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Abstract. An explicit pricing formula for inflation bond options is proposed in the Jarrow-Yildirim model. The formula resembles that for coupon bond options in the HJM model.

1. Introduction

Jarrow and Yildirim (2003) introduce a model for Treasury Inflation-Protected Securities (TIPS) and inflation derivatives based on the Heath-Jarrow-Morton (HJM) model. The Jarrow-Yildirim model describes the behavior of the *nominal* and *real yield curves* and the *inflation index*. Jarrow and Yildirim (2003) also propose a formula for *inflation index options*. Their results are extended by Mercurio (2005) to *zero-coupon inflation-indexed swap*, *year-on-year inflation-indexed swap* and *year-on-year inflation index cap*. Mercurio (2005) also studies a *market model* for inflation. Independently, Belgrade et al. (2004) also propose a market model approach to zero-coupon and year-on-year swaps.

In this brief note, using techniques similar to those used to price coupon bond options in Henrard (2003), the price of *options on capital-indexed inflation bonds* is derived. The formula obtained is explicit up to a parameter that is computed as the unique solution of a one-dimensional equation. In particular the results can be applied to TIPS options.

The description of capital-indexed inflation bonds can be found in (Deacon et al., 2004, Section 2.2.1). The real amounts paid at dates t_i ($1 \leq i \leq n$) are c_i, or in nominal terms the amount are $I_{t_i} c_i$[1]. The amounts c_i include the specific convention and frequeny of the bond and the principal at final date.

The discount factor linked to the real rates is denoted $P_2(t_0, T)$. It is the discount factor viewed from t_0 for a payment in T. The nominal value in t_0 of the bond described above is

$$I_{t_0} \sum_{i=1}^{n} c_i P_2(t_0, t_i). \tag{1}$$

2. Model and preliminary lemmas

The Jarrow-Yildirim model describes the behaviour of the *instantaneous forward* nominal (f_1) and real (f_2) interest rate. The forward rates viewed from t for the maturity T are denoted $f_i(t, T)$ ($1 \leq i \leq 2$). Throughout this paper the index 1 is related to the nominal rates, the index 2 to the real rates and the index 3 to the inflation. The (nominal and real) short-term rate are denoted $r_t^i = f_i(t, t)$. The cash accounts linked to the nominal and real rates are

$$N_v^i = \exp\left(\int_0^v r_s^i ds\right).$$

The rate volatilities σ_i are deterministic. The bond volatilities are $\nu_i(t, u) = \int_t^u \sigma_i(t, s) ds$. In the risk neutral world with numeraire N_s^1 the equations of the model are given by the equation

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[1]Without loss of generality, the reference inflation index used in this document is always 1.

(11)–(13) in Proposition 2 of Jarrow and Yildirim (2003) which are written below

$$df_1(t,T) = \sigma_1(t,T)\nu_1(t,T)dt + \sigma_1(t,T)dW_t^1 \tag{2}$$

$$df_2(t,T) = \sigma_2(t,T)\left(\nu_2(t,T) - \rho_{13}\sigma_3(t)\right)dt + \sigma_2(t,T)dW_t^2 \tag{3}$$

$$dI(t) = (r_t^1 - r_t^r)I_t dt + \sigma_3(t)I_t dW_t^3. \tag{4}$$

The covariation between the different Brownian motions are $[W_t^i, W_t^j] = \rho_{i,j}t$ $(1 \leq i,j \leq 3)$.

To obtain an explicit formula for the options on bonds, an extra condition on the real rate volatility is used. This is a separability condition which is satisfied by the extended Vasicek or Hull and White (1990) model and can be found in Henrard (2003) for options on coupon-bonds.

(H): the function σ_2 satisfies $\sigma_2(t,u) = g(t)h(u)$ for some positive functions g and h.

The following technical lemma on the cash accounts and bond prices will be useful. The formulas are equivalent to those for the HJM model obtained in Henrard (2006).

Lemma 1. *Let $0 \leq t \leq u \leq v$. In the Jarrow-Yildirim model, the real rate cash account and price of the zero-coupon bond can be written respectively as*

$$N_u^2(N_v^2)^{-1} = P_2(u,v)\exp\left(-\int_u^v \nu_2(s,v)dW_s^2 - \int_u^v \nu_2(s,v)(\nu_2(s,v)/2 - \rho_{23}\sigma_3(s))ds\right) \tag{5}$$

and

$$\begin{aligned} P_2(u,v) = {} & \frac{P_2(t,v)}{P_2(t,u)}\exp\left(-\frac{1}{2}\int_t^u \nu_2^2(s,v) - \nu_2^2(s,u)ds\right. \\ & \left. + \int_t^u (\nu_2(s,v) - \nu_2(s,u))\rho_{23}\sigma_3(s)ds - \int_t^u \nu_2(s,v) - \nu_2(s,u)dW_s^2\right) \end{aligned} \tag{6}$$

3. Option on inflation bond

The following result is obtained for a European call. The put value can be deduced by the (inflation) put/call parity.

The option *expiry* is t_0 and its *real strike* is K. In t_0 the call owner can receive the bond in exchange of the payment KI_{t_0}. Using the notation $c_0 = -K$, the value of the option at expiry is then

$$\max\left(I_{t_0}\sum_{i=0}^n c_i P_2(t_0,t_i), 0\right).$$

Theorem 1. *In the Jarrow-Yildirim model with the real rate volatility satisfying the condition (H) the value in 0 of a European call with real strike K and expiry t_0 is*

$$V_0 = I_0\sum_{i=0}^n c_i P_2(0,t_i) N\left(\frac{\kappa}{\sqrt{\tau_{11}}} - \frac{\tau_{12}}{\sqrt{\tau_{11}}} + g(t_i)\sqrt{\tau_{11}}\right). \tag{7}$$

where κ is the unique solution of

$$\sum_{i=0}^n c_i P_2(0,t_i)\exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)\kappa\right) = 0 \tag{8}$$

and

$$T = (\tau_{i,j}) = \begin{pmatrix} \int_0^{t_0} h^2(s)ds & \rho_{23}\int_0^{t_0} h(s)\sigma_3(s)ds \\ \rho_{23}\int_0^{t_0} h(s)\sigma_3(s)ds & \int_0^{t_0}\sigma_3^2(s)ds \end{pmatrix}.$$

Proof. Let $X_1 = \int_0^{t_0} h(s)dW_s^2$ and $X_2 = \int_0^{t_0}\sigma_3(s)dW_s^3$. The random variable X is normally distributed (Nielsen, 1999, Theorem 3.1) with mean 0 and variance T.

The generic value of the option obtained by Jarrow and Yildirim (2003) is

$$V_0 = \mathrm{E}\left(\max\left(I_{t_0}\sum_{i=0}^n c_i P_2(t_0,t_i), 0\right)(N_{t_0}^1)^{-1}\right).$$

The different building blocks of the problem are:

$$P_2(t_0,t_i) = \frac{P_2(0,t_i)}{P_2(0,t_0)} \exp\left(-\frac{1}{2}(g^2(t_i)-g^2(t_0))\tau_{11} + (g(t_i)-g(t_0))\tau_{12} - (g(t_i)-g(t_0))X_1\right). \tag{9}$$

$$I_{t_0} = N^1_{t_0} I_0 P_2(0,t_0) \exp\left(-\frac{1}{2}g^2(t_0)\tau_{11} + g(t_0)\tau_{12} - \frac{1}{2}\tau_{22} - g(t_0)X_1 + X_2\right). \tag{10}$$

Note that we are able to split the random variable X_1 from the dependency of the coupons $g(t_i)$ thanks to the hypothesis (H). This is the only place where the separability condition is used.

The option is exercised when

$$\sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)X_1\right) > 0,$$

or equivalently when $X_1 < \kappa$. Equation (8) has a unique and non-degenerate solution, as proved in Henrard (2003).

The expectation can be computed explicitely

$$\begin{aligned} V_0 &= \mathrm{E}\left(1\!\!1_{(X_1>\kappa)} I_0 \sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2}\tau_{22} - g(t_i)X_1 + X_2\right)\right) \\ &= I_0 \sum_{i=0}^{n} c_i P_2(0,t_i) \exp\left(-\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - \frac{1}{2}\tau_{22}\right) \\ &\quad \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{|\Sigma|}}\int_{-\infty}^{\kappa} \exp(-g(t_i)x_1)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(x_2 - \frac{1}{2}x\Sigma^{-1}x)dx_2\, dx_1 \end{aligned}$$

As noted in Henrard (2004), the inside integral is

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(x_2 - \frac{1}{2}x\Sigma^{-1}x)dx_2 = \frac{\sqrt{|T|}}{\sqrt{\tau_{11}}} \exp\left(-\frac{1}{2}\frac{1}{\tau_{11}}(x_1^2 - 2\tau_{12}x_1 - |T|)\right).$$

The result is obtained through a straightforward (but slightly tedious) computation. □

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