A sufficient condition for acyclic social choice in a single-profile world

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Abstract: In this paper we provide a sufficient condition for a social welfare relation to be a social decision relation (i.e. an acyclic social welfare relation) when the profile of individual preferences is given.

1. Introduction: Arrow's (1950, 1963) theorem shows that a "reasonable" rule that aggregates individual preferences does not exist (unless we are willing to make some compromises). The framework of Arrowian social choice theory assumes that the preference profile of individuals is a variable and an aggregation procedure assigns to each preference profile a social preference relation. In this framework along with a large body of impossibility results there is a sizeable literature that shows reasonable aggregation rules exist if we are willing to relax the requirement that social preference is transitive. A starting point for this alternative approach was provided in the work of Nakamura (1979) and Ferejohn and Fishburn (1979). A more recent paper and one that we shall be concerned with here is the one by Banks (1995), which also contains references to other results that were obtained in this direction.

In the literature concerned with Arrowian possibility theorems decisive sets play a crucial role. Suppose that an aggregation procedure is neutral (i.e. names of alternatives/candidates do not matter). A set of individuals is said to be decisive if given any two alternatives x and y, the fact that these agents prefer x to y is sufficient to guarantee that society prefers x to y. When individual preferences are allowed to be weak orders (i.e. complete and transitive) then instead of decisive sets we have decisive pairs. A decisive pair comprises two non-empty subsets of individuals with the first contained in the second such that if the smaller group strictly prefers x to y and the larger group considers x at least as good as y, then society prefers x to y, irrespective of what x and y are. Given a collection of decisive pairs its collegium is the set of individuals who belong to all the large sets and at least one small set. An aggregation procedure is said to be collegial if the collegium of every collection of decisive pairs is non-empty. The index of an aggregation procedure as defined in Banks (1995) is "plus infinity" if the procedure is collegial; *and* if the procedure is not collegial then its index is the cardinality of the smallest collection of decisive pairs whose collegium is empty. Note if a collection contains a single decisive pair then its collegium has got to be non-empty. Hence the index of an aggregation procedure has to be at least two. It can be shown that that in the Arrowian framework and under very innocuous assumptions, this index is at least three. Further, if the aggregation rule is a "voting rule" (i.e. satisfies an additional duality

condition in terms of the decisive pairs) then it is acyclic if and only if its index exceeds the cardinality of the set of alternatives. These results are available in Banks (1995).
The literature discussed above is based on assumptions that allow for preference profiles to vary. This flexibility allows considerable "degrees of freedom" that may not be available once we adhere to a fixed preference profile. Even with a fixed preference profile there are several ways of aggregating the individual preferences into a social welfare relation. Naturally we would want all such relations to be Paretian, i.e. if all individuals prefer x to y, then society should also prefer x to y and this should be reflected in the social welfare relation. However unless all individuals have identical preferences, the assumption that the social welfare relation is Paretian is not sufficient to identify a unique social preference. Thus questions similar to those that arise in the multi-profile set up can be addressed in a single profile world. Papers by Parks (1976), Hammond (1976), Kemp and Ng (1976), Pollack (1979), Roberts (1980), Rubinstein (1984) and more recently Feldman and Serrano (2008) show that if enough "diversity" is allowed in the given single preference profile, then under reasonable (or mild) assumptions impossibility of aggregation follows.
The starting point of our paper is the single profile framework as discussed in Feldman and Serrano (2008). In that paper it is shown that an assumption called "diversity under minimal decisiveness" leads to dictatorial social welfare relations, although unlike the multi-profile world dictatorship provides wider scope for interpretation in the single profile world. In our paper we ask the following question: Is there any known condition for aggregation procedures in multi-profile contexts to be acyclic that can be adapted to the single-profile framework and yield a similar result?
In order to answer this question we retain the Neutrality/monotonicity assumption in Feldman and Serrano (2008). This assumption says that if for all x,y,z,w agents who prefer x to y continue to prefer w to z, agents who prefer z to w also prefer y to x and society prefers x to y, then society also prefers w to z. We introduce the concept of a decisive pair along the lines suggested in Banks (1995). Further concepts such as collegium of a collection of decisive sets and collegiality of a social welfare relation are adapted from Banks (1995). Since there is limited maneuverability in a single profile world we modify the concept of a decisive set and call it "properly decisive". This is done to preempt both a pair of coalitions and the pair formed by its complements from being decisive. In fact there are straightforward examples that illustrate how mere decisiveness is vulnerable to ambiguities.
The index of a social welfare relation is defined as the cardinality of the smallest collection of decisive pairs whose collegium is empty. The index* of a social welfare relation is defined as the cardinality of the smallest collection of *properly* decisive pairs whose collegium is empty. If there is no such collection then the index* of the social welfare relation is plus infinity. Unlike the multi-profile context the single profile condition for a social welfare relation to be acyclic is only a sufficient condition and not a necessary one. We are able to show that if the index* of a social welfare relation is greater than the cardinality of the set of alternatives, then the social welfare relation is a social *decision* relation, i.e. it is acyclic. We achieve this by adjusting the first half of the proof of theorem 2 in Banks (1995), although our result relates more to theorem 4 in the paper just cited. We also show that the converse is not in general true, i.e. there are single-profile aggregation problems with an acyclic social welfare relation whose index*

is less than the cardinality of the set of alternatives. This is accomplished by using example 6 of Feldman and Serrano (2008).

2. The Model: Our model is motivated by the framework in Feldman and Serrano (2008). Thus we assume a society comprising $n \geq 2$ individuals and three or more alternatives. Let $N = \{1,2,\ldots,n\}$ denote the set of individuals and X denote the set of alternatives.

A binary relation R is said to be *asymmetric* if for all $x, y \in X$: [xRy] implies [*not* yRx].

The *asymmetric part* of (a binary relation) R denoted P(R) is defined as follows: for all $x,y \in X$: [xP(R)y] if and only if [xRy and *not* yRx].

The *symmetric part* of (a binary relation) R denoted I(R) is defined as follows: for all $x,y \in X$: [xI(R)y] if and only if [xRy and yRx].

Each agent $i \in N$ is assumed to have preferences over the set of alternatives which is represented by a binary relation R_i.

We assume that each R_i is reflexive (i.e. for all $x \in X$: xR_ix), connected (i.e. for all $x,y \in X$ with $x \neq y$: either xR_iy or yR_ix) and transitive (i.e. for all $x,y,z \in X$: [xR_iy, yR_iz] implies [xR_iz]). Note that a binary relation that is reflexive and connected is also said to be complete. R_i is said to be the *preference ordering* of i. The interpretation of xR_iy is that individual i considers alternative x to be "*at least as good as*" alternative y. The asymmetric part of R_i will be denoted P_i (instead of $P(R_i)$) and the symmetric part of R_i will be denoted I_i (instead of $I(R_i)$). The interpretation of xP_iy is that individual i "*prefers*" alternative x to alternative y; the interpretation of xI_iy is that individual i is "*indifferent between*" x and y.

The ordered n-tuple $(R_1,\ldots,R_n)$ is called the *preference profile* (of the society). In the current framework there is only one preference profile and we shall not be concerned with situations where society can (or needs to) contemplate alternative preference profiles.

A binary relation R is said to be *acyclic* if there does not exist a positive integer $K \geq 2$ and distinct alternatives $x_1, x_2,\ldots,x_K \in X$: (i) for all $i = 1,\ldots, K-1$ it is the case that $x_iP(R)x_{i+1}$; (ii) $x_KP(R)x_1$.

The problem we are concerned with in this paper concerns aggregating the given preference profile $(R_1,\ldots,R_n)$ into a binary relation that society may use in arriving at a decision. Apart from assuming that such a binary relation is "truly representative" of the society we shall also require that it is asymmetric and acyclic. In the present scenario we will refer (as in Lahiri (2009)) to an asymmetric binary relation on X as a *social welfare relation* and to an asymmetric and acyclic binary relations on X as a *social decision relation*.

In multi-profile contexts where preference profiles are variable, a function that assigns to each preference profile a asymmetric and acyclic binary relation (representing the preferences of society) is called a social decision function as for instance in Blau and Deb (1977) or Gaertner (2006). The reason why we refer to asymmetric and acyclic binary relations as social decision relations is because on every non-empty finite subset of X such a relation will have at least one maximal element.

While generic elements of X are represented by the letters x,y,z etc. specific elements will be represented by the letters a,b,c etc.

A social decision relation P is said to be *Paretian* if for all $x,y \in X$: [xP_iy for all $i \in N$] implies [xPy].

If P is Paretian then given any x,y in X if all individuals prefer x to y, it must be that society prefers x to y.

The following notation has been adapted from Banks (1995).
For $x,y \in X$, let $P(x,y) = \{i \in N/ xP_iy\}$ and $R(x,y) = \{i \in N/xR_iy\}$.

A social welfare relation P is said to satisfy *Neutrality/monotonicity* if for all $x,y,z,w \in X$: $[P(x,y) \subset P(w,z), R(x,y) \subset R(w,z)$ and $xPy]$ implies $[wPz]$.

In Feldman and Serrano (2008) a set of individuals V is said to be *decisive* (for a social welfare relation P) if it is nonempty and if, for all $x,y \in X$: $V \subset P(x,y)$, then xPy.
For our purposes the following definition adapted from Banks (1995) will turn out to be more convenient.

A pair (S,W) of sets of individuals (i.e. $S \subset N$ and $W \subset N$) is said to be a *decisive pair* (for a social welfare relation P) if the following is satisfied:
(i) $\phi \neq S \subset W$;
(ii) for all $x,y \in X$: $[S \subset P(x,y)$ and $W \subset R(x,y)]$ implies $[xPy]$.

If V is decisive (in the sense of Feldman and Serrano (2008)) then for any $W \subset N$ with $V \subset W$, it is the case that (V,W) is a decisive pair.

An individual i is said to be a dictator (for a social welfare relation P) if for all $x,y \in X$: $[xP_iy]$ implies $[xPy]$.
Thus if at the given profile it is the case that for all $x,y \in X$ we have xI_iy, then individual i is a dictator.

Given a social welfare relation P, let $\Delta_P = \{(S,W) \subset N \times N: (S,W)$ is a decisive pair for $P\}$.
If P is Paretian then Δ_P is non-empty since $(N,N) \in \Delta_P$. Further, if P satisfies Neutrality/monotonicity then for all $x,y \in X$ such that xPy we have $(P(x,y), R(x,y)) \in \Delta_P$.
In what follows we shall assume that a social welfare relation is Paretian and satisfies Neutrality/monotonicity.

3. Collegial social decision relations: Let P be a social welfare relation. Given a non-empty subset Δ of Δ_P, the *collegium* of Δ, denoted $\kappa(\Delta)$ is the set $(\bigcup_{(S,W)\in\Delta} S) \bigcap (\bigcap_{(S,W)\in\Delta} W)$.

Thus $\kappa(\Delta)$ is the set of those individuals who belong to every W and some S in Δ.
We say that a social decision relation P is *collegial* if for every non-empty subset Δ of Δ_P, the *collegium* of Δ is **non-empty**.
For a social welfare relation P, let $v(P) = +\infty$ if P is collegial
$= \min\{|\Delta|: \Delta \subset \Delta_P, \Delta \neq \phi$ and $\kappa(\Delta) = \phi\}$ otherwise.
v(P) is called the *index* of P.

for a voting rule to be acyclic is that the index of the voting rule is less than the number of alternatives, i.e. the cardinality of X. Our definition of collegium, collegial social welfare relations and index has been adapted from Banks (1995). So it is natural to try and see whether we get a result similar to the one by Banks in our context and if not the reasons for failure should be non-trivial. It turns out (as we shall see later) that the index of a social welfare relation being less than the cardinality of the set of alternatives is sufficient for social welfare relations to be social decision relation, i.e. acyclic. However for acyclicity it is not necessary that the index be less than the cardinality of X. The reason is quite trivial as the following example reveals.

Example 1: Let n = 3 and X = {a,b,c}. Suppose aP_1bP_1c, cP_2bP_2a and cP_3aP_3b. It is easy to see that majority rule is transitive leading to the social decision relation $P = P_3$. In fact individual 3 is a dictator in spite of majority rule since individual 1's preferences are exactly opposed to the preferences of individual 2. It is easy to observe that {1,2} as well as {3} are decisive in the sense of Feldman and Seranno (2008) and thus ({1,2}, {1,2}) as well as ({3}, {3}) belong to Δ_P.

Let Δ = ({1,2}, {1,2}), ({2}, {2})}. Clearly $\kappa(\Delta)$ is empty.

Thus $v(P) = 2 < 3 = |X|$, i.e. the index of P is less than the cardinality of X. However P is definitely acyclic.

The trouble with the example above is that the coalition {1,2} is vacuously decisive since there does not exist any $x,y \in X$ such that xP_iy for i = 1,2!

The above phenomenon leads to both {1,2} as well as its complement {3} being decisive.

In order to prevent both (S,W) as well as (S', W') with $S' \subset N \setminus W$ and $W' \subset N \setminus S$ from being in the reckoning we formulate the following concept similar to one available in Banks (1995).

Given a social welfare relation P, a decisive pair (S,W) for P is said to be *proper* if there exists $x,y \in X$ such that P(x,y) = S and R(x,y) = W. In such a situation we also say that (S,W) is properly decisive (for P).

Suppose (S,W) is properly decisive for P. Let S = P(a,b) and W = R(a,b). Thus aPb. Towards a contradiction suppose there exists (S', W') where $S' \subset N \setminus W = P(b,a)$ and $W' \subset N \setminus S = R(b,a)$ such that (S', W') is decisive. Since $S' \subset P(b,a)$ and $W' \subset R(b,a)$ we get bPa contradicting the asymmetry of P. Hence (S',W') cannot be decisive and hence cannot be properly decisive either.

It is worth noting that if instead of requiring P(x,y) = S and R(x,y) = W in the definition of properly decisive pairs, we had merely required the decisive pair (S,W) to satisfy $S \subset P(x,y)$ and $W \subset R(x,y)$ for some x,y in X then we would land up with the potentially problematic situation where ({i}, {i}) is a properly decisive pair for all $i \in N$, when $R_i = R_j$ for all $i,j \in N$. The collegium of {({i}, {i}), ({j}, {j})}for $i,j \in N$ with $i \neq j$ is empty. Thus $v(P) < 3 \leq |X|$. However given that the social welfare relation P is Paretian, we get $P = P_i$ for all $i \in N$ if $R_i = P_i \cup \{(x,x): x \in X\}$. Thus once again the necessity of (some version of) the index property for the existence of a social decision relation breaks down for a very trivial reason.

It is worth noting that (N,N) may not be properly decisive unless there exists $x,y \in X$ such that $P(x,y) = N$. In fact it is quite possible that (S,W) is properly decisive whereas (S', W') with $S \subset S' \subset W'$ is not simply because there does not exist $x,y \in X$ with $P(x,y) = S'$ and $R(x,y) = W'$. This of course is a drawback in our definition of proper decisive sets which does not appear to be easily remediable.

4. Properly decisive pairs and weakly collegial relations: Given a social welfare relation let $\Delta_P^0 = \{(S,W) \in \Delta_P: (S,W) \text{ is proper}\}$.

We say that a social welfare relation P is *weakly collegial* if for every non-empty subset Δ of Δ_P^0, the *collegium* of Δ is **non-empty**.

It is easy to see that if P is collegial then it is weakly collegial.

Claim 1: Let P be a social welfare relation (that is Paretian and satisfies Neutrality/monotonicity). Then for all $x,y \in X$: [xPy] implies [$P(x,y) \neq \phi$].

Proof of Claim 1: Let P be a social welfare relation and let xPy. Towards a contradiction suppose that $P(x,y) = \phi$. Thus $R(y,x) = N$. Thus $P(x,y) \subset P(y,x)$ and $R(x,y) \subset R(y,x)$. By Neutrality/monotonicity xPy implies yPx, contradicting the asymmetry of P. Thus $P(x,y) \neq \phi$. Q.E.D.

Claim 2: Let P be a social welfare relation (that is Paretian and satisfies Neutrality/monotonicity). Then for all $x,y \in X$: [xPy] implies [there exists $(S,W) \in \Delta_P$ such that $S \subset P(x,y)$ and $W \subset R(x,y)$].

Proof of Claim 2: Let P be a social welfare relation and let xPy. Let $S = P(x,y)$ and $W = R(x,y)$. By Neutrality/ monotonicity, for all $w,z \in X$: [$S \subset P(w,z)$, $W \subset R(w,z)$] implies [wPz]. By Claim 1, $S \neq \phi$ and hence $W \neq \phi$. Thus $(S,W) \in \Delta_P$ and since $S = P(x,y)$, $W = R(x,y)$, $(S,W) \in \Delta_P^0$. Q.E.D.

For a social welfare relation P, let $v^*(P) = +\infty$ if P is weakly collegial

$$= \min\{|\Delta|: \Delta \subset \Delta_P^0, \Delta \neq \phi \text{ and } \kappa(\Delta) = \phi\} \text{ otherwise.}$$

$v^*(P)$ is called the *index** of P.

If $v(P) = +\infty$ then so is $v^*(P)$. If $v(P) < +\infty$, then since $v(P) = \min\{|\Delta|: \Delta \subset \Delta_P, \Delta \neq \phi \text{ and } \kappa(\Delta) = \phi\} \leq \min\{|\Delta|: \Delta \subset \Delta_P^0, \Delta \neq \phi \text{ and } \kappa(\Delta) = \phi\} \leq v^*(P)$, we get that in any case $v(P) \leq v^*(P)$.

We can now obtain the following proposition.

Proposition 1: Let P be a (Paretian) social welfare relation (that satisfies Neutrality/monotonicity). If $v^*(P) > |X|$ then P is a social decision relation, i.e. P is acyclic. However the converse is not in general true.

In the statement of Claims 1 and 2 and Proposition 1 we have purposely put the two phrases in parenthesis, since given our blanket assumption that all social welfare relations being considered here are Paretian and satisfy Neutrality/monotonicity we do not need to mention them explicitly once again. Our purpose in mentioning the two properties explicitly in the statement of the proposition is purely for the purpose of emphasis and recall.

Proof of Proposition 1: Let P be a social welfare relation such that $v^*(P) > |X|$. Towards a contradiction suppose P is not acyclic. Thus there exists a positive integer $K \geq 2$ and distinct alternatives $x_1, x_2, \ldots, x_K \in X$: (i) for all $k = 1,\ldots, K-1$ it is the case that $x_k P x_{k+1}$; (ii) $x_K P x_1$.

Thus $K \leq |X| < v^*(P)$.

Let $S^k = P(x_k, x_{k+1})$ and $W^k = R(x_k, x_{k+1})$ for $k = 1,\ldots, K-1$; let $S^K = P(x_K, x_1)$ and $W^K = R(x_K, x_1)$.

By Claim 1, for $k = 1,\ldots,K$: $S^k \neq \phi$.

By Neutrality/ monotonicity (S^k, W^k) is a decisive pair for $k = 1,\ldots,K$ and by their definition, they are proper.

Let $\Delta = \{(S^k, W^k): k = 1,\ldots,K\}$.

Since $v^*(P) > |X| \geq K$, $\kappa(\Delta) = (\bigcup_{k=1}^{K} S^k) \cap (\bigcap_{k=1}^{K} W^k) \neq \phi$.

Let $i \in \kappa(\Delta)$.

Thus $x_k R_i x_{k+1}$ for $k = 1,\ldots,K-1$, $x_K R_i x_1$ **and** [*either* $x_k P(R_i) x_{k+1}$ for some $k = 1,\ldots,K-1$ *or* $x_K P(R_i) x_1$]. This contradicts the transitivity of R_i.

Hence P must be acyclic.

To show that the converse is not true, let n = 3 and X = {a,b,c,d}. Consider the following preference profile:

1) $aP_1bP_1cP_1d$;

2) $cP_2aP_2bP_2d$;

3) $aP_3cP_3bP_3d$.

Let P be the social welfare relation obtained by applying the pair-wise majority rule. Thus aPcPbPd, aPb, cPd, aPd.

Since P is transitive it is acyclic.

Observe the following:

(1) ({1,2}, {1,2}) is a properly decisive pair since it is a decisive pair and {1,2} = P(b,d) = R(b,d).

(2) ({1,3}, {1,3}) is a properly decisive pair since it is a decisive pair and {1,3} = P(a,c) = R(a,c).

(3) ({2,3}, {2,3}) is a properly decisive pair since it is a decisive pair and {2,3} = P(c,b) = R(b,d).

Let $\Delta = \{(\{1,2\}, \{1,2\}), (\{1,3\}, \{1,3\}), (\{2,3\}, \{2,3\})\}$.

Clearly $\kappa(\Delta) = \phi$.

Thus $v^*(P) \leq 3 < 4 = |X|$, although P is acyclic. Q.E.D.

An immediate consequence of proposition 1 and the fact that for all social welfare relations its index does not exceed index*, we have the following corollary

Corollary of Proposition 1: Let P be a (Paretian) social welfare relation (that satisfies Neutrality/monotonicity). If $v(P) > |X|$ then P is a social decision relation, i.e. P is acyclic. However the converse is not in general true.

Note : Although we require a social welfare relation to be Paretian and satisfy Neutrality/monotonicity, in neither the two claims or in proposition 1, do we make explicit use of the assumption that it is Paretian. However we do make use of the Neutrality/monotonicity assumption through out our analysis. If P is non-empty (i.e. there exists $x,y \in X$ such that xPy) then Neutrality/monotonicity of P implies that P is Paretian. For by Claim 1, $P(x,y) \neq \phi$ whenever xPy and then by Neutrality/monotonicity for all $w,z \in X$: [P(w,z) = N] implies [wPz], i.e. P is Paretian.

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