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Abstract. An explicit pricing formula for inflation bond options is proposed in the Jarrow-Yildirim model. The formula resembles that for coupon bond options in the HJM model.

1. Introduction

Jarrow and Yildirim (2003) introduce a model for Treasury Inflation-Protected Securities (TIPS) and inflation derivatives based on the Heath-Jarrow-Morton (HJM) model. The Jarrow-Yildirim model describes the behavior of the nominal and real yield curves and the inflation index. Jarrow and Yildirim (2003) also propose a formula for inflation index options. Their results are extended by Mercurio (2005) to zero-coupon inflation-indexed swap, year-on-year inflation-indexed swap and year-on-year inflation index cap. Mercurio (2005) also studies a market model for inflation. Independently, Belgrade et al. (2004) also propose a market model approach to zero-coupon and year-on-year swaps.

In this brief note, using techniques similar to those used to price coupon bond options in Henrard (2003), the price of options on capital-indexed inflation bonds is derived. The formula obtained is explicit up to a parameter that is computed as the unique solution of a one-dimensional equation. In particular the results can be applied to TIPS options.

The description of capital-indexed inflation bonds can be found in (Deacon et al., 2004, Section 2.2.1). The real amounts paid at dates \( t_i \) \((1 \leq i \leq n)\) are \( c_i \), or in nominal terms the amount are \( I_{t_i}c_i \). The amounts \( c_i \) include the specific convention and frequency of the bond and the principal at final date.

The discount factor linked to the real rates is denoted \( P_2(t_0, T) \). It is the discount factor viewed from \( t_0 \) for a payment in \( T \). The nominal value in \( t_0 \) of the bond described above is

\[
I_{t_0} \sum_{i=1}^{n} c_i P_2(t_0, t_i).
\]

2. Model and preliminary lemmas

The Jarrow-Yildirim model describes the behaviour of the instantaneous forward nominal \( (f_1) \) and real \( (f_2) \) interest rate. The forward rates viewed from \( t \) for the maturity \( T \) are denoted \( f_i(t, T) \) \((1 \leq i \leq 2)\). Throughout this paper the index 1 is related to the nominal rates, the index 2 to the real rates and the index 3 to the inflation. The (nominal and real) short-term rate are denoted \( r_i^t = f_i(t, t) \). The cash accounts linked to the nominal and real rates are

\[
N_i^t = \exp \left( \int_0^t r_i^s ds \right).
\]

The rate volatilities \( \sigma_i \) are deterministic. The bond volatilities are \( \nu_i(t, u) = \int_t^u \sigma_i(t, s) ds \). In the risk neutral world with numeraire \( N_{\Delta}^1 \) the equations of the model are given by the equation

\[
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\text{The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.}
\text{1Without loss of generality, the reference inflation index used in this document is always 1.}
(11)–(13) in Proposition 2 of Jarrow and Yildirim (2003) which are written below

\( df_1(t, T) = \sigma_1(t, T) \nu_1(t, T) dt + \sigma_1(t, T) dW_1^t \)
\( df_2(h, s) = \sigma_2(t, T) (\nu_2(h, s, T) - \rho_{13} \sigma_3(t)) dt + \sigma_2(t, T) dW_2^t \)
\( dI(t) = (r_1^i - r_1^i) Idt + \sigma_3(t) IdW_3^t. \)

The covariance between the different Brownian motions are \([W_i^t, W_j^t] = \rho_{i,j} t \) (1 ≤ i, j ≤ 3).

To obtain an explicit formula for the options on bonds, an extra condition on the real rate volatility is used. This is a separability condition which is satisfied by the extended Vasicek or Hull and White (1990) model and can be found in Henrard (2003) for options on coupon-bonds.

\textbf{(H):} the function \( \sigma_2 \) satisfies \( \sigma_2(t, u) = g(t)h(u) \) for some positive functions \( g \) and \( h \).

The following technical lemma on the cash accounts and bond prices will be useful. The formulas are equivalent to those for the HJM model obtained in Henrard (2006).

**Lemma 1.** Let \( 0 \leq t \leq u \leq v \). In the Jarrow-Yildirim model, the real rate cash account and price of the zero-coupon bond can be written respectively as

\[ N_u^2(N_v^2)^{-1} = P_2(u, v) \exp \left( -\int_u^v \nu_2(s, v) dW_s^2 - \int_u^v \nu_2(s, v)(\nu_2(s, v)/2 - \rho_{23} \sigma_3(s)) ds \right) \]

and

\[ P_2(u, v) = \frac{P_2(t, v)}{P_2(t, u)} \exp \left( -\frac{1}{2} \int_t^u \nu_2^2(s, v) - \nu_2^2(s, u) ds \right. \]

\[ \left. + \int_t^u (\nu_2(s, v) - \nu_2(s, u)) \rho_{23} \sigma_3(s) ds - \int_t^u \nu_2(s, v) - \nu_2(s, u) dW_s^2 \right) \]

### 3. Option on Inflation Bond

The following result is obtained for a European call. The put value can be deduced by the (inflation) put/call parity.

The option \textit{expiry} is \( t_0 \) and its \textit{real strike} is \( K \). In \( t_0 \) the call owner can receive the bond in exchange of the payment \( KI_{t_0} \). Using the notation \( c_0 = -K \), the value of the option at expiry is then

\[ \max \left( I_{t_0} \sum_{i=0}^n c_i P_2(t_0, t_i), 0 \right). \]

**Theorem 1.** In the Jarrow-Yildirim model with the real rate volatility satisfying the condition \( (H) \) the value in 0 of a European call with real strike \( K \) and expiry \( t_0 \) is

\[ V_0 = I_0 \sum_{i=0}^n c_i P_2(0, t_i) N \left( \frac{\kappa}{\sqrt{\tau_{11}}} - \frac{\tau_{12}}{\sqrt{\tau_{11}}} + g(t_i)\sqrt{\tau_{11}} \right) \]

where \( \kappa \) is the unique solution of

\[ \sum_{i=0}^n c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - g(t_i) \kappa \right) = 0 \]

and

\[ T = (\tau_{i,j}) = \begin{pmatrix} \int_0^t h^2(s) ds & \rho_{23} \int_0^t h(s) \sigma_3(s) ds \\ \rho_{23} \int_0^t h(s) \sigma_3(s) ds & \int_0^t \sigma_3^2(s) ds \end{pmatrix}. \]

**Proof.** Let \( X_1 = \int_0^t h(s) dW_2^s \) and \( X_2 = \int_0^t \sigma_3(s) dW_3^s \). The random variable \( X \) is normally distributed (Nielsen, 1999, Theorem 3.1) with mean 0 and variance \( T \).

The generic value of the option obtained by Jarrow and Yildirim (2003) is

\[ V_0 = \mathbb{E} \left( \max \left( I_{t_0} \sum_{i=0}^n c_i P_2(t_0, t_i), 0 \right) \left( N_{t_0}^1 \right)^{-1} \right). \]
The different building blocks of the problem are:

\[(9) \quad P_2(t_0, t_i) = \frac{P_2(0, t_i)}{P_2(0, t_0)} \exp \left( -\frac{1}{2} (g^2(t_i) - g^2(t_0))\tau_{11} + (g(t_i) - g(t_0))\tau_{12} - (g(t_i) - g(t_0))X_1 \right). \]

\[(10) \quad I_{t_0} = N_{t_0} I_0 P_2(0, t_0) \exp \left( -\frac{1}{2} g^2(t_0)\tau_{11} + g(t_0)\tau_{12} - \frac{1}{2} \tau_{22} - g(t_0)X_1 + X_2 \right). \]

Note that we are able to split the random variable \(X_1\) from the dependency of the coupons \(g(t_i)\) thanks to the hypothesis (H). This is the only place where the separability condition is used.

The option is exercised when

\[\sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)X_1 \right) > 0, \]

or equivalently when \(X_1 < \kappa\). Equation (8) has a unique and non-degenerate solution, as proved in Henrard (2003).

The expectation can be computed explicitly

\[V_0 = E\left( \mathbb{1}_{X_1 > \kappa} I_0 \sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2} \tau_{22} - g(t_i)X_1 + X_2 \right) \right) \]

\[= I_0 \sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2} \tau_{22} \right) \]

\[\quad \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{|\Sigma|} \int_{-\infty}^{\infty} \exp(-g(t_i)x_1) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x\Sigma^{-1}x) dx_2 dx_1. \]

As noted in Henrard (2004), the inside integral is

\[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x\Sigma^{-1}x) dx_2 = \frac{\sqrt{|T|}}{\sqrt{\tau_{11}}} \exp \left( -\frac{1}{2} \frac{1}{\tau_{11}} (x_1^2 - 2\tau_{12}x_1 - |T|) \right). \]

The result is obtained through a straightforward (but slightly tedious) computation. \(\square\)

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**References**


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