TIPS Options in the Jarrow-Yildirim model

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10 January 2006
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Abstract. An explicit pricing formula for inflation bond options is proposed in the Jarrow-Yildirim model. The formula resembles that for coupon bond options in the HJM model.

1. Introduction

Jarrow and Yildirim (2003) introduce a model for Treasury Inflation-Protected Securities (TIPS) and inflation derivatives based on the Heath-Jarrow-Morton (HJM) model. The Jarrow-Yildirim model describes the behavior of the nominal and real yield curves and the inflation index. Jarrow and Yildirim (2003) also propose a formula for inflation index options. Their results are extended by Mercurio (2005) to zero-coupon inflation-indexed swap, year-on-year inflation-indexed swap and year-on-year inflation index cap. Mercurio (2005) also studies a market model for inflation. Independently, Belgrade et al. (2004) also propose a market model approach to zero-coupon and year-on-year swaps.

In this brief note, using techniques similar to those used to price coupon bond options in Henrard (2003), the price of options on capital-indexed inflation bonds is derived. The formula obtained is explicit up to a parameter that is computed as the unique solution of a one-dimensional equation. In particular the results can be applied to TIPS options.

The description of capital-indexed inflation bonds can be found in (Deacon et al., 2004, Section 2.2.1). The real amounts paid at dates \( t_i \) \((1 \leq i \leq n)\) are \( c_i \), or in nominal terms the amount are \( I_{t_i} c_i \). The amounts \( c_i \) include the specific convention and frequency of the bond and the principal at final date.

The discount factor linked to the real rates is denoted \( P_{2i}(t_0, T) \). It is the discount factor viewed from \( t_0 \) for a payment in \( T \). The nominal value in \( t_0 \) of the bond described above is

\[
I_{t_0} \sum_{i=1}^{n} c_i P_{2i}(t_0, t_i).
\]

2. Model and preliminary lemmas

The Jarrow-Yildirim model describes the behaviour of the instantaneous forward nominal \( f_1 \) and real \( f_2 \) interest rate. The forward rates viewed from \( t \) for the maturity \( T \) are denoted \( f_i(t, T) \) \((1 \leq i \leq 2)\). Throughout this paper the index 1 is related to the nominal rates, the index 2 to the real rates and the index 3 to the inflation. The (nominal and real) short-term rate are denoted \( r^*_i = f_i(t, t) \). The cash accounts linked to the nominal and real rates are

\[
N^i_s = \exp \left( \int_0^s r^*_i ds \right).
\]

The rate volatilities \( \sigma_i \) are deterministic. The bond volatilities are \( \nu_i(t, u) = \int_t^u \sigma_i(t, s) ds \). In the risk neutral world with numeraire \( N^1_s \) the equations of the model are given by the equation

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\(^1\)Without loss of generality, the reference inflation index used in this document is always 1.
(11)–(13) in Proposition 2 of Jarrow and Yildirim (2003) which are written below

\begin{align}
(2) \quad df_1(t,T) &= \sigma_1(t,T)\nu_1(t,T)dt + \sigma_1(t,T)dW_1^1 \\
(3) \quad df_2(t,T) &= \sigma_2(t,T)(\nu_2(t,T) - \rho_{13}\sigma_3(t))dt + \sigma_2(t,T)dW_2^2 \\
(4) \quad dI(t) &= (r_1^i - r_1^j)I_t dt + \sigma_3(t)I_t dW_3^3.
\end{align}

The covariance between the different Brownian motions are $[W_i^1, W_j^2] = \rho_{i,j} t$ \(1 \leq i, j \leq 3\).

To obtain an explicit formula for the options on bonds, an extra condition on the real rate volatility is used. This is a separability condition which is satisfied by the extended Vasicek or Hull and White (1990) model and can be found in Henrard (2003) for options on coupon-bonds.

**(H):** the function $\sigma_2$ satisfies $\sigma_2(t,u) = g(t)h(u)$ for some positive functions $g$ and $h$.

The following technical lemma on the cash accounts and bond prices will be useful. The formulas are equivalent to those for the HJM model obtained in Henrard (2006).

**Lemma 1.** Let $0 \leq t \leq u \leq v$. In the Jarrow-Yildirim model, the real rate cash account and price of the zero-coupon bond can be written respectively as

\begin{align}
(5) \quad N_u^2(N_v^2)^{-1} &= P_2(u,v) \exp \left( -\int_u^v \nu_2(s,v)dW_s^2 - \int_u^v \nu_2(s,v)(\nu_2(s,v)/2 - \rho_{23}\sigma_3(s))ds \right) \\
\text{and} \\
(6) \quad P_2(u,v) &= \frac{P_2(t,v)}{P_2(t,u)} \exp \left( -\frac{1}{2} \int_t^u \nu_2(s,v) - \nu_2(s,u)ds \\
&\quad + \int_t^u (\nu_2(s,v) - \nu_2(s,u))\rho_{23}\sigma_3(s)ds - \int_t^u \nu_2(s,v) - \nu_2(s,u)dW_s^2 \right)
\end{align}

### 3. Option on Inflation Bond

The following result is obtained for a European call. The put value can be deduced by the (inflation) put/call parity.

The option expiry is $t_0$ and its real strike is $K$. In $t_0$ the call owner can receive the bond in exchange of the payment $KI_{t_0}$. Using the notation $c_0 = -K$, the value of the option at expiry is then

$$
\max \left( I_{t_0} \sum_{i=0}^n c_iP_2(t_0,t_i), 0 \right).
$$

**Theorem 1.** In the Jarrow-Yildirim model with the real rate volatility satisfying the condition (H) the value in 0 of a European call with real strike $K$ and expiry $t_0$ is

\begin{equation}
V_0 = I_0 \sum_{i=0}^n c_iP_2(0,t_i)N \left( \frac{\kappa}{\sqrt{\tau_{11}}} - \frac{\tau_{12}}{\sqrt{\tau_{11}}} + g(t_i)\sqrt{\tau_{11}} \right),
\end{equation}

where $\kappa$ is the unique solution of

\begin{equation}
\sum_{i=0}^n c_iP_2(0,t_i) \exp \left( -\frac{1}{2}g^2(t_i)\tau_{11} + g(t_i)\tau_{12} - g(t_i)\kappa \right) = 0
\end{equation}

and

$$
T = (\tau_{i,j}) = \begin{pmatrix}
\int_0^{t_0} h^2(s)ds \\
\rho_{23} \int_0^{t_0} h(s)\sigma_3(s)ds \\
\int_0^{t_0} \sigma_3^2(s)ds
\end{pmatrix}.
$$

**Proof.** Let $X_1 = \int_0^{t_0} h(s)dW_s^2$ and $X_2 = \int_0^{t_0} \sigma_3(s)dW_s^3$. The random variable $X$ is normally distributed (Nielsen, 1999, Theorem 3.1) with mean 0 and variance $T$.

The generic value of the option obtained by Jarrow and Yildirim (2003) is

$$
V_0 = E \left( \max \left( I_{t_0} \sum_{i=0}^n c_iP_2(t_0,t_i), 0 \right) (N_0^1)^{-1} \right).
$$
The different building blocks of the problem are:

\[(9) \quad P_2(t_0, t_i) = \frac{P_2(0, t_i)}{P_2(0, t_0)} \exp \left( -\frac{1}{2} (g^2(t_i) - g^2(t_0)) \tau_{11} + (g(t_i) - g(t_0)) \tau_{12} - (g(t_i) - g(t_0)) X_1 \right) . \]

\[(10) \quad I_{t_0} = N_{t_0} I_0 P_2(0, t_0) \exp \left( -\frac{1}{2} g^2(t_0) \tau_{11} + g(t_0) \tau_{12} - \frac{1}{2} \tau_{22} - g(t_0) X_1 + X_2 \right) . \]

Note that we are able to split the random variable \(X_1\) from the dependency of the coupons \(g(t_i)\) thanks to the hypothesis (H). This is the only place where the separability condition is used.

The option is exercised when

\[\sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - g(t_i) X_1 \right) > 0, \]

or equivalently when \(X_1 < \kappa\). Equation (8) has a unique and non-degenerate solution, as proved in Henrard (2003).

The expectation can be computed explicitly

\[V_0 = E \left( \mathbb{1}_{\{X_1 > \kappa\}} I_0 \sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - \frac{1}{2} \tau_{22} - g(t_i) X_1 + X_2 \right) \right) \]

\[= I_0 \sum_{i=0}^{n} c_i P_2(0, t_i) \exp \left( -\frac{1}{2} g^2(t_i) \tau_{11} + g(t_i) \tau_{12} - \frac{1}{2} \tau_{22} \right) \]

\[\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{|\Sigma|}} \int_{-\infty}^{\infty} \exp(-g(t_i)x_1) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x \Sigma^{-1} x) dx_2 \ dx_1 \]

As noted in Henrard (2004), the inside integral is

\[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(x_2 - \frac{1}{2} x \Sigma^{-1} x) dx_2 = \frac{\sqrt{|T|}}{\sqrt{\tau_{11}}} \exp \left( -\frac{1}{2} \frac{1}{\tau_{11}} (x_1^2 - 2\tau_{12} x_1 - |T|) \right). \]

The result is obtained through a straightforward (but slightly tedious) computation. \( \square \)

**Acknowledgement:** The author wishes to thank his colleagues for their valuable comments on this note.

**References**


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