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# TIPS OPTIONS IN THE JARROW-YILDIRIM MODEL

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ABSTRACT. An explicit pricing formula for inflation bond options is proposed in the Jarrow-Yildirim model. The formula resembles that for coupon bond options in the HJM model.

## 1. INTRODUCTION

Jarrow and Yildirim (2003) introduce a model for Treasury Inflation-Protected Securities (TIPS) and inflation derivatives based on the Heath-Jarrow-Morton (HJM) model. The Jarrow-Yildirim model describes the behavior of the *nominal* and *real yield curves* and the *inflation index*. Jarrow and Yildirim (2003) also propose a formula for *inflation index options*. Their results are extended by Mercurio (2005) to *zero-coupon inflation-indexed swap*, *year-on-year inflation-indexed swap* and *year-on-year inflation index cap*. Mercurio (2005) also studies a *market model* for inflation. Independently, Belgrade et al. (2004) also propose a market model approach to zero-coupon and year-on-year swaps.

In this brief note, using techniques similar to those used to price coupon bond options in Henrard (2003), the price of *options on capital-indexed inflation bonds* is derived. The formula obtained is explicit up to a parameter that is computed as the unique solution of a one-dimensional equation. In particular the results can be applied to TIPS options.

The description of capital-indexed inflation bonds can be found in (Deacon et al., 2004, Section 2.2.1). The real amounts paid at dates  $t_i$  ( $1 \leq i \leq n$ ) are  $c_i$ , or in nominal terms the amount are  $I_{t_i} c_i$ <sup>1</sup>. The amounts  $c_i$  include the specific convention and frequency of the bond and the principal at final date.

The discount factor linked to the real rates is denoted  $P_2(t_0, T)$ . It is the discount factor viewed from  $t_0$  for a payment in  $T$ . The nominal value in  $t_0$  of the bond described above is

$$(1) \quad I_{t_0} \sum_{i=1}^n c_i P_2(t_0, t_i).$$

## 2. MODEL AND PRELIMINARY LEMMAS

The Jarrow-Yildirim model describes the behaviour of the *instantaneous forward* nominal ( $f_1$ ) and real ( $f_2$ ) interest rate. The forward rates viewed from  $t$  for the maturity  $T$  are denoted  $f_i(t, T)$  ( $1 \leq i \leq 2$ ). Throughout this paper the index 1 is related to the nominal rates, the index 2 to the real rates and the index 3 to the inflation. The (nominal and real) short-term rate are denoted  $r_t^i = f_i(t, t)$ . The cash accounts linked to the nominal and real rates are

$$N_v^i = \exp \left( \int_0^v r_s^i ds \right).$$

The rate volatilities  $\sigma_i$  are deterministic. The bond volatilities are  $\nu_i(t, u) = \int_t^u \sigma_i(t, s) ds$ . In the risk neutral world with numeraire  $N_s^1$  the equations of the model are given by the equation

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<sup>1</sup>Without loss of generality, the reference inflation index used in this document is always 1.

(11)–(13) in Proposition 2 of Jarrow and Yildirim (2003) which are written below

$$(2) \quad df_1(t, T) = \sigma_1(t, T)\nu_1(t, T)dt + \sigma_1(t, T)dW_t^1$$

$$(3) \quad df_2(t, T) = \sigma_2(t, T)(\nu_2(t, T) - \rho_{13}\sigma_3(t))dt + \sigma_2(t, T)dW_t^2$$

$$(4) \quad dI(t) = (r_t^1 - r_t^r)I_t dt + \sigma_3(t)I_t dW_t^3.$$

The covariation between the different Brownian motions are  $[W_t^i, W_t^j] = \rho_{i,j}t$  ( $1 \leq i, j \leq 3$ ).

To obtain an explicit formula for the options on bonds, an extra condition on the real rate volatility is used. This is a separability condition which is satisfied by the extended Vasicek or Hull and White (1990) model and can be found in Henrard (2003) for options on coupon-bonds.

**(H):** the function  $\sigma_2$  satisfies  $\sigma_2(t, u) = g(t)h(u)$  for some positive functions  $g$  and  $h$ .

The following technical lemma on the cash accounts and bond prices will be useful. The formulas are equivalent to those for the HJM model obtained in Henrard (2006).

**Lemma 1.** *Let  $0 \leq t \leq u \leq v$ . In the Jarrow-Yildirim model, the real rate cash account and price of the zero-coupon bond can be written respectively as*

$$(5) \quad N_u^2(N_v^2)^{-1} = P_2(u, v) \exp\left(-\int_u^v \nu_2(s, v)dW_s^2 - \int_u^v \nu_2(s, v)(\nu_2(s, v)/2 - \rho_{23}\sigma_3(s))ds\right)$$

and

$$(6) \quad P_2(u, v) = \frac{P_2(t, v)}{P_2(t, u)} \exp\left(-\frac{1}{2}\int_t^u \nu_2^2(s, v) - \nu_2^2(s, u)ds\right. \\ \left. + \int_t^u (\nu_2(s, v) - \nu_2(s, u))\rho_{23}\sigma_3(s)ds - \int_t^u \nu_2(s, v) - \nu_2(s, u)dW_s^2\right)$$

### 3. OPTION ON INFLATION BOND

The following result is obtained for a European call. The put value can be deduced by the (inflation) put/call parity.

The option *expiry* is  $t_0$  and its *real strike* is  $K$ . In  $t_0$  the call owner can receive the bond in exchange of the payment  $KI_{t_0}$ . Using the notation  $c_0 = -K$ , the value of the option at expiry is then

$$\max\left(I_{t_0} \sum_{i=0}^n c_i P_2(t_0, t_i), 0\right).$$

**Theorem 1.** *In the Jarrow-Yildirim model with the real rate volatility satisfying the condition (H) the value in 0 of a European call with real strike  $K$  and expiry  $t_0$  is*

$$(7) \quad V_0 = I_0 \sum_{i=0}^n c_i P_2(0, t_i) N\left(\frac{\kappa}{\sqrt{\tau_{11}}} - \frac{\tau_{12}}{\sqrt{\tau_{11}}} + g(t_i)\sqrt{\tau_{11}}\right).$$

where  $\kappa$  is the unique solution of

$$(8) \quad \sum_{i=0}^n c_i P_2(0, t_i) \exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)\kappa\right) = 0$$

and

$$T = (\tau_{i,j}) = \begin{pmatrix} \int_0^{t_0} h^2(s)ds & \rho_{23} \int_0^{t_0} h(s)\sigma_3(s)ds \\ \rho_{23} \int_0^{t_0} h(s)\sigma_3(s)ds & \int_0^{t_0} \sigma_3^2(s)ds \end{pmatrix}.$$

*Proof.* Let  $X_1 = \int_0^{t_0} h(s)dW_s^2$  and  $X_2 = \int_0^{t_0} \sigma_3(s)dW_s^3$ . The random variable  $X$  is normally distributed (Nielsen, 1999, Theorem 3.1) with mean 0 and variance  $T$ .

The generic value of the option obtained by Jarrow and Yildirim (2003) is

$$V_0 = E\left(\max\left(I_{t_0} \sum_{i=0}^n c_i P_2(t_0, t_i), 0\right) (N_{t_0}^1)^{-1}\right).$$

The different building blocks of the problem are:

$$(9) \quad P_2(t_0, t_i) = \frac{P_2(0, t_i)}{P_2(0, t_0)} \exp \left( -\frac{1}{2} (g^2(t_i) - g^2(t_0)) \tau_{11} + (g(t_i) - g(t_0)) \tau_{12} - (g(t_i) - g(t_0)) X_1 \right).$$

$$(10) \quad I_{t_0} = N_{t_0}^1 I_0 P_2(0, t_0) \exp \left( -\frac{1}{2} g^2(t_0) \tau_{11} + g(t_0) \tau_{12} - \frac{1}{2} \tau_{22} - g(t_0) X_1 + X_2 \right).$$

Note that we are able to split the random variable  $X_1$  from the dependency of the coupons  $g(t_i)$  thanks to the hypothesis (H). This is the only place where the separability condition is used.

The option is exercised when

$$\sum_{i=0}^n c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - g(t_i) X_1 \right) > 0,$$

or equivalently when  $X_1 < \kappa$ . Equation (8) has a unique and non-degenerate solution, as proved in Henrard (2003).

The expectation can be computed explicitly

$$\begin{aligned} V_0 &= \mathbb{E} \left( \mathbb{1}_{\{X_1 > \kappa\}} I_0 \sum_{i=0}^n c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - \frac{1}{2} \tau_{22} - g(t_i) X_1 + X_2 \right) \right) \\ &= I_0 \sum_{i=0}^n c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - \frac{1}{2} \tau_{22} \right) \\ &\quad \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{|\Sigma|}} \int_{-\infty}^{\kappa} \exp(-g(t_i) x_1) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x \Sigma^{-1} x) dx_2 dx_1 \end{aligned}$$

As noted in Henrard (2004), the inside integral is

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x \Sigma^{-1} x) dx_2 = \frac{\sqrt{|T|}}{\sqrt{\tau_{11}}} \exp \left( -\frac{1}{2} \frac{1}{\tau_{11}} (x_1^2 - 2\tau_{12} x_1 - |T|) \right).$$

The result is obtained through a straightforward (but slightly tedious) computation.  $\square$

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