Iterated Potential and Robustness of Equilibria

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Abstract

For any given set-valued solution concept, it is possible to consider iterative elimination of actions outside the solution set. This paper applies such a procedure to define the concept of *iterated monotone potential maximizer* (*iterated MP-maximizer*). It is shown that under some monotonicity conditions, an iterated MP-maximizer is robust to incomplete information (Kajii and Morris, *Econometrica* **65** (1997)) and absorbing and globally accessible under perfect foresight dynamics for a small friction (Matsui and Matsuyama, *Journal of Economic Theory* **65** (1995)). Several simple sufficient conditions under which a game has an iterated MP-maximizer are also provided. *Journal of Economic Literature* Classification Numbers: C72, C73, D82.

KEYWORDS: equilibrium selection; robustness; incomplete information; perfect foresight dynamics; iteration; monotone potential; **p**-dominance.

Contents

1 Introduction

Economic modeling, by its nature, is based on simplified assumptions that schematize a given economic phenomenon. One way of assessing the role of the assumptions postulated is to compare the model with its "perturbed variants" based on slightly weakened assumptions. It is now well known in game theory that outcomes of a game may dramatically change when we allow for small departures from a given assumption (one may think of departure from the common knowledge assumption as demonstrated, among others, by Rubinstein (1989) or Carlsson and van Damme (1993)).[1] Let us say that an equilibrium is robust with respect to a given assumption if it is still an equilibrium when this assumption is slightly weakened.

The lack of robustness of some Nash equilibria has led game theorists to consider criteria that guarantee robustness. In bold strokes, two types of methods have proved to be powerful in identifying equilibria that are robust in various aspects: namely, the potential method (due to Monderer and Shapley (1996); see also Blume (1993), Hofbauer and Sorger (1999, 2002), Ui (2001)) and the risk-dominance method (due to Harsanyi and Selten (1988); see also Kandori, Mailath, and Rob (1993), Young (1993), Matsui and Matsuyama (1995), Morris, Rob, and Shin (1995), and Kajii and Morris (1997)). These criteria, however, are very demanding and such concepts fail to exist in many games. In this paper, we introduce a notion of *iterative construction* that enables us to enlarge the class of games where these approaches apply and hence to extend the existing sufficient conditions for equilibria to be robust.

Specifically, we consider two robustness tests. The first one is the so-called "robustness to incomplete information" test as originally defined by Kajii and Morris (1997). To motivate this approach, consider an analyst who plans to model some strategic situation by a particular complete information game. This analyst should be aware that his prediction might be (in some games) highly dependent on the assumption of complete information. Hence, if it is guaranteed that the analyst's prediction based on the complete information game is not qualitatively different from some equilibrium of the real incomplete information game being played, then he is justified in choosing the simplified assumption of complete information. To be more precise, robustness to incomplete information is defined as follows. A (pure) Nash equilibrium a^* of a complete information game $\mathbf{g}$ is *robust to incom-*

[1] Sensitivity to simplified assumptions has also been discussed in many economic applications. For instance, Morris and Shin (1998) and Goldstein and Pauzner (2005) respectively consider how the predictions of standard models on currency crises and on bank runs which rely on the existence of multiple self-fulfilling beliefs are modified when allowing for slight departure from the complete information assumption. In a series of papers, Matsuyama (1991, 1992a, 1992b) departs from the perfect reversibility assumption on action revisions such as career choice decisions and underlines its consequences in models of sectoral adjustment and economic development.

plete information if every "nearby" incomplete information elaboration of **g** has a Bayesian Nash equilibrium that generates an (ex-ante) distribution over actions assigning a weight close to one to a^*. "Nearby" incomplete information elaborations are incomplete information games such that the sets of players and actions are the same as in the complete information game **g**, and with high probability, each player knows that his payoffs are the same as in **g**. Thus, payoffs of the incomplete information elaboration are allowed to be very different of **g** with very low probability.

The second robustness test we consider is the one introduced by Matsui and Matsuyama (1995), namely, the perfect foresight dynamics approach. To motivate this approach, assume that an analyst considers a one-shot complete information game to predict the long-run outcome of a given repeated interaction. Consider a Nash equilibrium of this game and embed the game in a dynamic game with a large society of agents. If there is no link between time periods, then, regardless of the initial action distribution of the society, the Nash equilibrium is the limit of some equilibrium path in this dynamic game. But what if we slightly depart from such a simplified assumption and assume that there exists a small amount of irreversibility or friction in action revisions? If in this modified dynamic game, the Nash equilibrium is always the limit of an equilibrium path regardless of the initial action distribution, then the analyst can ignore the subtle complications induced by intertemporal effects through irreversibility. To be more precise, we consider a large society with continua of agents (one for each player position of **g**), in which a one-shot game **g** is played repeatedly in a random matching fashion. There is friction in action revisions: each agent cannot change his action at every point in time. Action revision opportunities follow independent Poisson processes. Agents, when given a revision opportunity, take actions that maximize their expected discounted payoffs. The degree of friction is then measured by the discounted average duration of a commitment. A *perfect foresight path* is a feasible path of action distribution along which each revising agent takes a best response to the future course of play. A Nash equilibrium a^* is *globally accessible* if for any initial action distribution, there exists a perfect foresight path that converges to a^*; a^* is *linearly absorbing* if the feasible path converging linearly to a^* is the unique perfect foresight path from each initial action distribution in a neighborhood of a^*. If a Nash equilibrium that is globally accessible is also absorbing, then it is the unique globally accessible equilibrium.

It has been known that even a strict Nash equilibrium may fail to be robust in each sense above. In 2×2 coordination games, for instance, while the risk-dominant equilibrium is robust in the above senses, the risk-dominated equilibrium is not: the risk-dominated equilibrium is never played in any Bayesian Nash equilibrium under some incomplete information structures

(Rubinstein (1989), Morris, Rob, and Shin (1995))[2] and it is never played along any equilibrium path for some initial action distributions (Matsui and Matsuyama (1995)). That is, even strict Nash equilibria which are often considered as being immune against most perturbations (see Kohlberg and Mertens (1986)) can be very sensitive to slight departure from some simplified assumptions.

In finding sufficient conditions for an equilibrium to be robust in each sense above, the two concepts of potential maximizer and **p**-dominance (the latter is a generalization of risk-dominance) have proved to be powerful. Kajii and Morris (1997) show that if the complete information game has a **p**-dominant equilibrium with low **p**, then it is robust to incomplete information,[3] while Ui (2001) shows that in potential games, the potential maximizer is robust to incomplete information. For perfect foresight dynamics, Hofbauer and Sorger (1999, 2002) show that a potential maximizer is stable for any small degree of friction, while the **p**-dominance condition is studied by Oyama (2002) (in a single population setting).[4] Furthermore, Morris and Ui (2005) introduce a generalization of potential and establishes the robustness of generalized potential maximizer to incomplete information. Oyama, Takahashi, and Hofbauer (2008, OTH henceforth) consider the stability of monotone potential maximizer (a special case of generalized potential maximizer) under the perfect foresight dynamics. The class of games with a monotone potential maximizer contains games with a **p**-dominant equilibrium with a low **p**, and therefore the results on generalized/monotone potential maximizer unify the potential maximizer and the **p**-dominance conditions.

This paper applies an *iterative construction* to potential and **p**-dominance methods to generate new sufficient conditions that are obtained by iterating the existing conditions above. Considering monotone potential, which unifies the two methods, we introduce *iterated monotone potential maximizer* (*iterated MP-maximizer*). Roughly speaking, our iterative procedure to build this concept can be described as follows. An action profile a^* is said to be an iterated MP-maximizer if there exists a sequence of subsets of action profiles $S^0 \supset S^1 \supset \cdots \supset S^m = \{a^*\}$ such that for all $k = 1, \cdots, m$, S^k is an MP-maximizer set in the game restricted to S^{k-1}, where S^0 is the set of all action profiles. We show that under certain monotonicity conditions, an iterated MP-maximizer is robust to incomplete information and globally accessible and linearly absorbing for a small friction. This is proved by ex-

[2] Kajii and Morris (1997) also provide a three-player three-action game where a unique Nash equilibrium, which is strict, is not robust to incomplete information.

[3] Tercieux (2006b) proves a set-valued extension of this result.

[4] Kojima (2006) considers another generalization of risk-dominance and establishes the stability result in a multiple population setting. Kim (1996) reports a similar result for binary games with many identical players. Tercieux (2006a) considers a set-valued extension of the **p**-dominance condition.

ploiting the similarity between the mathematical structures of incomplete information elaborations and perfect foresight dynamics, which may be of independent interest.[5]

Given our main results above, it remains to confirm their relevance in conceptual and practical aspects. While it is powerful enough to allow us, through iteration, to prove our results, the original MP-maximizer itself is an abstract concept so that no simple characterization has been known for a game to have an MP-maximizer (unless the game is a simple one such as a 2×2 game), and therefore it is in general a difficult task to find an MP-maximizer, and hence an iterated MP-maximizer, in a given game. This fact also makes it difficult to examine the additional bite iterated MP-maximizer has over MP-maximizer. We thus instead offer simpler concepts that remain easier to manipulate in identifying a robust Nash equilibrium. For these simpler concepts, we show by means of examples that the iterative construction considered in this paper indeed has an additional bite for both the potential and the $\mathbf{p}$-dominance methods. We also provide a simple application to demonstrate the practical use of our iterative construction in an economic context.

First, we consider iteration of $\mathbf{p}$-dominance by discussing the concept of *iterated* $\mathbf{p}$*-dominant equilibrium* defined by Tercieux (2006a). We prove that if a game has an iterated $\mathbf{p}$-dominant equilibrium with low $\mathbf{p}$, then this equilibrium is actually an iterated MP-maximizer and the relevant monotonicity conditions for our robustness results to hold are satisfied. It is also shown that iterated $\mathbf{p}$-dominance is strictly more general than $\mathbf{p}$-dominance. Second, as a specific form of MP-maximizer, we consider local potential maximizer (LP-maximizer) as introduced by Morris and Ui (2005). We define *iterated LP-maximizer* and verify that, in games with marginal diminishing returns, an iterated LP-maximizer is an iterated MP-maximizer. In contrast with MP-maximizers, Morris and Ui (2005) are able to give a simple characterization for LP-maximizers, which enables us to show that, for this specific form of MP-maximizers, our iterative construction leads to a strictly more general concept: we provide an example of a simple game that has an iterated LP-maximizer but no LP-maximizer. Restricting our attention to specific classes of games, we further give several other tools which are much easier to manipulate in finding robust Nash equilibria. In particular, for two-player supermodular coordination games, we introduce the concept of *iterated risk-dominance* which is based on (a generalization of) the pairwise risk-dominance concept considered by Kandori and Rob (1998) and thus relies only on local properties of the payoff structure.

Finally, we discuss a simple application to demonstrate that our itera-

[5]Takahashi (2008) reports a formal correspondence between perfect foresight dynamics and global games (with a certain class of noise structures) for games with linear payoff functions.

tive procedures can be applied to identify a robust prediction in an economic situation. Specifically, we consider a simple game of technology adoption inspired by Kandori and Rob (1998), which under certain assumptions falls into the class of supermodular coordination games. We identify an iteratively risk-dominant technology in this game, which in fact constitutes an iterated MP-maximizer and thus provides us with a robust prediction.

The paper is organized as follows. Section 2 introduces the concept of iterated MP-maximizer as well as other related concepts. Section 3 considers the informational robustness of iterated MP-maximizer, while Section 4 considers the stability of iterated MP-maximizer under the perfect foresight dynamics. Section 5 concludes.

2 Iterated Monotone Potential Maximizer

2.1 Underlying Game

Throughout our analysis, we fix the set of players, $I = \{1, 2, \cdots, N\}$, and the linearly ordered set of actions, $A_i = \{0, 1, \ldots, n_i\}$, for each player $i \in I$. We denote $\prod_{i\in I} A_i$ by A and $\prod_{j\neq i} A_j$ by A_{-i}. A one-shot complete information game is specified by, and identified with, a profile of payoff functions, $\mathbf{g} = (g_i)_{i\in I}$, where $g_i\colon A \to \mathbb{R}$ is the payoff function for player i. For $S = S_1 \times \cdots \times S_N$ where $S_i \subset A_i$, $g_i|_S$ denotes the restriction of g_i to S. We identify $\mathbf{g}|_S = (g_i|_S)_{i\in I}$ with the restricted game with the sets of actions S_i.

For any nonempty, at most countable set S, we denote by $\Delta(S)$ the set of all probability distributions on S. We sometimes identify each action in A_i with the element of $\Delta(A_i)$ that assigns one to the corresponding coordinate.

For $x_i, y_i \in \Delta(A_i)$, we write $x_i \precsim y_i$ if

$$\sum_{k=h}^{n_i} x_{ik} \leq \sum_{k=h}^{n_i} y_{ik}$$

for all $h \in A_i$. We write $x \precsim y$ for $x, y \in \prod_i \Delta(A_i)$ if $x_i \precsim y_i$ for all $i \in I$, and $x_{-i} \precsim y_{-i}$ for $x_{-i}, y_{-i} \in \prod_{j\neq i} \Delta(A_j)$ if $x_j \precsim y_j$ for all $j \neq i$. For $\pi_i, \pi'_i \in \Delta(A_{-i})$, we write $\pi_i \precsim \pi'_i$ if

$$\sum_{a_{-i}\in S_{-i}} \pi_i(a_{-i}) \leq \sum_{a_{-i}\in S_{-i}} \pi'_i(a_{-i})$$

for any increasing subset $S_{-i} \subset A_{-i}$.[6] The game $\mathbf{g}$ is said to be *supermodular* if whenever $h < k$, the difference $g_i(k, a_{-i}) - g_i(h, a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., if $a_{-i} \leq b_{-i}$, then

$$g_i(k, a_{-i}) - g_i(h, a_{-i}) \leq g_i(k, b_{-i}) - g_i(h, b_{-i}).$$

[6] $S_{-i} \subset A_{-i}$ is said to be increasing if $a_{-i} \in S_{-i}$ and $a_{-i} \leq b_{-i}$ imply $b_{-i} \in S_{-i}$.

It is well known that this property extends to $\Delta(A_{-i})$: if $h < k$ and $\pi_i \precsim \pi_i'$, then

$$g_i(k,\pi_i) - g_i(h,\pi_i) \le g_i(k,\pi_i') - g_i(h,\pi_i').$$

We endow $\prod_{i\in I}\Delta(A_i)$, $\Delta(A)$, and $\Delta(A_{-i})$, $i \in I$, with the sup (or max) norm: $|x| = \max_{i\in I}\max_{h\in A_i} x_{ih}$ for $x \in \prod_{i\in I}\Delta(A_i)$, $|\pi| = \max_{a\in A}\pi(a)$ for $\pi \in \Delta(A)$, and $|\pi_i| = \max_{a_{-i}\in A_{-i}}\pi_i(a_{-i})$ for $\pi_i \in \Delta(A_{-i})$. For $\varepsilon > 0$, denote $B_\varepsilon(x) = \{x' \in \prod_i \Delta(A_i) \mid |x' - x| < \varepsilon\}$ for $x \in \prod_{i\in I}\Delta(A_i)$, $B_\varepsilon(\pi) = \{\pi' \in \Delta(A) \mid |\pi' - \pi| < \varepsilon\}$ for $\pi \in \Delta(A)$, and $B_\varepsilon(\pi_i) = \{\pi_i' \in \Delta(A_{-i}) \mid |\pi_i' - \pi_i| < \varepsilon\}$ for $\pi_i \in \Delta(A_{-i})$. Write $B_\varepsilon(F) = \bigcup_{\pi\in F} B_\varepsilon(\pi)$ for $F \subset \Delta(A)$ and $B_\varepsilon(F_{-i}) = \bigcup_{\pi_i\in F_{-i}} B_\varepsilon(\pi_i)$ for $F_{-i} \subset \Delta(A_{-i})$.

Let f be a function from A to $\mathbb{R}$. With abuse of notion, $f(a_i,\cdot)$ are extended to $\prod_{j\neq i}\Delta(A_j)$ and $\Delta(A_{-i})$, and $f(\cdot)$ to $\prod_{j\in I}\Delta(A_j)$ and $\Delta(A)$ in the usual way. For $S_i \subset A_i$, let

$$br_f^i(x_{-i}|S_i) = \arg\max\{f(h,x_{-i}) \mid h \in S_i\}$$

for $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$, and

$$br_f^i(\pi_i|S_i) = \arg\max\{f(h,\pi_i) \mid h \in S_i\}$$

for $\pi_i \in \Delta(A_{-i})$. We also denote $br_f^i(x_{-i}) = br_f^i(x_{-i}|A_i)$ and $br_f^i(\pi_i) = br_f^i(\pi_i|A_i)$.

Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i\in I} S_i^*$. We say that S^* is a *best response set* of $\mathbf{g}$ if for all $i \in I$, $br_{g_i}^i(\pi_i) \cap S_i^* \neq \emptyset$ for all $\pi_i \in \Delta(S_{-i})$ and that S^* is a *strict best response set* of $\mathbf{g}$ if for all $i \in I$, $br_{g_i}^i(\pi_i) \subset S_i^*$ for all $\pi_i \in \Delta(S_{-i})$. An action profile $a^* \in A$ is a (strict) Nash equilibrium of $\mathbf{g}$ if $\{a^*\}$ is a (strict) best response set of $\mathbf{g}$.

2.2 Iterated MP-Maximizer

In this subsection, we define our main concept of iterated monotone potential maximizer (iterated MP-maximizer, in short). In the sequel, we denote $[\underline{a}_i, \overline{a}_i] = \{h \in A_i \mid \underline{a}_i \le h \le \overline{a}_i\}$, and for $\underline{a} = (\underline{a}_i)_{i\in I}$ and $\overline{a} = (\overline{a}_i)_{i\in I}$, $[\underline{a}, \overline{a}] = \prod_{i\in I}[\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{-i}, \overline{a}_{-i}] = \prod_{j\neq i}[\underline{a}_j, \overline{a}_j]$. We say that $S \subset A$ is an order interval, or simply an interval, if $S = [\underline{a}, \overline{a}]$ for some $\underline{a}, \overline{a} \in A$ such that $\underline{a}_i \le \overline{a}_i$ for all $i \in I$, and denote $S_i = [\underline{a}_i, \overline{a}_i]$ and $S_{-i} = [\underline{a}_{-i}, \overline{a}_{-i}]$.

We employ a refinement of the MP-maximizer concept due to Morris and Ui (2005).[7]

Definition 2.1. An interval $S^* \subset A$ is a *strict MP-maximizer set* of $\mathbf{g}$ if there exists a function $v\colon A \to \mathbb{R}$ such that $S^* = \arg\max_{a\in A} v(a)$, and for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br_v^i(\pi_i|[\min A_i, \min S_i^*]) \le \min br_{g_i}^i(\pi_i|[\min A_i, \max S_i^*]), \tag{2.1}$$

[7]This refinement has been introduced by OTH (2008, Definition 4.2) for action profiles (singleton sets).

and

$$\max br^i_v(\pi_i|[\max S^*_i, \max A_i]) \geq \max br^i_{g_i}(\pi_i|[\min S^*_i, \max A_i]). \tag{2.2}$$

Such a function v is called a *strict monotone potential function.*

Now our concept of iterated strict MP-maximizer is obtained by iteration of strict MP-maximizer.

Definition 2.2. An interval $S^* \subset A$ is an *iterated strict MP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \dots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict MP-maximizer set of $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \dots, m$.

An action profile $a^* \in A$ is an *iterated strict MP-maximizer* of $\mathbf{g}$ if $\{a^*\}$ is an iterated strict MP-maximizer set of $\mathbf{g}$.

For supermodular games, an iterated strict MP-maximizer is unique if it exists, due to Theorems 4.1 and 4.7 given in Section 4.

We also introduce a weaker, but more complicated, version of iterated MP-maximizer, which is sufficient to obtain the robustness to incomplete information and the stability under perfect foresight dynamics.

Definition 2.3. Let S^* and S be intervals such that $S^* \subset S \subset A$. S^* is an *MP-maximizer set* of $\mathbf{g}$ *relative to* S if there exist a function $v\colon A \to \mathbb{R}$ and a real number $\eta > 0$ such that $S^* = \arg\max_{a \in A} v(a)$, and for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}))$,

$$\min br^i_v(\pi_i|[\min S_i, \min S^*_i]) \leq \max br^i_{g_i}(\pi_i|[\min S_i, \max S^*_i]), \tag{2.3}$$

and

$$\max br^i_v(\pi_i|[\max S^*_i, \max S_i]) \geq \min br^i_{g_i}(\pi_i|[\min S^*_i, \max S_i]). \tag{2.4}$$

Such a function v is called a *monotone potential function relative to* $B_\eta(\Delta(S))$.

Notice the 'max' and the 'min' in the right hand sides of (2.3) and (2.4), respectively (cf. those of (2.1) and (2.2)). Notice also that v is defined on the whole set A and that (2.3) and (2.4) must be satisfied also for beliefs π_i that assign small probability (less than η) to actions outside S_{-i}, which is an indispensable requirement for the informational robustness and the stability; see Example 2.1.

Definition 2.4. An interval $S^* \subset A$ is an *iterated MP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \dots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is an MP-maximizer set relative to S^{k-1} for each $k = 1, \dots, m$.

An action profile $a^* \in A$ is an *iterated MP-maximizer* of $\mathbf{g}$ if $\{a^*\}$ is an iterated MP-maximizer set of $\mathbf{g}$.

For an iterated (strict) MP-maximizer set S^*, the sequence $S^0, S^1, \dots, S^m$ in the definition will be called *associated intervals* of S^*.

Remark 2.1. In Definition 2.3, let $\mathcal{P}_i = \{S_i^*\} \cup \{\{a_i\} \mid a_i \notin S_i^*\}$ and $\mathcal{P} = \{\prod_{i \in I} X_i \mid X_i \in \mathcal{P}_i \text{ for } i \in I\}$. If v is $\mathcal{P}$-measurable, then "$[\min S_i, \min S_i^*]$" in the left hand side of (2.1) and (2.3) and "$[\max S_i^*, \max S_i]$" in the left hand side of (2.2) and (2.4) can be replaced with "$[\min S_i, \max S_i^*]$" and "$[\min S_i^*, \max S_i]$", respectively. If S^* is an MP-maximizer set relative to A with v being $\mathcal{P}$-measurable, then it is an MP-maximizer (with respect to $\mathcal{P}$) in the sense of Morris and Ui (2005, Definition 8).

Here we show that iterated strict MP-maximizer is actually a refinement of iterated MP-maximizer.

Proposition 2.1. *An iterated strict MP-maximizer set is an iterated MP-maximizer set.*

It is sufficient to show the following.

Lemma 2.2. *Let S^* and S be intervals such that $S^* \subset S \subset A$. If S^* is a strict MP-maximizer set of $\mathbf{g}|_S$ with a strict monotone potential function $v\colon S \to \mathbb{R}$, then there exist a function $\tilde{v}\colon A \to \mathbb{R}$ and a real number $\eta > 0$ such that $S^* = \arg\max_{a \in A} \tilde{v}(a)$, and* (2.1) *and* (2.2) *with $A = S$ hold for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}))$.*

Moreover, if $v|_S$ is supermodular, then $\tilde{v}$ can be taken so that $\tilde{v}|_A$ is supermodular.

We call such a function $\tilde{v}$ a *strict monotone potential function relative to $B_\eta(\Delta(S))$*.

Proof. See Appendix. ∎

Finally, we report a useful fact for reference.

Lemma 2.3. *Suppose that $\mathbf{g}$ has an iterated MP-maximizer S^* with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ and $(v^k)_{k=1}^m$. Then, there exists $\eta > 0$ such that for all $k = 1, \dots, m$ and for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S_{-i}^k))$,*

$$br_{g_i}^i(\pi_i) \cap S_i^k \neq \emptyset.$$

Proof. Note first that for all $\ell = 1, \dots, k$, $S^\ell = \arg\max_{a \in S^{\ell-1}} v^\ell(a)$, and therefore we can take $\varepsilon^\ell > 0$ such that for all $i \in I$ and all $\pi_i \in B_{\varepsilon^\ell}(\Delta(S_{-i}^\ell))$,

$$br_{v^\ell}^i(\pi_i|[\min S_i^{\ell-1}, \min S_i^\ell]) = \min S_i^\ell,$$
$$br_{v^\ell}^i(\pi_i|[\max S_i^\ell, \max S_i^{\ell-1}]) = \max S_i^\ell$$

due to the continuity of $v^\ell(h,\pi_i)$ in π_i. By definition, for all $\ell = 1,\dots,k$, there exists $\eta^\ell > 0$ such that for all $i \in I$ and all $\pi_i \in B_{\eta^\ell}(\Delta(S^\ell_{-i}))$,

$$\max br^i_{g_i}(\pi_i|[\min S_i^{\ell-1}, \max S_i^\ell]) \geq \min br^i_{v^\ell}(\pi_i|[\min S_i^{\ell-1}, \min S_i^\ell]),$$
$$\min br^i_{g_i}(\pi_i|[\min S_i^{\ell}, \max S_i^{\ell-1}]) \leq \max br^i_{v^\ell}(\pi_i|[\max S_i^{\ell}, \max S_i^{\ell-1}]).$$

Setting $\eta = \min_\ell \varepsilon^\ell \wedge \min_\ell \eta^\ell$, we have that for all $\ell = 1,\dots,k$ and for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S^k_{-i}))$ $(\subset B_\eta(\Delta(S^\ell_{-i})))$,

$$\max br^i_{g_i}(\pi_i|[\min S_i^{\ell-1}, \max S_i^{\ell-1}]) \geq \min S_i^\ell,$$
$$\min br^i_{g_i}(\pi_i|[\min S_i^{\ell-1}, \max S_i^{\ell-1}]) \leq \max S_i^\ell,$$

and therefore,

$$br^i_{g_i}(\pi_i|S_i^{\ell-1}) \cap S_i^\ell \neq \emptyset.$$

An induction argument thus proves that

$$br^i_{g_i}(\pi_i) \cap S_i^k \neq \emptyset$$

for all $i \in I$ and all $\pi_i \in B_\eta(\Delta(S^k_{-i}))$, as claimed. ▮

Example 2.1. Consider the following 2×3 supermodular game:

	0	1	2
0	$1,0$	$1,1$	$0,0$
1	$0,0$	$1,1$	$1,0$

In this game, both $(0,1)$ and $(1,1)$ were iterated MP-maximizers if η in Definition 2.3 were allowed to be set to zero. But one can verify that none of them is robust to incomplete information or globally accessible under perfect foresight dynamics. Hence, this example shows that the requirement (in the definition of iterated MP-maximizer) that the conditions be satisfied for all $\pi_i \in B_\eta(\Delta(S^{k-1}))$ (where $\eta > 0$) is indispensable for robustness to incomplete information and stability under perfect foresight dynamics.

2.3 Iterated LP-Maximizer

Generally, finding an MP-maximizer or iterated MP-maximizer is a difficult task, since no full characterization (i.e., necessary and sufficient condition) has been known for a game to have an MP-maximizer and hence an iterated MP-maximizer (unless the game is a simple game such as a 2×2 game). In this subsection, we focus on a specific form of MP-maximizer, *local potential maximizer* (LP-maximizer) introduced by Morris (1999) and Morris and Ui (2005), for which several rather simple characterizations are available (e.g., Morris and Ui (2005) and Okada and Tercieux (2008)), and introduce the iterative notion of LP-maximizer as a specific form of iterated MP-maximizer. In Subsection 2.6, we will show, by means of an example, that the iterated LP-maximizer is strictly more general than the simple LP-maximizer, thereby demonstrating that, for such MP-maximizers, the iterative construction we offer in the present paper does have a bite.

We review the definition of LP-maximizer by Morris and Ui (2005) and then introduce its strict version which will in turn be applied for its iteration.

Definition 2.5. An interval $S^* \subset A$ is an *LP-maximizer set* of $\mathbf{g}$ if there exists a function $v\colon A \to \mathbb{R}$ such that $S^* = \arg\max_{a \in A} v(a)$, and for all $i \in I$ and all $a_i < \min S_i^*$,

$$\max_{a_i' \in a_i^+} \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i', a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i})$$

holds for all $\pi_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(a_i + 1, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(a_i, a_{-i}),$$

where $a_i^+ = \{a_i + 1\}$ if $a_i + 1 < \min S_i^*$ and $a_i^+ = S_i^*$ if $a_i + 1 = \min S_i^*$; and for all $i \in I$ and all $a_i > \max S_i^*$,

$$\max_{a_i' \in a_i^-} \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i', a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i})$$

holds for all $\pi_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(a_i - 1, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(a_i, a_{-i}),$$

where $a_i^- = \{a_i - 1\}$ if $a_i - 1 > \max S_i^*$ and $a_i^- = S_i^*$ if $a_i - 1 = \max S_i^*$. Such a function v is called a *local potential function*.

The strict version of LP-maximizer is defined as follows, where the weak inequalities in the previous definition are replaced with strict ones.[8]

Definition 2.6. An interval $S^* \subset A$ is a *strict LP-maximizer set* of $\mathbf{g}$ if there exists a function $v\colon A \to \mathbb{R}$ such that $S^* = \arg\max_{a \in A} v(a)$, and for all $i \in I$ and all $a_i < \min S_i^*$,

$$\max_{a_i' \in a_i^+} \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i', a_{-i}) > \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i})$$

[8]For action profiles (singleton sets), our definition is equivalent to that of OTH (2008, Definition 4.4(ii)).

holds for all $\pi_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})v(a_i+1, a_{-i}) > \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})v(a_i, a_{-i}),$$

where $a_i^+ = \{a_i+1\}$ if $a_i + 1 < \min S_i^*$ and $a_i^+ = S_i^*$ if $a_i + 1 = \min S_i^*$; and for all $i \in I$ and all $a_i > \max S_i^*$,

$$\max_{a_i'\in a_i^-} \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})g_i(a_i', a_{-i}) > \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})g_i(a_i, a_{-i})$$

holds for all $\pi_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})v(a_i-1, a_{-i}) > \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})v(a_i, a_{-i}),$$

where $a_i^- = \{a_i - 1\}$ if $a_i - 1 > \max S_i^*$ and $a_i^- = S_i^*$ if $a_i - 1 = \max S_i^*$. Such a function v is called a *strict local potential function.*

Now we define the notion of iterated strict LP-maximizer in a similar way as we defined iterated strict MP-maximizer.

Definition 2.7. An interval $S^* \subset A$ is an *iterated strict LP-maximizer set* of $\mathbf{g}$ if there exists a sequence of intervals $S^0, S^1, \dots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict LP-maximizer set of $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \dots, m$.

An action profile $a^* \in A$ is an *iterated strict LP-maximizer* of $\mathbf{g}$ if $\{a^*\}$ is an iterated strict LP-maximizer set of $\mathbf{g}$.

The game $\mathbf{g}$ is said to have *diminishing marginal returns* if for all $i \in I$, all $h \neq \min A_i, \max A_i$, and all $a_{-i} \in A_{-i}$,

$$g_i(h, a_{-i}) - g_i(h-1, a_{-i}) \geq g_i(h+1, a_{-i}) - g_i(h, a_{-i}).$$

As in Morris and Ui (2005) or OTH (2008, Lemma 4.2), one can show that if the game $\mathbf{g}$ or the local potential function v has diminishing marginal returns, then a strict LP-maximizer is a strict MP-maximizer. Therefore, in such games, an iterated strict LP-maximizer is always an iterated strict MP-maximizer.

Proposition 2.4. *If a^* is an iterated strict LP-maximizer of $\mathbf{g}$ with associated intervals $(S^k)_{k=0}^m$ and local potential functions $(v^k)_{k=1}^m$ and if for each $k = 1, \dots, m$, $\mathbf{g}|_{S^{k-1}}$ or $v^k|_{S^{k-1}}$ has marginal diminishing returns, then a^* is an iterated strict MP-maximizer of $\mathbf{g}$ with monotone potential functions $(v^k)_{k=1}^m$.*

2.4 Iterated p-Dominance

This subsection provides simple ways to find iterated monotone potentials using iteration of **p**-dominance as considered in Tercieux (2006a).

Let $\mathbf{p} = (p_i)_{i\in I} \in [0,1)^N$. Let us first review the definition of strict **p**-dominant equilibrium due to Kajii and Morris (1997).

Definition 2.8. An action profile $a^* \in A$ is a *strict* **p**-*dominant equilibrium* of **g** if for all $i \in I$,

$$\{a_i^*\} = br^i_{g_i}(\pi_i)$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a^*_{-i}) > p_i$.

Next we define strict **p**-best response set. This concept is a set-valued extension of the strict **p**-dominance concept (see Tercieux (2006a, 2006b)). The set $S = \prod_{i\in I} S_i$ ($S_i \subset A_i$, $i \in I$) is a strict **p**-best response set if, whenever any player i believes with probability strictly greater than p_i that the other players will play actions in S_{-i}, all of his best responses are contained in S_i.

Definition 2.9. Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i\in I} S_i^*$. The set S^* is a *strict* **p**-*best response set* of **g** if for all $i \in I$,

$$br^i_{g_i}(\pi_i) \subset S_i^*$$

holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(S^*_{-i}) > p_i$.

Now with the two steps procedure that we used to define an iterated MP-maximizer, we define iterated (strict) **p**-dominant equilibrium. Formally, this can be stated as follows.

Definition 2.10. Let S_i^* be a nonempty subset of A_i for each $i \in I$, and $S^* = \prod_{i\in I} S_i^*$. The set S^* is an *iterated strict* **p**-*best response set* of **g** if there exists a sequence $S^0, S^1, \dots, S^m$ with $A = S^0 \supset S^1 \supset \cdots \supset S^m = S^*$ such that S^k is a strict **p**-best response set in $\mathbf{g}|_{S^{k-1}}$ for each $k = 1, \dots, m$.

An action profile $a^* \in A$ is an *iterated strict* **p**-*dominant equilibrium* of **g** if $\{a^*\}$ is an iterated strict **p**-best response set of **g**.

For an iterated strict **p**-best response set S^*, the sequence $S^0, S^1, \dots, S^m$ in the definition will be called *associated subsets* of S^*.

We now prove a link between iterated **p**-dominant equilibrium and iterated MP-maximizer.

Proposition 2.5. *Let a^* be an iterated strict* **p**-*dominant equilibrium of* **g** *with $\sum_{i\in I} p_i < 1$, and $A = S^0 \supset S^1 \supset \cdots \supset S^m = \{a^*\}$ associated subsets. Then, there exists an order $<$ on A such that S^k's are intervals and a^* is* *an iterated strict MP-maximizer with monotone potential functions* $(v^k)_{k=1}^m$ *that are supermodular and of the form:*

$$v^k(a) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } a \in S^k, \\ -\sum_{i \in C^k(a)} p_i & \text{otherwise,} \end{cases} \tag{2.5}$$

where $C^k(a) = \{i \in I \mid a_i \in S_i^k\}$.

To have v^k's be supermodular, re-order the actions so that for all $i \in I$, for all $k = 1, \ldots, m$, and for all $a_i \in S_i^k$, $a_i' \in S_i^{k-1} \setminus S_i^k$, $a_i' < a_i$. Note that this implies that $a^* = \max A = \max S^1 = \cdots \max S^m$. One can verify that for all k, v^k is supermodular with respect to the new order.

Now Proposition 2.5 follows from the following lemma.

Lemma 2.6. *Let* $(S^k)_{k=0}^m$ *be intervals such that* $A = S^0 \supset S^1 \supset \cdots \supset S^m$ *and* $\max S^k = \max A$ *for all* $k = 1, \ldots, m$*. If for each* $k = 1, \ldots, m$*,* S^k *is a strict* $\mathbf{p}^k$*-best response set in* $\mathbf{g}|_{S^{k-1}}$ *with* $\sum_{i \in I} p_i^k < 1$*, then* S^m *is an iterated strict MP-maximizer set of* $\mathbf{g}$.

Proof. For each $k = 1, \ldots, m$, let v^k be given as in (2.5) with $p_i = p_i^k$. Consider any $k = 1, \ldots, m$ and any $i \in I$. It is now sufficient to show that v^k is a strict monotone potential functions for S^k in $\mathbf{g}|_{S^{k-1}}$. Denote $\underline{a}_j^\ell = \min S_j^\ell$ for each $j \in I$ and $\ell = k-1, k$. We want to show that for all $\pi_i \in \Delta(S_{-i}^{k-1})$,

$$\min br_{v^k}^i(\pi_i | S_i^{k-1}) \leq \min br_{g_i}^i(\pi_i | S_i^{k-1})$$

(note that $br_{v^k}^i(\pi_i | S_i^{k-1}) = br_{v^k}^i(\pi_i | [\underline{a}_i^{k-1}, \underline{a}_i^k])$ by construction).

Fix any $\pi_i \in \Delta(S_{-i}^{k-1})$. Observe that

$$v^k(h, \pi_i) = \sum_{a_{-i} \in S_{-i}^{k-1}} \pi_i(a_{-i}) v^k(h, a_{-i})$$

takes only two different values: one for $h < \underline{a}_i^k$ and another for $h \geq \underline{a}_i^k$. Hence,

$$\min br_{v^k}^i(\pi_i | S_i^{k-1}) \in \{\underline{a}_i^{k-1}, \underline{a}_i^k\}.$$

It is sufficient to consider the case where $\min br_{v^k}^i(\pi_i | S_i^{k-1}) = \underline{a}_i^k$. For such $\pi_i \in (S_{-i}^{k-1})$, we have

$$\begin{aligned} 0 < v^k(\underline{a}_i^k, \pi_i) - v^k(\underline{a}_i^{k-1}, \pi_i) &= \sum_{a_{-i} \in S_{-i}^k} \pi_i(a_{-i})(1 - p_i^k) - \sum_{a_{-i} \notin S_{-i}^k} \pi_i(a_{-i}) p_i^k \\ &= \sum_{a_{-i} \in S_{-i}^k} \pi_i(a_{-i}) - p_i^k, \end{aligned}$$

and thus $\pi_i(S_{-i}^k) > p_i^k$. Since S^k is a strict $\mathbf{p}^k$-best response set in $\mathbf{g}|_{S^{k-1}}$, $br_{g_i}^i(\pi_i | S_i^{k-1}) \subset S_i^k$. Therefore, we have $\min br_{g_i}^i(\pi_i | S_i^{k-1}) \geq \underline{a}_i^k = \min br_{v^k}^i(\pi_i | S_i^{k-1})$, completing the proof. ∎

In the case where $\mathbf{g}$ is supermodular, we have a simple characterization of iterated $\mathbf{p}$-dominant equilibrium by means of the notion of iterated pairwise $\mathbf{p}$-dominance.

Definition 2.11. An action profile $a^* \in A$ is an *iterated pairwise strict* $\mathbf{p}$*-dominant equilibrium* of $\mathbf{g}$ if there exists a sequence $0 = \underline{a}_i^0 \leq \underline{a}_i^1 \leq \cdots \leq \underline{a}_i^m = a_i^* = \overline{a}_i^m \leq \cdots \leq \overline{a}_i^1 \leq \overline{a}_i^0 = n_i$ for each $i \in I$ such that for all $k = 1, \ldots, m$, $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1}, \underline{a}^k]}$ and $\overline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\overline{a}^k, \overline{a}^{k-1}]}$.

Proposition 2.7. *Suppose that* $\mathbf{g}$ *is supermodular. If* a^* *is an iterated pairwise strict* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$*, then* a^* *is an iterated strict* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$*.*

Hence, by Proposition 2.5, if a^* is an iterated pairwise strict $\mathbf{p}$-dominant equilibrium of a supermodular game $\mathbf{g}$ with $\sum_{i \in I} p_i < 1$, then a^* is an iterated strict MP-maximizer of $\mathbf{g}$.

The proof utilizes the following fact.

Lemma 2.8. *Suppose that* $\mathbf{g}$ *is supermodular. Let S be an interval such that* $\max S = \max A$*. If* $\min S$ *is a strict* $\mathbf{p}$*-dominant equilibrium in* $\mathbf{g}|_{[0, \min S]}$*, then S is a strict* $\mathbf{p}$*-best response set of* $\mathbf{g}$*.*

Proof. Given S as above, denote $\underline{a}_i = \min S_i$ for each $i \in I$. Take any $i \in I$ and any $\pi_i \in \Delta(A_{-i})$ such that $\pi_i(S_{-i}) > p_i$. We want to show that $br^i_{g_i}(\pi_i) \subset S_i$. Define $\pi_i' \in \Delta(A_{-i})$ by

$$\pi_i'(a_{-i}) = \begin{cases} \pi_i(S_{-i}) & \text{if } a_{-i} = \underline{a}_{-i}, \\ 1 - \pi_i(S_{-i}) & \text{if } a_{-i} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi_i'(\underline{a}_i) > p_i$, we have $br^i_{g_i}(\pi_i'|[0, \underline{a}_i]) = \{\underline{a}_i\}$ by the assumption that $\underline{a}$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[0, \underline{a}]}$, so that $\min br^i_{g_i}(\pi_i') \geq \underline{a}_i$. On the other hand, since $\pi_i' \precsim \pi_i$, we have $\min br^i_{g_i}(\pi_i') \leq \min br^i_{g_i}(\pi_i)$ due to the supermodularity of $\mathbf{g}$. It thus follows that $\min br^i_{g_i}(\pi_i) \geq \underline{a}_i$, which implies that $br^i_{g_i}(\pi_i) \subset S_i$. ∎

Proof of Proposition 2.7. Suppose that a^* is an iterated pairwise $\mathbf{p}$-dominant equilibrium. It is sufficient to show that (a) for each $k = 1, \ldots, m$, $[\underline{a}^k, \overline{a}^0]$ is a strict $\mathbf{p}$-best response set in $\mathbf{g}|_{[\underline{a}^{k-1}, \overline{a}^0]}$, and (b) for each $k = 1, \ldots, m$, $[a^*, \overline{a}^k]$ is a strict $\mathbf{p}$-best response set in $\mathbf{g}|_{[a^*, \overline{a}^{k-1}]}$. But, since $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1}, \underline{a}^k]}$, (a) follows from Lemma 2.8 with $A = [\underline{a}^{k-1}, \overline{a}^0]$ and $S = [\underline{a}^k, \overline{a}^0]$. One can similarly prove (b) by Lemma 2.8 (by reversing the order on actions). ∎

Remark 2.2. For supermodular games, it is simple to check whether $\underline{a}^k$ is a strict $\mathbf{p}$-dominant equilibrium in $\mathbf{g}|_{[\underline{a}^{k-1},\underline{a}^k]}$ for some $\mathbf{p}$ with $\sum_{i\in I} p_i < 1$. Indeed, it is necessary and sufficient to check that for each $i \in I$, $br^i_{g_i}(\pi_i|[\underline{a}^{k-1}_i, \underline{a}^k_i]) = \{\underline{a}^k_i\}$ for $\pi_i \in \Delta([\underline{a}^{k-1}_{-i}, \underline{a}^k_{-i}])$ such that $\pi_i(\underline{a}^{k-1}_{-i}) = 1 - p_i$ and $\pi_i(\underline{a}^k_{-i}) = p_i$.

2.5 Iterated Risk-Dominance

In this subsection, we consider the class of two-player coordination games, where there are two players with the same action set $A_i = \{0, 1, \ldots, n\}$ for each $i = 1, 2$, and all the action profiles on the diagonal are strict Nash equilibria, i.e., (h, k) is a strict Nash equilibrium if and only if $h = k$.

We provide a simpler way to find iterated strict MP-maximizers in two-player supermodular coordination games. Let us first generalize the notion of pairwise risk-dominance by Kandori and Rob (1998) to asymmetric two-player games and then define our notion of iterated risk-dominance.

Definition 2.12. Let $\mathbf{g}$ be a two-player coordination game. We say that (h, h) *pairwise risk dominates* (k, k) in $\mathbf{g}$ if

$$\begin{aligned}\bigl(g_1(h,h) - g_1(k,h)\bigr) &\times \bigl(g_2(h,h) - g_2(k,h)\bigr) \\ &> \bigl(g_1(k,k) - g_1(h,k)\bigr) \times \bigl(g_2(k,k) - g_2(h,k)\bigr), \qquad (2.6)\end{aligned}$$

and write (h, h) PRD (k, k).

Definition 2.13. Let $\mathbf{g}$ be a two-player coordination game. (h^*, h^*) is an *iterated risk-dominant equilibrium* of $\mathbf{g}$ if

1. (h, h) PRD $(h-1, h-1)$ for each $h = 1, \ldots, h^*$, and
2. (h, h) PRD $(h+1, h+1)$ for each $h = h^*, \ldots, n-1$.

Proposition 2.9. *Suppose that* $\mathbf{g}$ *is a two-player supermodular coordination game. If* (h^*, h^*) *is an iterated risk-dominant equilibrium of* $\mathbf{g}$*, then it is an iterated strict MP-maximizer of* $\mathbf{g}$*.*

Proof. Suppose that (h^*, h^*) is an iterated risk-dominant equilibrium. In light of Lemma 2.6, it is sufficient to show that (a) for each $h = 1, \ldots, h^*$, $[h, n] \times [h, n]$ is a strict $\mathbf{p}^h$-best response set in $\mathbf{g}|_{[h-1,n]\times[h-1,n]}$ for some $\mathbf{p}^h$ such that $p^h_1 + p^h_2 < 1$, and (b) for each $k = h^*, \ldots, n-1$, $[h^*, k] \times [h^*, k]$ is a strict $\mathbf{p}^k$-best response set in $\mathbf{g}|_{[h^*,k+1]\times[h^*,k+1]}$ for some $\mathbf{p}^k$ such that $p^k_1 + p^k_2 < 1$. We only show (a).

Consider any $h = 1, \ldots, h^*$, and let

$$p^h_i = \frac{g_i(h-1,h-1) - g_i(h,h-1)}{g_i(h,h) - g_i(h-1,h) + g_i(h-1,h-1) - g_i(h,h-1)} > 0$$

and $\mathbf{p}^h = (p_1^h, p_2^h)$. Verify that $p_1^h + p_2^h < 1$ due to the condition (2.6) and that (h, h) is a strict $\mathbf{p}^h$-dominant equilibrium in $\mathbf{g}|_{[h-1,h]\times[h-1,h]}$. It therefore follows from Lemma 2.8 that $[h, n] \times [h, n]$ is a strict $\mathbf{p}^h$-best response set in $\mathbf{g}|_{[h-1,n]\times[h-1,n]}$. ∎

Example 2.2. Consider the following asymmetric supermodular game:

	0	1	2
0	$3, 1$	$0, 0$	$-2, -2$
1	$0, 0$	$2, 2$	$0, 0$
2	$-2, -2$	$0, 0$	$1, 3$

In this game, $(1, 1)$ is an iterated risk-dominant equilibrium and hence an iterated strict MP-maximizer. Note that this game has no iterated $\mathbf{p}$-dominant equilibrium for $p_1 + p_2 < 1$.

If we consider *symmetric* games (i.e., $g_2(k, h) = g_1(h, k)$ for all $h \in A_1$ and $k \in A_2$), the proof of Proposition 2.9 in fact shows also the following link between iterated $\mathbf{p}$-dominance and iterated risk-dominance.

Proposition 2.10. *Suppose that* $\mathbf{g}$ *is a symmetric two-player supermodular coordination game. If* (h^*, h^*) *is an iterated risk-dominant equilibrium of* $\mathbf{g}$*, then it is an iterated strict* (p, p)*-dominant equilibrium of* $\mathbf{g}$ *for some* $p < 1/2$*.*

Example 2.3. Consider the following symmetric supermodular game:

	0	1	2
0	$1, 1$	$0, 0$	$-3, -6$
1	$0, 0$	$2, 2$	$0, 0$
2	$-6, -3$	$0, 0$	$3, 3$

In this game, $(2, 2)$ is an iterated risk-dominant equilibrium and indeed an iterated strict $(2/5, 2/5)$-dominant equilibrium. Observe that this game has no (p, p)-dominant equilibrium for any $p < 1/2$.[9]

2.6 An Example: Potential versus Iterated Potential

In this subsection, while focusing on LP-maximizer, a specific form of MP-maximizer, we provide a numerical example to demonstrate that our iterative construction leads to a strictly more general concept. The following symmetric 3×3 game $\mathbf{g}$ will show that the iterated strict LP-maximizer is strictly more general than the strict LP-maximizer:

[9]Note also that this game has no globally risk-dominant equilibrium as defined by Kandori and Rob (1998).

	0	1	2
0	$-1,-1$	$-1,3$	$1,0$
1	$3,-1$	$1,1$	$0,0$
2	$0,1$	$0,0$	$2,2$

In fact, we show that this game has no LP-maximizer while it does have an iterated strict LP-maximizer with supermodular strict local potential functions.

It is easy to check that $\{1,2\}\times\{1,2\}$ is a strict $(0,0)$-best response set of $\mathbf{g}$ since action 0 is strictly dominated by action 2. In addition, $\{2\}\times\{2\}$ is a strict $(1/3,1/3)$-best response set of $\mathbf{g}|_{\{1,2\}\times\{1,2\}}$. Tercieux (2006b) shows that a $\mathbf{p}$-best response set with $\sum_{i\in I} p_i < 1$ is an LP-maximizer set with a supermodular local potential function. One can show that this relationship extends to the strict versions of these notions. Hence, we have the following.

Claim 2.11. $(2,2)$ *is an iterated strict LP-maximizer with associated intervals* $S^1 = \{1,2\}\times\{1,2\}$ *and* $S^2 = \{2\}\times\{2\}$ *and with strict local potential functions* v^1 *and* $v^2|_{\{1,2\}\times\{1,2\}}$ *that are both supermodular.*

Indeed, the strict local potential functions v^1 and v^2 can be taken respectively as follows:

	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

	1	2
1	1	0
2	0	2

This iterated strict LP-maximizer $(2,2)$ is also an iterated strict MP-maximizer with the same potential functions v^1 and v^2 and thus is robust to incomplete information as well as globally accessible (for small frictions) and linearly absorbing according to our main results.

In what follows, we show that $(2,2)$ is not an LP-maximizer.

Claim 2.12. $(2,2)$ *is not an LP-maximizer of* $\mathbf{g}$.

To show this, we use the following simple characterization of LP-maximizers provided by Morris and Ui (2005, Lemma 9).

Lemma 2.13. *Action profile* a^* *is an LP-maximizer of* $\mathbf{g}$ *if and only if there exists a function* $v\colon A\to\mathbb{R}$ *such that* $\{a^*\} = \arg\max_{a\in A} v(a)$, *and for all* $i\in I$, *there exists a function* $\mu_i\colon A_i\to\mathbb{R}_+$ *such that if* $a_i < a_i^*$, *then for all* $a_{-i}\in A_{-i}$,

$$\mu_i(a_i)\,(v(a_i+1,a_{-i}) - v(a_i,a_{-i})) \le g_i(a_i+1,a_{-i}) - g_i(a_i,a_{-i}), \qquad (2.7)$$

and if $a_i > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(a_i)\left(v(a_i - 1, a_{-i}) - v(a_i, a_{-i})\right) \leq g_i(a_i - 1, a_{-i}) - g_i(a_i, a_{-i}). \tag{2.8}$$

Proof of Claim 2.12. To prove by contradiction, assume that $(2,2)$ is an LP-maximizer of $\mathbf{g}$. Let v be a local potential function for $(2,2)$ with weight functions $\{\mu_i(\cdot)\}_{i \in I}$ as in Lemma 2.13. Note first that because $g_1(2,1) - g_1(1,1) = -1 < 0$, the inequality (2.7) above implies that $\mu_1(1) > 0$. In addition, $g_1(1,2) - g_1(0,2) = -1 < 0$, which here again implies $\mu_1(0) > 0$. Symmetrically, we must have that $\mu_2(0), \mu_2(1) > 0$. Now again using the inequality (2.7), we must have

$$\begin{aligned}
v(2,2) - v(1,2) &\leq \frac{1}{\mu_1(1)}\big(g_1(2,2) - g_1(1,2)\big) = \frac{2}{\mu_1(1)}, \\
v(1,2) - v(0,2) &\leq \frac{1}{\mu_1(0)}\big(g_1(1,2) - g_1(0,2)\big) = \frac{-1}{\mu_1(0)}, \\
v(0,2) - v(0,1) &\leq \frac{1}{\mu_2(1)}\big(g_2(0,2) - g_2(0,1)\big) = \frac{-3}{\mu_2(1)}.
\end{aligned}$$

Summing up these inequalities, we have

$$0 < v(2,2) - v(0,1) \leq \frac{2}{\mu_1(1)} - \frac{1}{\mu_1(0)} - \frac{3}{\mu_2(1)}, \tag{2.9}$$

where the first inequality follows from the fact that $\{(2,2)\} = \arg\max_{a \in A} v(a)$. In a similar way, we must have

$$\begin{aligned}
v(2,2) - v(2,1) &\leq \frac{1}{\mu_2(1)}\big(g_2(2,2) - g_2(2,1)\big) = \frac{2}{\mu_2(1)}, \\
v(2,1) - v(2,0) &\leq \frac{1}{\mu_2(0)}\big(g_2(2,1) - g_2(2,0)\big) = \frac{-1}{\mu_2(0)}, \\
v(2,0) - v(1,0) &\leq \frac{1}{\mu_1(1)}\big(g_1(2,0) - g_1(1,0)\big) = \frac{-3}{\mu_2(1)}.
\end{aligned}$$

Summing up these inequalities, we have

$$0 < v(2,2) - v(1,0) \leq \frac{2}{\mu_2(1)} - \frac{1}{\mu_2(0)} - \frac{3}{\mu_1(1)}. \tag{2.10}$$

Now summing up (2.9) and (2.10), we have $0 < -(1/\mu_1(0) + 1/\mu_1(1) + 1/\mu_2(0) + 1/\mu_2(1))$, a contradiction since all these weights must be strictly positive numbers. ▮

Remark 2.3. In this paper, we use a linear order over action sets. Morris and Ui (2005) define LP-maximizers for more general orders. It is not difficult to show that even for such orders, $(2,2)$ is not an LP-maximizer of $\mathbf{g}$.

Remark 2.4. In the above game, there is no **p**-dominant equilibrium for $p_1 + p_2 < 1$ while there is an iterated (p, p)-dominant equilibrium for some $p < 1/2$. Hence, this example also shows, as does Example 2.3, that iterated **p**-dominance is strictly more general than **p**-dominance. The same can be said also in the 3×3 example of Young (1993); see Tercieux (2006a, Example 1).

Remark 2.5. Morris and Ui (2005) have also shown that a **p**-dominant equilibrium with $\sum_{i \in I} p_i < 1$ is an LP-maximizer. Hence, this example also demonstrates that an iterated **p**-dominant equilibrium with $\sum_{i \in I} p_i < 1$ is not necessarily an LP-maximizer.

Remark 2.6. One can nevertheless find, possibly by guesswork, a monotone potential function to show that $(2, 2)$ is actually a strict MP-maximizer in this game. A monotone potential function is given for example by

	0	1	2
0	0	0	-3
1	0	1	0
2	-3	0	2

We emphasize, however, that no systematic way to directly find an MP-maximizer has been known beyond 2×2 games.

2.7 An Application: Technology Adoption

In this subsection, we discuss a simple application in which an iterated strict MP-maximizer exists and hence helps to identify a robust prediction. Our purpose here is to demonstrate that our iterative procedures, in particular iterated risk-dominance, can in fact be applied to an economic situation to single out a unique equilibrium outcome as a robust prediction. We consider the following technology choice game inspired by Kandori and Rob (1998). There are two players $i = 1, 2$. Each player i chooses a technology to adopt from a set of available technologies, denoted by $\{0, 1, \ldots, n\}$ and ordered by quality (net of price). The payoff of a player choosing technology h when the other player chooses technology k is given by

$$g(h, k) = q(h) - c(h, k),$$

where $q(h)$ is the inherent quality of technology h and thus is increasing in h, and $c(h, k)$ is the cost due to incompatibility with technology k. We assume that $c(h, k) > 0$ if $h \neq k$, while $c(h, h) = 0$. One way to interpret $c(h, k) > 0$ is that a technology-h user has to buy an adapter which enables him to work with a technology-k user. We assume that the cost of incompatibility is of significance so that (h, h) is a strict Nash equilibrium for all $h = 1, \dots, n$. Let denote this symmetric coordination game by $\mathbf{g}$.

We impose several restrictions on the functions q and c. First, we assume decreasing differences in the inherent quality of technology h, i.e., the marginal gain from adopting technology h over technology $h-1$ is strictly decreasing in h. This is reminiscent of the standard assumption of diminishing marginal returns.

Assumption 2.1. $q(h) - q(h-1)$ is strictly decreasing in h.

Second, consider a situation where consumers 1 and 2 "miscoordinate" where consumer 1 adopts technology h while consumer 2 adopts technology $h-1$. We assume that the difference between the cost of the consumer with the higher technology and that of the other consumer is increasing in h. In other words, the relative cost of being the leader (i.e., the consumer with the higher technology) upon miscoordination is larger when the miscoordination occurs for higher technological standards.

Assumption 2.2. $c(h, h-1) - c(h-1, h)$ is nondecreasing in h.

This assumption is satisfied, in particular, when the cost function is symmetric so that $c(h, k) = c(k, h)$ for all h and k.

Third, we assume that the cost function c is submodular. That is, the marginal cost of adopting higher technologies is smaller when the other consumer chooses higher technological standards. Hence, the consumer will have a larger incentive to adopt higher standard when the other does so. Thus, under this assumption, the game $\mathbf{g}$ will indeed be supermodular.

Assumption 2.3. For all h, $c(h+1, k) - c(h, k)$ is nonincreasing in k.

We here introduce functions that will be useful in utilizing iterated risk-dominance. For $h = 1, \dots, n$, let

$$\begin{aligned} r_h &= \big(g(h,h) - g(h-1,h)\big) - \big(g(h-1,h-1) - g(h,h-1)\big) \\ &= 2 \times [q(h) - q(h-1)] - [c(h,h-1) - c(h-1,h)] \end{aligned} \tag{2.11}$$

and

$$R_h = \sum_{\ell=1}^{h} r_\ell, \tag{2.12}$$

and $R_0 = 0$. Observe that $r_h > 0$ if and only if (h, h) PRD $(h-1, h-1)$, while $r_h < 0$ if and only if $(h-1, h-1)$ PRD (h, h) (recall that "PRD" stands for "pairwise risk-dominates"). Accordingly, r_h can be seen as a measure of risk-dominance, and thus, R_h as the *cumulative risk-dominance*.

By the definition of r_h, we have the following.

Lemma 2.14. *Under Assumptions 2.1 and 2.2, r_h is strictly decreasing in h.*

Now we show that for a generic choice of payoffs satisfying Assumptions 2.1 and 2.2, an iteratively risk-dominant technology exists and maximizes the cumulative risk-dominance R_h. If in addition Assumption 2.3 is satisfied, this technology indeed constitutes an iterated strict MP-maximizer.

Proposition 2.15. *Assume that the game* $\mathbf{g}$ *satisfies* $r_h \neq 0$ *for all* h. *Under Assumptions 2.1 and 2.2, if* h^* *maximizes* R_h, *then* (h^*, h^*) *is a (unique) iterated risk-dominant equilibrium of* $\mathbf{g}$.

If, in addition, Assumption 2.3 is satisfied, then (h^*, h^*) *is a (unique) iterated strict MP-maximizer of* $\mathbf{g}$.

Proof. If $r_h < 0$ for all $h \geq 1$, then $h^* = 0$ maximizes R_h and $(h^*, h^*) = (0, 0)$ is the iterated risk-dominant equilibrium. If instead $r_h \geq 0$ for some $h \geq 1$, then under Assumptions 2.1 and 2.2 and by the generic choice of payoffs, it follows from Lemma 2.14 that there is a unique $h^* \geq 1$ such that $r_h > 0$ if and only if $h \leq h^*$. Clearly, such h^* maximizes R_h and (h^*, h^*) is the iterated risk-dominant equilibrium.

If Assumption 2.3 holds, then $\mathbf{g}$ becomes supermodular. From Proposition 2.9, it therefore follows that under Assumptions 2.1–2.3 the equilibrium (h^*, h^*) obtained above is an iterated strict MP-maximizer. ▮

Our main results together with the proposition above show that the iterated strict MP-maximizer (h^*, h^*) is a unique equilibrium that is robust to incomplete information as well as globally accessible and linearly absorbing under perfect foresight dynamics with small frictions. Thus (h^*, h^*) can be seen as a unique, robust prediction, which allows comparative statics analysis. For example, the technology h^* is larger when, for each technological standard h, the marginal productivity, $q(h)-q(h-1)$, is larger or the relative cost of miscoordination for the leader, $c(h, h-1) - c(h-1, h)$, is smaller.

3 Robustness to Incomplete Information

3.1 ε-Elaborations and Robust Equilibria

Given the game $\mathbf{g}$, we consider the following class of incomplete information games. Each player $i \in I$ has a countable set of types, denoted by T_i. We write $T = \prod_{i \in I} T_i$ and $T_{-i} = \prod_{j \neq i} T_i$. The prior probability distribution on T is given by P. We assume that P satisfies that $\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) > 0$ for all $i \in I$ and $t_i \in T_i$. Let $\Delta_0(T)$ be the set of such probability distributions on T. Under this assumption, the conditional probability of t_{-i} given t_i, $P(t_{-i}|t_i)$, is well-defined by $P(t_{-i}|t_i) = P(t_i, t_{-i}) / \sum_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})$. An event $T' \subset T$ is said to be a *simple event* if it is a product of sets of types of each player, i.e., $T' = \prod_{i \in I} T'_i$ where each $T'_i \subset T_i$. Given a simple event T', we write $T'_{-i} = T'_1 \times \cdots \times T'_{i-1} \times T'_{i+1} \times \cdots \times T'_N$ and $P(T'_{-i}|t_i) = \sum_{t_{-i} \in T'_{-i}} P(t_{-i}|t_i)$. The payoff function for player $i \in I$ is a bounded function $u_i\colon A \times T \to \mathbb{R}$. Denote $\mathbf{u} = (u_i)_{i\in I}$. Fixing type space T, we represent an incomplete information game by $(\mathbf{u}, P)$.

A (behavioral) strategy for player i is a function $\sigma_i\colon T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i. Denote by Σ_i the set of strategies for player i, and let $\Sigma = \prod_{i\in I} \Sigma_i$, $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma$, $\Sigma_{-i} = \prod_{j\neq i} \Sigma_j$, and $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n) \in \Sigma_{-i}$. For a strategy σ_i, we denote by $\sigma_i(a_i|t_i)$ the probability that $a_i \in A_i$ is chosen at $t_i \in T_i$. We write $\sigma(a|t) = \prod_{i\in I} \sigma_i(a_i|t_i)$ and $\sigma_{-i}(a_{-i}|t_{-i}) = \prod_{j\neq i} \sigma_j(a_j|t_j)$. We also write $\sigma_P(a) = \sum_{t\in T} P(t)\sigma(a|t)$. We endow Σ with the topology of uniform convergence on finite subsets of T.[10] The set Σ is convex, and compact with respect to this topology.

We define $\sigma_i \precsim \sigma'_i$ for $\sigma_i, \sigma'_i \in \Sigma_i$ by $\sigma_i(t_i) \precsim \sigma'_i(t_i)$ for all $t_i \in T_i$; $\sigma \precsim \sigma'$ for $\sigma, \sigma' \in \Sigma$ by $\sigma_i \precsim \sigma'_i$ for all $i \in I$; and $\sigma_{-i} \precsim \sigma'_{-i}$ for $\sigma_{-i}, \sigma'_{-i} \in \Sigma_{-i}$ by $\sigma_j \precsim \sigma'_j$ for all $j \neq i$.

The expected payoff to player i with type $t_i \in T_i$ playing $h \in A_i$ against strategy profile σ_{-i} is given by

$$U_i(h, \sigma_{-i})(t_i) = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\, u_i\big((h, \sigma_{-i}(t_{-i})), (t_i, t_{-i})\big),$$

where $u_i((h, \sigma_{-i}(t_{-i})), t) = \sum_{a_{-i}\in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i}) u_i((h, a_{-i}), t)$. Let $BR^i\colon \Sigma_{-i} \times T_i \to A_i$ be defined for each i by

$$BR^i(\sigma_{-i})(t_i) = \arg\max\{U_i(h, \sigma_{-i})(t_i) \mid h \in A_i\}.$$

Note that for each $i \in I$, the correspondence BR^i is upper semi-continuous since U_i is continuous.

Definition 3.1. A strategy profile $\sigma \in \Sigma$ is a *Bayesian Nash equilibrium* of $(\mathbf{u}, P)$ if for all $i \in I$, all $h \in A_i$, and all $t_i \in T_i$,

$$\sigma_i(h|t_i) > 0 \Rightarrow h \in BR^i(\sigma_{-i})(t_i).$$

Let $\beta^i\colon \Sigma_{-i} \to \Sigma_i$ be player i's best response correspondence in $(\mathbf{u}, P)$, defined by

$$\begin{aligned}\beta^i(\sigma_{-i}) = \big\{\xi_i \in \Sigma_i \,\big|\, &\forall\, h \in A_i,\ \forall\, t_i \in T_i : \\ &\big[\xi_i(h|t_i) > 0 \Rightarrow h \in BR^i(\sigma_{-i})(t_i)\big]\big\},\end{aligned} \tag{3.1}$$

and $\beta\colon \Sigma \to \Sigma$ be given by $\beta(\sigma) = \prod_{i\in I} \beta^i(\sigma_{-i})$. A Bayesian Nash equilibrium of $(\mathbf{u}, P)$, $\sigma \in \Sigma$, is a fixed point of β, i.e., $\sigma \in \beta(\sigma)$. Since β is nonempty-, convex-, and compact-valued and upper semi-continuous, the existence of Bayesian Nash equilibria then follows from Kakutani's fixed point theorem.

[10] This topology is metrizable by the metric d_μ defined by $d_\mu(\sigma, \sigma') = \sup_{t\in T} \mu(t)\big|\sigma(t) - \sigma'(t)\big|$ for $\mu \in \Delta(T)$ such that $\mathrm{supp}(\mu) = T$.

Given $\mathbf{g}$, let $T_i^{g_i}$ be the set of types t_i such that payoffs of player i of type t_i is given by g_i and he knows his payoffs:

$$T_i^{g_i} = \{t_i \in T_i \mid u_i(a, (t_i, t_{-i})) = g_i(a) \\ \text{for all } a \in A \text{ and all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0\}.$$

Denote $T^{\mathbf{g}} = \prod_i T_i^{g_i}$.

Definition 3.2. Let $\varepsilon \in [0, 1]$. An incomplete information game $(\mathbf{u}, P)$ is an *ε-elaboration* of $\mathbf{g}$ if $P(T^{\mathbf{g}}) = 1 - \varepsilon$.

Following Kajii and Morris (1997), we say that a^* is robust if, for small $\varepsilon > 0$, every ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium σ with $\sigma_P(a^*)$ close to 1.

Definition 3.3. Action profile $a^* \in A$ is *robust to all elaborations* in $\mathbf{g}$ if for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$, any ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that $\sigma_P(a^*) \geq 1 - \delta$.

Given $P \in \Delta_0(T)$, we write for any function $f \colon A \to \mathbb{R}$

$$BR_f^i(\sigma_{-i}|S_i)(t_i) = \arg\max_{h \in S_i} \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) f(h, \sigma_{-i}(t_{-i})),$$

where $S_i \subset A_i$, $\sigma_{-i} \in \Sigma_{-i}$, and $t_i \in T_i$. Note that this can be written as

$$BR_f^i(\sigma_{-i}|S_i)(t_i) = br_f^i(\pi_i^{t_i}(\sigma_{-i})|S_i)$$

where $\pi_i^{t_i}(\sigma_{-i}) \in \Delta(A_{-i})$ is given by

$$\pi_i^{t_i}(\sigma_{-i})(a_{-i}) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}).$$

Thus, if $f|_{S_i \times A_{-i}}$ is supermodular, then whenever $\sigma_{-i} \precsim \sigma'_{-i}$, we have

$$\min BR_f^i(\sigma_{-i}|S_i)(t_i) \leq \min BR_f^i(\sigma'_{-i}|S_i)(t_i),$$
$$\max BR_f^i(\sigma_{-i}|S_i)(t_i) \leq \max BR_f^i(\sigma'_{-i}|S_i)(t_i).$$

3.2 Informational Robustness of Iterated MP-Maximizer

In this subsection, we state and prove our first main result, which shows that under certain monotonicity conditions, an iterated MP-maximizer is robust to incomplete information.

Theorem 3.1. *Suppose that $\mathbf{g}$ has an iterated MP-maximizer a^* with associated intervals $(S^k)_{k=0}^m$ and monotone potential functions $(v^k)_{k=1}^m$. If for each $k = 1, \dots, m$, $g_i|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$ or $v^k|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$, then a^* is robust to all elaborations in $\mathbf{g}$.*

Due to Lemma 2.2, we immediately have the following.

Corollary 3.2. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and strict monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \dots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S^{k-1}}$ *is supermodular, then* a^* *is robust to all elaborations in* $\mathbf{g}$*.*

Suppose that a^* is an iterated MP-maximizer of $\mathbf{g}$ with monotone potential functions $(v^k)_{k=1}^m$ that are relative to $B_{2\eta}(S^{k-1})$ respectively for $k = 1, \dots, m$, where $\eta > 0$ is sufficiently small so that for all $i \in I$ and all $k = 1, \dots, m$,

$$br^i_{g_i}(\pi_i) \cap S^k_i \neq \emptyset,$$

and therefore,

$$br^i_{g_i}(\pi_i | S^k_i) \subset br^i_{g_i}(\pi_i)$$

hold for $\pi_i \in B_{2\eta}(S^k_{-i})$ (see Lemma 2.3). For each $k = 0, 1, \dots, m$ and $i \in I$, write $S^k_i = [\underline{a}^k_i, \overline{a}^k_i]$, where $0 = \underline{a}^0_i \leq \underline{a}^1_i \leq \dots \leq \underline{a}^m_i = a^*_i = \overline{a}^m_i \leq \dots \leq \overline{a}^1_i \leq \overline{a}^0_i = n_i$. We assume without loss of generality that for all $k = 1, \dots, m$, $S^k \neq S^{k-1}$, i.e., for some $i \in I$, $\underline{a}^k_i \neq \underline{a}^{k-1}_i$ or $\overline{a}^k_i \neq \overline{a}^{k-1}_i$.

Now, given $P \in \Delta_0(T)$, define $J^k_P \colon \Sigma \to \mathbb{R}$ for each $k = 1, \dots, m$ to be

$$J^k_P(\sigma) = \sum_{t \in T} P(t) v^k(\sigma(t)),$$

and for any $\xi, \zeta \in \Sigma$ such that $\xi(t) \in \prod_i \Delta([\underline{a}^0_i, \underline{a}^{k-1}_i])$ and $\zeta(t) \in \prod_i \Delta([\overline{a}^{k-1}_i, \overline{a}^0_i])$ for all $t \in T$, and any simple event $T' \subset T$, let

$$\begin{aligned}
\Sigma^{k,-}_{\xi,T'} &= \{\sigma \in \Sigma \,|\, \forall i \in I : \sigma_i(t_i) = \xi_i(t_i) \ \forall t_i \in T_i \setminus T'_i, \\
&\qquad \sigma_i(t_i) \in \Delta([\underline{a}^{k-1}_i, \underline{a}^k_i]) \ \forall t_i \in T'_i\}, \\
\Sigma^{k,+}_{\zeta,T'} &= \{\sigma \in \Sigma \,|\, \forall i \in I : \sigma_i(t_i) = \zeta_i(t_i) \ \forall t_i \in T_i \setminus T'_i, \\
&\qquad \sigma_i(t_i) \in \Delta([\overline{a}^k_i, \overline{a}^{k-1}_i]) \ \forall t_i \in T'_i\}.
\end{aligned}$$

Consider the maximization problems:

$$\max J^k_P(\sigma) \quad \text{s.t. } \sigma \in \Sigma^{k,-}_{\xi,T'}, \tag{3.2}$$

$$\max J^k_P(\sigma) \quad \text{s.t. } \sigma \in \Sigma^{k,+}_{\zeta,T'}. \tag{3.3}$$

Since J^k_P is continuous, and $\Sigma^{k,-}_{\xi,T'}$ and $\Sigma^{k,+}_{\zeta,T'}$ are compact, the above maximization problems admit solutions.

Lemma 3.3. (1) *For each* $k = 1, \dots, m$ *and for any* $P \in \Delta_0(T)$*, any simple event* $T' \subset T$*, and any* $\xi, \zeta \in \Sigma$ *such that* $\xi(t) \in \prod_i \Delta([\underline{a}^0_i, \underline{a}^k_i])$ *and* $\zeta(t) \in \prod_i \Delta([\overline{a}^k_i, \overline{a}^0_i])$ *for all* $t \in T$*: there exists a solution* $\sigma^{k,-}$ *to the maximization problem* (3.2) *such that*

$$\sigma^{k,-}_i(t_i) = \min BR^i_{v^k}(\sigma^{k,-}_{-i} | [\underline{a}^{k-1}_i, \underline{a}^k_i])(t_i) \tag{3.4}$$

for all $i \in I$ and all $t_i \in T_i'$; and there exists a solution $\sigma^{k,+}$ to the maximization problem (3.3) *such that*

$$\sigma_i^{k,+}(t_i) = \max BR_{v^k}^i(\sigma_{-i}^{k,+}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \tag{3.5}$$

for all $i \in I$ and all $t_i \in T_i'$.

(2) *For each $k = 1, \dots, m$, there exists $\kappa^k > 0$ such that for any $P \in \Delta_0(T)$, any simple event $T' \subset T$, and any $\xi, \zeta \in \Sigma$ such that $\xi(t) \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ and $\zeta(t) \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$ for all $t \in T$: any solution σ to the maximization problem* (3.2) *satisfies*

$$\sigma_P(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T');$$

and any solution σ to the maximization problem (3.3) *satisfies*

$$\sigma_P(\overline{a}^k) \geq 1 - \kappa^k P(T \setminus T').$$

Proof. (1) We only show the existence of a solution that satisfies (3.4) (the existence of a solution that satisfies (3.5) is proved similarly). First note that for each i,

$$\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) v^k(\sigma(t_i, t_{-i})) \\ = \Big(\sum\nolimits_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})\Big) \sum\nolimits_{h \in A_i} \sigma_i(h|t_i) U_i^k(h, \sigma_{-i})(t_i) \tag{3.6}$$

for all $t_i \in T_i'$, where

$$U_i^k(h, \sigma_{-i})(t_i) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\, v^k\big((h, \sigma_{-i}(t_{-i})), (t_i, t_{-i})\big).$$

Therefore, any solution to (3.2), σ^k, satisfies, for all $i \in I$,

$$\sigma_i^k(h|t_i) > 0 \Rightarrow h \in BR_{v^k}^i(\sigma_{-i}^k|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \tag{3.7}$$

for all $t_i \in T_i'$.

Since J_P^k is continuous on $\Sigma_{\xi,T'}^{k,-}$, the set of maximizers is a nonempty, closed, and hence compact, subset of $\Sigma_{\xi,T'}^{k,-}$. Hence, a minimal optimal solution (with respect to the order $\precsim$ on Σ) exists by Zorn's lemma (see Lemma A.2.2 in OTH (2008)). Let $\sigma^{k,-}$ be such a minimal solution.

Take any $i \in I$, and consider the strategy σ_i given by

$$\sigma_i(t_i) = \begin{cases} \xi_i(t_i) & \text{for all } t_i \in T_i \setminus T_i' \\ \min BR_{v^k}^i(\sigma_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) & \text{for all } t_i \in T_i'. \end{cases}$$

By the definition of σ_i together with equation (3.7), we have $\sigma_i \precsim \sigma_i^{k,-}$. On the other hand, by equation (3.6)

$$J_P^k(\sigma_i, \sigma_{-i}^{k,-}) \geq J_P^k(\sigma^{k,-}),$$

meaning that $(\sigma_i, \sigma_{-i}^{k,-}) \in \Sigma_{\xi,T'}^{k,-}$ is also optimal. Hence, the minimality of $\sigma^{k,-}$ implies that $\sigma_i(t_i) = \sigma_i^{k,-}(t_i)$ for all $t_i \in T_i$. Thus, we have (3.4).

(2) Let $v_{\max}^k = v^k(\underline{a}^k) = v^k(\overline{a}^k)$, $\overline{v}^k = \max_{a \in A \setminus [\underline{a}^k, \overline{a}^k]} v^k(a)$, and $\underline{v}^k = \min_{a \in A} v^k(a)$. Note that $v_{\max}^k > \overline{v}^k \geq \underline{v}^k$. Set $\kappa^k = (v_{\max}^k - \underline{v}^k)/(v_{\max}^k - \overline{v}^k)$. Then, the same argument in the proof of Theorem 3 in Ui (2001) will establish the conclusion. Let $\tilde{\sigma} \in \Sigma_{\xi,T'}^{k,-}$ be such that, $\tilde{\sigma}(\underline{a}^k|t) = 1$ for all $t \in T'$. Let σ be any solution to the maximization problem (3.2). Hence we have

$$\begin{aligned} J_P^k(\sigma) \geq J_P^k(\tilde{\sigma}) &= \sum_{t \in T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) + \sum_{t \in T \setminus T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) \\ &= P(T')v_{\max}^k + \sum_{t \in T \setminus T'} \sum_{a \in A} P(t)\tilde{\sigma}(a|t)v^k(a) \\ &\geq P(T')v_{\max}^k + [1 - P(T')]\underline{v}^k. \end{aligned}$$

We also have

$$\begin{aligned} J_P^k(\sigma) &= \sum_{a \in A} \left[\sum_{t \in T} P(t)\sigma(a|t) \right] v(a) \\ &= \sum_{a \in A} \sigma_P(a)v(a) \\ &= \sigma_P(\underline{a}^k)v_{\max}^k + \sum_{a \neq \underline{a}^k} \sigma_P(a)v(a) \\ &\leq \sigma_P(\underline{a}^k)v_{\max}^k + (1 - \sigma_P(\underline{a}^k))\overline{v}^k. \end{aligned}$$

Combining the above inequalities, we have:

$$\sigma_P(\underline{a}^k)v_{\max}^k + (1 - \sigma_P(\underline{a}^k))\overline{v}^k \geq P(T')v_{\max}^k + [1 - P(T')]\underline{v}^k$$

and thus,

$$\sigma_P(\underline{a}^k) \geq 1 - \frac{v_{\max}^k - \underline{v}^k}{v_{\max}^k - \overline{v}^k} P(T \setminus T'),$$

as claimed. ▮

We will need the following lemma, the proof of which mimics that of Lemma B in Kajii and Morris (1997).

Lemma 3.4. *Given any simple event $S \subset T$, let*

$$T_i' = S_i \cap \{t_i \in T_i \mid P(S_{-i}|t_i) \geq 1 - \eta\}$$

for $i \in I$, and $T' = \prod_{i \in I} T_i'$. Then,

$$1 - P(T') \leq \gamma(1 - P(S)),$$

where $\gamma = 1 + N(1 - \eta)/\eta > 0$.

Proof. Let $B_i = \{t_i \in T_i \,|\, P(S_{-i}|t_i) \geq 1-\eta\}$ and $B = \prod_{i\in I} B_i$. By Kajii and Morris (1997, Lemma A), we have

$$P(S \cap (B_i^c \times T_{-i})) \leq \frac{1-\eta}{\eta} P((B_i^c \times T_{-i}) \setminus S)$$

for all $i \in I$. Note then that

$$P(S \setminus B) \leq \sum_{i\in I} P(S \cap (B_i^c \times T_{-i})) \leq N \frac{1-\eta}{\eta} P((B_{i'}^c \times T_{-i'}) \setminus S)$$

for some $i' \in I$. We therefore have

$$\begin{aligned} 1 - P(T') &= P(S \setminus B) + P(T \setminus S) \\ &\leq N\frac{1-\eta}{\eta} P((B_{i'}^c \times T_{-i'}) \setminus S) + P(T \setminus S) \\ &\leq N\frac{1-\eta}{\eta} P(T \setminus S) + P(T \setminus S) \\ &= \gamma P(T \setminus S), \end{aligned}$$

as claimed. ▮

In the following, we let $\sigma^{0,-}, \sigma^{0,+} \in \Sigma$ be such that $\sigma^{0,-}(t) = \underline{a}^0$ and $\sigma^{0,+}(t) = \overline{a}^0$ for all $t \in T$, respectively.

Lemma 3.5. *There exist $c^1, \dots, c^m > 0$ such that for any $P \in \Delta_0(T)$ and any simple event $T^0 \subset T$, there exist $\sigma^{1,-}, \dots, \sigma^{m,-}, \sigma^{1,+}, \dots, \sigma^{m,+} \in \Sigma$ and simple events $T^1, \dots, T^{m-1} \subset T$ with $T^0 \supset T^1 \supset \cdots \supset T^{m-1}$ such that for each $k = 1, \dots, m$,*

($*_k^-$) *for all $i \in I$, $\sigma_i^{k,-}(t_i) = \sigma_i^{k-1,-}(t_i)$ for all $t_i \in T_i \setminus T_i^{k-1}$,*

$$\sigma_i^{k,-}(t_i) = \min BR^i_{v^k}(\sigma_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \tag{3.8}$$

and

$$\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) \geq 1-\eta \quad \textit{for all } t_i \in T_i^{k-1}, \tag{3.9}$$

and $\sigma_P^{k,-}(\underline{a}^k) \geq 1 - c^k P(T \setminus T^0)$.

and

($*_k^+$) *for all $i \in I$, $\sigma_i^{k,+}(t) = \sigma_i^{k-1,+}(t)$ for all $t_i \in T_i \setminus T_i^{k-1}$,*

$$\sigma_i^{k,+}(t_i) = \max BR^i_{v^k}(\sigma_{-i}^{k,+}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \tag{3.10}$$

and

$$\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}^{k,+}([a_{-i}^*, \overline{a}_{-i}^{k-1}]|t_{-i}) \geq 1-\eta \quad \textit{for all } t_i \in T_i^{k-1}, \tag{3.11}$$

and $\sigma_P^{k,+}(\overline{a}^k) \geq 1 - c^k P(T \setminus T^0)$.

Proof. Let $\kappa^1, \dots, \kappa^m > 0$ be as in Lemma 3.3(2) and γ as in Lemma 3.4. Set $c^k = (2\gamma)^{k-1}\kappa^1 \cdots \kappa^k$ for $k = 1, \dots, m$. Fix any $P \in \Delta_0(T)$ and any simple event $T^0 \subset T$. First, by Lemma 3.3 for (3.2) and (3.3) with $k = 1$, $\xi = \sigma^{0,-}$, $\zeta = \sigma^{0,+}$, and $T' = T^0$, we have $\sigma^{1,-}$ and $\sigma^{1,+}$ that satisfy $(*_1^-)$ and $(*_1^+)$, respectively.

Next, for $k \geq 2$ assume that there exist $T^1, \dots, T^{k-2}$, $\sigma^{1,-}, \dots, \sigma^{k-1,-}$, and $\sigma^{1,+}, \dots, \sigma^{k-1,+}$ that satisfy $(*_1^-), \dots, (*_{k-1}^-)$ and $(*_1^+), \dots, (*_{k-1}^+)$, respectively. We can assume that there is no redundancy in $T^1, \dots, T^{k-2}$ (if $k \geq 3$); i.e., for all $\ell = 2, \dots, k-1$, if $\underline{a}_i^\ell = \underline{a}_i^{\ell-1}$ and $\overline{a}_i^\ell = \overline{a}_i^{\ell-1}$, then $T_i^{\ell-1} = T_i^{\ell-2}$. Let

$$S_i^{k-1} = T_i^{k-2} \cap \{t_i \in T_i \mid \sigma_i^{k-1,-}(t_i) = \underline{a}_i^{k-1} \text{ and } \sigma_i^{k-1,+}(t_i) = \overline{a}_i^{k-1}\}$$

for each $i \in I$, and $S^{k-1} = \prod_{i \in I} S_i^{k-1}$. Let also

$$T_i^{k-1} = S_i^{k-1} \cap \{t_i \in T_i \,|\, P(S_{-i}^{k-1}|t_i) \geq 1 - \eta\} \tag{3.12}$$

for each $i \in I$, and $T^{k-1} = \prod_{i \in I} T_i^{k-1}$. Note that $T^{k-1} \subset T^{k-2}$.

Now consider the maximization problems (3.2) and (3.3) with $\xi = \sigma^{k-1,-}$, $\zeta = \sigma^{k-1,+}$, and $T' = T^{k-1}$. Then by Lemma 3.3, we have $\sigma^{k,-}$ and $\sigma^{k,+}$ that satisfy (3.8) and (3.10), and $\sigma_P^{k,-}(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1})$ and $\sigma_P^{k,+}(\overline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1})$, respectively. Since $\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) = \sigma_{-i}^{k,+}([a_{-i}^*, \overline{a}_{-i}^{k-1}]|t_{-i}) = 1$ for all $t_{-i} \in S_{-i}^{k-1}$ (by the definition of S_{-i}^{k-1} and the maximization problems), it follows that

$$\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) \\
&\qquad \geq \sum_{t_{-i} \in S_{-i}^{k-1}} P(t_{-i}|t_i)\sigma_{-i}^{k,-}([\underline{a}_{-i}^{k-1}, a_{-i}^*]|t_{-i}) = P(S_{-i}^{k-1}|t_i) \geq 1 - \eta
\end{aligned}$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$, where the last inequality follows from the definition of T_i^{k-1}, (3.12). This means that $\sigma^{k,-}$ satisfies (3.9). Note that since $\sigma^{k-1,-}$ and $\sigma^{k-1,+}$ are pure strategies, $\sigma_P^{k-1,-}(\underline{a}^{k-1}) = P(\{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1}\})$ and $\sigma_P^{k-1,+}(\overline{a}^{k-1}) = P(\{t \in T \mid \sigma^{k-1,+}(t) = \overline{a}^{k-1}\})$. Since, by the no-redundancy assumption, for all $t \in T \setminus T^{k-2}$, there exists an $i \in I$ such that $\sigma_i^{k-1,-}(t_i) < \underline{a}_i^{k-1}$ or $\sigma_i^{k-1,+}(t_i) > \overline{a}_i^{k-1}$, it follows that $S^{k-1} = \{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1} \text{ and } \sigma^{k-1,+}(t) = \overline{a}^{k-1}\}$. Hence,

$$\begin{aligned}
P(T \setminus S^{k-1}) &\leq P(T \setminus \{t \in T \mid \sigma^{k-1,-}(t) = \underline{a}^{k-1}\}) \\
&\qquad\qquad + P(T \setminus \{t \in T \mid \sigma^{k-1,+}(t) = \overline{a}^{k-1}\}) \\
&= \big(1 - \sigma_P^{k-1,-}(\underline{a}^{k-1})\big) + \big(1 - \sigma_P^{k-1,+}(\overline{a}^{k-1})\big) \\
&\leq 2c^{k-1} P(T \setminus T^0).
\end{aligned} \tag{3.13}$$

Thus, we have

$$\sigma_P^{k,-}(\underline{a}^k) \geq 1 - \kappa^k P(T \setminus T^{k-1}) \geq 1 - \kappa^k \times \gamma P(T \setminus S^{k-1})$$
$$\geq 1 - \kappa^k \gamma \times 2c^{k-1} P(T \setminus T^0) = 1 - c^k P(T \setminus T^0),$$

where the first inequality follows from Lemma 3.3, the second inequality follows from Lemma 3.4, and the third inequality follows from (3.13). The same argument applies to $\sigma^{k,+}$. ∎

Lemma 3.6. *For every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for any ε-elaboration $(\mathbf{u}, P)$ with $\varepsilon \leq \bar{\varepsilon}$, there exist $\sigma^-, \sigma^+ \in \Sigma$ and simple events $T^1, \ldots, T^{m-1} \subset T$ with $T^{\mathbf{g}} = T^0 \supset T^1 \supset \cdots \supset T^{m-1} \supset T^m = \emptyset$ such that*

$(*^-)$ *for all $i \in I$, $\sigma_i^-(t_i) = \underline{a}_i^0$ for all $t_i \in T_i \setminus T_i^{g_i}$,*

$$\sigma_i^-(t_i) = \min BR^i_{v^k}(\sigma^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \setminus T_i^k \quad (3.14)$$

and

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma^-_{-i}([\underline{a}_{-i}^{k-1}, a^*_{-i}]|t_{-i}) \geq 1 - \eta \quad \textit{for all } t_i \in T_i^{k-1} \quad (3.15)$$

for each $k = 1, \ldots, m$, and $\sigma_P^-(a^*) \geq 1 - \delta$,

and

$(*^+)$ *for all $i \in I$, $\sigma_i^+(t_i) = \overline{a}_i^0$ for all $t_i \in T_i \setminus T_i^{g_i}$,*

$$\sigma_i^+(t_i) = \max BR^i_{v^k}(\sigma^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t_i) \quad \textit{for all } t_i \in T_i^{k-1} \setminus T_i^k \quad (3.16)$$

and

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma^+_{-i}([a^*_{-i}, \overline{a}_{-i}^{k-1}]|t_{-i}) \geq 1 - \eta \quad \textit{for all } t_i \in T_i^{k-1} \quad (3.17)$$

for each $k = 1, \ldots, m$, and $\sigma_P^+(a^*) \geq 1 - \delta$.

Proof. Take $c^1, \ldots, c^m > 0$ as in Lemma 3.5. Given any $\delta > 0$, let $\bar{\varepsilon} = \delta / c^m$. Fix any ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$ with $\varepsilon \leq \bar{\varepsilon}$, and let $T^0 = T^{\mathbf{g}}$. Then take $\sigma^{0,-}, \ldots, \sigma^{m,-}$ and $\sigma^{0,+}, \ldots, \sigma^{m,+}$ that satisfy $(*_k^-)$ and $(*_k^+)$ for $k = 1, \ldots, m$, respectively, with $T^1, \ldots, T^{m-1} \subset T$. Set $\sigma^- = \sigma^{m,-}$ and $\sigma^+ = \sigma^{m,+}$. We only verify that σ^- satisfies $(*^-)$.

By construction, we have (3.15) for each $k = 1, \ldots, m$. We also have $\sigma_P^-(a^*) \geq 1 - \delta$ by $(*_m^-)$.

Consider any $k = 1, \ldots, m-1$. Note from (3.12) that

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma^{k,-}_{-i}(\underline{a}^k_{-i}|t_{-i}) \geq 1 - \eta,$$

for all $t_i \in T_i^k$. It follows by the choice of η that for all $i \in I$,

$$\sigma_i^{k,-}(t_i) = \min BR^i_{v^k}(\sigma_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) = \underline{a}_i^k$$

for all $t_i \in T_i^k$ $(\subset T_i^{k-1})$, so that $\sigma^{k,-}(t) = \underline{a}^k$ and hence $\sigma^-(t) \in [\underline{a}^k, a^*]$ for all $t \in T^k$. Note also that $\sigma^-(t) = \sigma^{k,-}(t)$ for all $t \in T \setminus T^k$. Since $v^k(a) = v^k(a')$ for all $a, a' \in [\underline{a}^k, \overline{a}^k]$, it follows that for all $i \in I$ and all $t_i \in T_i^{k-1}$, $BR^i_{v^k}(\sigma_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) = BR^i_{v^k}(\sigma_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i)$. Therefore, for all $i \in I$ and all $t_i \in T_i^{k-1} \setminus T_i^k$,

$$\begin{aligned}\sigma_{-i}^-(t_i) = \sigma_{-i}^{k,-}(t_i) &= \min BR^i_{v^k}(\sigma_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) \\ &= \min BR^i_{v^k}(\sigma_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i),\end{aligned}$$

which means that σ^- satisfies (3.14). ▮

Proof of Theorem 3.1. Suppose that v^k's are monotone potential functions for a^* relative to $B_{2\eta}([\underline{a}^{k-1}, \overline{a}^{k-1}])$. Let $\delta > 0$ be given Take $\bar{\varepsilon}$ as in Lemma 3.6. Fix any ε-elaboration $(\mathbf{u}, P)$ with $\varepsilon \leq \bar{\varepsilon}$, and take σ^-, σ^+, and $T^0, T^1, \ldots, T^m$ that satisfy $(*^-)$ and $(*^+)$, respectively. Let $\tilde{\Sigma} = \{\sigma \in \Sigma \mid \sigma^- \precsim \sigma \precsim \sigma^+\}$. We will show that $\tilde{\beta}(\sigma) = \beta(\sigma) \cap \tilde{\Sigma}$ is nonempty for any $\sigma \in \tilde{\Sigma}$, where β is the best response correspondence of $(\mathbf{u}, P)$ defined in (3.1). Then, since $\tilde{\Sigma}$ is convex and compact, it follows from Kakutani's fixed point theorem that the nonempty-, convex-, and compact-valued upper semi-continuous correspondence β has a fixed point $\sigma^* \in \tilde{\beta}(\sigma^*) \subset \tilde{\Sigma}$, which is a Bayesian Nash equilibrium of $(\mathbf{u}, P)$ and satisfies $\sigma^- \precsim \sigma^* \precsim \sigma^+$. Since both σ^- and σ^+ satisfy $\sigma_P^-(a^*) \geq 1 - \delta$ and $\sigma_P^+(a^*) \geq 1 - \delta$, respectively, σ^* satisfies $\sigma_P^*(a^*) \geq 1 - 2\delta$.

Take any $\sigma \in \tilde{\Sigma}$. For $t_i \in T_i \setminus T_i^0$, $BR^i_{g_i}(\sigma)(t_i) \subset [\sigma_i^-(t_i), \sigma_i^+(t_i)]$ holds. Consider any $k = 1, \ldots, m$. Note that

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}([\underline{a}_{-i}^{k-1}, \overline{a}_{-i}^{k-1}]|t_{-i}) \geq 1 - 2\eta$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$.

Suppose first that $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\sigma_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t_i) &\leq \max BR^i_{g_i}(\sigma_{-i}^-|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t_i) \\ &\leq \max BR^i_{g_i}(\sigma_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$, where the second inequality follows from the assumption that v^k is a monotone potential function relative to $B_{2\eta}([\underline{a}^{k-1}, \overline{a}^{k-1}])$, and the third inequality follows from the supermodularity of $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\sigma^+_{-i}|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\geq \min BR^i_{g_i}(\sigma^+_{-i}|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\\ &\geq \min BR^i_{g_i}(\sigma_{-i}|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$.

Suppose next that $v^k|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned}\min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i) &\leq \min BR^i_{v^k}(\sigma|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i)\\ &\leq \max BR^i_{g_i}(\sigma|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$, where the second inequality follows from the supermodularity of $v^k|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$, and the third inequality follows from the assumption that v^k is a monotone potential function relative to $B_{2\eta}([\underline{a}^{k-1},\overline{a}^{k-1}])$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\geq \max BR^i_{v^k}(\sigma|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\\ &\geq \min BR^i_{g_i}(\sigma|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\end{aligned}$$

for all $t_i \in T_i^{k-1} \setminus T_i^k$.

Therefore, in each case, we have for all $t_i \in T_i^{k-1} \setminus T_i^k$,

$$\begin{aligned}\max BR^i_{g_i}(\sigma|[\underline{a}_i^{k-1},\overline{a}_i^k])(t_i) &\geq \min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i),\\ \min BR^i_{g_i}(\sigma|[\underline{a}_i^k,\overline{a}_i^{k-1}])(t_i) &\leq \max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i).\end{aligned}$$

Since

$$\sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}([\underline{a}_{-i}^{k-1},\overline{a}_{-i}^{k-1}]|t_{-i}) \geq 1-2\eta$$

for all $i \in I$ and all $t_i \in T_i^{k-1}$ and hence

$$BR^i_{g_i}(\sigma)(t_i) \cap [\underline{a}_i^{k-1},\overline{a}_i^{k-1}] \neq \emptyset$$

by the choice of η, it follows that

$$\begin{aligned}&BR^i_{g_i}(\sigma)(t_i)\\ &\quad \cap \left[\min BR^i_{v^k}(\sigma^-|[\underline{a}_i^{k-1},\underline{a}_i^k])(t_i), \max BR^i_{v^k}(\sigma^+|[\overline{a}_i^k,\overline{a}_i^{k-1}])(t_i)\right] \neq \emptyset.\end{aligned}$$

This implies the nonemptiness of $\tilde{\beta}(\sigma)$. ▮

By Proposition 2.5, we immediately have the following.

Corollary 3.7. *If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i\in I} p_i < 1$, then a^* is robust to all elaborations in $\mathbf{g}$.*

3.3 Uniqueness of Robust Equilibrium and Iterated p-Dominance

Our first theorem, together with our results provided in Subsection 2.4, shows that an iterated **p**-dominant equilibrium with low **p** is actually robust to incomplete information. In this subsection, we prove a stronger result: when an iterated strict **p**-dominant equilibrium with low **p** exists, it is the *unique* robust equilibrium.

Proposition 3.8. *An iterated strict* **p***-dominant equilibrium of* **g** *with* $\sum_{i\in I} p_i < 1$ *is the unique robust equilibrium in* **g**.

This proposition is a corollary to the following lemma.

Lemma 3.9. *Suppose that a^* is an iterated strict* **p***-dominant equilibrium of* **g** *with $\sum_{i\in I} p_i \leq 1$. Then, for all $\varepsilon > 0$, there exists an ε-elaboration where the strategy profile σ^* such that $\sigma^*(t) = a^*$ for all $t \in T$ is the unique Bayesian Nash equilibrium.*

Proof. Let a^* be an iterated strict **p**-dominant equilibrium with $\sum_{i\in I} p_i \leq 1$ and $(S^0, \dots, S^m)$ an associated sequence. Let $q_i = (p_i / \sum_{j\in I} p_j) \geq p_i$ for each $i \in I$ (we can assume without loss of generality that $p_i > 0$ for all i). Note that $\sum_{i\in I} q_i = 1$. Now let $T_i = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ for each $i \in I$. For each $\varepsilon > 0$, we construct an ε-elaboration $(\mathbf{u}, P^\varepsilon)$ as follows. Define $P^\varepsilon \in \Delta_0(T)$ by

$$P^\varepsilon(t_1, \dots, t_N) = \begin{cases} \varepsilon(1-\varepsilon)^\tau q_i & \text{if } t_i = \tau + 1 \text{ and } t_j = \tau \text{ for all } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and $u_i \colon A \times T \to \mathbb{R}$ for each $i \in I$ by

$$u_i(a; t) = \begin{cases} g_i(a) & \text{if } t_i \neq 0, \\ 1 & \text{if } t_i = 0 \text{ and } a_i = a_i^*, \\ 0 & \text{if } t_i = 0 \text{ and } a_i \neq a_i^*. \end{cases}$$

Fix any $\varepsilon > 0$, and let us now study the set of Bayesian Nash equilibria of $(\mathbf{u}, P^\varepsilon)$.

Consider the sequence of modified incomplete information games $\{(\mathbf{u}|_{S^k}, P^\varepsilon)\}_{k=0}^{m-1}$ where in $(\mathbf{u}|_{S^k}, P^\varepsilon)$, the set of actions available to player $i \in I$ is S_i^k and player i's payoff function $u_i|_{S_i^k} \colon S^k \times T \to \mathbb{R}$ is given by the restriction of u_i to $S^k \times T$. We want to show that any Bayesian Nash equilibrium of $(\mathbf{u}, P^\varepsilon)$, σ^*, satisfies $\sigma^*(t) = a^*$ for all $t \in T$.

First note that if σ^* is a Bayesian Nash equilibrium of $(\mathbf{u}, P^\varepsilon)$ such that for $k = 0, \dots, m-1$, $\mathrm{supp}(\sigma^*(t)) \subset S^k$ for all $t \in T$, then σ^* is an equilibrium of $(\mathbf{u}|_{S^k}, P^\varepsilon)$. It is therefore sufficient to show that for each

$k = 0, \ldots, m-1$, any Bayesian Nash equilibrium σ^* of $(\mathbf{u}|_{S^{k-1}}, P^\varepsilon)$ is such that $\mathrm{supp}(\sigma^*(t)) \subset S^k$ for all $t \in T$. We proceed by induction.

Let σ^* be a Bayesian Nash equilibrium of $(\mathbf{u}|_{S^{k-1}}, P^\varepsilon)$. We show that for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau) = 1$ for all $\tau \geq 0$. By construction, for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|0) = 1$. Our inductive hypothesis is that for all $i \in I$, $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau) = 1$. Take any $i \in I$ and consider the type $t_i = \tau + 1$. By construction of the type space, we have

$$\begin{aligned} P^\varepsilon((t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_N)|\tau + 1) &= \frac{\varepsilon(1-\varepsilon)^\tau q_i}{\varepsilon(1-\varepsilon)^\tau q_i + \sum_{j \neq i} \varepsilon(1-\varepsilon)^{\tau+1} q_j} \\ &> q_i \geq p_i \end{aligned}$$

if $t_j = \tau$ for all $j \neq i$. Thus by the inductive hypothesis, each agent i assigns a probability strictly above p_i to the other players playing actions in S_{-i}^k. But since S^k is a strict $\mathbf{p}$-best response set of $\mathbf{g}|_{S^{k-1}}$ and since $\tau + 1 \in T_i^{u_i}$, this implies that $\sum_{a_i \in S_i^k} \sigma_i^*(a_i|\tau + 1) = 1$. Thus our inductive hypothesis holds for $\tau + 1$. ▮

Proof of Proposition 3.8. If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium with $\sum_{i \in I} p_i < 1$, then it is an iterated MP-maximizer with supermodular monotone potential functions by Proposition 2.5 and hence is robust to all elaborations by Theorem 3.1. But by Lemma 3.9, no action profile other than a^* is played in any robust equilibrium. ▮

4 Stability under Perfect Foresight Dynamics

4.1 Perfect Foresight Paths and Stability Concepts

Given the game $\mathbf{g}$, we consider the following dynamic societal game. Society consists of N continua of agents, one for each role in $\mathbf{g}$. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play $\mathbf{g}$. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate $\lambda > 0$. The processes are independent across agents. We choose without loss of generality the unit of time in such a way that $\lambda = 1$.

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by $\phi_i(t) = (\phi_{ih}(t))_{h \in A_i} \in \Delta(A_i)$, where $\phi_{ih}(t)$ is the fraction of agents who are committing to action $h \in A_i$ at time t. Let $\phi(t) = (\phi_i(t))_{i \in I} \in \prod_{i \in I} \Delta(A_i)$ and $\phi_{-i}(t) = (\phi_j(t))_{j \neq i} \in \prod_{j \neq i} \Delta(A_j)$. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_{ih}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \geq 0$.

Definition 4.1. A path $\phi\colon \mathbb{R}_+ \to \prod_{i\in I} \Delta(A_i)$ is said to be *feasible* if it is Lipschitz continuous, and for all $i \in I$ and almost all $t \geq 0$, there exists $\alpha_i(t) \in \Delta(A_i)$ such that

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t). \tag{4.1}$$

Denote by Φ^i the set of feasible paths for population i, and let $\Phi = \prod_{i\in I} \Phi^i$ and $\Phi^{-i} = \prod_{j\neq i} \Phi^j$. For $x \in \prod_{i\in I} \Delta(A_i)$, the set of feasible paths starting from x is denoted by $\Phi_x = \prod_i \Phi^i_x$. We endow Φ_x with the topology of uniform convergence on compact intervals.[11] The set Φ_x is convex, and compact with respect to this topology.

We define $\phi_i \precsim \psi_i$ for $\phi_i, \psi_i \in \Phi^i$ by $\phi_i(t) \precsim \psi_i(t)$ for all $t \geq 0$; $\phi \precsim \psi$ for $\phi, \psi \in \Phi$ by $\phi_i \precsim \psi_i$ for all $i \in I$; and $\phi_{-i} \precsim \psi_{-i}$ for $\phi_{-i}, \psi_{-i} \in \Phi^{-i}$ by $\phi_j \precsim \psi_j$ for all $j \neq i$. Note that if $\phi(0) \precsim \psi(0)$ and $\dot{\phi}(t) + \phi(t) \precsim \dot{\psi}(t) + \psi(t)$ for almost all $t \geq 0$, then $\phi \precsim \psi$.

A revising agent in population i anticipates the future evolution of the action distribution, and commits to an action that maximizes his expected discounted payoff. The expected discounted payoff of committing to action $h \in A_i$ at time t with a given anticipated path $\phi_{-i} \in \Phi^{-i}$ is given by

$$\begin{aligned} V_{ih}(\phi_{-i})(t) &= (1+\theta) \int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} g_i(h, \phi_{-i}(z))\, dz\, e^{-s} ds \\ &= (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} g_i(h, \phi_{-i}(s))\, ds, \end{aligned}$$

where $\theta > 0$ is a common discount rate. Following Matsui and Matsuyama (1995), we view $\theta/\lambda = \theta$ as the *degree of friction*.

Let $BR^i_{g_i}\colon \Phi^{-i} \times \mathbb{R}_+ \to A_i$ be defined for each i by

$$BR^i_{g_i}(\phi_{-i})(t) = \arg\max\{V_{ih}(\phi_{-i})(t) \mid h \in A_i\}.$$

Note that for each $i \in I$, the correspondence $BR^i_{g_i}$ is upper semi-continuous since V_i is continuous.

Definition 4.2. A feasible path ϕ is said to be a *perfect foresight path* in $\mathbf{g}$ if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t).$$

Let $\beta^i_x\colon \Phi^{-i}_x \to \Phi^i_x$ be defined by

$$\beta^i_x(\phi_{-i}) = \{\psi_i \in \Phi^i_x \mid \dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t) \ \text{ a.e.}\}, \tag{4.2}$$

and $\beta_x\colon \Phi_x \to \Phi_x$ be given by $\beta_x(\phi) = \prod_i \beta^i_x(\phi_{-i})$. A perfect foresight path ϕ with $\phi(0) = x$ is a fixed point of $\beta_x\colon \Phi_x \to \Phi_x$, i.e., $\phi \in \beta_x(\phi)$. Verify that

[11] This topology is metrizable by the metric d_r defined by $d_r(\phi, \phi') = \sup_{t\geq 0} e^{-rt}|\phi(t) - \phi'(t)|$ for $r > 0$.

β_x is nonempty-, convex-, and compact-valued and upper semi-continuous (see, e.g., OTH (2008, Remark 2.1)). The existence of perfect foresight paths then follows from Kakutani's fixed point theorem.

Following Matsui and Matsuyama (1995) and OTH (2008), we employ the following stability concepts.

Definition 4.3. (a) $a^* \in A$ is *globally accessible* in $\mathbf{g}$ if for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path from x that converges to a^*.

(b) $a^* \in A$ is *absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that any perfect foresight path from any $x \in B_\varepsilon(a^*)$ converges to a^*.

(c) $a^* \in A$ is *linearly absorbing* in $\mathbf{g}$ if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(a^*)$, the linear path to a^* is a unique perfect foresight path from x.

Given $\theta > 0$, we write for any function $f\colon A \to \mathbb{R}$

$$BR^i_f(\phi_{-i}|S_i)(t) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} f(h, \phi_{-i}(s))\, ds, ,$$

where $S_i \subset A_i$, $\phi_{-i} \in \Phi_{-i}$, and $t \geq 0$. Note that this can be written as

$$BR^i_f(\phi_{-i}|S_i)(t) = br^i_f(\pi_i^{t_i}(\phi_{-i})|S_i)$$

where $\pi_i^{t_i}(\phi_{-i}) \in \Delta(A_{-i})$ is given by

$$\pi_i^{t_i}(\phi_{-i})(a_{-i}) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} \left(\prod\nolimits_{j\neq i} \phi_{ja_j}(s)\right) ds.$$

Thus, if $f|_{S_i \times A_{-i}}$ is supermodular, then whenever $\phi_{-i} \precsim \phi'_{-i}$, we have

$$\min BR^i_f(\phi_{-i}|S_i)(t) \leq \min BR^i_f(\phi'_{-i}|S_i)(t),$$
$$\max BR^i_f(\phi_{-i}|S_i)(t) \leq \max BR^i_f(\phi'_{-i}|S_i)(t).$$

4.2 Global Accessibility of Iterated MP-Maximizer

In this subsection, we move to our second main result. We show that under the same monotonicity conditions as in the incomplete information case, an iterated MP-maximizer is selected by the perfect foresight dynamics approach.

In addition, as will become clear, by exploiting the similarity between the mathematical structures of incomplete information elaborations and perfect foresight dynamics, we provide a proof of this result that is strongly related to the proof of our first main result.

Theorem 4.1. *Suppose that $\mathbf{g}$ has an iterated MP-maximizer a^* with associated intervals $(S^k)_{k=0}^m$ and monotone potential functions $(v^k)_{k=1}^m$. If for each $k = 1, \ldots, m$, $g_i|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$ or $v^k|_{S_i^{k-1} \times A_{-i}}$ is supermodular for all $i \in I$, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible in $\mathbf{g}$ for all $\theta \in (0, \bar{\theta})$.*

Due to Lemma 2.2, we immediately have the following.

Corollary 4.2. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and strict monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \dots, m$*,* $g_i|_{S_i^{k-1} \times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S^{k-1}}$ *is supermodular, then there exists* $\bar{\theta} > 0$ *such that* a^* *is globally accessible in* $\mathbf{g}$ *for all* $\theta \in (0, \bar{\theta})$*.*

Suppose that a^* is an iterated MP-maximizer of $\mathbf{g}$ with monotone potential functions $(v^k)_{k=1}^m$ that are relative to $B_\eta(S^{k-1})$ respectively for $k = 1, \dots, m$, where $\eta > 0$ is sufficiently small so that for all $i \in I$ and all $k = 1, \dots, m$,

$$br^i_{g_i}(\pi_i) \cap S_i^k \neq \emptyset,$$

and therefore,

$$br^i_{g_i}(\pi_i | S_i^k) \subset br^i_{g_i}(\pi_i)$$

hold for $\pi_i \in B_\eta(S_{-i}^k)$ (see Lemma 2.3). For each $k = 0, 1, \dots, m$ and $i \in I$, write $S_i^k = [\underline{a}_i^k, \overline{a}_i^k]$, where $0 = \underline{a}_i^0 \leq \underline{a}_i^1 \leq \cdots \leq \underline{a}_i^m = a_i^* = \overline{a}_i^m \leq \cdots \leq \overline{a}_i^1 \leq \overline{a}_i^0 = n_i$.

For each $k = 1, \dots, m$, define $J_\theta^k \colon \Phi \to \mathbb{R}$ to be

$$J_\theta^k(\phi) = \int_0^\infty \theta e^{-\theta t} v^k(\phi(t))\, dt,$$

and for any $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^{k-1}])$ and $y \in \prod_i \Delta([\overline{a}_i^{k-1}, \overline{a}_i^0])$, let

$$\begin{aligned}
\Phi_x^{k,-} = \{\phi \in \Phi \mid \phi(0) = x,& \\
&\dot{\phi}_i(t) + \phi_i(t) \in \Delta([\underline{a}_i^{k-1}, \underline{a}_i^k])\ \forall i \in I,\ \text{a.a.}\, t \geq 0\}, \\
\Phi_y^{k,+} = \{\phi \in \Phi \mid \phi(0) = y,& \\
&\dot{\phi}_i(t) + \phi_i(t) \in \Delta([\overline{a}_i^k, \overline{a}_i^{k-1}])\ \forall i \in I,\ \text{a.a.}\, t \geq 0\}.
\end{aligned}$$

Consider the maximization problems:

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_x^{k,-}, \tag{4.3}$$

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_y^{k,+}. \tag{4.4}$$

Since J_θ^k is continuous, and $\Phi_x^{k,-}$ and $\Phi_y^{k,+}$ are compact, the above maximization problems admit solutions.

Lemma 4.3. (1) *For each* $k = 1, \dots, m$*, and for any* $\theta > 0$ *and any* $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ *and* $y \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$*: there exists a solution to the maximization problem* (4.3)*,* $\phi^{k,-}$*, such that*

$$\dot{\phi}_i^{k,-}(t) = \min BR^i_{v^k}(\phi_{-i}^{k,-} | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^{k,-}(t) \tag{4.5}$$

for all $i \in I$ and almost all $t \geq 0$; there exists a solution to the maximization problem (4.4), $\phi^{k,+}$, *such that*

$$\dot{\phi}_i^{k,+}(t) = \min BR^i_{v^k}(\phi_{-i}^{k,+}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^{k,+}(t) \tag{4.6}$$

for all $i \in I$ and almost all $t \geq 0$.

(2) *For each $k = 1, \ldots, m$, there exists $\bar{\theta}^k > 0$ such that for any $\theta \in (0, \bar{\theta}^k)$ and any $x \in \prod_i \Delta([\underline{a}_i^0, \underline{a}_i^k])$ ($y \in \prod_i \Delta([\overline{a}_i^k, \overline{a}_i^0])$, resp.), any solution to the maximization problem* (4.3) *((4.4), resp.) converges to $\underline{a}^k$ ($\overline{a}^k$, resp.).*

Proof. (1) We only show the existence of a solution that satisfies (4.5) (the existence of a solution that satisfies (4.6) is proved similarly). First note that for each $i \in I$,

$$\begin{aligned}(1+\theta)e^{-\theta t}v^k(\phi(t)) &= \sum_{h \in A_i} e^t \phi_{ih}(t) \frac{d}{dt}\left(-e^{-(1+\theta)t} V_{ih}^k(\phi_{-i})(t)\right) \\ &= \frac{d}{dt}\left(-e^{-\theta t}\sum\nolimits_{h \in A_i} \phi_{ih}(t) V_{ih}^k(\phi_{-i})(t)\right) \\ &\qquad + e^{-\theta t}\sum\nolimits_{h \in A_i}\left(\dot{\phi}_{ih}(t) + \phi_{ih}(t)\right) V_{ih}^k(\phi_{-i})(t)\end{aligned}$$

for almost all $t \geq 0$, where

$$V_{ih}^k(\phi_{-i})(t) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} v^k(h, \phi_{-i}(s))\, ds.$$

Therefore, any solution to (4.3), ϕ^k, satisfies

$$\dot{\phi}_{ih}^k(t) > -\phi_{ih}^k \Rightarrow h \in BR^i_{v^k}(\phi_{-i}^k|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) \tag{4.7}$$

for all $i \in I$ and almost all $t \geq 0$. It then follows from Lemma A.1.3 in OTH (2008) that there exists a feasible path $\phi_i^{k,-}$ that satisfies (4.5).

(2) We show that there exists $\bar{\theta}^k > 0$ such that for any $\theta \in (0, \bar{\theta}^k)$, any solution to (4.3) ((4.4), resp.) approaches arbitrarily close to $\underline{a}^k$ ($\overline{a}^k$, resp.). Here, $\bar{\theta}^k$ can be taken independently of x and y. Then, by following the proofs of Lemmas 3 and 4 in HS (1999) (see also Theorem 4.1 in HS (2002)) for the potential game $v|_{[\underline{a}^0, \underline{a}^k]}$, one can show that once any feasible path that satisfies (4.7) gets close enough to the potential maximizer $\underline{a}^k$, it must converge to $\underline{a}^k$. A dual argument applies to solutions to (4.4)

Let $v^k_{\max} = v^k(\underline{a}^k) = v^k(\overline{a}^k)$, $\overline{v}^k = \max_{a \in A \setminus [\underline{a}^k, \overline{a}^k]} v^k(a)$, and $\underline{v}^k = \min_{a \in A} v^k(a)$. Note that $v^k_{\max} > \overline{v}^k \geq \underline{v}^k$. Let ϕ be any solution to (4.3), and ψ the linear path from x to $\underline{a}^k$: i.e., for all $i \in I$ and $t \geq 0$, $\psi_{ih}(t) = 1 - (1 - x_{ih})e^{-t}$ if $h = \underline{a}_i^k$ and $\psi_{ih}(t) = x_{ih}e^{-t}$ otherwise. Denote $\phi(a|t) = \prod_{i \in I} \phi_{ia_i}(t)$ and $\psi(a|t) = \prod_{i \in I} \psi_{ia_i}(t)$. We first have

$$\begin{aligned}&J_\theta^k(\phi) \geq J_\theta^k(\psi) \\ &\quad = \int_0^\infty \theta e^{-\theta t}\psi(\underline{a}^k|t)\, dt\, v^k_{\max} + \sum_{a \neq \underline{a}^k} \int_0^\infty \theta e^{-\theta t}\psi(a|t)\, dt\, v^k(a)\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^\infty \theta e^{-\theta t}\psi(\underline{a}^k|t)\,dt\,v^k_{\max} + \left[1 - \int_0^\infty \theta e^{-\theta t}\psi(\underline{a}^k|t)\,dt\right]\underline{v}^k \\
&= v^k_{\max} - \left[1 - \int_0^\infty \theta e^{-\theta t}\prod_{i\in I}\left\{1 - \left(1 - x_{i\underline{a}^k_i}\right)e^{-t}\right\}dt\right]\left(v^k_{\max} - \underline{v}^k\right) \\
&\geq v^k_{\max} - \left[1 - \int_0^\infty \theta e^{-\theta t}\left(1 - e^{-t}\right)^N dt\right]\left(v^k_{\max} - \underline{v}^k\right).
\end{aligned}
$$

We also have

$$
\begin{aligned}
J^k_\theta(\phi) &= \int_0^\infty \theta e^{-\theta t}\phi(\underline{a}^k|t)\,dt\,v^k_{\max} + \sum_{a\neq\underline{a}^k}\int_0^\infty \theta e^{-\theta t}\phi(a|t)\,dt\,v^k(a) \\
&\leq \int_0^\infty \theta e^{-\theta t}\phi(\underline{a}^k|t)\,dt\,v^k_{\max} + \left[1 - \int_0^\infty \theta e^{-\theta t}\phi(\underline{a}^k|t)\,dt\right]\overline{v}^k.
\end{aligned}
$$

Combining these inequalities, we have

$$
\int_0^\infty \theta e^{-\theta t}\phi(\underline{a}^k|t)\,dt \geq 1 - \frac{v^k_{\max} - \underline{v}^k}{v^k_{\max} - \overline{v}^k}\left[1 - \int_0^\infty \theta e^{-\theta t}\left(1 - e^{-t}\right)^N dt\right].
$$

The integral in the right hand side converges to one as θ goes to zero. Therefore, given $\delta > 0$ we have $\overline{\theta}^k > 0$ such that for all $\theta \in (0, \overline{\theta}^k)$,

$$
\int_0^\infty \theta e^{-\theta t}\phi(\underline{a}^k|t)\,dt \geq 1 - \delta,
$$

which implies that there exists $t \geq 0$ such that $\phi(\underline{a}^k|t) \geq 1 - \delta$, and hence, $\phi_{i\underline{a}^k_i}(t) \geq 1 - \delta$ for all $i \in I$. ▮

In the following, we set $T^0 = 0$, and $\phi^{0,-}$ and $\phi^{0,+}$ to be such that $\phi^{0,-}(t) = \underline{a}^0$ and $\phi^{0,+}(t) = \overline{a}^0$ for all $t \geq 0$, respectively.

Lemma 4.4. *There exists $\bar{\theta} > 0$ such that for any $\theta \in (0, \bar{\theta})$, there exist $T^1, \ldots, T^{m-1}$ with $T^1 \leq \cdots \leq T^{m-1} < \infty$ and feasible paths $\phi^{1,-}, \ldots, \phi^{m,-}$ and $\phi^{1,+}, \ldots, \phi^{m,+}$ such that for each $k = 1, \ldots, m$,*

$(*^-_k)$ $\phi^{k,-}(t) = \phi^{k-1,-}(t)$ *for all* $t \in [0, T^{k-1}]$, $\phi^{k,-}(T^{k-1}) \in B_\eta(\underline{a}^{k-1})$,

$$
\dot{\phi}^{k,-}_i(t) = \min BR^i_{v^k}(\phi^{k,-}_{-i}|[\underline{a}^{k-1}_i, \underline{a}^k_i])(t) - \phi^{k,-}_i(t)
$$

for all $i \in I$ and almost all $t \in [T^{k-1}, \infty)$, and $\lim_{t\to\infty}\phi^{k,-}(t) = \underline{a}^k$,

and

$(*^+_k)$ $\phi^{k,+}(t) = \phi^{k-1,+}(t)$ *for all* $t \in [0, T^{k-1}]$, $\phi^{k,+}(T^{k-1}) \in B_\eta(\overline{a}^{k-1})$,

$$
\dot{\phi}^{k,+}_i(t) = \max BR^i_{v^k}(\phi^{k,+}_{-i}|[\overline{a}^k_i, \overline{a}^{k-1}_i])(t) - \phi^{k,+}_i(t)
$$

for all $i \in I$ and almost all $t \in [T^{k-1}, \infty)$, and $\lim_{t\to\infty}\phi^{k,+}(t) = \overline{a}^k$.

Proof. Take $\bar{\theta}^1, \dots, \bar{\theta}^m$ as in Lemma 4.3, and set $\bar{\theta} = \min\{\bar{\theta}^1, \dots, \bar{\theta}^m\}$. Fix any $\theta \in (0, \bar{\theta})$. First, by Lemma 4.3 for (4.3) and (4.4) with $k = 1$, $x = \underline{a}^0$, and $y = \overline{a}^0$, we have feasible paths $\phi^{1,-}$ and $\phi^{1,+}$ that satisfy $(*_1^-)$ and $(*_1^+)$, respectively.

Next, for $k \geq 2$ assume that there exist $T^0, \dots, T^{k-2}$, $\phi^{1,-}, \dots, \phi^{k-1,-}$, and $\phi^{1,+}, \dots, \phi^{k-1,+}$ that satisfy $(*_1^-), \dots, (*_{k-1}^-)$ and $(*_1^+), \dots, (*_{k-1}^+)$. Let $T^{k-1} \geq T^{k-2}$ be such that $\phi^{k-1,-}(t) \in B_\eta(\underline{a}^{k-1})$ and $\phi^{k-1,+}(t) \in B_\eta(\overline{a}^{k-1})$ for all $t \geq T^{k-1}$. Then, consider the maximization problems:

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_{T^{k-1}}^{k,-}, \tag{4.8}$$

$$\max\ J_\theta^k(\phi) \quad \text{s.t. } \phi \in \Phi_{T^{k-1}}^{k,+}, \tag{4.9}$$

where

$$\begin{aligned}
\Phi_{T^{k-1}}^{k,-} = \{\phi \in \Phi \mid\ & \phi(t) = \phi^{k-1,-}(t)\ \forall t \in [0, T^{k-1}], \\
& \dot{\phi}_i(t) + \phi_i(t) \in \Delta([\underline{a}_i^{k-1}, \underline{a}_i^k])\ \forall i \in I,\ \text{a.a.}\, t \in [T^{k-1}, \infty)\}, \\
\Phi_{T^{k-1}}^{k,+} = \{\phi \in \Phi \mid\ & \phi(t) = \phi^{k-1,+}(t)\ \forall t \in [0, T^{k-1}], \\
& \dot{\phi}_i(t) + \phi_i(t) \in \Delta([\overline{a}_i^k, \overline{a}_i^{k-1}])\ \forall i \in I,\ \text{a.a.}\, t \in [T^{k-1}, \infty)\}.
\end{aligned}$$

Observe that (4.8) and (4.9) are equivalent to (4.3) with $x = \phi^{k-1,-}(T^{k-1})$ and (4.4) with $y = \phi^{k-1,+}(T^{k-1})$, respectively. Therefore, by Lemma 4.3 we have feasible paths $\phi^{k,-}$ and $\phi^{k,+}$ that satisfy $(*_k^-)$ and $(*_k^+)$, respectively. ∎

Let $T^m = \infty$.

Lemma 4.5. *There exists $\bar{\theta} > 0$ such that for any $\theta \in (0, \bar{\theta})$, there exist $T^1, \dots, T^{m-1}$ with $T^1 \leq \cdots < T^{m-1} \leq \infty$ and feasible paths ϕ^- and ϕ^+ such that*

$(*^-)$ *$\phi^-(0) = \underline{a}^0$, $\lim_{t\to\infty} \phi^-(t) = a^*$, and for each $k = 1, \dots, m$, $\phi^-(t) \in B_\eta([\underline{a}^{k-1}, a^*])$ for all $t \in [T^{k-1}, \infty)$ and*

$$\dot{\phi}_i^-(t) = \min BR_{v^k}^i(\phi_{-i}^- | [\underline{a}_i^{k-1}, \underline{a}_i^k])(t) - \phi_i^-(t)$$

for all $i \in I$ and almost all $t \in [T^{k-1}, T^k)$,

and

$(*^+)$ *$\phi^+(0) = \overline{a}^0$, $\lim_{t\to\infty} \phi^+(t) = a^*$, and for each $k = 1, \dots, m$, $\phi^+(t) \in B_\eta([a^*, \overline{a}^{k-1}])$ for all $t \in [T^{k-1}, \infty)$ and*

$$\dot{\phi}_i^+(t) = \max BR_{v^k}^i(\phi_{-i}^+ | [\overline{a}_i^k, \overline{a}_i^{k-1}])(t) - \phi_i^+(t)$$

for all $i \in I$ and almost all $t \in [T^{k-1}, T^k)$.

Proof. Take $\bar{\theta}$ as in Lemma 4.4. Fix any $\theta \in (0, \bar{\theta})$, and let $\phi^{1,-}, \dots, \phi^{m,-}$, and $\phi^{1,+}, \dots, \phi^{m,+}$ satisfy $(*_k^-)$ and $(*_k^+)$ for $k = 1, \dots, m$, respectively. Set $\phi^- = \phi^{m,-}$ and $\phi^+ = \phi^{m,+}$. We only verify that ϕ^- satisfies $(*^-)$.

For each $k = 1, \dots, m$, we have $\phi_i^-(t) \in B_\eta([\underline{a}_i^{k-1}, a_i^*])$ for all $i \in I$ and all $t \geq T^{k-1}$. We also have $\lim_{t\to\infty} \phi^-(t) = a^*$. Observe that T^k's can be taken sufficiently large so that for each $k = 1, \dots, m-1$ and $i \in I$, $\phi_{ih}^-(t) = \phi_{ih}^{k,-}(t) = e^{-(t-T^k)}\phi_{ih}^{k,-}(T^k)$ for all $h \notin [\underline{a}_i^k, a_i^*]$ and all $t \geq T^k$. Note that by construction, $\phi^-(t) = \phi^{k,-}(t)$ for all $t \leq T^k$. Since $v^k(a) = v^k(a')$ for all $a, a' \in [\underline{a}^k, \bar{a}^k]$, it follows that for each $k = 1, \dots, m-1$ and $i \in I$, $BR_{v^k}^i(\phi_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) = BR_{v^k}^i(\phi_{-i}^{k,-}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t)$ for all $t \leq T^k$. ▮

Proof of Theorem 4.1. Suppose that v^k's are monotone potential functions for a^* relative to $B_\eta([\underline{a}^{k-1}, \bar{a}^{k-1}])$. Take $\bar{\theta}$ as in Lemma 4.5. Fix any $\theta \in (0, \bar{\theta})$ and let ϕ^- and ϕ^+ satisfy $(*^-)$ and $(*^+)$, respectively.

Fix any $x \in \prod_i \Delta(A_i)$. Let β_x be the best response correspondence defined in (4.2). Let $\tilde{\Phi}_x = \{\phi \in \Phi_x \,|\, \phi^- \precsim \phi \precsim \phi^+\}$. We will show that $\tilde{\beta}_x(\phi) = \beta_x(\phi) \cap \tilde{\Phi}_x$ is nonempty for any $\phi \in \tilde{\Phi}_x$. Then, since $\tilde{\Phi}_x$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^* \in \tilde{\beta}_x(\phi^*) \subset \tilde{\Phi}_x$, which is a perfect foresight path in $\mathbf{g}$ and satisfies $\phi^- \precsim \phi^* \precsim \phi^+$. Since both ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*.

Take any $\phi \in \tilde{\Phi}_x$. Consider any $k = 1, \dots, m$. Note that $\phi(t) \in B_\eta([\underline{a}^{k-1}, \bar{a}^{k-1}])$ for all $t \geq T^{k-1}$.

Suppose first that $g_i|_{[\underline{a}_i^{k-1}, \bar{a}_i^{k-1}] \times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned} \min BR_{v^k}^i(\phi_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) &\leq \max BR_{g_i}^i(\phi_{-i}^-|[\underline{a}_i^{k-1}, \bar{a}_i^k])(t) \\ &\leq \max BR_{g_i}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \bar{a}_i^k])(t) \end{aligned}$$

for all $t \in [T^{k-1}, T^k)$, where the second inequality follows from the assumption that v^k is a monotone potential function relative to $B_\eta([\underline{a}^{k-1}, \bar{a}^{k-1}])$, and the third inequality follows from the supermodularity of $g_i|_{[\underline{a}_i^{k-1}, \bar{a}_i^{k-1}] \times A_{-i}}$. Similarly, for all $i \in I$,

$$\begin{aligned} \max BR_{v^k}^i(\phi_{-i}^+|[\bar{a}_i^k, \bar{a}_i^{k-1}])(t) &\geq \min BR_{g_i}^i(\phi_{-i}^+|[\underline{a}_i^k, \bar{a}_i^{k-1}])(t) \\ &\geq \min BR_{g_i}^i(\phi_{-i}|[\underline{a}_i^k, \bar{a}_i^{k-1}])(t) \end{aligned}$$

for all $t \in [T^{k-1}, T^k)$.

Suppose next that $v^k|_{[\underline{a}_i^{k-1}, \bar{a}_i^{k-1}] \times A_{-i}}$ are supermodular for all $i \in I$. Then, for all $i \in I$,

$$\begin{aligned} \min BR_{v^k}^i(\phi_{-i}^-|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) &\leq \min BR_{v^k}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) \\ &\leq \max BR_{g_i}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \bar{a}_i^k])(t) \end{aligned}$$

for all $t \in [T^{k-1}, T^k)$, where the second inequality follows from the supermodularity of $v^k|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$, and the third inequality follows from the assumption that v^k is a monotone potential function relative to $B_\eta([\underline{a}^{k-1}, \overline{a}^{k-1}])$. Similarly, for all $i \in I$,

$$\begin{aligned}\max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) &\geq \max BR^i_{v^k}(\phi_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t) \\ &\geq \min BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^k, \overline{a}_i^{k-1}])(t)\end{aligned}$$

for all $t \in [T^{k-1}, T^k)$.

Therefore, in each case, we have for all $t \in [T^{k-1}, T^k)$,

$$\begin{aligned}\max BR^i_{g_i}(\phi_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t) &\geq \min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t), \\ \min BR^i_{g_i}(\phi_{-i}|\underline{a}_i^k, \overline{a}_i^{k-1}])(t) &\leq \max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t).\end{aligned}$$

Since $\phi(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ for all $t \geq T^{k-1}$ and hence

$$BR^i_{g_i}(\phi_{-i})(t) \cap [\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \neq \emptyset$$

by the choice of η, it follows that

$$\begin{aligned}&BR^i_{g_i}(\phi_{-i})(t) \\ &\qquad \cap \left[\min BR^i_{v^k}(\phi^-_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t), \max BR^i_{v^k}(\phi^+_{-i}|[\overline{a}_i^k, \overline{a}_i^{k-1}])(t)\right] \neq \emptyset,\end{aligned}$$

Let $\tilde{F}_i(\phi_{-i})(t)$ be the convex hull of the above set. Then the differential inclusion

$$\dot{\psi}(t) \in \tilde{F}(\phi)(t) - \psi(t), \qquad \psi(0) = x$$

has a solution ψ (see OTH (2008, Remark 2.1)). Since $\tilde{F}_i(\phi_{-i})(t) \subset F_i(\phi_{-i})(t) = \{\alpha_i \in \Delta(A_i) \,|\, \alpha_{ih} > 0 \Rightarrow h \in BR^i_{g_i}(\phi_{-i})(t)\}$, we have $\psi \in \beta_x(\phi)$. By the construction of ϕ^-, ϕ^+, and ψ, we have $\phi^- \precsim \psi \precsim \phi^+$. Thus, we have $\psi \in \tilde{\beta}_x(\phi) = \beta_x(\phi) \cap \tilde{\Phi}_x$, implying the nonemptiness of $\tilde{\beta}_x(\phi)$. ▮

By Proposition 2.5, we immediately have the following.

Corollary 4.6. *If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i \in I} p_i < 1$, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible in $\mathbf{g}$ for all $\theta \in (0, \bar{\theta})$.*

4.3 Linear Absorption of Iterated Strict MP-Maximizer

In this subsection, we prove that under the same monotonicity condition as in the informational robustness and the global accessibility results, an iterated strict MP-maximizer is linearly absorbing (regardless of the degree of friction), and therefore, it is the unique equilibrium that is globally accessible and linearly absorbing for any small degree of friction.[12]

Theorem 4.7. *Suppose that* $\mathbf{g}$ *has an iterated strict MP-maximizer* a^* *with associated intervals* $(S^k)_{k=0}^m$ *and strict monotone potential functions* $(v^k)_{k=1}^m$*. If for each* $k = 1, \dots, m$*,* $g_i|_{S_i^{k-1}\times A_{-i}}$ *is supermodular for all* $i \in I$ *or* $v^k|_{S^{k-1}}$ *is supermodular, then* a^* *is linearly absorbing in* $\mathbf{g}$ *for all* $\theta > 0$*.*

We will use the following result due to Hofbauer and Sorger (2002) and OTH (2008).

Lemma 4.8. *Suppose that* $v|_S$ *is a potential game with a unique potential maximizer* $a^* \in S$*. Then,* a^* *is absorbing in* $v|_S$ *for all* $\theta > 0$*. If in addition,* $v|_S$ *is supermodular, then* a^* *is linearly absorbing in* $v|_S$ *for all* $\theta > 0$*.*

Suppose that a^* is an iterated strict MP-maximizer of $\mathbf{g}$ with associated intervals $(S^k)_{k=0}^m$ and strict monotone potential functions $(v^k)_{k=1}^m$. Due to Lemma 2.2, we can have $(\tilde{v}^k)_{k=1}^m$ and $\eta > 0$ such that for each $k = 1, \dots, m$, $\tilde{v}^k \colon A \to \mathbb{R}$ is a strict monotone potential function relative to $B_\eta(\Delta(S^{k-1}))$. For each $k = 0, 1, \dots, m$ and $i \in I$, write $S_i^k = [\underline{a}_i^k, \overline{a}_i^k]$, where $0 = \underline{a}_i^0 \le \underline{a}_i^1 \le \cdots \le \underline{a}_i^m = a_i^* = \overline{a}_i^m \le \cdots \le \overline{a}_i^1 \le \overline{a}_i^0 = n_i$. In defining such $(\tilde{v}^k)_{k=1}^m$ and $\eta > 0$, we extend v^k ($k = 1, \dots, m$) to A so that $[\underline{a}^{k-1}, \underline{a}^k]$ and $[\overline{a}^k, \overline{a}^{k-1}]$ are strict best response sets in the games $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, respectively, and take $\eta > 0$ to be sufficiently small so that for all $k = 1, \dots, m$ and all $i \in I$,

$$br^i_{\tilde{v}^k}(\pi_i | [\underline{a}_i^0, \underline{a}_i^k]) \subset [\underline{a}_i^{k-1}, \underline{a}_i^k]$$

for all $\pi_i \in B_\eta(\Delta([\underline{a}_{-i}^{k-1}, \underline{a}_{-i}^k]))$ and

$$br^i_{\tilde{v}^k}(\pi_i | [\overline{a}_i^k, \overline{a}_i^0]) \subset [[\overline{a}_i^k, \overline{a}_i^{k-1}]$$

for all $\pi_i \in B_\eta(\Delta([\overline{a}_{-i}^k, \overline{a}_{-i}^{k-1}]))$. In the case where $v^k|_{[\underline{a}^{k-1}, \overline{a}^{k-1}]}$ is supermodular, v^k is extended so that $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$ are supermodular. We assume without loss of generality that in each potential game $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, resp.), any perfect foresight path from $B_\eta(\underline{a}^k)$ ($B_\eta(\overline{a}^k)$, resp.) converges (linearly, in the case where the game is also supermodular) to $\underline{a}^k$ ($\overline{a}^k$, resp.).

For an interval $S \subset A$, we say that a feasible path ϕ is an *S-perfect foresight path* if for all $i \in I$, all $h \in A_i$, and almost all $t \ge 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i_{g_i}(\phi_{-i} | S_i)(t). \tag{4.10}$$

Note that if ϕ is an S-perfect foresight path with $\phi(0) = x$, then for all $i \in A_i$ and all $h \notin S_i$, $\phi_{ih}(t) = x_{ih} e^{-t}$ for all $t \ge 0$.

[12] Because of the supermodularity conditions, it can be shown that the same stability properties in fact hold under rationalizable foresight (Matsui and Oyama (2006)); see OTH (2008, Subsection 3.3).

Lemma 4.9. *For each $k = 1, \dots, m$, if $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ is supermodular for all $i \in I$ or $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$ are supermodular, then* (1) *for any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) \in B_\eta(\Delta([\underline{a}^k, \overline{a}^k]))$,*

$$\lim_{t \to \infty} \sum_{h \in [\underline{a}_i^k, \overline{a}_i^k]} \phi_{ih}^*(t) = 1 \tag{4.11}$$

for all $i \in I$, and (2) *there exists $\eta^k \in (0, \eta]$ such that for any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) \in B_{\eta^k}(\Delta([\underline{a}^k, \overline{a}^k]))$,*

$$BR_{g_i}^i(\phi_{-i}^*|[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}])(t) \subset [\underline{a}_i^k, \overline{a}_i^k] \tag{4.12}$$

for all $i \in I$ and $t \geq 0$.

Proof. (1) Take any $x \in B_\eta(\Delta([\underline{a}^k, \overline{a}^k]))$ and any $[\underline{a}^{k-1}, \overline{a}^{k-1}]$-perfect foresight path ϕ^* with $\phi^*(0) = x$. Note that $\phi^*(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ for all $t \geq 0$. Let

$$x_i^{k,-} = \eta \underline{a}_i^0 + (1 - \eta) \underline{a}_i^k, \qquad x_i^{k,+} = \eta \overline{a}_i^0 + (1 - \eta) \overline{a}_i^k,$$

and denote $x^{k,-} = \left(x_i^{k,-}\right)_{i \in I}$ and $x^{k,+} = \left(x_i^{k,+}\right)_{i \in I}$. We will find perfect foresight paths $\phi^{k,-}$ and $\phi^{k,+}$ for $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, respectively, such that $\phi^{k,-}(0) = x^{k,-}$, $\phi^{k,+}(0) = x^{k,+}$, and $\phi^{k,-}(t) \precsim \phi^*(t) \precsim \phi^{k,+}(t)$ for all $t \geq 0$. Then, since the potential maximizer $\underline{a}^k$ ($\overline{a}^k$, resp.) is absorbing in $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k, \overline{a}^0]}$, resp.), and hence $\phi^{k,-}$ ($\phi^{k,+}$, resp.) converges to $\underline{a}^k$ ($\overline{a}^k$, resp.), ϕ^* must satisfy (4.11).

The argument below follows that in OTH (2008, Appendix A.3). We show the existence of $\phi^{k,-}$; the existence of $\phi^{k,+}$ can be shown similarly. Let $\tilde{\Phi}_{x^{k,-}}$ be the set of feasible paths $\phi \in \Phi_{x^{k,-}}$ such that for all $i \in I$ and all $t \geq 0$, $\phi_i(t) \in \Delta([\underline{a}_i^0, \underline{a}_i^k])$, $\phi_i(t) \precsim \phi_i^*(t)$, and $\phi_{ih}(t) = x_{ih}^{k,-} e^{-t}$ for all $h < \underline{a}_i^{k-1}$. Consider the best response correspondence $\beta_{\tilde{v}^k}^-$ for the stage game $\tilde{v}^k|_{[\underline{a}^0, \underline{a}^k]}$. We will show that $\tilde{\beta}_{\tilde{v}^k}^-(\phi) = \beta_{\tilde{v}^k}^-(\phi) \cap \tilde{\Phi}_{x^{k,-}}$ is nonempty for any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. Then, since $\tilde{\Phi}_{x^{k,-}}$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^{k,-} \in \tilde{\beta}_{\tilde{v}^k}^-(\phi^{k,-}) \subset \tilde{\Phi}_{x^{k,-}}$, as desired.

Take any $\phi \in \tilde{\Phi}_{x^{k,-}}$. Note that $\phi(t) \in B_\eta(\Delta([\underline{a}^{k-1}, \underline{a}^k]))$ for all $t \geq 0$, and therefore $BR_{\tilde{v}^k}^i(\phi_{-i}|[\underline{a}_i^0, \underline{a}_i^k])(t) = BR_{\tilde{v}^k}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t)$ by the choice of η. In the case where $g_i|_{[\underline{a}_i^{k-1}, \overline{a}_i^{k-1}] \times A_{-i}}$ is supermodular for all $i \in I$, we have, for all $i \in I$ and all $t \geq 0$,

$$\begin{aligned}\min BR_{\tilde{v}^k}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \underline{a}_i^k])(t) &\leq \min BR_{g_i}^i(\phi_{-i}|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t) \\ &\leq \min BR_{g_i}^i(\phi_{-i}^*|[\underline{a}_i^{k-1}, \overline{a}_i^k])(t),\end{aligned}$$

where the first inequality follows from the assumption that $\tilde{v}^k$ is a strict monotone potential relative to $B_\eta(\Delta([\underline{a}^{k-1}, \overline{a}^{k-1}]))$ and the second inequality follows from the supermodularity of $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$. In the case where v^k is supermodular, we have, for all $i \in I$ and all $t \geq 0$,

$$\begin{aligned}\min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}_i^{k-1},\underline{a}_i^k])(t) &\leq \min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}_i^{k-1},\underline{a}_i^k])(t)\\ &\leq \min BR^i_{g_i}(\phi^*_{-i}|[\underline{a}_i^{k-1},\overline{a}_i^k])(t),\end{aligned}$$

where the first inequality follows from the supermodularity of $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ and the second inequality follows from the assumption that $\tilde{v}^k$ is a strict monotone potential relative to $B_\eta(\Delta([\underline{a}^{k-1},\overline{a}^{k-1}]))$. Therefore, in each case, we have, for all $i \in I$ and all $t \geq 0$,

$$\min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}_i^0,\underline{a}_i^k])(t) \leq \min BR^i_{g_i}(\phi^*_{-i}|[\underline{a}_i^{k-1},\overline{a}_i^k])(t).$$

It follows that the solution ψ to

$$\dot{\psi}_i(t) = \min BR^i_{\tilde{v}^k}(\phi_{-i}|[\underline{a}_i^0,\underline{a}_i^k])(t) - \psi_i(t), \qquad \psi(0) = x_i^{k,-},$$

which is a best response to ϕ in the game $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$, satisfies $\psi \in \tilde{\Phi}_{x^{k,-}}$. This implies the nonemptiness of $\tilde{\beta}^-_{\tilde{v}^k}(\phi)$.

(2) If $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$ is supermodular for all $i \in I$, then arguments analogous to those in OTH (2008, Appendix A.1) show that (1) implies (2). If $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$ are supermodular, then $\underline{a}^k$ ($\overline{a}^k$, resp.) is linearly absorbing in $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ ($\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$, resp.) and hence $\phi^{k,-}$ ($\phi^{k,+}$, resp.) converges linearly to $\underline{a}^k$ ($\overline{a}^k$, resp.). Therefore, for all i and all $h \notin [\underline{a}_i^k,\overline{a}_i^k]$, $\phi^*(t) = x_{ih}e^{-t}$ for all $t \geq 0$. Since $[\underline{a}^k,\overline{a}^k]$ is a strict best response set in $\mathbf{g}$, it follows that ϕ^* must satisfy (4.12). ∎

Proof of Theorem 4.7. Suppose that $\tilde{v}^k$'s are strict monotone potential functions relative to $B_\eta(\Delta([\underline{a}^{k-1},\overline{a}^{k-1}]))$ and that for each $k = 1,\dots,m$, $g_i|_{[\underline{a}_i^{k-1},\overline{a}_i^{k-1}]\times A_{-i}}$ is supermodular for all $i \in I$ or $\tilde{v}^k|_{[\underline{a}^0,\underline{a}^k]}$ and $\tilde{v}^k|_{[\overline{a}^k,\overline{a}^0]}$ are supermodular. Take $\eta^1,\dots,\eta^m$ as in Lemma 4.9, and let $\varepsilon = \min\{\eta^1,\dots,\eta^m\}$.

Fix any $x \in B_\varepsilon(a^*)$ and any perfect foresight path ϕ^* in $\mathbf{g}$ with $\phi^*(0) = x$. It is sufficient to prove that for all $k = 1,\dots,m$,

$$BR^i_{g_i}(\phi^*_{-i}|[\underline{a}_i^{k-1},\overline{a}_i^{k-1}])(t) \subset [\underline{a}_i^k,\overline{a}_i^k] \qquad (*_k)$$

holds for all $i \in I$ and all $t \geq 0$, which can be done by applying Lemma 4.9 iteratively. Indeed, since ϕ^* is an $[\underline{a}^0,\overline{a}^0]$-perfect foresight path, $(*_1)$ is true by Lemma 4.9. If $(*_1)$–$(*_{k-1})$ are true, then ϕ^* is an $[\underline{a}^{k-1},\overline{a}^{k-1}]$-perfect foresight path, so that $(*_k)$ is also true by Lemma 4.9. ∎

By Proposition 2.5, we immediately have the following.

Corollary 4.10. *If a^* is an iterated strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i\in I} p_i < 1$, then a^* is linearly absorbing in $\mathbf{g}$ for all $\theta > 0$.*

5 Conclusion

For any given set-valued solution concept, in principle, it is possible to consider iterative elimination of actions outside the solution set. In this paper, we applied such an iterative construction to two refinements of Nash equilibrium: p-dominant equilibrium (Morris, Rob, and Shin (1995) and Kajii and Morris (1997)) or p-best response set (Tercieux (2006a)); and potential maximizer (Monderer and Shapley (1996)) or MP-maximizer (Morris and Ui (2005)). We showed that the iterative construction preserves their robustness to incomplete information (Kajii and Morris (1997)) as well as stability under perfect foresight dynamics (Matsui and Matsuyama (1995)): *iterated p-dominant equilibria* as well as *iterated MP-maximizers* (under some monotonicity conditions) are both robust to incomplete information and globally accessible (for a small degree of friction) and linearly absorbing under perfect foresight dynamics. We also proposed simple procedures, for some special classes of games, to find an iterated p-dominant equilibrium or an iterated MP-maximizer. In particular, we introduced iterated pairwise p-dominance and iterated risk-dominance for general supermodular games and two-player supermodular coordination games, respectively. An iterated MP-maximizer is shown to exist and to be easy to find in an economically relevant class of games. However, generally, finding an MP-maximizer or iterated MP-maximizer is a difficult task. We see these simpler procedures as natural first steps to check whether our main theorems apply.

We provided numerical examples to assess the relevance of the iterative construction for both the p-dominance and the potential maximization approaches. In particular, the example in Subsection 2.6 shows, first, that for the p-dominance approach, our iterative construction leads to a strictly more general concept. In this example, the game has no (p_1, p_2)-dominant equilibrium such that $p_1 + p_2 < 1$, but has an iterated strict (p, p)-dominant equilibrium for some $p < 1/2$ and hence our results show that it is robust to incomplete information and stable under perfect foresight dynamics. The same example also shows that, for LP-maximizers, iteration has a bite in the potential approach as well. For this specific form of MP-maximizers, we could exploit an existing characterization that is rather easy to manipulate in verifying that the game has no LP-maximizer while having an iterated LP-maximizer. Note, by contrast, that no such characterization has been known for MP-maximizers. This fact makes it difficult to examine the additional bite of iterated MP-maximizer over MP-maximizer. Identifying a full characterization, in particular a (tight) necessary condition, for a game to have an MP-maximizer or an iterated MP-maximizer is left an open problem for future research.

Finally, we considered a simple application of our concepts in a context of technology adoption, to demonstrate that our iterative procedure can be used in identifying a robust prediction in an economic situation. Given that potential maximization methods are found in various fields in economics, such as industrial organization (Slade (1994), Monderer and Shapley (1996)), mechanism design and implementation (Sandholm (2007), Bergemann and Morris (2008)) and economic geography (Oyama (2006)), as well as in transportation science (Beckmann *et al.* (1956), Rosenthal (1973)), it is hoped that the theory developed in this paper will be of use also in other applications than the simple one discussed here.

Appendix

A.1 Proof of Lemma 2.2

Let $S^*, S \subset A$, and $v\colon S \to \mathbb{R}$ be as in the statement. For $i \in I$ and $a_i \in A_i$, let

$$\Pi^-_{ia_i}(g_i) = \{\pi_i \in \Delta(A_{-i}) \mid \min br^i_{g_i}(\pi_i|[\min S_i, \max S^*_i]) \leq a_i\},$$
$$\Pi^+_{ia_i}(g_i) = \{\pi_i \in \Delta(A_{-i}) \mid \max br^i_{g_i}(\pi_i|[\min S^*_i, \max S_i]) \geq a_i\};$$

and for $f \in \mathbb{R}^A$,

$$\widehat{\Pi}^-_{ia_i}(f) = \{\pi_i \in \Delta(A_{-i}) \mid \min br^i_f(\pi_i|[\min S_i, \min S^*_i]) \leq a_i\},$$
$$\widehat{\Pi}^+_{ia_i}(f) = \{\pi_i \in \Delta(A_{-i}) \mid \max br^i_f(\pi_i|[\max S^*_i, \max S_i]) \geq a_i\}.$$

Observe that $\Pi^-_{ia_i}(g_i)$ and $\widehat{\Pi}^-_{ia_i}(f)$ ($\Pi^+_{ia_i}(g_i)$ and $\widehat{\Pi}^+_{ia_i}(f)$, resp.) are closed (in $\Delta(A_{-i})$) due to the lower (upper, resp.) semi-continuity of $\min br^i_{g_i}$ and $\min br^i_f$ ($\max br^i_{g_i}$ and $\max br^i_f$, resp.). Note that these sets may be empty. Here we give a characterization of strict MP-maximizer in terms of these sets.

Lemma A.1.1. *S^* is a strict MP-maximizer set of $\mathbf{g}|_S$ with a strict monotone potential function v if and only if $S^* = \arg\max_{a\in S} v(a)$, and for all $i \in I$,*

$$\Pi^-_{ia_i}(g_i) \cap \Delta(S_{-i}) \subset \widehat{\Pi}^-_{ia_i}(v) \cap \Delta(S_{-i})$$

for all $a_i \in [\min S_i, \min S^*_i]$ and

$$\Pi^+_{ia_i}(g_i) \cap \Delta(S_{-i}) \subset \widehat{\Pi}^+_{ia_i}(v) \cap \Delta(S_{-i})$$

for all $a_i \in [\max S^*_i, \max S_i]$.

Now, extend v arbitrarily to A (i.e., consider a function defined on A that coincides with v on S, and denote it again by v) satisfying $S^* = \arg\max_{a\in S} v(a)$. In the case where v is supermodular, extend v so that $v|_A$ is supermodular.

For $\gamma > 0$, define $c_\gamma \colon A \to \mathbb{R}$ by

$$c_\gamma(a) = \gamma \sum_{i \in I} |a_i - S_i^*|,$$

where

$$|a_i - S_i^*| = \begin{cases} 0 & \text{if } a_i \in S_i^*, \\ \min S_i^* - a_i & \text{if } a_i < \min S_i^*, \\ a_i - \max S_i^* & \text{if } a_i > \max S_i^*. \end{cases}$$

Observe that if $h < k \leq \min S_i^*$ or $h > k \geq \max S_i^*$, then for all $a_{-i} \in A_{-i}$,

$$c_\gamma(k, a_{-i}) - c_\gamma(h, a_{-i}) = -|k - h|\gamma \leq -\gamma.$$

Fix any $\gamma > 0$ such that $\gamma < \bigl(\max_{a \in A} v(a) - \max_{a \notin S^*} v(a)\bigr) / \sum_{i \in I} n_i$. Then define $\tilde{v} \colon A \to \mathbb{R}$ by

$$\tilde{v}(a) = v(a) + c_\gamma(a). \tag{A.1}$$

By the choice of γ, $S^* = \arg\max_{a \in A} \tilde{v}(a)$. Verify also that if $v|_S$ is supermodular, then so is $\tilde{v}$. The following lemma shows that the transformation above expands $\widehat{\Pi}^-_{ia_i}(v)$ and $\widehat{\Pi}^+_{ia_i}(v)$.

Lemma A.1.2. *Given $v \colon A \to \mathbb{R}$, let $\tilde{v} \colon A \to \mathbb{R}$ be defined by* (A.1). *For each $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$, there exists an open set $U^-_{ia_i} \subset \Delta(A_{-i})$ such that*

$$\widehat{\Pi}^-_{ia_i}(v) \subset U^-_{ia_i} \subset \widehat{\Pi}^-_{ia_i}(\tilde{v}).$$

Similarly, for each $i \in I$ and $a_i \in [\max S_i^*, \max S_i]$, there exists an open set $U^+_{ia_i} \subset \Delta(A_{-i})$ such that

$$\widehat{\Pi}^+_{ia_i}(v) \subset U^+_{ia_i} \subset \widehat{\Pi}^+_{ia_i}(\tilde{v}).$$

Proof. Fix $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$. Take any $\pi_i \in \widehat{\Pi}^-_{ia_i}(v)$: i.e., $\min br^i_v(\pi_i | [\min S_i, \min S_i^*]) \leq a_i$. Take $\varepsilon(\pi_i) > 0$ such that if $\pi_i' \in B_{\varepsilon(\pi_i)}(\pi_i)$, then

$$\max_{h,k \in A_i} \bigl| \bigl(\tilde{v}(k, \pi_i') - \tilde{v}(h, \pi_i')\bigr) - \bigl(\tilde{v}(k, \pi_i) - \tilde{v}(h, \pi_i)\bigr) \bigr| < \gamma.$$

Let us show that $B_{\varepsilon(\pi_i)}(\pi_i) \subset \widehat{\Pi}^-_{ia_i}(\tilde{v})$. Take any $\pi_i' \in B_{\varepsilon(\pi_i)}(\pi_i)$, and let $\underline{a}_i = \min br^i_{\tilde{v}}(\pi_i' | [\min S_i, \min S_i^*])$. We want to show that $\underline{a}_i \leq a_i$. It is sufficient to show that $\underline{a}_i \leq \min br^i_v(\pi_i | [\min S_i, \min S_i^*])$. If $h < \underline{a}_i$, then

$$\begin{aligned} v(\underline{a}_i, \pi_i) - v(h, \pi_i) &= \bigl(\tilde{v}(\underline{a}_i, \pi_i) - c_\gamma(\underline{a}_i, \pi_i)\bigr) - \bigl(\tilde{v}(h, \pi_i) - c_\gamma(h, \pi_i)\bigr) \\ &= \tilde{v}(\underline{a}_i, \pi_i) - \tilde{v}(h, \pi_i) + (\underline{a}_i - h)\gamma \\ &\geq \tilde{v}(\underline{a}_i, \pi_i) - \tilde{v}(h, \pi_i) + \gamma \\ &> \tilde{v}(\underline{a}_i, \pi_i') - \tilde{v}(h, \pi_i') > 0. \end{aligned}$$

This means that $\underline{a}_i \leq \min br_v^i(\pi_i|[\min S_i, \min S_i^*])$, which implies that $\pi_i' \in \widehat{\Pi}_{ia_i}^-(\tilde{v})$.

Then set $U_{ia_i}^- = \bigcup_{\pi_i \in \widehat{\Pi}_{ia_i}^-(v)} B_{\varepsilon(\pi_i)}(\pi_i)$. ▮

Proof of Lemma 2.2. Given $v\colon A \to \mathbb{R}$, let $\tilde{v}\colon A \to \mathbb{R}$ be defined by (A.1). Then, $S^* = \arg\max_{a \in A} \tilde{v}(a)$; and if $v|_S$ is supermodular, then so is $\tilde{v}$. For each $i \in I$ and $a_i \in [\min S_i, \min S_i^*]$ such that $\Pi_{ia_i}^-(g_i) \neq \emptyset$, take an open set $U_{ia_i}^-$ as in Lemma A.1.2. Note that $\Pi_{ia_i}^-(g_i) \cap \Delta(S_{-i}) \subset U_{ia_i}^-$. Since $\Pi_{ia_i}^-(g_i)$ and $\Delta(S_{-i})$ are closed in a compact set $\Delta(A_{-i})$, there exists $\eta^-(i, a_i) > 0$ such that

$$\Pi_{ia_i}^-(g_i) \cap B_{\eta^-(i,a_i)}(\Delta(S_{-i})) \subset U_{ia_i}^-.$$

Apply the same argument to each $i \in I$ and $a_i' \in [\max S_i^*, \max S_i]$ such that $\Pi_{ia_i'}^+(g_i) \neq \emptyset$ to obtain $\eta^+(i, a_i') > 0$ such that

$$\Pi_{ia_i'}^+(g_i) \cap B_{\eta^+(i,a_i')}(\Delta(S_{-i})) \subset U_{ia_i'}^+,$$

where $U_{ia_i'}^+$ is as in Lemma A.1.2.

Finally, set $\eta = \min_{i,a_i} \eta^-(i, a_i) \wedge \min_{i,a_i'} \eta^+(i, a_i')$. ▮

References

Beckmann, M., C. B. McGuire, and C. B. Winsten (1956). *Studies in the Economics of Transportation*, Yale University Press, New Haven.

Bergemann, D. and S. Morris (2008). "The Role of the Common Prior in Robust Implementation," *Journal of the European Economic Association* **6**, 551-559.

Blume, L. E. (1993). "The Statistical Mechanics of Strategic Interaction," *Games and Economic Behavior* **5**, 387-424.

Carlsson, H. and E. van Damme (1993). "Global Games and Equilibrium Selection," *Econometrica* **61**, 989-1018.

Goldstein, I. and A. Pauzner (2005). "Demand-Deposit Contracts and the Probability of Bank Runs," *Journal of Finance* **60**, 1293-1327.

Harsanyi, J. C. and R. Selten (1988). *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge.

Hofbauer, J. and G. Sorger (1999). "Perfect Foresight and Equilibrium Selection in Symmetric Potential Games," *Journal of Economic Theory* **85**, 1-23.

Hofbauer, J. and G. Sorger (2002). "A Differential Game Approach to Evolutionary Equilibrium Selection," *International Game Theory Review* **4**, 17-31.

Kajii, A. and S. Morris (1997). "The Robustness of Equilibria to Incomplete Information," *Econometrica* **65**, 1283-1309.

Kandori, M., G. J. Mailath, and R. Rob (1993). "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica* **61**, 29-56.

Kandori, M. and R. Rob (1998). "Bandwagon Effects and Long Run Technology Choice," *Games and Economic Behavior* **22**, 30-60.

Kim, Y. (1996). "Equilibrium Selection in n-Person Coordination Games," *Games and Economic Behavior* **15**, 203-227.

Kohlberg, E. and J.-F. Mertens (1986). "On the Strategic Stability of Equilibria," *Econometrica* **54**, 1003-1037.

Kojima, F. (2006). "Risk-Dominance and Perfect Foresight Dynamics in N-Player Games," *Journal of Economic Theory* **128**, 255-273.

Matsui, A. and K. Matsuyama (1995). "An Approach to Equilibrium Selection," *Journal of Economic Theory* **65**, 415-434.

Matsui, A. and D. Oyama (2006). "Rationalizable Foresight Dynamics," *Games and Economic Behavior* **56**, 299-322.

Matsuyama, K. (1991). "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium," *Quarterly Journal of Economics* **106**, 617-650.

Matsuyama, K. (1992a). "A Simple Model of Sectoral Adjustment," *Review of Economic Studies* **59**, 375-388.

Matsuyama, K. (1992b). "The Market Size, Entrepreneurship, and the Big Push," *Journal of the Japanese and International Economies* **6**, 347-364.

Monderer, D. and L. Shapley (1996). "Potential Games," *Games and Economic Behavior* **14**, 124-143.

Morris, S. (1999). "Potential Methods in Interaction Games," mimeo.

Morris, S., R. Rob, and H. S. Shin (1995). "p-Dominance and Belief Potential," *Econometrica* **63**, 145-157.

Morris, S. and H. S. Shin (1998). "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *American Economic Review* **88**, 587-597.

Morris, S. and T. Ui (2005). "Generalized Potentials and Robust Sets of Equilibria," *Journal of Economic Theory* **124**, 45-78.

Okada, D. and O. Tercieux (2008). "Log-linear Dynamics and Local Potential," mimeo.

Oyama, D. (2002). "p-Dominance and Equilibrium Selection under Perfect Foresight Dynamics," *Journal of Economic Theory* **107**, 288-310.

Oyama, D. (2006). "Agglomeration under Forward-Looking Expectations: Potentials and Global Stability," mimeo.

Oyama, D., S. Takahashi, and J. Hofbauer (2008). "Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics," *Theoretical Economics* **3**, 155-192.

Rosenthal, R. W. (1973). "A Class of Games Possessing Pure-Strategy Nash Equilibria," *International Journal of Game Theory* **2**, 65-67.

Rubinstein, A. (1989). "The Electronic Mail Game: Strategic Behavior under 'Almost Common Knowledge'," *American Economic Review* **79**, 385-391.

Sandholm, W. H. (2007). "Pigouvian Pricing and Stochastic Evolutionary Implementation," *Journal of Economic Theory* **132**, 47-69.

Slade, M. E. (1994). "What Does an Oligopoly Maximize?," *Journal of Industrial Economics* **42**, 45-61.

Takahashi, S. (2008). "Perfect Foresight Dynamics in Games with Linear Incentives and Time Symmetry," *International Journal of Game Theory* **37**, 15-38.

Tercieux, O. (2006a). "p-Best Response Set," *Journal of Economic Theory* **131**, 45-70.

Tercieux, O. (2006b). "p-Best Response Set and the Robustness of Equilibria to Incomplete Information," *Games and Economic Behavior* **56**, 371-384.

Ui, T. (2001). "Robust Equilibria of Potential Games," *Econometrica* **69**, 1373-1380.

Young, P. (1993). "The Evolution of Conventions," *Econometrica* **61**, 57-84.