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22 February 2009

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February 2009.
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1. Introduction: Aggregation of individual preferences into a social preference is a major issue in welfare economics. The least that any such aggregation procedure is required to guarantee is what is known in the literature as the Pareto principle. The Pareto principle says that if a social state is perceived as "no better than" a second social state by all individuals and strictly worse than the second by some individual then the second social state is socially preferred to the first. The Pareto principle gives rise to the Pareto relation in a natural manner: a social state is said to be "Pareto superior to" or "Pareto dominate" a second social state if and only if the first is at least as good as the second for all individuals and strictly better than the second for some. Had it not been that the Pareto relation is possibly incomplete (i.e. there may be social states which are not comparable via the Pareto relation) the problem of preference aggregation would have been practically non-existent. If the Pareto relation were always complete then we could identify social preferences with the Pareto relation. However, since the Pareto relation usually admits non-comparable pairs the problem of preference aggregation arises. Why do we insist on complete comparability of all pairs of social states for an aggregation rule? This is done simply to ensure that undominated choices from a set of social states are indeed the socially "best" choices from the same set of social states. This result appears in Sen [1970]

Most of preference aggregation theory is concerned with individual preferences which are extremely well-behaved, i.e. reflexive, transitive and connected. Such individual preferences are known as orderings. A consequence of individual preferences being orderings is that the Pareto relation becomes transitive. However there is little justification in economics for individuals to exhibit the kind of consistency that aggregation theory demands. It is important that a decision maker is capable of taking decisions when confronted with a choice problem. For decision making to be possible it is not necessary that preferences be transitive. It is well known that a necessary and sufficient condition for decision making from finite sets of alternatives to be possible is that the preference of the decision maker is acyclic. Further if all individuals have acyclic preferences then the Pareto relation (interpreted in the "strong" sense and discussed later) is acyclic and not necessarily transitive.

Conventional aggregation theory also requires that the social preference that is derived from individual preferences is an ordering over social states. Once again for social decision making acyclicity rather than transitivity is the required feature. Unlike the problems that social choice theory has to contend with when individual and social

preferences are assumed to be orderings, there are fewer problems and meaningful aggregation procedures when social preferences are required to be merely acyclic. A good exposition of the aggregation theory when social welfare is assumed to be acyclic can be found in Moulin (1988). More up-to-date results can be found in Banks (1995).
There are two very important questions that arise in the context of Pareto relations and social welfare relations. The first question is the following: Is the set of all Pareto optimal social states the same as the set of social states chosen by some Pareto consistent social welfare relation? The second question in the same context is: How well do Pareto consistent approximations of a social welfare relation perform in choosing the social states that a given Pareto consistent social welfare relation would?
Before we proceed to answer these two questions it is important to point out that the set of all Pareto relations when individual preferences are assumed to be reflexive and acyclic is identical to the set of all reflexive and acyclic binary relations. To see this note that if all individual preferences are reflexive then the Pareto relation is clearly reflexive. Further if we interpret Pareto superiority in the strong sense (i.e. a social state is (strongly) Pareto superior to a second only if all individuals prefer the first to the second) then the existence of a Pareto preference cycle would have to coincide with the same preference cycle arising for each individual separately, the latter being ruled out if individual preferences are acyclic. Thus acyclicity of individual preferences implies acyclicity of the Pareto relation. On the other hand given any reflexive and acyclic binary relation, we can interpret that as the Pareto relation of a society where every individual's preference coincides with the given relation.
Of the two questions mentioned above, the first question has been dealt with by Banerjee and Pattanaik (1996) in the case when the Pareto relation is assumed to be reflexive and transitive and social welfare is an ordering. In such an environment they show that the set of all Pareto optimal social states does indeed coincide with the set of all social states chosen by Pareto consistent social welfare orderings. In this paper we show that such an identity prevails if the Pareto relation, as well as all Pareto consistent social welfare relations is acyclic. The proof of this result is much simpler than the corresponding proof in Banerjee and Pattanaik (1996). This is largely because it is easy to show that given any reflexive and acyclic binary relation it is possible to extend it to a connected, reflexive and acyclic binary relation by simply including all non-comparable pairs along with the given relation. A consequence of our result and that of Banerjee and Pattanaik (1996) is that if the Pareto relation is transitive then the set of all social states chosen by social welfare orderings is the same as the set of all social states chosen by acyclic social welfare relations.
The second question that we posed has been discussed by Suzumura (1999) in a setting similar to the one in Banerjee and Pattanaik (1996). In Suzumura (1999) it is shown that the social states chosen by a Pareto consistent social welfare relation coincides with the set of all social states that are chosen by every binary relation that strictly includes the Pareto relation and is strictly included in the social welfare relation. The requirement that the "approximate" relations are strictly included in the given social welfare relation is to ensure that the result is non-trivial. The requirement that the "approximate" relations strictly include the Pareto relation is to ensure the veracity of the assertion. However, these strict inclusions require the assumption that the set of such approximations is non-empty. Our corresponding result tells a similar story when the Pareto relation and the

Pareto consistent social welfare relation is acyclic. We need to make the same non-emptiness("non-triviality") assumption that appears in Suzumura (1999). Further our proof is almost identical to the proof of the corresponding result (called "Recoverabilty Theorem") in Suzumura (1999). Since transitive binary relations are always acyclic, the Recoverability theorem follows from our second theorem.

It is hoped that the extensions of existing results for social welfare orderings to the situation where we merely require acyclicity of social welfare relations will be a small but fruitful step towards broadening the horizons of "applicable" welfare economics.

2. The Model: Let X be a non-empty finite set of social states and $<X>$ the set of all non-empty subsets of X.

A binary relation Q on X is a subset of $X \times X$; if $(x,y) \in Q$, we often represent it as xQy.

A binary relation Q is said to be *reflexive* if for all $x \in X$ it is the case that xQx; it is said to be *connected* if for all $x,y \in X$ with $x \neq y$ it is the case that either xQy or yQx.

Given a binary relation Q, its *asymmetric* part denoted P(Q) is the binary relation $\{(x,y) \in Q: (y,x) \notin Q\}$; its *symmetric* part denoted I(Q) is the binary relation $\{(x,y) \in Q: (y,x) \in Q\}$.

Given a binary relation Q a P(Q)-*cycle* is a non-empty finite subset set $\{x_1, x_2, \ldots, x_K\}$of X for some positive integer $K > 1$ such that: (i) $x_i\, P(Q) x_{i+1}$ for $i = 1, \ldots, K-1$; (ii) $x_K P(Q) x_1$.

A binary relation Q is said to be *acyclic* if it does not have a P(Q)-cycle.

Given a binary relation Q and $S \in <X>$, let $M(S,Q) = \{x \in S$: for all $y \in S$ it is the case that $(y,x) \notin P(Q)\}$. M(S,Q) is said to be the set of Q-*undominated* social states in S.

It is well known that the set of Q- undominated social states in S is non-empty for all $S \in <X>$ if and only if Q is acyclic.

If Q is reflexive and connected then for all $S \in <X>$: $M(S,Q) = \{x \in S$: for all $y \in S$ it is the case that $xQy\}$, where the latter set is called the set of Q-*best* alternatives (social states) in S. This result can be found in Sen [1970] for instance.

A binary relation Q' is said to *extend* (or be *an extension of*) a binary relation Q if $Q \subset Q'$ and $P(Q) \subset P(Q')$. In such a situation Q is said to be a *sub-relation* of Q'.

Given a binary relation Q let $\overline{E}$(Q) denote the set of all reflexive, connected and acyclic binary relations that extend Q.

For the purpose of this paper a reflexive and acyclic binary relation will be called a *Pareto* relation.

In what follows we will assume that we are **given** a Pareto relation R^0.

Further, for the purpose of this paper any reflexive, connected and acyclic binary relation that extends R^0 will be referred to as a *Paretian Social Welfare Relation.*

Let $\Delta(X) = \{(x,x): x \in X\}$. $\Delta(X)$ is called the diagonal of X. A binary relation is reflexive if and only if it contains $\Delta(X)$.

3. Reflexive, connected and acyclic extensions: The following proposition is significant for the purposes of this paper.

Proposition 1: Given a binary relation Q, $\overline{E}$(Q) is nonempty if and only if Q is acyclic.

Proof: First suppose $\overline{E}$(Q) is nonempty. Then since for all Q'$\in$ $\overline{E}$(Q) it is the case that P(Q) $\subset$ P(Q') it follows that any P(Q)-cycle is a P(Q')-cycle for all such Q'. If Q'$\in$ $\overline{E}$(Q) then there are no P(Q')-cycles. Hence the non-emptiness of $\overline{E}$(Q) implies the absence of P(Q)-cycles. Thus Q must be acyclic.

Now suppose Q is acyclic. If Q$\in$ $\overline{E}$(Q) we are done. If not then the set N(Q) $\equiv$ $\{(x,y)\in X\times X: (x,y)\notin Q \;\&\; (y,x)\notin Q\}$ is non-empty. Note if (x,y)$\in$N(Q) then so does (y,x). Further if (x,x)$\notin$Q for some x$\in$X, then (x,x)$\in$N(Q).

Let Q^0 = Q$\cup$N(Q). Clearly xI(Q^0)y for all (x,y)$\in$N(Q) and Q^0 is reflexive as well as connected.

Towards a contradiction suppose there is a P(Q^0)-cycle. Then for some positive integer K > 1 there exists a non-empty finite subset $\{x_1,\ldots,x_K\}$of X such that: (i) x_i P(Q^0)x_{i+1} for i = 1,…, K-1; (ii) x_KP(Q^0)x_1.

But this implies (i) x_i P(Q)x_{i+1} for i = 1,…, K-1; (ii) x_KP(Q)x_1, giving rise to a P(Q)-cycle and contradicting the acyclicity of Q.

Thus $Q^0\in$ $\overline{E}$(Q), i.e. $\overline{E}$(Q) is non-empty. Q.E.D.

We can use proposition 1 to prove the following result which has interesting consequences for Paretian social welfare relations.

Proposition 2: Let Q be a binary relation. Then for all S$\in$<X> it is the case that M(S,Q) = $\bigcup_{R\in\overline{E}(Q)} M(S,R)$ if and only if Q is acyclic.

Proof: First suppose that for all S$\in$<X> it is the case that M(S,Q) = $\bigcup_{R\in\overline{E}(Q)} M(S,R)$.

Towards a contradiction suppose that Q is not acyclic. Then by proposition 1, $\overline{E}$(Q) is empty.

Thus $\bigcup_{R\in\overline{E}(Q)} M(S,R)$ is empty for all S$\in$<X> although for all x$\in$X, it is the case that M({x}, Q) = {x}$\neq\phi$, leading to a violation of the assumed equality. Thus Q is acyclic.

Now suppose Q is acyclic. By proposition 1, $\overline{E}$(Q) is non-empty.

Let S$\in$<X>. Suppose x$\in$ $\bigcup_{R\in\overline{E}(Q)} M(S,R)$. Towards a contradiction suppose there exists y$\in$S such that yP(Q)x. Thus yP(R)x for all R$\in$ $\overline{E}$(Q) contradicting x$\in$ $\bigcup_{R\in\overline{E}(Q)} M(S,R)$.

Thus x$\in$M(S,Q). Thus $\bigcup_{R\in\overline{E}(Q)} M(S,R)\subset$ M(S,Q).

Now suppose x$\in$M(S,Q).

As in the proof of proposition 1, let N(Q) $\equiv$ $\{(w,z)\in X\times X: (w,z)\notin Q \;\&\; (z,w)\notin Q\}$.

Let Q' = Q$\cup$N(Q). Clearly Q'$\in$ $\overline{E}$(Q) and P(Q') = P(Q).

Thus x$\in$M(S,Q'), i.e. x$\in$ $\bigcup_{R\in\overline{E}(Q)} M(S,R)$.

Thus $M(S,Q) \subset \bigcup_{R\in \overline{E}(Q)} M(S,R)$.

Combining the two inclusions we get $M(S, Q) = \bigcup_{R\in \overline{E}(Q)} M(S,R)$. Q.E.D.

Let $\overline{E}^0$ denote $\overline{E}(R^0)$. By proposition 1, $\overline{E}^0$ is non-empty, i.e. the set of Paretian social welfare relations is non-empty. By proposition 2 we obtain the following theorem which is similar to a corresponding result available in Banerjee and Patttanaik [1996].

Theorem 1: For all $S\in <X>$: $M(S, R^0) = \bigcup_{R\in \overline{E}^0} M(S,R)$.

Theorem 1 says that that the set of Pareto optimal social states coincides with the set of social states that are chosen by some Paretian social welfare relation.

4. Approximating a Paretian social welfare relation: Now suppose as in Suzumura [1999] that R is a given Paretian social welfare relation. Unlike Suzumura [1999] transitivity or even consistency of binary relations plays no role in our analysis. Given a set of social states, it is easy to see that the R-undominated social states (or the social choice set) from S will be Pareto optimal for S(i.e. undominated by the Pareto relation in S). The question that we are interested in is the following: what is the precise structure of the social choice set from S in terms of reflexive and acyclic binary relations that satisfies the Pareto principle and approximates R, i.e. is an extension of R^0 and a sub-relation of R?

Let $\Theta(R^0, R) = \{Q\subset X\times X: R^0 \subset\subset Q \subset\subset R \text{ and } P(R^0)\subset P(Q) \subset P(R)\}$.

Thus if $Q\in \Theta(R^0, R)$ then Q is reflexive (since R^0 is) and acyclic (since R is).

Neither R^0 nor R belongs to $\Theta(R^0, R)$ since we require each binary relation in $\Theta(R^0, R)$ to be a strict superset of R^0 and a strict subset of R.

The remarkable fact is that the following result similar to the “Recoverability” theorem in Suzumura [1999] holds in this acyclic scenario.

Theorem 2: Suppose $\Theta(R^0, R)$ is non-empty. Then for all $S\in <X>$: $M(S,R) = \bigcap_{Q\in\Theta(R^0,R)} M(S,Q)$.

Proof: Let $S\in <X>$.

First suppose $x\in M(S,R)$. Thus for all $y\in S$ it is the case that $(y,x)\notin P(R)$.

Towards a contradiction suppose there exists $Q'\in \Theta(R^0, R)$ such that $x\notin M(S,Q')$.

Thus there exists $y\in S$ such that $yP(Q')x$.

Since $P(Q') \subset P(R)$ we get that $yP(R)x$ contradicting $x\in M(S,R)$.

Thus $x\in \bigcap_{Q\in\Theta(R^0,R)} M(S,Q)$.

Hence $M(S,R) \subset \bigcap_{Q\in\Theta(R^0,R)} M(S,Q)$.

Now suppose $x\in \bigcap_{Q\in\Theta(R^0,R)} M(S,Q)$.

Towards a contradiction suppose $x\notin M(S,R)$.

Thus there exists $y\in S$ such that $yP(R)x$. However for all $Q\in \Theta(R^0, R)$ it is the case that

$(y,x) \notin P(Q)$.
Consider $Q \cup \{(y,x)\}$ for $Q \in \Theta(R^0, R)$.
Suppose for some Q^0 there exists a positive integer K>1 and a non-empty subset $\{x_1, \ldots, x_K\}$ such that: (i) $x_1 = x$, $x_K = y$; (ii) $x_i P(Q^0) x_{i+1}$ for i = 1,…,K-1.
Then $x_i P(R) x_{i+1}$ for i = 1,…,K-1. This combined with yP(R)x contradicts the acyclicity of R.
Hence since Q^0 is acyclic $Q^0 \cup \{(y,x)\}$ is acyclic.
If xQ^0y, then it must be the case that xRy contradicting yP(R)x.
Thus for no $Q^0 \in \Theta(R^0, R)$ is it the case that xQ^0y.
Thus $yP(Q^0 \cup \{(y,x)\})x$.
Since $Q^0 \subset R$, $P(Q^0) \subset P(R)$ and yP(R)x, we get that R extends $Q^0 \cup \{(y,x)\}$.
Further $R^0 \subset Q^0 \cup \{(y,x)\}$ and $P(R^0) \subset P(Q^0) \subset P(Q^0 \cup \{(y,x)\})$.
If $Q^0 \cup \{(y,x)\} \subset\subset R$, then $Q^0 \cup \{(y,x)\} \in \Theta(R^0, R)$.
But $x \notin M(S, Q^0 \cup \{(y,x)\})$ contradicting $x \in \bigcap_{Q \in \Theta(R^0, R)} M(S,Q)$.
Hence it must be the case that $Q \cup \{(y,x)\} = R$ for all $Q \in \Theta(R^0, R)$.
Thus there exists a unique Q* such that $\Theta(R^0, R) = \{Q^*\}$.
Since $R^0 \subset\subset Q^*$, there exists $(w,z) \in Q^* \backslash R^0$.
For reasons identical to those that lead to $Q^0 \cup \{(y,x)\} \in \Theta(R^0, R)$ we get that both $R^0 \cup \{(y,x)\}$ and $R^0 \cup \{(w,z)\}$ belong to $\Theta(R^0, R)$ contradicting that $\Theta(R^0, R)$ is a singleton.
Thus $x \in M(S,R)$.
Thus $\bigcap_{Q \in \Theta(R^0, R)} M(S,Q) \subset M(S,R)$.
Combining the two inclusions we get that $\bigcap_{Q \in \Theta(R^0, R)} M(S,Q) = M(S,R)$. Q.E.D.

Acknowledgment: I would like to thank S. Chandramohan for much help with the references of this paper.

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