}_{2} \ldots\ldots\ \Sigma_{i\varepsilon m}\, (\Pi\, \alpha^{i}_{2}\, \omega^{i}_{n-1} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
.\qquad\qquad .\qquad\qquad . \\
\Sigma_{i\varepsilon m}\, (\Pi\, (\alpha^{i}_{n-1}\omega^{i}_{n-1}) \ldots \Sigma_{i\varepsilon m}\, (\Pi\, \alpha^{i}_{n-1}\omega^{i}_{n-1})
\end{bmatrix}
$$

In this standard representation, the matrix $\mathbf{M}_k$ is derived from utility maximization. Hence, the coefficients are necessarily constants, k = 1, making (1) a positive linear homogeneous time-invariant system. As $\mathbf{M}_k$ is a Metzler (1945) matrix ($m_{ij} \geq 0$, $\forall i \neq j$), it can be argued that if the system did start at any $\mathbf{p}(0) > \mathbf{0}$, it would have preserved the non-negativity of the state vector $\mathbf{p}$. According to the Frobenius-Perron Theorem (see references), that would have been necessary and sufficient for the existence and the local stability of the fixed-point.

To determine the asymptotic stability of the fixed-point it suffices to expand $\zeta\,(\mathbf{p})$ about $\mathbf{p}^*$ in a Taylor's series expansion:

(2) $$(d\mathbf{p}/dt)^T = (\zeta\,(\mathbf{p}^*))^T + \mathbf{J}_{p^*}\,(\mathbf{p} - \mathbf{p}^*)^T + \ldots,$$

where $\mathbf{J}$ is the Jacobian matrix evaluated at $\mathbf{p}^*$, $\zeta\,(\mathbf{p}^*) = \mathbf{0}$, $(\mathbf{p} - \mathbf{p}^*)$ is the Euclidean distance from the equilibrium. Differentiating the squared distance with respect to time gives:

(3) $$d\,(\mathbf{p} - \mathbf{p}^*)^2/dt = -\,2\,\mathbf{p}^*\,\zeta\,(\mathbf{p}\,(t)).$$

If Equations (1) – (3) were a realistic representation of the pure exchange economy, then the assumption that price movements follow a 'random walk' would be justified, but they are not. In the next section, I will reexamine system (1) in view of showing why it is not a true representation of the actual system, but the reader might have already noticed that the problem stems from the assumption of constant coefficients.

III- AN ALTERNATIVE VIEWPOINT

Each i, being a price taker in the market, acts independently while being unable to recognize equilibrium prices as such. He or she observes only relative and transient prices and adjusts according to the process:

(4) $$d\,p_j / dt \rightarrow \left\{\begin{smallmatrix} > \\ = \\ < \end{smallmatrix}\right\} 0 \ \text{ if } \ \zeta_j\,\mathbf{p}) \rightarrow \left\{\begin{smallmatrix} > \\ = \\ < \end{smallmatrix}\right\} 0, \ \ \forall j\ \varepsilon\ (n-1).$$

At this juncture, the interesting question is: What can i really adjust? I have shown elsewhere (Dominique, 2008a) that i does not maximize any unobservable utility function. Instead, it is the coefficients of $\mathbf{M}_k$ that adjust to the destruction or the creation of excess demands. Indeed, returning to the matrix $\mathbf{B}$ above, it is easily seen that i can carry out the process of adjustment described in (4) either by modifying his budget share distribution ($\Sigma_j\ \alpha^{i}_{j} = 1$) and /or the quantities of initial endowments ($\Sigma_j\ \omega^{i}_{j}$) brought to the market in response to relative price variations. And precisely that uncoordinated adjustment process destroys the linearity of (1).

To see how linearity comes to be lost, consider the following. I first denote a homomorphism as Hom (.), an endomorphism as End (.), | . | is the norm of vector space, and R as the real line. Then I appeal to two theorems without proofs since they can be found in Taylor and Mann (1972, 315 - 24):

Theorem 1: *Each m x n matrix is the standard representation of a unique linear transformation from R^n to R^m and, conversely, every element of Hom (R^n , R^m) has a unique standard representation as an m x n matrix.*

Furthermore, if Tε Hom (R^n , R^m) and Fε Hom (R^m, R^p), then $(F \circ T)(\mathbf{p})$ ε Hom (R^n, R^p) = $F(T(\mathbf{p}))$, $\forall \mathbf{p}$ ε n.

Theorem 2 : *If T is a linear transformation from R^n to R^m, then there exists some number M such that:* $|T\mathbf{p}| \leq \mathbf{M}\,|\mathbf{p}|,\ \forall \mathbf{p}\ \varepsilon R^n$.

It should also be noticed that T is continuous, and End (R^n) = Hom (R^n, R^n) denotes the vector space of all linear transformations from R^n to R^n, representable by n x n matrices. Now, let Γ ε End (R^n) be the open subset of all invertible transformations in End (R^n) such that for each $\zeta(\mathbf{p})$ ε R^n, there is a $\mathbf{p}$ in R^n, such that

(1') $$T_k = \zeta(\mathbf{p}_k) = \mathbf{M}_k\,\mathbf{p}_k,\ k = 1, 2, \ldots$$

Equation (1') is conform to Equation (1) in its general form. However, if $\zeta(\mathbf{p}) \neq \mathbf{0}$, the uncoordinated adjustment process is ongoing. Let Δ be the set of all possible uncoordinated adjustments:

(5) $$\Delta = \{\Delta_k : \Delta_k\ \varepsilon\ \Delta\},\ \ k = 1, 2, \ldots,$$

And let ∇ be the set of all resulting $\mathbf{M}_k$:

(6) $$\nabla = \{\nabla_k : \nabla_k\ \varepsilon\ \nabla\},\ \ k = 1, 2, \ldots.$$

In the pure exchange model, all prices are relative as the nth good may be selected as the numéraire. Then, the set of all invertible transformations in End (R^{n-1}) is:

(7) $$\Gamma = \{T_k : T_k\ \varepsilon\ \Gamma\},\ \ k = 1, 2, \ldots$$

By Theorem 2, $T_k : \zeta(\mathbf{p}) \rightarrow \mathbf{p}_k^*$. Since $\mathbf{M}_k$ is a representation of T_k, we may write $\mathbf{M}_k$: $\zeta(\mathbf{p}) \rightarrow \mathbf{p}_k^*$, $\forall$k. Then, there exist mappings g, h, and ϕ such that:

(8) $$\begin{gathered} g : \Delta \rightarrow \nabla; \\ h: \nabla \rightarrow \mathbf{p}_k^* \\ \phi\ (=h \circ g) : \Delta \rightarrow \mathbf{p}_k^*. \end{gathered}$$

This is to say that each time an adjustment is carried out, the coefficients of $\mathbf{M}$ undergo a change, implying the selection of a different T_k ε End (R^{n-1}, R^{n-1}). Then, as the mapping ϕ (.) is continuous, we may write:

ϕ_t (**p**): $\Delta \rightarrow E_k \varepsilon$ End (R^{n-1}), where E_k mimics a compact invariant set of equilibrium prices. This would mean that E_k may be a strange attractor if there is a neighborhood N of E_k such that if $\phi_t(\mathbf{p}_k)$ is in N at $t \geq 0$, then $\phi_t(\mathbf{p}_k) \rightarrow E_k$ as $t \rightarrow \infty$ for all $\mathbf{p}_k$ in N. The union of all N's of E_k is the domain of attraction and is the stable manifold E_k to which all orbits are attracted. This happens to be the definition of a fractal attractor.

If E_k is indeed a fractal attractor, then it is made of an infinite number of branched surfaces which are interleaved and which intersect, although trajectories do not. Instead, trajectories would move from one branched surface to another. Hence, E_k would contain a countable set of periodic orbits of arbitrarily long periods, an uncountable set of aperiodic orbits, and a dense orbit. Having these properties would make E_k a strange or fractal attractor. The question now is: How can we be sure of this?

IV- THE TEST

From the kind of aperiodic motions described above, one might be tempted at first sight to conclude the behavior of E_k is conformable to that of a Snap-back Repeller (see, Morotto, 1979). While it is true that a Snap-back Repeller displays periodic orbits of all integer periods, but it also displays an uncountable number of exponentially diverging aperiodic orbits, which imply chaos or period-3 in R^n. But, a close examination of the Jacobian matrix of (1'), supported by many empirical studies, rules that out.

The other possibility is a fractal attractor (see Mandelbrot, 1983a; 1983b). Fortunately, there is a simple test to verify that supposition. One of the main properties of fractal systems, which happen to be more common in nature, is the notion of self-similarity. Self-similarity refers to the number of copies of itself an object of any size contains, and obviously, except for scale, the copies are identical to the object itself. This then allows a generalization of the concept of Hausdorff's (1919) dimensions, d. By definition, a Hausdorff measure is the ratio of the logarithm of the increase in the length of a curve, say, to the logarithm of the decrease in the scale used to measure the curve. For smooth manifolds, Hausdorff dimensions coincide with the Euclidean dimensions, but for irregular shapes, the Hausdorff measures are fractional or non-integer valued. Mandelbrot (1983a) refers to such measures as fractal dimensions, while Grassberger and Proccacia (1983) prefer the term 'correlation dimension'. Regardless of preference though, the main point is that knowing the number of smaller copies of itself the object contains and their relative sizes, logarithms allows the determination of the fractal dimension of the object.

Let the number of smaller copies be n, and let s be the length reduction factor, then

(9) $$n = (-1)(1/s)^d;$$

expanding (9), we have :

(10) $$d = \log n / \log s.$$

For example, the Cantor's (1883) set is a well-known fractal object. It consists of two smaller copies of itself of sizes 1/ 3 each. Then its fractal dimension is log 2 / log 3. The von Koch (1904) curve snow flake is another fractal object that contains 4 smaller copies of itself of sizes 1 /3 each; its fractal dimension is log 4 / log 3. Obviously, some shapes are more difficult to measure, but there is always a way to arrive at the fractal dimension of any shape.

Applying this result to the Walrasian economy, we must first recall that money is neutral in the sense that it only determines the relativity of prices. Therefore, the size of the whole Walrasian economy is n – 1. Its smallest size is $\zeta_j(\mathbf{p})$, the excess demand for good j, determining the relative price p_j. The size of the smallest market is then (1/ (n-1)). Hence, the fractal dimension (3) of (1'), d_M, is:

(11) $$1 \leq d_M = \log n / \log (n-1) \leq 2,$$

This then demonstrates that the Walrasian economy is indeed a fractal structure. It is easily seen also that it approaches linearity only as $n \to \infty$.

We could also turn to the data for a confirmation of the above conclusion. All actual markets operate on the same principles. It suffices therefore to look at capital markets where 'Rescaled Range' analyses are commonly used as tools to determine persistence or anti-persistence that might be present in these markets. To do so, the analyst just calculates the Hurst's (1951) exponent e^H for a particular market. If it is equal to 1/ 2, it is concluded that the capital market follows a random walk, following Albert Einstein's work on Brownian motion. If it is $\neq$ 1/ 2, that indicates a biased random walk or a fractal structure. Casti (1994) reports earlier results of e^H for various international capital markets. The values range from 0.68 to 0.78, thus confirming two of my main points. First, these markets, being actual markets, are clearly much more complicated than a pure exchange market; yet, they preserve the property of non-linearity and, of course, they too do not follow a random walk. The second point is that once the Hurst's coefficient for a market is known, its fractal dimension is easily calculated from the approximation given in (12). As the data used to calculate e^H exist in 2-D space, the fractal dimensions of the capital markets examined by Casti can be obtained from the formula:

(12) $$d_c = 2 - e^H.$$

This means their fractal dimensions range from 1.22 and 1.32.

There are other studies that lend support to this conclusion. For example, Peters (1989, 1990) has demonstrated that a fractal attractor of dim 2.33 lurks at the heart of the S&P-500 Index. The calculated value of the S&P-500 from Casti's data is d_c = 1.22, but it is from data covering the period of January 1950 to July 1988. Peter's value implies that the S&P-500 has become a more jagged index over the years. Furthermore, Gensay and Stengos (1988) and Brock (1986) have found that, on the whole, the Canadian and U.S. economies, respectively, are characterized by low deterministic chaos, which is another name for fractal attractors. This is not at all surprising. Fractals are everywhere in nature and in the universe, from the coast line (d = 1.26), to the typical cloud (d = 1.35), to complex protein surfaces (d =2.4), to the human brain (d = 2.72), on to the observable universe (4.02 ± 0.10), etc. One of the most intriguing fractals to date is a mathematical object created by Mandelbrot, known as the Mandelbrot set. That set which results from a countable infinity of iterations on a powerful computer gives a multitude of spokes. Counting them from the largest to the smallest gives the Fibonacci sequence. That sequence is one of the most ubiquitous and important constructs in all of mathematics. It is: $\{x_i\} = x_{i-2} + x_{i-1},$ for $x_0 = x_1 = 0$, $i \geq 2$ and the nth number in the sequence, is given by Binet's formula as $x_n = (1/\sqrt{5})\,[(1+\sqrt{5})^n / 2 - (1-\sqrt{5})^n / 2]$. The first term in Binet's formula is the famous 'golden ratio' raised to the power of n. As it can be seen, as the values become large, the second term vanishes, and we are left with the golden ratio raised to the power of n. The golden ratio itself is omnipresent in nature, arising in a bewildering variety of settings, ranging from pinecones, to sun flower seeds, to the branching of trees, and on to the spiral patterns of the DNA molecule and galactic arms, etc. It is not surprising therefore that markets, as human constructs, are unable to escape the ascendancy of the organizing principle of fractals.

A few words of caution with regard to statistical analyses are in order here, however. First, as it can be seen, a high fractal value indicates a highly jagged structure, but by definition to be a fractal it suffices to have a non integer value no matter how low. In other words, a value of 1 would indicate that the log-log graph of the variable in question is a straight line, indicating linearity; whereas a value between 1 and 2 indicates a graph that lives in between 1-D and 2-D spaces. Second, the power of empirical methods used to determine whether or not a time series is characterized by a fractal attractor seem to have low power over series with few entries. Instead, very long series are recommended, and weekly entries are to be preferred over monthly or yearly entries.

IV CONCLUDING REMARKS

Behind the collapse of the world capital market in October 2008 obviously lie many factors. The neoliberal orthodoxy that gained the upper hand in governments, corporations, and international economic institutions during the late 1980s is one. For it gave rise to two American legislative decisions, namely the Repeal of the Glass-Steagall Act (GSA) in 1999 and the passage of the Commodity Futures Modernization Act (CFMA) of 2000. The repeal of the GSA allowed commercial banks to engage in speculation over unviable mortgages drawn up by unscrupulous individuals, and permitted non commercial institutions to over-leverage themselves in the frenzy to create virtual wealth. The CFMA, on the other hand, sanctioned an atmosphere which encouraged pundits to behave swaggeringly in risky situations.

A second factor is the ascendance of the Bayesian viewpoint on the uses of statistics. It should be recalled that, according to Reverend Bayes, uncertainty could be quantified and treated as bets placed on a roulette wheel, or reduced to probability distribution, eschewing all judgment. The admonitions of Bernoulli, Keynes, Frank Knight, Wheeler, etc. as regards the difference between chance games and the natural events, were simply brushed aside on Wall Street on the erroneous assumption that mathematical modeling devised by the so-called financial experts could tame uncertainty. Bankers and CEOs became oblivious to the frenzy of reckless bets as long as they themselves were handsomely rewarded for making short term profits. Blinded by greed, they were adamant in converting bogus mortgages into collateralized debt obligations (CDO's) and in taking or giving Credit Default Swaps (CDS's). When the deception was finally uncovered, the market swung to a cycle of long period, i. e. to a black swan.

The twin assumption of linearity and normally distributed price movements was the main focus of this note. It begins by taking a brief look at the neo-classical economic theory which led its adherents to believe that economies are time invariant positive linear systems. This in turn is the reason why the Bayesian viewpoint so easily gained the upper hand in the early 1940s. These two assumptions ignore another important admonition. That is that of Henri Poincaré, one of the fathers of dynamical analyses, to the effect that we should aim at qualitative rather than quantitative solutions when faced with non-linearity. That too was eschewed by the quants. Thus, ignoring all cautions and admonitions, these Wall Street actors precipitated the collapse of the U.S capital market and that of the world economy by extension.

The assumption of linearity is a carry over from Walras (1883, 1900) which obviously has always been at variance with empirical observations. The paper proposes a reformulation which shows that the continuous destruction and creation of excess demands by self-interested agents with strong monotone preferences is the law of motion in the non-linear Walrasian system. Instead of fixed-point equilibrium, it leads instead to a compact invariant set of prices; in other words, to a fractal attractor. In that reformulation, constant market fluctuations are explained by aperiodic cycles. And what Professor Taleb has coined 'black swan' is explained by cycles of arbitrarily long periods. These findings are reminiscent of the difference between fractal and normal distributions. As it is now well-known, a fractal distribution assigns a higher probability of occurrence to events considered extremely rare in the normal distribution. Together they provide a good explanation as to why the financial experts of Wall Street strayed so far in their assessment of market risks.

Our results are not only in line with empirical investigations, but they also show why the die-hard Walrasian linearity assumption must be revisited. Indeed, had Walras himself known about fractal structures, and had today's modern statistical techniques been available to him, economic analysis would have been put on a different trajectory. And may be, just may be, we would not have been so keen on holding to untenable notions to the effect that markets are efficient and self-correcting linear systems.

NOTES

1 To derive (1), no utility maximization is necessary. It suffices to know the budget share distribution and the distribution of initial emdowments.

2 For ease of interpretation of matrix **B**, note that $b_{11} = (\alpha^1{}_1\, \omega^1{}_1 + \alpha^2{}_1\, \omega^2{}_1 + \ldots + \alpha^m{}_1\, \omega^m{}_1)$.

3 If n is the number of copies, while (n-1) is the smallest size obtainable, then log n = d log (n – 1), Dividing by log (n – 1) gives Equation (10).

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