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SZPILRAJN-TYPE THEOREMS IN ECONOMICS

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Abstract. The Szpilrajn "constructive type" theorem on extending binary relations, or its generalizations by Dushnik and Miller [10], is one of the best known theorems in social sciences and mathematical economics. Arrow [1], Fishburn [11], Suzumura [22], Donaldson and Weymark [8] and others utilize Szpilrajn's Theorem and the Well-ordering principle to obtain more general "existence type" theorems on extending binary relations. Nevertheless, we are generally interested not only in the existence of linear extensions of a binary relation R, but in something more: the conditions of the preference sets and the properties which R satisfies to be "inherited" when one passes to any member of some "interesting" family of linear extensions of R. Moreover, in extending a preference relation R, the problem will often be how to incorporate some additional preference data with a minimum of disruption of the existing structure or how to extend the relation so that some desirable new condition is fulfilled. The key to addressing these kinds of problems is the szpilrajn constructive method. In this paper, we give two general "constructive type" theorems on extending binary relations, a Szpilrajn type and a Dushnik-Miller type theorem, which generalize and give a "constructive type" version of all the well known extension theorems in the literature.

JEL Classification Codes: C60, D00, D60, D71.

Key words and Phrases: Consistent binary relations, extension theorems, intersection of binary relations.

1. Introduction

One of the most fundamental results on extensions of binary relations is due to Szpilrajn [23] who shows that any transitive and asymmetric relation has a transitive, asymmetric and complete extension. The original proof of Szpilrajn uses Zermelo's Well-Ordering theorem which is equivalent to the Lemma of Zorn. The result remains true if asymmetry is replaced with reflexivity, that is, any quasi-ordering has an ordering extension. Arrow [1, page 64] states this generalization of Szpilrajn without a proof and Hansson [13] provides a proof on the basis of Szpilrajn's original theorem. Fishburn [11] also gives a proof that utilizes Szpilrajn's Theorem. While the property of being a quasi-ordering is sufficient for the existence of an ordering extension of a relation, this is not necessary. As shown by Suzumura [22], consistency is necessary and sufficient for the existence of an ordering extension.

Dushnik and Miller [10] strengthen the Szpilrajn theorem by proving that every strict partial order is the intersection of its strict linear order extensions. The Szpilrajn theorem and its strengthening by Dushnik and Miller belong to the most quoted theorems in order theory, mathematical logic, mathematical social sciences, mathematical economics and other fields in pure and applied mathematics. The sufficient part of Suzumuras's result, described above, was subsequently used by Donaldson and Weymark [8] in their proof that every quasi-ordering is the intersection of a collection of orderings; this result extends Dushnik and Miller's fundamental observation on intersections of strict linear orders. Duggan [9] proves a general extension theorem from which those mentioned above (Dushnik-Miller and Donaldson-Weymark) results -and several new ones- can be obtained as special cases.

The existence of extension theorems of Szpilrajn type have played an important role in the theory of choice. One way of assessing whether a preference relation is rational[1] is to check whether it can be extended to a transitive and complete relation (see [7] and [19][2]). Another example is the problem of the existence of maximal elements of binary relations[3]. For example, if R is a binary relation on a compact space (X,τ) and R has a continuous[4] linear extension[5] $\leq$, then the compactness of (X,τ) implies that $\leq$ is a complete linear order. This means, in particular, that $\leq$ has maximal elements. Of course, any maximal element of $\leq$ is also a maximal element of R. Problems of Szpilrajn-type also include the general continuous utility representation problem[6], (see [6] and [14]). In a very general sense, a binary relation R on a topological

[1]It is well known that the economic approach to rational behavior traditionally begins with a preference relation R and determines the optimal choice function F from R. Revealed preference theory provides another axiomatic approach to rational behavior by reversing the above procedure.

[2]In particular Szpilrajn theorem is the main tool for proving a known theorem of Richter that establishes the equivalence between rational and congruous consumers.

[3]The existence of maximal elements means that there exists a choice for which there exist no strictly better choices. In the case of considering the set of alternatives as a topological space, conditions for the existence of maximal elements are given by using topological conditions on the alternative's set as well as continuity assumptions on the relation. The usual topological condition that have been used to provide the existence of maximal elements is that of compactness.

[4]A linear order on some topological space (X,τ) is continuous if for every point $x \in X$ both sets $d(x) = \{y \in X | y \leq x\}$ and $i(x) = \{z \in X | x \leq z\}$ are closed subsets of X.

[5]In case that X is endowed with some topology τ one mainly is interested in continuous linear orders instead of only linear orders. This motivates the problem of generalizing the Szpilrajn Theorem to the continuous case.

[6]From a classical normative point of view, a standard key assumption in preference modeling is to assume that preferences define a strict linear order. In this way, preferences can be represented in a real line.

space (X, τ) has a continuous utility representation if there exists some topological space (Y, τ) that is endowed with a continuous linear order $\leq$ and, in addition, some continuous order preserving function u on X the codomain of which is Y. The problem of the existence of $(Y, \tau, \leq)$ and u is equivalent to the problem of finding necessary and sufficient conditions for the existence of some continuous linear extension of R. In addition, the existence of Szpilrajn type Theorems are applied (i) By Stehr [21] to characterize the global orientability; (ii) By Sholomov [20] to characterize ordinal relations; (iii) By Nehring and Puppe [16] on a unifying structure of abstract choice theory (iv) By Blackorby, Bossert and Donaldson [3] in pure population problems e.t.c. Dushnik-Miller type extension theorems have played an important role in welfare economics. For example, on the existence of social welfare ordering for a fixed profile in the sense of Bergson and Samuelson.[7] Weymark [24] applies Dushnik and Miller extension theorem in order to prove a generalization of Moulin's Pareto extension theorem. Many quasi-orderings are obtained as the intersection of a finite number of orders.[8]

It is clear that the results concerning the existence of linear extensions of a binary relation R that we have been considering are useful in social sciences and welfare economics. But, in extending a binary relation R, it is interesting to see whether the conditions of the underlying space X or the properties which R satisfies should be "inherited" when one passes to any member of some family of linear extensions of R. On the other hand, the problem will often be how to incorporate some additional preference data with a minimum of disruption of the existing structure or how to extend the relation so that some desirable new condition is fulfilled. For example, we might wish to adjoin the pair (x, y) to R that does not already relates x and y. If we are to preserve transitivity, we must also adjoin all other pairs of the form (u, v) where $(u, x) \in R$ and $(y, v) \in R$. It is also interesting to see when a binary relation R has a

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[7]Let $(R_1, R_2, ..., R_n)$ be a fixed profile of the individual preference relations. A binary relation Q, is called *Pareto unanimity relation*, if

$$xQy \Leftrightarrow xR_iy \text{ for all } i \in \{1, 2, ..., n\} \text{ and all } x, y \in X.$$

Social welfare ordering in the sense of Bergson and Samuelson is an ordering R such that

$$xQy \Rightarrow xRy \quad \text{and} \quad xP(Q)y \Rightarrow xP(R)y \quad \text{for all} \quad x, y \in X$$

If $R_1, R_2, ..., R_n$ are transitive then Q is quasi-transitive. By the corollary of Szpilrajn Theorem proved by Fishburn we have the existence of an ordering extension R. Hence, R is a Social Welfare Ordering for $(R_1, R_2, ..., R_n)$. If $R_1, R_2, ..., R_n$ are non-transitive and Q is consistent, then by the theorem of Suzumura we have the existence of an ordering extension.

[8]Donaldson and Weymark [8], utilize their generalization of Dushnik-Miller extension theorem to prove that the strong Pareto quasi-ordering on a set of alternatives X, constructed by using a single profile $U = (U_1, U_2, ..., U_n)$ of real valued utility functions on X, is the intersection of the dictatorial orderings constructed by U.

linear extension which preserves the maximal elements. In this case we can identify the Nash equilibria of a game with incomplete preferences to the familiar problem of obtaining Nash equilibria of a collection of games with complete preferences (see [2] and Theorem 9 below). Hansson [13] uses a Szpilrajn's type construction of a linear order, in a way, to prove that there is an underlying preference relation for a choice structure (V, R). In fact, he construct an extension (V', f') to (V, f), such that V' is closed with respect to finite unions. In case that X is endowed with some topology τ, Herden and Pallack [14] and Bosi and Herden [4] utilize the "constructive"method of Szpilrajn to find the conditions under which τ is preserved in the extended relation. In conclusion, there are many types of conditions that one may wish to preserve, or to achieve, in an extension process. They include:

(i) Order theoretic conditions (consistency, acyclicity, transitivity, completeness, e.t.c.);

(ii) Topological conditions (continuity, openness or closedness of the preference sets);

(iii) linear-space conditions (convexity, homogeneity, translation-invariance).

On the other hand, the Dushnik-Miller's theorem tell us when a binary relation R has a *realizer.* This means that, there exists a collection of linear extensions $\mathcal{F}$ of R whose intersection is R and for every pair of elements $x, y \in X$ with x incomparable to y (neither (x, y) nor (y, x) is a member of R), there exists an $L_R \in \mathcal{F}$ with $(x, y) \in L_R$. Much of economic and social behavior observed is either group behavior or that of an individual acting for a group. Group preferences may be regarded as derived from individual preferences, by means of some process of aggregation. For example, if all voters agree that some alternative x is preferred to another alternative y, then the majority rule will return this ranking. In this case, there is one simple condition that is nearly always assumed called the *principle of unanimity* or *Pareto principle.* This declares that the preference relation for a group of individuals should include the intersection of their individual preferences. Another example of the use of intersections is in the description of simple games[9] which can be represented as the intersection of weighted majority games [12].[10]

[9]A *simple game* is a pair (N, v) where $N = \{1, 2, ..., n\}$ is called the set of players or voters. Every $S \subseteq N$ is a *coalition*, $\mathcal{C}(N)$ is the set of all coalitions, $v : \mathcal{C}(N) \to \{0, 1\}; v(\emptyset) = 0$ is the *characteristic function*, which satisfies $v(N) = 1$ and $v(S) \subset v(T)$ if $S \subset T$. A coalition S is *winning* if $v(S) = 1$ and *losing* otherwise. The set of winning coalitions is denoted by $\mathcal{W}$ and the set of losing coalitions is denoted by $\mathcal{L}$.

[10]A simple game (N, v) is a *weighted majority game* (WMG) if it admits a representation by means of the $n + 1$ nonnegative real numbers $[q; w_1, ..., w_n]$ such that

The existence of Dushnik-Miller's type theorems is a tool for telling us when a binary relation can be represented as the intersection of linear orders. But, the Dushnik-Miller's Theorem refers to something more specific, that is, the size of a family of linear order extensions (realizer) of R whose intersection is R. The concept of a realizer $\mathcal{F}$ of R leads to the definition of dimension of R. According to Dushnik and Miller, the *dimension* of a partial order $\succ$ is defined to be the minimum size of a realizer of $\succ$. In fact, the Dushnik-Miller's theorem provides a constructive procedure that represent binary relations as an intersection of a number of linear order extensions equal to its dimension. Using this fact, one can obtain a binary relation R with the intersection of a reduced number of its linear order extensions. For example, in the games that are compositions of m individualist games[11] (N, u_i) $(i = 1, ..., m)$ via unanimity, the usual description of the game, by means of minimal winning coalitions, requires $n_1 \cdot ... \cdot n_m$ coalitions (with $n_i = |N_i|$) and each one of them has m players, i.e., $m \cdot n_1 \cdot ... \cdot n_m$ digits are needed to describe the game. Using [12, Theorem 3.1],[12] $(n+1) \cdot (m-p)$ $(p < m)$ digits are required to describe the game. This latter number is generally much smaller than the former, and so, the description of the game is much shorter. Many other interesting applications of the dimension of a binary relation are obtained in Economics. For example, Ok [17, Proposition 1] shows that if $(X, \succ)$ is a preordered set with X countable and $dim(X, \succ) < \infty$, then $\succ$ is representable by means of a real function u in such a way that $x \succ y$ if and only if $u(x) > u(y)$. From a multicriteria point of view, classical crisp[13] dimension refers to a minimal representation of crisp partial orders as the intersection of linear

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$$v(S) = \begin{cases} 1 & \text{if } W(S) \geq q, \\ 0 & \text{if } W(S) > q. \end{cases}$$

where, for each coalition $S \subseteq N$, $w(S) = \sum_{i \in S} w_i$. The number q is called the *quota* and w_i the weight of player i.

[11] If a game with player set $N = \{1, ..., n\}$ admits a partition $N_1, ..., N_m$ in such a way that

$$\mathcal{W} = \{S \subseteq N : |S \cap N_i| \geq 1, \text{ for all } i = 1, ..., m\}$$

we shall say that this game is a *composition of m individualist games via unanimity.*

[12] Let $(N, \mathcal{W})$ be a composition of m individualist games (N_i, u_i) $(i = 1, ..., m)$ with $1 \leq n_1 \leq ... \leq n_m$ via unanimity and let $p < m$ such that either $n_p = 1$, $n_{p+1} > 1$ or $p = 0$ if $n_1 > 1$. Then the dimension of $(N, \mathcal{W})$ is $m - p$.

[13] Given a finite set of alternatives $X = \{x_1, x_2, ..., x_n\}$, a crisp partial order set $R \subseteq X \times X$ is characterized by a mapping

$$\mu : X \times X \longrightarrow \{0, 1\}$$

being

(i) irreflexive: $\mu(x_i, x_i) = 0 \ \ \forall x_i \in X$,
(ii) antisymmetric: $\mu(x_i, x_j) = 1 \Rightarrow \mu(x_j, x_i) = 0$,
(iii) transitive: $\mu(x_i, x_j) = \mu(x_j, x_k) = 1 \Rightarrow \mu(x_i, x_k) = 1$.

orders, in the sense that each of one of these linear orders is a possible underlying criterion. Brightwell and Scheinerman [5], on the basis of Dushnik-Miller's original theorem, prove that the fractional dimension[14] of a partially ordered set $(X, \succ)$ arises naturally when considering a particular two-person game on $(X, \succ)$, e.t.c.

In this paper, we give two general "constructive type"extension theorems: (i) if R is a binary relation on a set X and x, y any two non-comparable elements of R, then there exists a linear order extension R_{L_1} such that $xR_{L_1}y$ if and only if R is Δ-consistent and (ii) the transitive closure of a binary relation R has as realizer the set of linear order extensions of R if and only if R is Δ-consistent. It seems that the weakening of the axiom of partial order to that of Δ-consistency is the best we can for, as a binary relation satisfies the classical theorems of Szpilrajn and Dushnik and Miller if and only if it is a Δ-consistent binary relation. The above results generalize all the well known extension theorems in the literature.

2. The extension theorems

Let X be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary relation on X. We sometimes abbreviate $(x, y) \in R$ as xRy. Let $P(R)$ and $I(R)$ denote, respectively, the *asymmetric part* of R and the *symmetric part* of R, which are defined, respectively, by $P(R) = \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \notin R\}$ and $I(R) = \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \in R\}$. Let also $\Delta = \{(x, x) | x \in X\}$ denotes the diagonal ox X. We say that R on X is (i) *reflexive* if for each $x \in X$ $(x, x) \in R$; (ii) *irreflexive* if we never have $(x, x) \in R$; (iii) *transitive* if for all $x, y, z \in X$, $[(x, z) \in R \text{ and } (z, y) \in R] \Longrightarrow (x, y) \in R$; (iv) *antisymmetric* if for each $x, y \in X$, $[(x, y) \in R \text{ and } (y, x) \in R] \Longrightarrow x = y$; (v) *complete* if for each $x, y \in X$, $x \neq y$ we have xRy or yRx. (vi) *strongly complete* if for each $x, y \in X$, we have xRy or yRx. The *transitive closure* of a relation R is denoted by $\overline{R}$, that is for all $x, y \in X, (x, y) \in \overline{R}$ if there exist $m \in \mathbb{N}$ and $z_0, ..., z_m \in X$

It is therefore assumed that $\mu(x_i, x_j) = 1$ means that alternative x_i is strictly better than x_j ($\mu(x_i, x_j) = 0$ otherwise).

[14] Brightwell and Scheinerman [5] introduce the notion of *fractional dimension* of a poset $(X, \succ)$. Let $\mathcal{F} = \{L_1, L_2, ..., L_t\}$ be a nonempty multiset of linear extensions of $(X, \succ)$. The authors in [5] call $\mathcal{F}$ a k-fold realizer of $(X, \succ)$ if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., —$\{i = 1, ..., t : y < x$ in $L_i\}| \geq k$. We call a k-fold realizer of size t a $-t$-realizer. The *fractional dimension* of $(X, \succ)$ is then the least rational number $fdim(X, \succ) \geq 1$ for which there exists a $k - t$-realizer of $(X, \succ)$ so that $\frac{k}{t} \geq \frac{1}{fdim(X, \succ)}$. Using this terminology, the dimension of $(X, \succ)$, is the least t for which there exists a 1-fold realizer of $(X, \succ)$.

such that $x = z_0, (z_k, z_{k+1}) \in R$ for all $k \in \{0, ..., m-1\}$ and $z_m = y$. Clearly, $\overline{R}$ is transitive and, because the case $m = 1$ is included, it follows that $R \subseteq \overline{R}$. *Acyclicity* says that there do not exist m and $z_0, z_1, ..., z_m \in X$ such that $x = z_0$, $(z_k, z_{k+1}) \in R$ for all $k \in \{0, ..., m-1\}$ and $z_m = x$. The relation R is *consistent*, if for all $x, y \in X$, for all $m \in \mathbb{N}$, and for all $z_0, z_1, ..., z_m \in X$, if $x = z_0, (z_k, z_{k+1}) \in R$ for all $k \in \{0, ..., m-1\}$ and $z_m = y$, we have that $(y, x) \notin P(R)$. In case where $(y, x) \in P(R)$ we say that R contains a $P(R)$-*cycle*. A consistent binary relation R is called Δ-consistent if $I(\overline{R}) \subseteq \Delta$. The following combination of properties are considered in the next theorems. A binary relation R on X is (i) *quasi-order* if R is reflexive and transitive; (ii) *order* if R is a strongly complete quasi-order; (iii) *partial order* if R is an antisymmetric quasi-order; (iv) *linear order* if R is a strongly complete partial order; (v) *strict partial order* if R is irreflexive and transitive. (vi) *strict linear order* if R is a complete strict partial order. A binary relation R_L is an *extension* of a binary relation R if and only if $R \subseteq R_L$ and $P(R) \subseteq P(R_L)$. In other words, an extension R_L of R subsumes all the pairwise information provided by R, and possibly further information. Let $inc(R) = \{(x, y) \in X \times X | (x, y) \notin R$ and $(y, x) \notin R\}$ be the set of incomparable pairs of R. The set of all of the linear extensions of R is denoted by $\mathcal{L}(R)$. Any subset $\mathcal{F} \subseteq \mathcal{L}(R)$ is a realizer of R if, for every $(x, y) \in inc(R)$, we have $xR_{L_i}y$ in some $R_{L_i} \in \mathcal{F}$ and $yR_{L_j}x$ in some $R_{L_j} \in \mathcal{F}$. The minimum cardinality of a realizer of R is called the *dimension* of R and is denoted by $dim(R)$.

In order to prove the results of this paper, we need the following proposition which is a simplification of Suzumura's definition (see [9]).

Proposition 1. Let R be a consistent binary relation on X. Then $P(R) \subseteq P(\overline{R})$.

The following theorem generalize the Szpilrajn's extension Theorem.

Theorem 2. Let R be a binary relation on X and x, y are any two non-comparable elements of R. Then, R has a linear order extension R_{L_1} in which $xR_{L_1}y$ and a linear order extension R_{L_2} in which $yR_{L_2}x$ if and only if R is Δ-consistent.

Proof. Let R be a Δ-consistent binary relation and x and y are any two non-comparable elements of R. We put $R^* = R \cup \Delta \cup \{(x, y)\}$. Clearly, R^* is an extension of R such that xR^*y. Proposition 1 implies that $\overline{R^*}$ is an extension of R^*. Clearly, $\overline{R^*}$ is a Δ-consistent extension of R such that $x\overline{R^*}y$. Suppose that $\widetilde{\mathcal{R}} = \{\widetilde{R}_i | i \in I\}$ denote the set of Δ-consistent extensions of R such that $x\widetilde{R}_i y$. Since $\overline{R^*} \in \widetilde{\mathcal{R}}$ we have that

$\widetilde{\mathcal{R}} \neq \emptyset$. Let $\mathcal{Q} = (Q_i)_{i\in I}$ be a chain in $\widetilde{\mathcal{R}}$, and let $\widehat{Q} = \bigcup_{i\in I} Q_i$. We show that $\widehat{Q}$ is a Δ-consistent extension of R. First, we prove that $\widehat{Q}$ is an extension of R. To verify that $P(R) \subseteq P(\widehat{Q})$, take any $(s,t) \in P(R)$ and suppose $(s,t) \notin P(\widehat{Q}) = P(\bigcup_{i\in I} Q_i)$. Clearly, $s \neq t$ and for each $i \in I$, $(s,t) \in Q_i$. Since $(s,t) \notin P(\bigcup_{i\in I} Q_i)$ we conclude that $(t,s) \in \bigcup_{i\in I} Q_i$. Hence, $(s,t) \in I(Q_{i^*}) \subseteq I(\overline{Q_{i^*}}) \subseteq \Delta$ for some $i^* \in I$, a contradiction to $s \neq t$. It remains to prove that $\widehat{Q}$ is Δ-consistent. First, we observe that $\widehat{Q}$ is consistent, for suppose otherwise there exist a natural number m, $\lambda \in X$ and alternatives $z_0, z_1, ..., z_m \in X$ such that

$$\lambda = z_0 \widehat{Q} z_1 ... z_{m-1} \widehat{Q} z_m P(\widehat{Q}) z_0 = \lambda.$$

Consider the largest i for which there exist such $\lambda, m, z_0, ..., z_m$. It follows that Q_i is non consistent, a contradiction. We now prove that $I(\overline{\widehat{Q}}) \subseteq \Delta$. Indeed, let $(s,t) \in I(\overline{\widehat{Q}})$. Then, there must exist natural numbers m, n, alternatives $s_0, s_1, ..., s_m, t_0, t_1, ..., t_m \in X$ and $Q_{s_1}, ..., Q_{s_m}, Q_{t_1}, ..., t_{x_n} \in \mathcal{Q}$ such that

$$s = s_0 Q_{s_1} s_1 ... Q_{s_m} s_m = t \;\text{ and } t = t_0 Q_{t_1} t_1 ... Q_{t_n} t_n = s$$

If $|Q| = \max\{Q_{s_1}, ..., Q_{s_m}, Q_{t_1}, ..., Q_{t_n}\}$ with respect to set inclusion, then $(s,t) \in I(\overline{|Q|}) \subseteq \Delta$. Hence, $s = t$ which implies that $I(\overline{\widehat{Q}}) \subseteq \Delta$. Therefore, $\widehat{Q}$ is Δ-consistent. Since $(x,y) \in \widehat{Q}$ we conclude that $\widehat{Q} \in \widetilde{\mathcal{R}}$. By Zorn's lemma $\widetilde{\mathcal{R}}$ possesses an element, say Q^*, that is maximal with respect to set inclusion. Since Q^* is Δ-consistent, Proposition 1 implies that $P(Q^*) \subseteq P(\overline{Q^*})$. Hence, $\overline{Q^*}$ is a Δ-consistent extension of R such that $x\overline{Q^*}y$. Hence, by maximality of Q^*, we conclude that $Q^* = \overline{Q^*}$. Thus, $Q^* = R_{L_1}$ is a transitive extension of R. To prove that R_{L_1} is complete, take any $(s,t) \notin R_{L_1}$. Then, $(s,t) \notin Q^*$. Define $Q' = Q^* \cup \{(s,t)\}$. Since $(x,y) \in Q^*$, we have $(x,y) \neq (s,t)$. Therefore, by maximality of Q^*, it must be that Q' contains a $P(Q')$-cycle, so there exist a natural number m and alternatives $z_0, ..., z_m \in X$ such that

$$\nu = z_0 Q' z_1 ... z_{m-1} Q' z_m P(Q') z_0 = \nu$$

Since Q^* is consistent, there must exists $k = 0, ..., m-1$ such that $(z_k, z_{k+1}) = (s,t)$ and for all $i \in \{0, ..., m-1\}$ with $i \neq k$, $(z_i, z_{i+1}) \in Q'$ if and only if $(z_i, z_{i+1}) \in Q^*$. It follows that $(t,s) \in \overline{Q^*} = Q^* = R_{L_1}$.

Finally, since Q^* is Δ-consistent we deduce that $R_{L_1} \cap R_{L_1}^{-1} = I(Q^*) = I(\overline{Q^*}) \subseteq \Delta$. Hence, R_{L_1} is a linear order extension of R^*. Clearly, R_{L_1} is also a linear order extension of R such that $xR_{L_1}y$.

Conversely, suppose that R has a linear order extension R_{L_1}. Suppose to the contrary that R is non Δ-consistent. Then, either $I(\overline{R}) \supset \Delta$ or R contains a $P(R)$-cycle. In the first case, there exist $s, t \in X$, such that $(s,t) \in I(\overline{R}) = I(R_{L_1})$ and $(s,t) \notin \Delta$, which contradicts the fact that R_{L_1} is antisymmetric. For the second case, there must exist an integer $m \geq 1$ and $z_0, ..., z_m \in X$ such that

$$\mu = z_0 R z_1 ... z_{m-1} R z_m P(R) z_0 = \mu.$$

Then, we have

$$\mu P(R_{L_1}) \mu$$

which is impossible. □

An irreflexive variant of Theorem 2 is the following corollary.

Corollary 3. Let R be a binary relation on X and x, y are any two non-comparable elements of R. Then, R has a strict linear order extension R_{L_1} in which $xR_{L_1}y$ and a strict linear order extension R_{L_2} in which $yR_{L_2}x$ if and only if R is acyclic.

Proof. If R is acyclic, then R is consistent and $I(\overline{R}) = \emptyset \subset \Delta$. Hence, Theorem 2 implies that R has a linear order extension S such that xSy. Clearly, $R_{L_1} = S \setminus \Delta$ is a strict linear order extension of R satisfying $xR_{L_1}y$. Conversely, suppose that R has a strict linear order extension R_{L_1}. Suppose to the contrary that R is non-acyclic. Hence, there must exist an integer $m \geq 1$ and $z_0, ..., z_m \in X$ such that

$$\lambda = z_0 R z_1 ... z_{m-1} R z_m R z_0 = \lambda.$$

Then, we have

$$\lambda = z_0 R_{L_1} z_1 ... z_{m-1} R_{L_1} z_m = z_0 = \lambda$$

which contradicts the fact that R_{L_1} is irreflexive and transitive. □

As Corollary to Theorem 2, we obtain the following well known extension theorem.

Corollary 4. (Szpilrajn's Extension Theorem [23]). Every (strict) partial order R possesses a (strict) linear order extension R_L. Moreover, if x and y are any two non-comparable elements of R, then there exists a (strict) linear order extension R_{L_1} in which $xR_{L_1}y$ and a (strict) linear order extension R_{L_2} in which $yR_{L_2}x$.

Corollary 5. Let R be a binary relation on X and x, y are any two non-comparable elements of R. Then, R has an ordering extension R_{L_1} in which $xR_{L_1}y$ and a linear order extension R_{L_2} in which $yR_{L_2}x$ if and only if R is consistent.

Proof. We prove the "sufficiency"part. The proof of necessity is similar to that of the proof of necessity in Theorem 2. Let X be a non-empty set, $x, y \in X$ and let R be a consistent binary relation. For all $s \in X$, let $[s] = \{t \in X | (s,t) \in I(\overline{R} \cup \Delta)\}$. Let $\mathcal{X} = \{[s] | s \in X\}$. $\mathcal{X}$ is non-empty because X is non-empty and $\overline{R} \cup \Delta$ reflexive. Define the relation R^* on $\mathcal{X}$ as follows:

For all $[s], [t] \in \mathcal{X}$

$$([s],[t]) \in R^* \text{ if and only if there exist } s' \in [s] \text{ and } t' \in [t] \text{ such that } (s', t') \in P(R).$$

Clearly, if $(s,t) \in P(R)$ for some $s, t \in X$, then $([s],[t]) \in R^*$. Now, by way of contradiction, we prove that the relation R^* is Δ-consistent. Suppose that

$$[z_1]R^*[z_2]...R^*[z_n]P(R^*)[z_1] \text{ for some } z_1, z_2, ..., z_n \in X.$$

Therefore, there exist, $z_1', z_1'', z_2', z_2'', ..., z_n', z_n'' \in X$ such that

$$z_1' P(R) z_2' I(\overline{R}) z_2'' P(R) z_3', ..., P(R) z_n' I(\overline{R}) z_n'' P(R) z_1''.$$

Since $z_1 I(\overline{R}) z_1'$ and $z_1'' I(\overline{R}) z_1$, by Proposition 1, we obtain $z_1 P(\overline{R}) z_1$, a contradiction. On the other hand, $I(\overline{R^*}) = \emptyset \subseteq \{([s],[s]) | s \in X\}$. Hence, by Theorem 2, R^* has a linear extension $\widehat{R}$ such that $[x]\widehat{R}[y]$.

Define the relation R_{L_1} on X by letting, for all $s, t \in X$,

$$(s,t) \in R_{L_1} \text{ if and only if } (s,t) \in I(\overline{R}) \text{ or } ([s],[t]) \in \widehat{R}.$$

By construction, we conclude that $(x,y) \in R_{L_1}$. It remains to prove that R_{L_1} is an ordering extension of R. Clearly, R_{L_1} is an ordering. To complete the proof, it is sufficient to prove that $R \subseteq R_{L_1}$ and $P(R) \subseteq P(R_{L_1})$.

Let $(s,t) \in R$. If $(s,t) \in I(R) \subseteq I(\overline{R})$, then $(s,t) \in R_{L_1}$. If $(s,t) \in P(R)$, it follows that $([s],[t]) \in R^*$ and, because $\widehat{R}$ is a linear extension of R^*, $([s],[t]) \in \widehat{R}$. Hence, $(s,t) \in R_{L_1}$. Finally, let $(s,t) \in P(R) \subseteq P(\overline{R})$. Therefore, $(s,t) \in R$ and, by the above argument, $(s,t) \in R_{L_1}$. Because $(s,t) \notin I(\overline{R})$, the definition of R_{L_1} implies $([s],[t]) \in \widehat{R}$. Since $(s,t) \in P(\overline{R})$, it must be that $([t],[s]) \notin R^*$. Otherwise, there exist $t' \in [t]$ and $s' \in [s]$ such that $(t',s') \in P(R) \subseteq P(\overline{R})$. But then $(t,s) \in P(\overline{R})$, which is impossible. Hence, $([s],[t]) \in P(R^*) \subseteq P(\widehat{R})$. Hence, $([t],[s]) \notin \widehat{R}$. Since $(t,s) \notin I(\overline{R})$, it follows that $(t,s) \notin R_{L_1}$. Hence, $(s,t) \in P(R_{L_1})$, which completes the proof. □

Corollary 5 also generalizes Suzumura's existence extension theorem. It is worth to note that Suzumura's proof utilizes Szpilrajn's result.

Corollary 6. (Suzumura's Extension Theorem [22]). A binary relation R has an ordering extension if and only if R is consistent.

Since a quasi-ordering is a reflexive and consistent binary relation, the following corollary is obvious.

Corollary 7. (Arrow [1]; Hanson [13]; Fishburn [11]). Every quasi-ordering has an ordering extension.

We now give a proposition, as an example, which shows the usefulness for the Szpilrajn's constructive procedure of extending binary relations.

Proposition 8. Let R be a Δ-consistent binary relation on some non-empty set X, and let x^* be a maximal element of R in X. Then, there exists a complete extension Q of R such that x^* is a maximal element of Q in X.

Proof. Let R be a Δ-consistent binary relation and let x^* be a maximal element of R in X. Clearly, $\overline{R}$ is a Δ-consistent extension of R. To show that x^* is a maximal element of $\overline{R}$, suppose to the contrary that $(y,x^*) \in P(\overline{R}) \subseteq \overline{R}$ for some $y \in \overline{R}$. It then follows that, there exists a natural number n and alternatives $t_1, t_2, ..., t_{n-1}, t_n$ such that $yRt_1...t_{n-1}Rt_nRx^*$. Since $(t_n, x^*) \notin P(R)$, we conclude that $(t_n, x^*) \in I(R) \subseteq I(\overline{R})$. Hence, because of Δ-consistency, we conclude that $t_n = x^*$. Similarly, $(t_{n-1}, x^*) \in R$, and an induction argument based on this logic yields $y = x^*$, a contradiction to $(y,x) \in P(\overline{R})$. Hence, x^* is a maximal element of $\overline{R}$. Suppose that $\widetilde{\mathcal{R}} = \{\widetilde{R}_i | i \in I\}$ denote the set of Δ-consistent extensions of R which has x^* as maximal element. Since $\overline{R} \in \widetilde{\mathcal{R}}$ we have that $\widetilde{\mathcal{R}} \neq \emptyset$. Let $\mathcal{Q} = (Q_i)_{i \in I}$ be a chain in $\widetilde{\mathcal{R}}$, and let $\widehat{Q} = \bigcup_{i\in I} Q_i$. We show that $\widehat{Q} \in \widetilde{\mathcal{R}}$. As in the proof of Theorem 2, we conclude that $\widehat{Q}$ is a Δ-consistent extension of R. To verify that x^* is a maximal element of $\widehat{Q}$, take any $y \in X$ and suppose $(y, x^*) \in P(\widehat{Q}) = P(\bigcup_{i\in I} Q_i)$. Clearly, $y \neq x^*$ and $(y, x^*) \in Q_{i^*}$ for some $i^* \in I$. Since $(x^*, y) \notin \bigcup_{i\in I} Q_i$ we conclude that $(x^*, y) \notin Q_i$ for each $i \in I$. Hence, $(y, x^*) \in P(Q_{i^*})$, a contradiction to $Q_{i^*} \in \widetilde{\mathcal{R}}$. Therefore, $\widehat{Q} \in \widetilde{\mathcal{R}}$. By Zorn's lemma $\widetilde{\mathcal{R}}$ possesses an element, say Q^*, that is maximal with respect to set inclusion. Since Q^* is consistent, we have $P(Q^*) \subseteq P(\overline{Q^*})$. Hence, $\overline{Q^*}$ is an extension of R which has x^* as maximal element. By maximality of Q^* we conclude that $Q^* = \overline{Q^*}$. Hence, Q^* is a transitive extension of R. We prove that Q^* is complete. Indeed, take any $s, t \in X$ such that $(s, t) \notin Q^*$ and $(t, s) \notin Q^*$. Clearly, one of t and s is different from x^*. Let $s \neq x^*$. Define $Q = Q^* \cup \{(t, s)\}$. Since x^* is a maximal element of Q, by maximality of Q^*, it must be that Q contains a $P(Q)$-cycle, so there exist a natural number m and alternatives $z_0, ..., z_m \in X$ such that

$$s = z_0 Q z_1 ... z_{m-1} Q z_m P(Q) z_0 = s$$

Since Q^* is consistent, there must exists $k = 0, ..., m-1$ such that $(z_k, z_{k+1}) = (t, s)$ and for all $i \in \{0, ..., m-1\}$ with $i \neq k$, $(z_i, z_{i+1}) \in Q$ if and only if $(z_i, z_{i+1}) \in Q^*$. It follows that $(s, t) \in \overline{Q^*} = Q^*$, a contradiction. Hence, Q^* is a complete extension of R which has x^* as maximal element. □

As consequence of the previous result we have a generalization of Sophie Bade's result in [2, Theorem 1](she uses transitive binary relations) which shows that the set of Nash equilibria of any game[15] with incomplete preferences can be characterized in terms of certain derived games with complete preferences. More general, it is shown a fundamental similarity between the theory of games with incomplete preferences and the existing theory of games with complete preferences. I put in mind the following definition:

Definition. We say that a game $G' = \{(A_i, R'_i) | i \in I\}$ is a completion of a game $G = \{(A_i, R_i) | i \in I\}$ if R'_i is a complete extension of R_i for

[15] In this case, $G = \{(A_i, R_i) | i \in I\}$ is an arbitrary (normal-form) game. Where I is a set of players, player i's nonempty action space is denoted by A_i and R_i is player i's preference relation on the outcome space $A = \prod_{i\in I} A_i$.

each i. In what follows, we denote the set of all Nash equilibria[16] of a game G by $N(G)$. In the following theorem, each preference relation R_i is assumed to be Δ-consistent.

Theorem 9. Let $G = \{(A_i, R_i)|i \in I\}$ be any game. Then
$$N(G) = \bigcup\{N(G')|G' \text{ is a completion of } G\}.$$

Proof. Clearly, $\bigcup\{N(G')|G' \text{ is a completion of } G\} \subseteq N(G)$. Conversely, let $a^* \in N(G)$, that is, a^* is a Nash equilibrium of G. Let us define $B_i = \{(a_i, a^*_{-i})|a_i \in A_i\}$ for all players i. Then, for any player i, a^* is a maximal element of R_i in B_i. By Proposition 8, there exists a completion R'_i of R_i for each player i such that a^* is maximal point of R'_i in B_i. Consequently a^* is a Nash equilibrium of the completion $G' = \{(A_i, R'_i)|i \in I\}$. Hence, $N(G) \subseteq \bigcup\{N(G')|G' \text{ is a completion of } G\}$. □

3. Refinements of Szpilrajn's type theorems

In this paragraph, we give a general extension theorem in which all the well known extended theorems of Dushnik-Miller type are obtained as special cases.

Theorem 10. The transitive closure of a binary relation R has as realizer the set of linear order extensions of R if and only if R is Δ-consistent.

Proof. Suppose that R is a Δ-consistent binary relation on X and let $x, y \in X$ such that $(x, y) \in inc(\overline{R})$. Let $\mathcal{Q}$ be the set of all linear order extensions of R. By Theorem 2, $\mathcal{Q}$ is non-empty. We prove that $\mathcal{Q}$ is a realizer of $\overline{R}$. We first show that $\overline{R} \subseteq Q$ and $P(\overline{R}) \subseteq P(Q)$ for all $Q \in \mathcal{Q}$. Clearly, the transitivity of Q implies that $Q = \overline{Q} \supseteq \overline{R}$. To verify that $P(\overline{R}) \subseteq P(Q)$, take any $(s, t) \in P(\overline{R})$ and suppose $(s, t) \notin P(Q)$. Clearly, $s \neq t$. Since $(s, t) \in \overline{R} \subseteq Q$, this means that $(t, s) \in Q$. Hence, $(t, s) \in I(Q)$ which is impossible because of antisymmetry of Q. Hence, Q is a linear order extension of $\overline{R}$.

We now prove that $\overline{R} = \bigcap_{Q \in \mathcal{Q}} Q$. To prove the sufficient part, note that $\overline{R} \subseteq \bigcap_{Q \in \mathcal{Q}} Q$ follows from $\overline{R}$ being a subrelation of each $Q \in \mathcal{Q}$. Thus, we have only to show that $\bigcap_{Q \in \mathcal{Q}} Q \subseteq \overline{R}$. Suppose that there exists

[16] An action profile $a = (a_1, ..., a_{|I|})$ is a *Nash equilibrium* if for no player i there exists an action $a'_i \in A_i$ such that $(a'_i, a_{-i}) R_i (a_i, a_{-i})$.

an $(s,t) \in \bigcap_{Q \in \mathcal{Q}} Q$ with $(s,t) \notin \overline{R}$. Hence, $(s,t) \notin R$. It follows that $(t,s) \notin R$. Indeed, suppose to the contrary that $(t,s) \in R$, so that $(t,s) \in P(R)$. Since each $Q \in \mathcal{Q}$ is an extension of R, we must then have $(t,s) \in P(Q)$ which is in contradiction with $(s,t) \in Q$ for all $Q \in \mathcal{Q}$. Hence, s and t are not ranked by R. Now consider $R' = R \cup \{(t,s)\}$. Clearly, $(t,s) \in P(R')$. Then, as in the proof of Theorem 2, we can prove that R' is Δ-consistent. By Theorem 2, R' has a linear order extension $\widehat{R}$. Since $(t,s) \in P(R')$, we must then have $(t,s) \in P(\widehat{R})$. Note that R' is an extension of R. Hence, $\widehat{R}$ is also a linear order extension of R. Therefore $\widehat{R} \in \mathcal{Q}$. Noting that $(s,t) \notin \widehat{R}$, this is a contradiction with the assumption that $(s,t) \in \bigcap_{Q \in \mathcal{Q}} Q$. Hence, $\bigcap_{Q \in \mathcal{Q}} Q \subseteq \overline{R}$ which implies that $\overline{R} = \bigcap_{Q \in \mathcal{Q}} Q$. Since $(x,y) \in inc(\overline{R}) \subseteq inc(R)$, Theorem 2 implies that $\mathcal{Q}$ is a realizer of $\overline{R}$.

Conversely, suppose that $\overline{R}$ has as realizer the set of all linear order extensions of R, let $\mathcal{Q}$. We prove that R is Δ-consistent. Indeed, since $\bigcap_{Q \in \mathcal{Q}} Q = \overline{R}$ we have $P(R) \subseteq \bigcap_{Q \in \mathcal{Q}} P(Q) \subseteq P(\bigcap_{Q \in \mathcal{Q}} Q) = P(\overline{R})$. By [9, Definition 4], we conclude that R is consistent. It remains to prove that $I(\overline{R}) \subseteq \Delta$. Suppose to the contrary that $I(\overline{R}) \supset \Delta$. This implies that, for all $Q \in \mathcal{Q}$ there exist $s,t \in X$, $s \neq t$, such that

$$(s,t) \in I(\overline{R}) \subseteq I(Q) \text{ and } (s,t) \notin \Delta$$

which contradicts the antisymmetry of Q. □

In light of the proof of Theorem 10, the following theorem is now obvious.

Theorem 11. Let R be a Δ-consistent binary relation on a set X. If X is finite, then the dimension of $\overline{R}$ is finite. If $|X| = m$, where m is a transfinite cardinal, then the dimension of $\overline{R}$ is $\leq m$.

The following corollary is an irreflexive variant of Theorem 10.

Corollary 12. The transitive closure of a binary relation R has as realizer the set of strict linear order extensions of R if and only if R is acyclic.

Proof. If R is acyclic, then $R \cup \Delta$ is Δ-consistent. Hence, $\overline{R \cup \Delta} = \overline{R} \cup \Delta$ is the intersection of all linear order extensions of R. Let $\mathcal{Q}$ be the class of linear order extensions of R such that $\overline{R} \cup \Delta = \bigcap_{Q \in \mathcal{Q}} Q$. Then, $\overline{R} = \bigcap_{Q \in \mathcal{Q}} Q \setminus \Delta$, where $Q \setminus \Delta$ is a strict linear order extension of R. Conversely, suppose that $\overline{R}$ is the intersection of all strict linear order extensions of R. Then it follows that $\overline{R}$ is irreflexive from which we conclude that R is acyclic. □

Next is a result due to Dushnik and Miller [10, Theorem 2.32].

Corollary 13. If R is any (strict) partial order on a set X, then there exists a collection $\mathcal{Q}$ of (strict) linear orders on X which realize R.

Proof. This follows immediately from Theorem 10, by letting R to be (strict) partial order. □

The following corollary strengthen Suzumura's fundamental Theorem in a "constructive type" version (see Corollary 5).

Corollary 14. The transitive closure of a binary relation R has as realizer the set of ordering extensions of R if and only if R is consistent.

Proof. Let X be a non-empty set and let R be a consistent binary relation on X. Let also $x, y \in X$ be such that $(x, y) \in inc(R)$. Define $\mathcal{X}$ and R^* as in the proof of Corollary 5. Then, R^* is a Δ-consistent binary relation. Let $\widehat{\mathcal{R}}$ be the set of linear order extensions of R^* in $\mathcal{X}$. By Theorem 10, the transitive closure of R^* is the intersection of all the members of $\widehat{\mathcal{R}}$ such that $[x]\widehat{R}_L[y]$ for some $\widehat{R}_L \in \widehat{\mathcal{R}}$. For every $\widehat{R} \in \widehat{\mathcal{R}}$, we define the relation $R'(\widehat{R})$ on X as follows:

$$(s, t) \in R'(\widehat{R}) \text{ if and only if } \ (s, t) \in I(\overline{R}) \text{ or } ([s], [t]) \in \widehat{R}.$$

Let $\mathcal{R}'$ be the collection of relations $\{R'(\widehat{R}) | \widehat{R} \in \widehat{\mathcal{R}}\}$. By Corollary 5, $R'(\widehat{R})$ is an ordering extension of R for all $\widehat{R} \in \widehat{\mathcal{R}}$. To complete the proof, it is sufficient to show that $\overline{R}$ is the intersection of all orderings in $\mathcal{R}'$ and $xR'(\widehat{R})y$ for some $R'(\widehat{R}) \in \mathcal{R}'$. By transitivity of $R'(\widehat{R})$, we conclude that $\overline{R} \subseteq R'(\widehat{R})$ for all $\widehat{R} \in \widehat{\mathcal{R}}$. Hence, $\overline{R} \subseteq \bigcap_{\widehat{R} \in \widehat{\mathcal{R}}} R'(\widehat{R})$. To prove the converse, let $(s, t) \in \bigcap_{\widehat{R} \in \widehat{\mathcal{R}}} R'(\widehat{R})$. This implies that $(s, t) \in I(\overline{R})$ or $([s], [t]) \in \widehat{R}$ for all $\widehat{R} \in \widehat{\mathcal{R}}$. If $(s, t) \in I(\overline{R})$, $(s, t) \in \overline{R}$ follows immediately. If $(s, t) \notin I(\overline{R})$, we have $([s], [t]) \in \widehat{R}$ for all $\widehat{R} \in \widehat{\mathcal{R}}$. Because the transitive closure of R^* is the intersection of all orderings in $\widehat{\mathcal{R}}$, this implies $([s], [t]) \in \overline{R^*}$. Hence,

$[s]R^*[z_1]R^*[z_2]R^*...R^*[z_{n-1}]R^*[z_n]R^*[t]$ for some $z_1, z_2, ..., z_{n-1}, z_n \in X$. Therefore, there exist $s', z_1', z_1'', z_2', ..., z_{n-1}'', z_n', z_n'', t' \in X$ such that

$$s'P(R)z_1'I(\overline{R})z_1''P(R)z_2'...z_{n-1}''P(R)z_n'I(\overline{R})z_n''P(R)t'.$$

Since $sI(\overline{R})s'$ and $t'I(\overline{R})t$, we conclude that $(s,t) \in \overline{R}$. Finally, it is easy to check that $xR'(\widehat{R}_L)y$. The proof of the converse is similar to that of the proof of the converse in Theorem 10. $\square$

The next result, proved by Donaldson and Weymark [8], strengthens Fishburn's Lemma 15.4 in [11] and Suzumura's Theorem A(4) in [22].

Corollary 15. Every quasi-ordering is the intersection of a collection of orderings.

Proof. It is an immediate consequence of the sufficient part of Corollary 14. $\square$

We recall the following definitions from [9].

Definition 16. A class $\mathcal{R}$ is *closed upward* if, for all chains $\mathcal{C}$ in $\mathcal{R}$,

$$\bigcup\{R|R \in \mathcal{C}\} \in \mathcal{R}.$$

Definition 17. A class $\mathcal{R}$ is *arc-receptive* if, for all distinct s and t and for all transitive $R \in \mathcal{R}$, $(t,s) \notin R$ implies $\overline{R \cup \{(s,t)\}} \in \mathcal{R}$.

Corollary 18. (Duggan's General Extension Theorem [9]). Assume $\mathcal{R}$ is closed upward and arc-receptive. If R is consistent and $\overline{R} \in \mathcal{R}$, then

$$\overline{R} = \bigcap\{R' \in \mathcal{R}|R' \text{ is a strongly complete, transitive extension of } R\}.$$

Proof. To prove the corollary, let R be a consistent binary relation such that $\overline{R} \in \mathcal{R}$. It follows from Corollary 14 that,

$$\overline{R} = \bigcap\{R' \ |R' \text{ is a strongly complete, transitive extension of } R\}.$$

It remains to prove that $R' \in \mathcal{R}$. Because $R \subseteq R'$ by transitivity of R', we obtain $\overline{R} \subseteq R'$. If $R' = \overline{R}$, then $R' \in \mathcal{R}$. Suppose that $\overline{R} \subset R'$. We first show that there exists a transitive extension of R, let Q, such that $Q \in \mathcal{R}$ and $\overline{R} \subset Q \subseteq R'$. Indeed, assume that $s,t \in X$ are such that $(s,t) \in R' \setminus \overline{R}$. There are two cases to consider: (i) $(t,s) \in R'$; (ii) $(t,s) \notin R'$.

Case (i). $(t,s) \in R'$. In this case, since $\mathcal{R}$ is arc-receptive, $\overline{R} \in \mathcal{R}$ and $(s,t) \notin \overline{R}$ we conclude that $Q = \overline{\overline{R} \cup \{(t,s)\}} \in \mathcal{R}$. We now prove that Q is a transitive extension of R. Since Q is transitive, by Proposition 1, it suffices to show that Q is an extension of $\overline{R}$. Clearly, $\overline{R} \subset Q$. To verify that $P(\overline{R}) \subset P(Q)$, take any $(p,q) \in P(\overline{R})$ and suppose $(p,q) \notin P(Q)$. Since $(p,q) \in \overline{R} \subset \overline{\overline{R} \cup \{(t,s)\}}$, this means that $(q,p) \in \overline{\overline{R} \cup \{(t,s)\}}$. Hence, there exists $z_0, z_1, ..., z_m \in X$ such that

$$q = z_0 \overline{R} \cup \{(t,s)\} z_1 ... \overline{R} \cup \{(t,s)\} z_m = p.$$

Thus, there exists at least one $k \in \{0, 1, ..., m-1\}$ such that $(z_k, z_{k+1}) = (t,s)$, for otherwise $(q,p) \in \overline{R}$, a contradiction. Let z_λ be the first occurrence of t and let z_μ the last occurrence of s. Then, since $(p,q) \in P(\overline{R}) \subseteq \overline{R}$,

$$s = z_\mu \overline{R} z_{\mu+1} ... \overline{R} z_m = p \overline{R} q \overline{R} z_0 ... \overline{R} z_\lambda = t.$$

Hence, $(s,t) \in \overline{R}$, a contradiction. Since R' is transitive, $\overline{R} \subset Q \subseteq R'$.

Case (ii). $(t,s) \notin R'$. In this case, we must have $(t,s) \notin \overline{R}$, since otherwise, we must have $(t,s) \in R'$, a contradiction.
Let $Q = \overline{\overline{R} \cup \{(s,t)\}}$. Then, as in the case (i), we obtain $Q \in \mathcal{R}$ and $\overline{R} \subset Q \subsetneq R'$.

Let $\widehat{\mathcal{Q}} = (\widehat{Q_i})_{i \in I}$ be the set of transitive extensions of R such that $\overline{R} \subset \widehat{Q_i} \subseteq R'$ and $\widehat{Q_i} \in \mathcal{R}$. Let $\mathcal{C}$ be a chain in $\widehat{\mathcal{Q}}$, and $\widehat{C} = \bigcup \mathcal{C}$. Clearly, $\overline{R} \subset \widehat{C} \subsetneq R'$. Since $\mathcal{R}$ is closed upward, $\widehat{C} \in \mathcal{R}$. Therefore, by Zorn's lemma, $\widehat{\mathcal{Q}}$ has an element, say $\widetilde{Q}$, that is maximal with respect to set inclusion. Then, $R' = \widetilde{Q} \in \mathcal{R}$. Otherwise, there exists $(s,t) \in R' \setminus \widetilde{Q}$ such that $Q' = \overline{\widetilde{Q} \cup \{(s,t)\}}$ or $Q' = \overline{\widetilde{Q} \cup \{(t,s)\}}$ is a transitive extension of R satisfying $\overline{R} \subset \widetilde{Q} \subset Q' \subseteq R'$, which is impossible by maximality of $\widetilde{Q}$. This completes the proof. □

Clearly, Theorem 10 concludes all the extension theorems referred to Duggan [9, pp. 13-14].

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