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Abstract

This paper presents an axiomatization of residual income, aka excess profit, and illustrates how it may univocally engenders fixed-income or variable-income assets. In the first part it is shown that, depending on the relations between excess profit and the investor's excess wealth, a well-specified theory of residual income is generated: one is the *standard* theory, which historically traces back to Hamilton (1777) and Marshall (1890) and is a deep-rooted notion in economic theory, finance, and accounting. Another one is the *systemic value added* or *lost-capital paradigm*: introduced in Magni (2000, 2003), the theory is enfolded in Keynes's (1936) notion of user cost and is naturally generated by an arbitrage-theory perspective. In the second part, the paper reverts the usual analysis: instead of computing residual incomes profits from a pattern of cash flows, residual incomes are fixed first to derive vectors of cash flows. It is shown that variable- or fixed-income assets may be constructed on the basis of either theory starting from pre-determined growth rates for excess profit. In particular, zero-coupon bonds and coupon bonds traded in a capital market are shown to be deducted as *equilibrium* vectors of residual-income-based assets.

Keywords and phrases. Residual income, excess profit, capital, arbitrage, bond.

JEL: C00, C60, D53, G00, G12, G31, M21, M41.

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1 Introduction

Excess profit is profit in excess of a normal profit. The notion is usually traced back to Marshall (1890), presumably inspired by Hamilton (1777) who writes of "excess of gross profits above the interest of his stock ... if the profit of his trade be less than his stock would have yielded at common interest, he may properly account it a losing one" (Hamilton, 1777, vol. II , p. 246, as quoted in Arnold, 2000, p. 14. Also quoted in Mepham, 1980, p. 14). The concept has been used since the first years of the twentieth century in several disciplines: in business valuation (Leake, 1921; Sloan, 1929. See also Carsberg, 1966, and Goetzmann and Gartska, 1999), accounting finance (Preinreich, 1936, 1937, 1938), business economics (Edey, 1957; Edwards and Bell, 1961), management accounting (Solomons, 1965), corporate finance (Bodenhorn, 1959, 1964. See also the investment opportunities approach in Miller and Modigliani's, 1961). The idea of using a normal profit as a benchmark is widespread (Edey, 1957, Bodenhorn, 1964, Carsberg, 1966; Archer and D'Ambrosio, 1972; Begg, Fisher and Dornbusch, 1984): the normal profit acts as a threshold, as a *norm* in the sense of Kahneman and Miller (1986); it represents the foregone profit and is often called *opportunity cost*, a concept that has been thoroughly investigated by Austrian economists (in particular Ludwig von Mises) and by economists of the London School of Economics (Hayek, Coase, Thirlby); it is now a cornerstone in finance, economic theory and decision theory (see Buchanan, 1969, for a survey on cost and choice).

A huge number of synonyms have been coined to denote excess profit. Among others, we cite the use of "abnormal earnings" (Ohlson, 1989, 1995; Francis, Olsson and Oswald, 2000; Revsine, Collins and Johnson, 2005), "economic profit" (Cnossen, 1998, Ehrbar, 1998), "economic value added" (Stewart, 1991; Rogerson, 1997; Ehrbar, 1998; Magni, 2003, 2005; Stoughton and Zechner, 2007). Most notably, the term "residual income" is used since Solomons (1965), who refers to the income that residues after covering a capital charge (opportunity cost). Actually, "residual income" is by far the most widely used term in the relevant literature (e.g. Ohlson, 1989, 1995; Peasnell, 1981, 1982; Martin, Petty and Rich, 2003; Magni, 2003), so we will make interchangeable use of the terms "excess profit" and "residual income".

In recent years residual income has given a renewed interest by several authors working in such different fields as accounting finance (Peasnell, 1981, 1982; Ohlson, 1995; O'Hanlon and Peasnell, 2002), applied finance (Stewart, 1991; Young and O'Byrne, 2001; Martin, Petty and Rich, 2003); management accounting (Rogerson, 1997; Baldenius and Reichelstein, 2005; Pfeiffer, 2004), financial mathematics (Peccati, 1989; Magni, 2000; Ghiselli Ricci and Magni, 2006), economic modelling (Magni, 2005). The notion has proved significant in its relations with the notion of value and the net-present-value concept: by summing the discounted value of the residual incomes of an economic activity one gets the net present value (NPV) (see Preinreich, 1938; Edwards and Bell, 1961; Lücke, 1955; Bodenhorn, 1964; Miller and Modigliani, 1961; Peasnell, 1981, 1982; Peccati, 1987, 1989), which implies that the economic value of the asset itself may be computed with no recourse to cash flows, but only to excess profits. This NPV-consistency (aka *conservation property*) has been extensively used for constructing performance measures signalling wealth creation or wealth destruction in an economic activity, such as a company or a project. A performance measure is often used to reduce agency problems (Jensen, 1986; Meckling and Jensen, 1976), so residual income has been widely adopted as a key concept for devising management compensation plans aimed at inducing managers' optimal behaviour (see Rogerson, 1997; Mohnen, 2003; Pfeiffer, 2000; Baldenius and Reichelstein, 2005; Mohnen and Bareket, 2007). The use of this notion has also been advocated for tax policies: under the *allowance-for-corporate-equity* (also known as the *imputed income method*) only excess profits are taxed, whereas normal returns to capital are exempt from corporate taxes (Boadway and Bruce, 1984; Rose and Wiswesser, 1998; Andersson et al., 1998. See also Sørensen, 1994, 1998 on the Dual Income Tax). A recent use of the notion regards portfolio optimization: excess profit is used for extracting implied expected returns from analysts' forecasts: Frankel and Lee (1998) employ residual income for forecasting equity premium, an essential component of the cost of capital; Hagemeister and Kempf (2006) use expected returns implied by residual income for Markowitz-optimization; the same authors use expected returns implied by expected residual incomes to test different versions of the Capital Asset Pricing Model (Hagemesiter and Kempf, 2007). Barniv and Myring (2006) use residual incomes for assessing the explanatory power for security prices in seventeen countries; Desroisiers, Lemaire and L'Her (2007) use excess profit to deduce the implicit expected rates of return of nineteen countries.

The present work aims at offering a contribution which is original in three senses:

(i) despite the impressive amount of contributions carried out in one century, no axiomatization has been given so far for this concept. There is need of investigating the implicit *starting points* of the notion. Such a need is more compelling after the recent proposal of a new definition of residual income, introduced and investigated in several papers by Magni (2000, 2003, 2005). This notion is implicitly subsumed in a value-based metric proposed by Drukarczyck and Schueler (2000) and has been anticipated by Pressacco and Stucchi (1997) in the final remarks of their paper. While the standard notion traces back to Marshall (1890), the new notion is naturally embraced in arbitrage theory and is enfolded in Keynes's (1936) notion of user cost (see Magni, 2009a). This paper aims at providing an axiomatization of residual income, showing that different axioms subsume the two definitions of residual income, and highlighting some interesting formal properties;

(ii) the literature on residual income is disparate and varied as for the perspectives taken, the objectives pursued, and the language used, but a feature seems to be shared by all contributions: a given sequence of (realized or estimated) cash flows is pre-determined (project, firm, etc.) and residual income is consequently computed. In the second part of the paper we make a conceptual and formal shift by starting from a given sequence of pre-determined excess profits and consequently construct sequences of cash flows. This suggests a possible use of this notion for financial purposes: construction of contracts that explicitly guarantee the creditor a given path for excess profit, rather than contracts that guarantee a given pattern of cash flows. The use of the notion of excess profit rather than cash flow may be a significant one: given that excess profit (and not cash flows) measures wealth creation, the explicit declaration of a given pattern of excess profits may in some cases be more substantial for the investor than mere knowledge of a given pattern of cash flows;

(iii) securities such as zero-coupon bonds and coupon bonds are commonly traded in the capital markets. The usual definition is given in terms of cash flows: a zero-coupon bond guarantees a single final payoff (nominal or par value), a coupon bond pays off a sequence of coupons computed by applying the market rate to the nominal value plus, at the terminal date, the payment of the face value. This paper shows that these bonds may be defined in terms of excess profits: theoretically, one considers contracts where the residual-income sequence is fixed, and then assumes that such assets are introduced in the market, where any disequilibrium is arbitraged away.

The structure of the paper is as follows. Section 2 presents the terminology and some important preliminary results that will be used in the paper. Section 3 provides an axiomatization of the notion of residual income, starting from a given pattern of cash flows: it shows that different axioms generate different paradigms. Section 4 reverts the analysis: residual incomes are fixed first, and cash flows are univocally determined; it is shown that, depending on the paradigm followed and the pattern of excess profits determined, either fixed-income or variable-income assets may be generated. Section 5 makes use of the results found in the previous section to provide a definition of zero-coupon and coupon bonds as equilibrium vectors of residual-income-based assets.

2 Terminology and preliminary results

Throughout the paper, let $n > 1$ represent the length of an economic activity[1] and let r and $\vec{\imath} = (i_1, i_2, \ldots, i_n)$ be respectively a real number and a vector of real numbers representing rates of return. We assume that $r \in \,]-1, \infty[$, $i_1, ..., i_n > 0$ and $r \neq i_k$ for all $k \in \{2, ..., n\}$.

Definition 1. Fixed $\vec{a}, \vec{\tau} \in \mathbb{R}^n$ and a positive real number c, denote with $\vec{w} = \vec{w}(\vec{a}, \vec{\tau}) \in \mathbb{R}^{n+1}$ the vector recursively defined as

$$\begin{cases} w_k = w_{k-1}(1 + \tau_k) - a_k, & k \in \mathbb{N}_n; \\ w_0 = c \end{cases} \tag{1}$$

where $\mathbb{N}_n := \{1, ..., n\}$.

Using induction one finds, for any $k \in \mathbb{N}_n$,

$$w_k(\vec{a}, \vec{\tau}) = c\, \nu_{0,k}(\vec{\tau}) - \sum_{j=1}^{k} a_j\, \nu_{j,k}(\vec{\tau}), \tag{2}$$

where

$$\nu_{j,k}(\vec{\tau}) := \begin{cases} \prod_{s=j+1}^{k} (1 + \tau_s), & \text{for } j < k; \\ 1, & \text{for } j = k \end{cases} \tag{3}$$

for all $0 \leq j \leq k$. When $\tau_k = \tau$ for all $k \in \mathbb{N}_n$, we will omit the vectorial notation and (1) boils down to

$$\begin{aligned} w_k(\vec{a}, \tau) &= c \cdot \nu_{0,k}(\tau) - \sum_{j=1}^{k} a_j\, \nu_{j,k}(\tau) \\ &= c(1 + \tau)^k - \sum_{j=1}^{k} a_j\, (1 + \tau)^{k-j}. \end{aligned} \tag{4}$$

Finally, interpreting $\vec{a}$ as a vector of cash flows, we denote by

$$A_n(c, r) := \left\{ \vec{a} \in \mathbb{R}^n \;:\; \sum_{k=1}^{n} \frac{a_k}{(1 + r)^k} = c \right\}$$

[1] We will use interchangeably the terms "economic activity" and "asset". With such terms we refer to any situation that may be described by a sequence of cash flows (be it a project, a firm, a financial contract, etc.).

the set of cash-flow vectors such that r is an internal rate of return. We will often refer to asset $\mathcal{A}$ to designate the economic activity consisting of outlay c and cash-flow vector $\vec{a} \in A_n(c, r)$. For any $k \in \mathbb{N}_n$, if τ_k is a period return rate, $w_k(\vec{a}, \vec{\tau})$ represents the capital invested in the economic activity $\mathcal{A}$ in the k-th period, i.e. in the interval between time $k-1$ and time k. Note that $\vec{a} \in A_n(c, r)$ if and only if $w_n(\vec{a}, r) = 0$. In the sequel, $\vec{\imath}$ represents the vector of the *normal* rates of return determined by the market. We call $w_k(\vec{a}, r)$ and $w_k(\vec{a}, \vec{\imath})$, respectively, the *outstanding capital* (or *balance*) and the *normal capital* of $\mathcal{A}$ at time k. Reshaping eq. (1), one may highlight the remuneration of the economic agent $\tau_k \cdot w_{k-1}$. The expression

$$\tau_k \cdot w_{k-1} = w_k - w_{k-1} + a_k$$

is a most general framework shared by several economic domains: in economic theory it defines income or profit as the maximum that can be consumed by an individual in a determined period without impairing her wealth or capital (see Fetter, 1937; Hicks, 1946; Lee, 1985. See also Samuelson, 1964, eq. (1')); in accounting it gives voice to the so-called *clean surplus relation* which connects earnings and dividends (see Canning, 1929; Brief and Peasnell, 1995; Penman, 2007); in the theory of financial contracts and in capital budgeting, as well as in actuarial sciences, it is essential in the construction of amortization tables and project balances as a function of cash flows and return rates (see Robichek and Myers, 1965; Teichroew, Robichek and Montalbano, 1965a,b; Francis, 2004; Van de Mieroop, 2005; Fabozzi, 2006; Promislow, 2006; Werner and Sotkov, 2006).

We will also make use of the notion of *internal return vector* $\vec{r}$, introduced by Weingartner (1966), which generalizes the notion of internal rate of return and enables one to overcome the problems related to the existence and uniqueness of internal rate of return. For a given asset described by an arbitrary cash-flow vector $\vec{a}$, consider a pre-determined sequence $\{w_0, w_1, \ldots, w_{n-1}, w_n\}$ of the project balances such that $w_0 = c$ and $w_n = 0$. Under the natural assumption (here systematically followed throughout the paper) that any outstanding capital is nonzero for $k < n$, define the k-th period rate as

$$r_k := \frac{a_k + w_k}{w_{k-1}} - 1 \quad \text{for all } k \in \mathbb{N}_n. \tag{5}$$

(see the notions of *accounting rate* and *internal discount function* in Peasnell, 1982, eq. (13). See also the notion of *internal financial law* in Peccati, 1989, and Gallo and Peccati, 1993).

Remark 1. It is worth noting that eq. (5) is just a reframing of the fundamental equation (1) (the latter is the inverse function of the former). Given that eq. (5) implies $\sum_{k=1}^{n} a_k/\nu_{0,k}(\vec{r}) = c$ (see

Peasnell, 1982, Theorem 2, p. 367), we may consider the set

$$A_n(c,\vec{r}) := \left\{ \vec{a} \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} \frac{a_k}{\nu_{0,k}(\vec{r})} = c \right\}$$

which includes any asset (still denoted by $\mathcal{A}$) consisting of an outlay c and a cash-flow vector $\vec{a}$.[2] Evidently, the balance w_k may be either fixed *a priori*, as in (5), or derived through its companion (1) once an internal return vector $\vec{r}$ has been assigned.

Definition 2. The *market value* of project $\mathcal{A}$ at time k is

$$v_k(\vec{a},\vec{\imath}) := \sum_{j=k+1}^{n} \frac{a_j}{\nu_{k,j}(\vec{\imath})} \qquad k \in \{0, 1, \dots, n-1\}. \tag{6}$$

Asset $\mathcal{A}$'s net present value (NPV) is therefore

$$\mathrm{NPV}(\mathcal{A}) = v_0(\vec{a},\vec{\imath}) - c.$$

It is worth noting that $\vec{a} \in A_n(v_0, \vec{\imath})$.

Definition 3. Denote with $F_k(\vec{a})$ the *excess capital* of project $\mathcal{A}$ at time k:

$$F_k(\vec{a}) := w_k(\vec{a},\vec{r}) - w_k(\vec{a},\vec{\imath}) \qquad k \in \mathbb{N}_n^*$$

where $\mathbb{N}_n^* := \mathbb{N}_n \cup \{0\}$. Excess capital is capital in excess of normal capital. Note that for $k = n$, one finds $F_n(\vec{a})$=NPV$(\mathcal{A}) \cdot \nu_{0,n}(\vec{\imath})$, which is also known as Net Final Value, whereas, in general, $F_k(\vec{a}) \neq \mathrm{NPV}(\mathcal{A}) \cdot \nu_{0,k}(\vec{\imath})$ if $k < n$.

It is worth underlining the relation between excess capital and excess investor's wealth: for any $k \in \mathbb{N}_n$, let $E_k(\vec{a},\vec{r})$ be the investor's wealth if he invests c in asset $\mathcal{A}$ at the period rate r_k, and let $E_k(\vec{\imath})$ be the investor's wealth if he invests c at the period rate i_k. Fixed any initial wealth E_0, $E_k(\vec{a},\vec{r})$ is computed considering the outstanding capital employed in the economic activity plus the economic value (at the normal rate) of the residual funds:

$$E_k(\vec{a},\vec{r}) = w_k(\vec{a},\vec{r}) + \big[(E_0 - c)\nu_{0,k}(\vec{\imath}) + \sum_{j=1}^{k} a_j \nu_{j,k}(\vec{\imath})\big].$$

If, by contrast, the investor invested funds at the normal rate i_k, he would own a *normal* wealth $E_k(\vec{\imath}) = E_0 \cdot \nu_{0,k}(\vec{\imath})$. Consistently with the notion of excess capital, *excess wealth* may be naturally defined as wealth in excess of normal wealth: $E_k(\vec{a},\vec{r}) - E_k(\vec{\imath})$. It coincides with *excess return*

$$[E_k(\vec{a},\vec{r}) - E_0] - [E_k(\vec{\imath}) - E_0],$$

[2] Note that $A_n(c,r) \subset A_n(c,\vec{r})$.

which is an excess profit measured in the whole interval between time 0 and time k. Owing to Definition 3 one finds

$$E_k(\vec{a},\vec{r}) - E_k(\vec{\imath}) = F_k(\vec{a}) \qquad k \in \mathbb{N}_n^* \tag{7}$$

and, in particular, $E_n(\vec{a},\vec{r}) - E_n(\vec{\imath}) = w_n(\vec{a},\vec{r}) - w_n(\vec{a},\vec{\imath}) = -w_n(\vec{a},\vec{\imath}) = F_n(\vec{a})$. The notions of excess wealth, excess return, excess capital are therefore equivalent.[3]

For any economic activity $\mathcal{A}$, the associated vector of residual incomes will be hereafter identified with a mapping $\vec{\phi} : A_n(c,\vec{r}) \to \mathbb{R}^n$ fulfilling certain properties. The k-th component of $\vec{\phi}$ refers to the excess profit generated between time $k-1$ and time k.

Definition 4. A mapping $\vec{\phi} : A_n(c,\vec{r}) \to \mathbb{R}^n$ is called NPV-*consistent* if there exists a vector $\vec{\sigma} \in (\mathbb{N}_n)^n$ such that, for any asset $\mathcal{A}$,

$$\sum_{k=1}^{n} \frac{\phi_k(\vec{a})}{\nu_{0,\sigma_k}(\vec{\imath})} = \mathrm{NPV}(\mathcal{A}).$$

Definition 4 is a generalization of the well-known conservation property, which states that

$$\sum_{k=1}^{n} \frac{\phi_k(\vec{a})}{\nu_{0,k}(\vec{\imath})} = \mathrm{NPV}(\mathcal{A})$$

(see Pfeiffer, 2004). In this case, the vector $\vec{\sigma}$ is given by $(1,2,\ldots,n)$.

3 From cash flows to residual income: An axiomatization

This section provides an axiomatization for either the standard paradigm of residual income and the more recent proposal, which has been named *systemic value added* (Magni, 2000, 2003) and *lost-capital paradigm* (Magni, 2009a).

We begin by considering a first fundamental definition of residual income, which relates income and capital charge.

Definition 5. *Given any economic activity $\mathcal{A}$, excess profit (residual income) $\phi_k(\vec{a})$ in the k-th period is profit in excess of a capital charge $\tau_k w_{k-1}(\vec{a},\vec{r})$, where τ_k is an interest rate representing an appropriate cost of capital.*

[3]Note that $F_k(\vec{a})$ could have been alternatively defined as

$$F_k(\vec{a}) := [w_k(\vec{a},\vec{r}) + a_k - w_0] - [w_k(\vec{a},\vec{\imath}) + a_k - w_0]$$

which is actually an excess return (difference between alternative returns) as referred to the interval $[0,k]$.

While excess profit refers to wealth creation in the k-th period, excess return refers to wealth creation in the interval between time 0 and time k. We then state the rather natural condition that the sum of the first k excess profits should coincide with the excess return generated in the first k periods. As seen, excess capital is equivalent to excess return, so we have the following

Axiom 1 (Additive coherence) *For any asset $\mathcal{A}$, the sum of the first k excess profits is equal to excess return generated in the first k periods:*

$$\sum_{j=1}^{k} \phi_j(\vec{a}) = F_k(\vec{a}) \quad \textit{for all } k \in \mathbb{N}_n. \tag{8}$$

Definition 5 and Axiom 1 induce a well-determined notion of residual income.

Proposition 1. *Given an economic activity $\mathcal{A}$, Definition 5 and Axiom 1 imply the following NPV-consistent excess profit:*

$$\phi_k(\vec{a}) = w_{k-1}(\vec{a}, \vec{r})(r_k - i_k^*) \quad \textit{for all } k \in \mathbb{N}_n, \tag{9}$$

where $i_k^* = i_k^*(\vec{r}, \vec{\imath})$ is given by

$$i_k^*(\vec{r}, \vec{\imath}) = \frac{a_k + w_k(\vec{a}, \vec{\imath}) - w_{k-1}(\vec{a}, \vec{\imath})}{w_{k-1}(\vec{a}, \vec{r})}. \tag{10}$$

Proof. Fixed any $k \in \mathbb{N}_n$, Definition 5 implies

$$\phi_k(\vec{a}) = r_k w_{k-1}(\vec{a}, \vec{r}) - \tau_k w_{k-1}(\vec{a}, \vec{r}). \tag{11}$$

Axiom 1 implies

$$\phi_k(\vec{a}) = F_k(\vec{a}) - F_{k-1}(\vec{a}). \tag{12}$$

Combining (11) and (12), we obtain

$$F_k(\vec{a}) - F_{k-1}(\vec{a}) = r_k w_{k-1}(\vec{a}, \vec{r}) - \tau_k w_{k-1}(\vec{a}, \vec{r}).$$

Applying Definition 3 one gets

$$w_k(\vec{a}, \vec{r}) - w_k(\vec{a}, \vec{\imath}) - w_{k-1}(\vec{a}, \vec{r}) + w_{k-1}(\vec{a}, \vec{\imath}) = r_k w_{k-1}(\vec{a}, \vec{r}) - \tau_k w_{k-1}(\vec{a}, \vec{r}).$$

Using eq. (1), solving for τ_k, and denoting with i_k^* the solution, we get (9), where i_k^* is as in eq. (10). Finally, NPV-consistency follows from Axiom 1: picking $\vec{\sigma} = (n, n, \ldots, n)$,

$$\sum_{k=1}^{n} \frac{\phi_k(\vec{a})}{\nu_{0,n}(\vec{\imath})} = \frac{F_n(\vec{a})}{\nu_{0,n}(\vec{\imath})} = \text{NPV}(\mathcal{A}).$$

$\square$

Proposition 1 tells us that the appropriate cost of capital is not the normal rate i_k and that the capital charge is not $i_k w_{k-1}(\vec{a}, \vec{r})$, but $i_k^* \cdot w_{k-1}(\vec{a}, \vec{r}) = [a_k + w_k(\vec{a}, \vec{\imath})] - w_{k-1}(\vec{a}, \vec{\imath})$. To interpret the latter, it may be noted that if the investor invested c at the interest rate i_k, and periodically withdrew the amount a_k from asset $\mathcal{A}$, his profit would just be given by the end-of-period amount $a_k + w_k(\vec{a}, \vec{\imath})$ (cash flow plus end-of-period capital) minus the capital invested at the beginning of the period $w_{k-1}(\vec{a}, \vec{\imath})$. Surprising as it is, this capital charge is the *only one* that guarantees additive coherence.

Remark 2. It is worth noting that if Axiom 1 is fulfilled, any permutation of $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_k)$ leads to the same excess return $F_k(\vec{a})$. Consider any permutation $\vec{\phi}_p = (\phi_{p_1}, \phi_{p_2}, \ldots, \phi_{p_n})$ of $\vec{\phi}$: one finds $\sum_{j=1}^{k} \phi_{p_j} = \sum_{j=1}^{k} \phi_j$. In particular, for $k = n$, one finds

$$\sum_{j=1}^{n} \frac{\phi_{p_j}(\vec{a})}{\nu_{0,n}(\vec{\imath})} = \frac{F_n(\vec{a})}{\nu_{0,n}(\vec{\imath})} = \text{NPV}(\mathcal{A}).$$

This invariance with respect to permutation reflects what Penman (1992) calls an "aggregation property", and means that residual incomes, unlike cash flows, aggregate in a *value* sense. That is, residual incomes are analogous to values, so the order of summation is immaterial. Thus, for valuation purposes, one does not have to worry about timing. The evaluator does not have to predict excess profits each period, but only the grand total excess profit that an economic activity will deliver to the investor.

The following proposition shows that such a residual income is naturally obtained in an arbitrage theory perspective.

Proposition 2. *Suppose a capital market exists and let* $\big(\nu_{0,1}(\vec{\imath}), \nu_{0,2}(\vec{\imath}), \ldots, \nu_{0,n}(\vec{\imath})\big)$ *represents the yield term structure. Suppose an arbitrageur assumes a long (respectively, short) position on project* $\mathcal{A}$ *and takes on a short (respectively, long) position on a portfolio* p *replicating project* $\mathcal{A}$*'s cash flows. If* $\mathcal{A}$ *is a nonzero-NPV activity (i.e. arbitrage opportunities exist), then the arbitrage gain in the* k*-th period is given by (the absolute value of) eq.* (9).

Proof. The unit price of a k-period zero-coupon bond (zcb) issued at time 0 is $\nu_{0,k}(\vec{\imath})$. Consider a portfolio p of k-period zcb's, $k \in \mathbb{N}_n$, replicating $\vec{a}$; its market value is $p_0 = \sum_{k=1}^{n} a_k/\nu_{0,k}(\vec{\imath})$. By assumption, $p_0 \neq c$. Suppose $p_0 > c$: the arbitrageur may sell short the portfolio, go long on $\mathcal{A}$, and invest $(p_0 - c)$ in n-period zcb's. The resulting cash flows will be zero at time $k < n$. At time n, the arbitrageur receives a grand total arbitrage gain of $(p_0 - c)\nu_{0,n}(\vec{\imath}) = F_n(\vec{a})$ (see Table 1). To isolate the arbitrage gain generated in the k-th period, just net out the positions on zcb's (see Table 2). For each $k \in N_n$, $w_{k-1}(\vec{a}, \vec{r})$ represents the invested capital of the long position, whereas $w_{k-1}(\vec{a}, \vec{\imath})$ represents the financing capital of the net short position. Therefore, $r_k w_{k-1}(\vec{a}, \vec{r})$ is the profit from the arbitrage strategy, and $i_k w_{k-1}(\vec{a}, \vec{\imath})$ is the cost (interest expense) of the arbitrage strategy. The difference between the two amounts provides the net payoff from the arbitrage strategy:

$$r_k w_{k-1}(\vec{a}, \vec{r}) - i_k w_{k-1}(\vec{a}, \vec{\imath}) = w_{k-1}(\vec{a}, \vec{r})(r_k - i_k^*) > 0.$$

Suppose now $p_0 < c$. The arbitrageur may reverse long and short positions and the arbitrage gain in the k-th period is equal to

$$i_k w_{k-1}(\vec{a}, \vec{\imath}) - r_k w_{k-1}(\vec{a}, \vec{r}) = |w_{k-1}(\vec{a}, \vec{r})(r_k - i_k^*)| > 0$$

$\square$

Table 1. Arbitrage strategy

	Cash flows				
Time	0	1	2	...	n
Long position on A	$-c$	a_1	a_2	...	a_n
Short position on portfolio p	p_0	$-a_1$	$-a_2$	...	$-a_n$
Long position on zcb's	$-(p_0 - c)$	0	0	...	$F_n(\vec{a})$
Total	0	0	0	...	$F_n(\vec{a})$

Table 2. Arbitrage strategy: netting out positions on p

	Cash flows				
Time	0	1	2	$\cdots$	n
Long position on A	$-c$	a_1	a_2	$\cdots$	a_n
Net short position on zcb's	c	$-a_1$	$-a_2$	$\cdots$	$F_n(\vec{a}) - a_n$
Total	0	0	0	...	$F_n(\vec{a})$

Remark 3. The arbitrage perspective may also be interpreted as a choice between investing c in asset $\mathcal{A}$ or investing c at the market rates i_k with periodically withdrawals of a_k, $k \in \mathbb{N}_n$. In the former case, the cash-flow vector is $(-c, a_1, a_2, \dots, a_n)$, in the latter case it is $(-c, a_1, a_2, \dots, a_n - F_n(\vec{a}))$. The normal capital $w_k(\vec{a}, \vec{\imath})$ represents the capital *lost* (foregone) by the investor if he decides to invest in asset $\mathcal{A}$. The profit $i_k w_{k-1}(\vec{a}, \vec{\imath})$ is then the profit *lost* by the investor if he invests in $\mathcal{A}$. The excess profit is therefore the difference between actual profit and lost profit. Based as it is on lost capital, the arbitrage-based residual income may be called *lost capital* residual income. We will use the symbol $\vec{\phi}^{\mathcal{L}}$ to denote it ($\mathcal{L}$:=lost capital).

Remark 4. If we explicitly consider the dependence of $\vec{\phi}^{\mathcal{L}}$ upon the rates $\vec{r}$ and $\vec{\imath}$, it is straightforward from the definition that, for any asset $\mathcal{A}$,

$$\phi_k^{\mathcal{L}}(\vec{a}, \vec{r}, \vec{\imath}) = -\phi_k^{\mathcal{L}}(\vec{a}, \vec{\imath}, \vec{r}) \qquad \text{for all } k \in \mathbb{N}_n.$$

This property of antisymmetry has a natural interpretation in the arbitrage perspective: consider a contract whereby two agents take on opposite positions on A and p (as in Table 2). In each period, the arbitrage gain of one agent is the arbitrage loss of the other agent: the contract is a zero-sum game.[4]

Remark 5. The choice between using funds for a purpose and using it for another and the result in terms of capital represents a depreciation in the value of the asset. Such a depreciation represents the "opportunity cost of putting goods and resources to a certain use" (Scott, 1953, p. 369), and is therefore an economic measure of "the opportunity *lost* when another decision is carried through" (Scott, 1953, p. 375, italics added). Therefore, the excess capital $F_k(\vec{a}) = w_k(\vec{a}, \vec{r}) - w_k(\vec{a}, \vec{\imath})$ measures a depreciation with respect to the use of the funds. It is a *depreciation through use* (Coase, 1938). Using eqs. (1) and (9) one gets

$$\phi_k^{\mathcal{L}}(\vec{a}) = \big(w_{k-1}(\vec{a}, \vec{\imath}) - w_{k-1}(\vec{a}, \vec{r})\big) - \big(w_k(\vec{a}, \vec{\imath}) - w_k(\vec{a}, \vec{r})\big) \qquad k \in \mathbb{N}_n. \tag{13}$$

Eq. (13) tells us that the $\mathcal{L}$ residual income may be expressed as the depreciation through time of depreciation through use, which means that both kinds of depreciation are incorporated in

[4] A domain where this property may have practical relevance is value-based management: consider two assets (business units, firms, financial securities etc.) and suppose that periodic performance of one is evaluated on the basis of the performance of the other. Antisymmetry means that if K is the profit of one asset above the other one's, then $-K$ is the profit of the latter below the former's (see also Magni, 2009b, section 7).

the definition of the $\mathcal{L}$ residual income. This is an interesting result: it is usually thought that "economists cannot afford to lump together, as 'depreciation', changes in present value caused by the passage of time, and by use" (Scott, 1953, p. 371); on the contrary, the excess profit generated by Definition 5 and Axiom 1 enables one to lump together *depreciation through time* and *depreciation through use*. If one picks market values as outstanding capitals, depreciation through use is just Keynes's (1936, 1967) *user cost* (see Magni, 2009c). In other words, depreciation through time of Keynes's user cost is a particular case of eq. (9).

Remark 6. Equivalent notions of *lost capital* have been around in the recent literature with different uses and purposes: *external financing* (Pressacco and Stucchi, 1997), *invested capital* (Schueler, 2000, Drukarczyk and Schueler, 2000), *initial investment not recovered* (Vélez-Pareja, 2001), *adjusted invested capital* (Young and O'Byrne, 2001), *unrecovered capital* (O'Hanlon and Peasnell, 2002). While enfolded in Keynes's reasoning as depreciation of user cost (see remark above), and while it naturally stems from an arbitrage perspective, the lost-capital residual income is not the standard notion of residual income in the literature, and has been introduced only in recent years (Magni, 2000, 2003) and thoroughly investigated from several points of view (Magni, 2005, 2009b).[5] In the applicative literature, a performance metric has been proposed by Drukarczyk and Schueler (2000) named *Net Economic Income*, which may be shown to be just the depreciation through time of Keynes's user cost (see Magni, 2009c). It may be easily shown that the notion of *adjusted EVA* is also equivalent to an $\mathcal{L}$ residual income under the assumption that the outstanding balance keeps constant: $w_{k-1}(\vec{a},\vec{r}) = w_k(\vec{a},\vec{r})$ (see Young and O'Byrne, 2001, p. 42).

Suppose now a different axiomatization is given for residual income. While Axiom 1 aggregates excess profits in a *value* sense, we now suppose that aggregation is made in a *cash-flow* sense. We then have the following

Axiom 1' (Adjusted additive coherence) *For any asset $\mathcal{A}$, the capitalized sum of the first k excess profits is equal to excess return generated in the first k periods:*

$$\sum_{j=1}^{k} \phi_j(\vec{a}) \cdot \nu_{j,k}(\vec{\imath}) = F_k(\vec{a}) \quad \textit{for all } k \in \mathbb{N}_n. \tag{14}$$

Proposition 3. *Given an economic activity $\mathcal{A}$, Definition 5 and Axiom 1' imply the following NPV-consistent excess profit:*

$$\phi_k(\vec{a}) = w_{k-1}(\vec{a},\vec{r})(r_k - i_k) \quad \textit{for all } k \in \mathbb{N}_n. \tag{15}$$

[5] Pressacco and Stucchi (1997) anticipate the theory with a hint in the final remarks of their paper.

Proof. Fixed any $k \in \mathbb{N}_n$, from Axiom 1' we get $\phi_k(\vec{a}) = F_k(\vec{a}) - (1+i_k)F_{k-1}(\vec{a})$; then, applying Definition 5, we derive that $(r_k - \tau_k)w_{k-1}(\vec{a},\vec{r}) = F_k(\vec{a}) - (1+i_k)F_{k-1}(\vec{a})$. Using eq. (1), one finds $\tau_k = i_k$. NPV-consistency follows by picking $\sigma_k = k$, because

$$\sum_{k=1}^{n} \frac{w_{k-1}(\vec{a},\vec{r})(r_k - i_k)}{\nu_{0,k}(\vec{\imath})} = \sum_{k=1}^{n} \frac{-w_{k-1}(\vec{a},\vec{r})(1+i_k) + w_k(\vec{a},\vec{r}) + a_k}{\nu_{0,k}(\vec{\imath})} = -c + \sum_{k=1}^{n} \frac{a_k}{\nu_{0,k}(\vec{\imath})} = \text{NPV}(\mathcal{A}).$$

□

Remark 7. Proposition 3 generates the notion of residual income as historically developed and investigated in the economic literature up to recent times. We will henceforth use the symbol $\vec{\phi}^{\mathcal{S}}$ to denote this *standard* excess profit ($\mathcal{S}$=standard). While the implicit assumption in the $\mathcal{L}$ paradigm is that a genuine additive coherence should hold, the $\mathcal{S}$ definition requires an *adjusted* additive coherence (Axiom 1') to guarantee NPV-consistency, where adjustment is made for timing. In the case of the $\mathcal{L}$ paradigm, we may note that two rates are put into place: a cost of capital i^*_k and a discount rate i_k, which are different. In the calculation of *Goodwill*, it was (and is) a common practice to use different rates for discounting excess profits, and long since "writers have noted the temptation to use different rates" (Carsberg, 1966, p. 14). While regarded as a wrong practice from theoreticians, the $\mathcal{L}$ residual income corroborates this practice, if discounting is made with the discount factor $1/\nu_{0,n}(\vec{\imath})$. Note that the use of an adjusted additive coherence, while obviously leading to NPV-consistency, prevents the standard residual income to enjoy the two properties of invariance with respect to permutations (see Remark 2) and antisymmetry with respect to the rates (see Remark 4).

Remark 8. Note that eq. (1) implies that standard and lost-capital residual income coincide in the first period: $\phi^{\mathcal{S}}_1 = \phi^{\mathcal{L}}_1 = c(r_1 - i_1)$.

Remark 9. It is worth noting that the cost of capital in the $\mathcal{L}$ paradigm is a multiple of the market rate of return: from eqs. (1) and (10) one finds, for $k \in \mathbb{N}_n$

$$i^*_k = i_k + i_k \cdot \frac{w_{k-1}(\vec{a},\vec{\imath}) - w_{k-1}(\vec{a},\vec{r})}{w_{k-1}(\vec{a},\vec{r})}.$$

The benchmark for signalling value creation in the $\mathcal{L}$ paradigm is equal to the market rate i_k plus (minus) the return on the relative excess capital. The factor $(w_{k-1}(\vec{a},\vec{\imath}) - w_{k-1}(\vec{a},\vec{r}))/w_{k-1}(\vec{a},\vec{r})$ just represents the relative differential capital owned by the investor, on which she earns (or foregoes if negative) interest at the rate i_k. It is just the consideration of this interest income that enables aggregation of residual incomes in a value sense: they are (net) *values* referred to time n, and their sum is the net final value $F_n(\vec{a})$. Conversely, the use of the market rate in the $\mathcal{S}$ paradigm makes it necessary a proper adjustment: residual incomes sum to the net final value only if they are capitalized with a suitable market interest factor.

4 From residual income to cash flows

In this section we reverse the analysis by taking excess profit (either standard or lost-capital) as an independent variable. Given a vector of market rates $\vec{\imath}$, we deduct an unambiguous asset $\mathcal{A}$ consisting of outlay c and a cash-flow vector $\vec{a}$ such that $\vec{a} \in A_n(c, r)$, with $r \neq i_k$, $k \geq 2$ (note that, according to Remark 8, r is uniquely determined by the vector of market rates and the choice of the first residual income). To this end, assume that a given vector $\vec{\alpha} \in \mathbb{R}^n$ is assigned. The question is: does there exist, possibly uniquely determined, a vector $\vec{a} \in A_n(c, r)$ such that $\vec{\phi}(\vec{a}) = \vec{\alpha}$ (where $\vec{\phi} = \vec{\phi}^{\mathcal{L}}$ or $\vec{\phi} = \vec{\phi}^{\mathcal{S}}$)?

Definition 6. We will say that $\vec{s} = \vec{s}(\vec{\alpha}) \in A_n(c, r)$ is $\mathcal{S}$-induced if $\vec{\phi}^{\mathcal{S}}(\vec{s}) = \vec{\alpha}$.

Proposition 4. *Given any $\vec{\alpha} \in \mathbb{R}^n$, there exists a uniquely determined $\mathcal{S}$-induced $\vec{s}(\vec{\alpha})$.*

Proof. Consider the system of equations in the variables $a_1, ..., a_{n-1} \in \mathbb{R}$ given by $\phi_k^{\mathcal{S}}(\vec{a}) = \alpha_k$ for all $k \in \mathbb{N}_n$. By Remark 8, we may omit the equation with index $k = 1$ and, after some calculations, it is possible to write the above system in the equivalent form

$$\sum_{j=1}^{k} a_j (1+r)^{k-j} = \eta_k \quad \text{for all } k \in \mathbb{N}_{n-1} \tag{16}$$

where $\eta_k := c(1+r)^k - \alpha_{k+1}/(r - i_{k+1})$ for any $k \in \mathbb{N}_{n-1}$. It is clear that (16) is a linear system in the variables $a_1, ..., a_{n-1}$, whose associated matrix is in lower triangular form with determinant equal to 1; hence, there exists a unique solution, denoted with $(s_1, \ldots, s_{n-1})$. Finally, by setting

$$s_n := c(1+r)^n - \sum_{j=1}^{n-1} s_j (1+r)^{n-j},$$

we deduce that $\vec{s} = (s_1, ..., s_n) \in A_n(c, r)$ and is $\mathcal{S}$-induced. □

Definition 7. We will say that $\vec{l} = \vec{l}(\vec{\alpha}) \in A_n(c, r)$ is $\mathcal{L}$-induced if $\vec{\phi}^{\mathcal{L}}(\vec{l}) = \vec{\alpha}$.

Proposition 5. *Given any $\vec{\alpha} \in \mathbb{R}^n$, there exists a uniquely determined $\mathcal{L}$-induced $\vec{l}(\vec{\alpha})$.*

Proof. Consider the system of equations in the variables $a_1, ..., a_{n-1} \in \mathbb{R}$ given by $\phi_k^{\mathcal{L}}(\vec{a}) = \alpha_k$ for all $k \in \mathbb{N}_n$. By Remark 8, we may omit the equation with index $k = 1$, so, for $k > 1$, after some calculations, it is possible to write the above system in the equivalent form

$$\sum_{j=1}^{k-1} q_{j,k}(\vec{\imath}) \cdot a_j = c \cdot q_{0,k}(\vec{\imath}) + \alpha_k \quad \text{for all } k \in \{2, ..., n\} \tag{17}$$

where

$$q_{j,k}(\vec{\imath}) := i_k \cdot \nu_{j,k-1}(\vec{\imath}) - r \cdot \nu_{j,k-1}(r) \tag{18}$$

for $0 \le j \le k-1$. It is clear that (17) is a linear system in the variables $a_1, ..., a_{n-1}$, whose associated matrix is in lower triangular form. Its determinant is given by

$$\prod_{k=2}^{n} q_{k-1,k}(\vec{\imath}),$$

which, by (18) and (3), is equal to

$$\prod_{k=2}^{n} (i_k - r).$$

Therefore, the assumption $r \neq i_k$ for all $k \in \{2, ..., n\}$ directly implies that system (17) admits a unique solution, denoted with $(l_1, ..., l_{n-1})$. Finally, by setting

$$l_n := c(1+r)^n - \sum_{j=1}^{n-1} l_j (1+r)^{n-j},$$

we deduce that $\vec{l} = (l_1, ..., l_n) \in A_n(c, r)$ and is $\mathcal{L}$-induced. $\square$

Remark 10. The complexity of an arbitrary linear system, even if in triangular form as (16) or (17), does not generally make it possible to provide an explicit, compact formula of the solution. For instance, with regard to system (17), it is not hard to see that, for any n, we find

$$l_1(\vec{\alpha}) = \frac{\alpha_2 + c \cdot q_{0,2}(\vec{\imath})}{q_{1,2}(\vec{\imath})}$$

and

$$l_2(\vec{\alpha}) = \frac{q_{1,2}(\vec{\imath})(\alpha_3 + c \cdot q_{0,3}(\vec{\imath})) - q_{1,3}(\vec{\imath})(\alpha_2 + c \cdot q_{0,3}(\vec{\imath}))}{q_{1,2}(\vec{\imath}) \cdot q_{2,3}(\vec{\imath})},$$

but the formulae for the solutions of higher dimension are generally not easy to write. Fortunately, this is not the case of system (16), which has a straightforward solution: let $\vec{s}(\vec{\alpha})$ be an $\mathcal{S}$-induced asset, and note that the linear system may be rewritten as $w_{k-1}(\vec{s}, r) = \alpha_k/(r - i_k)$, $k \in \{2, \dots, n\}$. Hence, equation (1) implies the unique solution

$$s_k = \frac{\alpha_k}{r - i_k}(1 + r) - \frac{\alpha_{k+1}}{r - i_{k+1}} \quad \text{for all } k \in \mathbb{N}_n \tag{19}$$

(for $k = n$, we obviously have $\alpha_{n+1} := 0$, given that $w_n(\vec{s}, r)$=0).

Definition 8. Given any $\vec{a}, \vec{b} \in A_n(c, r)$, we will say that $\vec{a}, \vec{b}$ are *profit-siblings* if $\vec{\phi^{\mathcal{S}}}(\vec{a}) = \vec{\phi^{\mathcal{L}}}(\vec{b})$.

Remark 11. If $\vec{a}, \vec{b} \in A_n(c, r)$ are profit-siblings, then $\vec{a}(\vec{\alpha})$ and $\vec{b}(\vec{\alpha})$ are $\mathcal{S}$- and $\mathcal{L}$-induced respectively, where $\alpha_k = \phi_k^{\mathcal{S}}(\vec{a}) = \phi_k^{\mathcal{L}}(\vec{b})$ for all $k \in \mathbb{N}_n$. For this reason, if two assets are profit-siblings, in accordance with the notations adopted in Definitions 6 and 7, they will be denoted with $\vec{s}(\vec{\alpha})$ and $\vec{l}(\vec{\alpha})$, where $\vec{\alpha} = \vec{\phi^{\mathcal{S}}}(\vec{s}) = \vec{\phi^{\mathcal{L}}}(\vec{l})$.

Two profit-siblings assets are such that the $\mathcal{S}$ excess profit of one project coincides with the $\mathcal{L}$ excess profit of the other one. Note that this does not imply that they are equally attractive, given that $\sum_{k=1}^n \alpha_k/\nu_{0,k}(\vec{\imath}) \neq \sum_{k=1}^n \alpha_k/\nu_{0,n}(\vec{\imath})$ for any n. In the sequel, we will analyze in detail the deep connections between two profit-siblings assets in the special case

$$i_1 = i_2 = \cdots = i_n = i.$$

The following lemma shows that if two economic activities are profit-siblings, then the cash flows of the $\mathcal{S}$-induced asset provide information about the depreciation of the normal capital of the $\mathcal{L}$-induced asset.

Lemma 1. *Let $\vec{s}(\vec{\alpha})$ and $\vec{l}(\vec{\alpha})$ be profit-siblings. Then*

$$s_k = w_{k-1}(\vec{l}, i) - w_k(\vec{l}, i)$$

for all $k \in \mathbb{N}_{n-1}$.

Proof. Being $\vec{s}$ and $\vec{l}$ profit-siblings, one gets $\phi_{k+1}^{\mathcal{S}}(\vec{s}) = \phi_{k+1}^{\mathcal{L}}(\vec{l})$ for all $k \in \{0, 1, ..., n-1\}$, which means

$$w_k(\vec{s}, r)(r - i) = r w_k(\vec{l}, r) - i w_k(\vec{l}, i) \tag{20}$$

for all $k \in \{0, 1, ..., n-1\}$. In particular, for $k \geq 1$, eqs. (1) and (20) imply

$$(1 + r)w_{k-1}(\vec{s}, r)(r - i) - s_k(r - i) = r(1 + r)w_{k-1}(\vec{l}, r) - i(1 + i)w_{k-1}(\vec{l}, i) - l_k(r - i).$$

Owing to (20), the above equation boils down to

$$i(1+r)w_{k-1}(\vec{l},i)+s_k(r-i)=i(1+i)w_{k-1}(\vec{l},i)+l_k(r-i),$$

whence $s_k = l_k - iw_{k-1}(\vec{l},i) = w_{k-1}(\vec{l},i) - w_k(\vec{l},i)$, where the last equality is due to the usual outstanding capital recurrence equation. □

Thanks to the above lemma, the problems lying in the resolution of the system (17) are overcome by the fact that any $\mathcal{L}$-induced project has its own profit-sibling $\mathcal{S}$-induced asset.

Remark 12. The notion of *shadow project* investigated in Magni (2000, 2003, 2005) bears relations to the notion of profit-sibling assets. Asset $\vec{s} \in \mathbb{R}^n$ is a *shadow* (project) of asset $\vec{l} \in \mathbb{R}^n$ if $\vec{\phi^{\mathcal{S}}}(\vec{s}) = \vec{\phi^{\mathcal{L}}}(\vec{l})$. Contrary to profit-siblings assets here defined, shadow projects are not required to have the same internal return vector, nor is it required that the period internal rate r_k is constant (in other words, two shadow projects may have no internal rate of return and they do not necessarily belong to the same set $A_n(c,\vec{r})$). In his papers, Magni focuses on a particular shadow project $\vec{s}$, such that $w_k(\vec{s},\vec{r}) = w_k(\vec{l},i)$ for all k, which implies $\vec{s} = \vec{l} + \phi^{\mathcal{L}}(\vec{l})$, which in turn implies that the net final value is obtained by subtracting the (noncapitalized) respective cash flows and summing over k: $F_n(\vec{l},\vec{r}) = \sum_{k=1}^{n} \phi_k^{\mathcal{L}}(\vec{l}) = \sum_{k=1}^{n}(s_k - l_k)$ (e.g., see Magni, 2003, eq. (25)).

Lemma 2. *Let $\vec{s}(\vec{\alpha})$ and $\vec{l}(\vec{\alpha})$ be profit-siblings. Then*

$$l_k = cr + s_k - \alpha_k + r\sum_{j=1}^{k-1}(\alpha_j - s_j) \tag{21}$$

for all $k \in \mathbb{N}_{n-1}$, where we set $\sum_{j=1}^{0}(\alpha_j - s_j) := 0$,

Proof. From eq. (13),

$$\alpha_k = \phi_k^{\mathcal{L}}(\vec{l}) = (w_k(\vec{l},r) - w_{k-1}(\vec{l},r)) - (w_k(\vec{l},i) - w_{k-1}(\vec{l},i))$$

for all $k \in \mathbb{N}_n$; hence, by Lemma 1, one finds

$$w_k(\vec{l},r) = w_{k-1}(\vec{l},r) + \alpha_k - s_k \qquad k \in \mathbb{N}_{n-1}. \tag{22}$$

The repeated application of the previous equation for all the indices $k-1, k-2, ..., 1$ yields

$$w_k(\vec{l},r) = c + \sum_{j=1}^{k}(\alpha_j - s_j). \tag{23}$$

Therefore, applying the outstanding capital recurrence equation, one may rewrite (23) as

$$c + \sum_{j=1}^{k} (\alpha_j - s_j) = \Big(c + \sum_{j=1}^{k-1} (\alpha_j - s_j)\Big)(1+r) - l_k$$

and finally, solving the above equation for l_k, the claim is shown. □

Now we show how a pre-fixed pattern of residual incomes may generate specific pattern of cash-flow vectors. We will then assume that excess profit grows at a certain rate. We start with the significant case of constant growth rate.[6]

Proposition 6. *Given any growth rate $g \in \mathbb{R} \setminus \{-1\}$,[7] suppose that $\vec{\alpha} = \phi^{\mathcal{S}}(\vec{s})$ is governed by the following dynamic recurrence equation:*

$$\alpha_{k+1} = (1+g)\alpha_k \quad \textit{for all } k \in \mathbb{N}_{n-1}. \tag{24}$$

Then,

$$\begin{cases} s_k = c(r-g)(1+g)^{k-1} & k \in \mathbb{N}_{n-1} \\ s_n = c(r-g)(1+g)^{n-1} + c(1+g)^n. \end{cases} \tag{25}$$

Proof. Equation (24) implies $\alpha_k = (1+g)^{k-1}\alpha_1$, whence $w_k(\vec{s}, r) = c(1+g)^k$. Then, $s_k = w_{k-1}(\vec{s}, r)(1+r) - w_k(\vec{s}, r) = c(r-g)(1+g)^{k-1}$. Also, $s_n = w_{n-1}(\vec{s}, r)(1+r) = c(1+g)^{n-1}(1+r)$, which may be reshaped as $c(r-g)(1+g)^{n-1} + c(1+g)^n$. □

Proposition 6 tells us that the first $n-1$ cash flows of the $\mathcal{S}$-induced asset are equal to the standard residual incomes that would be obtained if the market rate of return were equal to the growth rate, while the last cash flow also includes the (would-be) time-n market value of c. The resulting asset is a variable-income asset, where coupons increase at the rate g, the first one being $c(r-g)$. At the final date, beside the last coupon, the investors receives the capital compounded at the rate g.

[6] This implies that the difference between consecutive residual incomes is proportional to residual income. Such a difference is known in the relevant literature as "abnormal earnings growth" (Ohlson, 2005, Ohlson and Juettner-Nauroth, 2005; Penman, 2005; Brief, 2007); the latter is just the "EVA-improvement" introduced by O'Byrne (1996, 1997. See also Young and O'Byrne, 2001) and anticipated in Walter (1956), in Bodenhorn (1959), and in Miller and Modigliani's (1961) investment opportunities approach to valuation.

[7] If $g = -1$, then either $r = i$ or $c = 0$. Our assumptions $r \neq i$ and $c \neq 0$ forbid the case $g = -1$.

Remark 13. Note that, for $k \in \{2, \ldots, n-1\}$ and $g \neq r$, the cash flows and the actual residual incomes grow at the same rate g:

$$\frac{\phi_k^{\mathcal{S}}(\vec{s})}{\phi_{k-1}^{\mathcal{S}}(\vec{s})} = \frac{s_k}{s_{k-1}} = 1 + g.$$

For k=n, the last growth factor is a multiple of $(1+g)$: $\frac{s_n}{s_{n-1}} = (1+g)\frac{1+r}{r-g}$. This implies that to obtain an asset where $\mathcal{S}$ residual incomes grow at the rate g, one just has to fix cash flows so that they grow at the rate g as well, starting from $s_1 = c(r-g)$.

Suppose now that one is willing to construct a bond whereby the holder will receive $\mathcal{S}$ residual incomes increasing at the rate g. Among the infinite cases, consider the following significant ones:

Case 1. $g = i$.

If the growth rate is equal to the market rate, the cash-flow vector is

$$\vec{s} = \big(\alpha_1, \alpha_1(1+i), \ldots, \alpha_1(1+i)^{n-2}, \alpha_1(1+i)^{n-1} + c(1+i)^n\big).$$

Note that the cash flows coincide with the excess profits: $s_k = \alpha_k$ for all $k \in \mathbb{N}_{n-1}$ and the final cash flows is bigger than the last excess profit by the compounded value, at the market rate, of the initial investment c: $s_n = \alpha_n + c(1+i)^n$. Therefore, when residual income grows at i, the $\mathcal{S}$-induced asset may be interpreted as a portfolio of two assets: the first one is equivalent to a security replicating the residual-income sequence, the other one is a zero-coupon bond with face value equal to $c(1+i)^n$:

$$\vec{s} = \big(\alpha_1, \alpha_2, \ldots, \alpha_n\big) + \big(0, 0, \ldots, 0,\ c(1+i)^n\big).$$

The market value of the security is $\sum_{j=1}^{n} \alpha_j (1+i)^{-j}$=NPV($\vec{s}$), the value of the zero-coupon bond is c (which implies that the value of the portfolio is NPV($\vec{s}$)+c=$v_0(\vec{a}, \vec{\imath})$, as expected). Such a portfolio gives the holder a vector of gains $\phi^{\mathcal{S}}(\vec{s})$ over the holder of a zero-coupon bond traded in the market with face value $c(1+i)^n$.

Case 2. $g = 0$.

In this case, excess profit is kept constant over time, that is, $\alpha_k = \alpha_1$. The asset generated is

$$\vec{s} = \big(cr, cr, \ldots, cr, cr + c\big).$$

It is a fixed-income asset with a coupon equal to cr and repayment of the capital at the final date. In this case, given that a coupon bond traded in a market is described by $(ci, ci, \ldots, ci, ci + c)$, the

$\mathcal{S}$-induced asset may be interpreted as a portfolio of a coupon bond traded in the market and a fixed-income asset with constant cash flows equal to residual incomes:

$$\vec{s} = (ci, ci, \ldots, ci, ci + c) + (\alpha_1, \alpha_1, \ldots, \alpha_1).$$

In other terms, such an asset may be interpreted as giving the holder a constant arbitrage gain α_1 over a coupon bond priced at par value in the market.

Case 3. $g = r$.

The asset induced is

$$\vec{s} = \big(0, 0, \ldots, c(1+r)^n\big).$$

It is a zero-coupon asset which may be reinterpreted as a zero-coupon bond traded in the market plus an arbitrage surplus:

$$\vec{s} = (0, 0, \ldots, c(1+i)^n) + \big(0, 0, \ldots, c[\nu_{0,n}(r) - \nu_{0,n}(i)]\big).$$

The following proposition generalizes Proposition 6 assuming variable growth rates g_k, $k \in \mathbb{N}_n$.

Proposition 7. *Let* $\vec{g} = (g_1, g_2, \ldots, g_n) \in \mathbb{R}^n \setminus \big\{(-1, -1, \ldots, -1)\big\}$ *be a growth vector, and let* $\nu_{0,k}(\vec{g})$ *be the corresponding growth factor, for any* $k \in \mathbb{N}_n$. *Suppose that* $\vec{\alpha} = \vec{\phi}^{\mathcal{S}}(\vec{s})$ *is governed by the following dynamic recurrence equation:*

$$\alpha_{k+1} = (1 + g_k)\alpha_k \quad \textit{for all } k \in \mathbb{N}_{n-1}. \tag{26}$$

Then

$$\begin{cases} s_k = c(r - g_k)\nu_{0,k-1}(\vec{g}) & k \in \mathbb{N}_{n-1} \\ s_n = c(r - g_n)\nu_{0,n-1}(\vec{g}) + c\nu_{0,n}(\vec{g}). \end{cases} \tag{27}$$

Proof. The proof is analogous to that of Proposition 6, with $\nu_{0,k}(\vec{g})$ replacing $(1 + g)^k$ (note that the result holds for any g_n, so the latter is just a "phantom" scalar). □

Proposition 8. *Let* $\vec{s}(\vec{\alpha})$ *and* $\vec{l}(\vec{\alpha})$ *be profit-siblings. Given any growth rate* $g \in \mathbb{R} \setminus \{-1\}$, *suppose that* $\vec{\alpha} = \vec{\phi}^{\mathcal{L}}(\vec{l})$ *is governed by the following dynamic recurrence equation:*

$$\alpha_{k+1} = (1 + g)\alpha_k \quad \textit{for all } k \in \mathbb{N}_{n-1}. \tag{28}$$

Then,

$$\begin{cases} l_k = cr\frac{i}{g} + \frac{c(1+g)^{k-1}(i-g)(g-r)}{g} & k \in \mathbb{N}_{n-1} \\ l_n = c\frac{1+r}{g}\left[(1+g)^{n-1}(g - i) + i\right] \end{cases} \tag{29}$$

for $g \neq 0$, *and*

$$\begin{cases} l_k = cr + ci(1-(k-1)r) & k \in \mathbb{N}_{n-1} \\ l_n = c(1+r)^n - c(r+i)\frac{(1+r)^n-(1+r)}{r} + ic\frac{(n-2)(1+r)-(n-1)(1+r)^2+(1+r)^n}{r} \end{cases} \tag{30}$$

for $g = 0$.

Proof. By Proposition 6, we have $s_k = c(r-g)(1+g)^{k-1}$ and $\alpha_k = \alpha_1(1+g)^{k-1}$. Therefore, Lemma 2 implies

$$l_k = cr + c(r-g)(1+g)^{k-1} - c(r-i)(1+g)^{k-1} + \frac{cr(r-i)}{1+g}\sum_{j=1}^{k-1}(1+g)^j - \frac{cr(r-g)}{1+g}\sum_{j=1}^{k-1}(1+g)^j. \tag{31}$$

Let $g \neq 0$; using the well-known identity $\sum_{j=1}^{n} v^j = (v - v^{n+1})/(1-v)$ for $v \neq 1$ and $n \in \mathbb{N}$, one finds, after some passages, the first equality in eq. (29). As for the second equality, $l_n \in A_n(c,r)$ implies $l_n = c(1+r)^n - \sum_{j=1}^{n-1} l_j(1+r)^{n-j}$, which in turn implies

$$l_n = c(1+r)^n - cr\frac{i}{g}\sum_{j=1}^{n}(1+r)^{n-j} - c\frac{(i-g)(g-r)}{g}\frac{(1+r)^n}{1+g}\sum_{j=1}^{n-1}\left(\frac{1+g}{1+r}\right)^j$$

and employing again the well-known identity, after some manipulations we get

$$l_n = c\frac{1+r}{g}\left[(1+g)^{n-1}(g-i)+i\right].$$

Let $g = 0$. Then, equation (31) goes to

$$l_k = cr + ci(1-(k-1)r) \tag{32}$$

so showing the first part of (30). The second part follows from $l_n = c(1+r)^n - \sum_{j=1}^{n-1} l_j(1+r)^{n-j}$ and the identity

$$\sum_{j=1}^{n}(j-1)v^j = \frac{(n-1)v^{n+2} - kv^{n+1} + v^2}{(1-v)^2}, \quad \text{for all } v \in \mathbb{R} \setminus \{1\} \text{ and } n \in \mathbb{N},$$

after appropriate manipulations. □

Remark 14. The $\mathcal{L}$-induced cash flows, for $g \neq 0$, may be rewritten, due to (28) and (29), as

$$l_k = c\frac{ir}{g} + (s_k - \phi_k^{\mathcal{L}}(\vec{l}))\frac{g-r}{g}. \tag{33}$$

The difference $s_k - \phi_k^{\mathcal{L}}(\vec{l})$ is particularly significant in management accounting and is linked to the notion of *allocation rule*. An allocation rule is a way of allocating investment c over the asset's life span; formally, it is a vector $(\theta_1, \theta_2, \ldots, \theta_n)$ such that $\theta_k{\cdot}c$ denotes the investment cost allocated to the k-th period and $\Pi_k := s_k - \theta_k \cdot c$ may be taken as a performance measure (Rogerson, 1997; Pfeiffer, 2004; Mohnen 2003). Taking $\Pi_k := \phi_k^{\mathcal{L}}(\vec{l})$, we have $\theta_k = (s_k - \phi_k^{\mathcal{L}}(\vec{l}))/c$ and (33) may be written as $l_k = c\frac{ir}{g} + \theta_k c\frac{r-g}{g}$.

Among the infinite cases, consider the following significant ones:

Case 1. $g = i$.

The cash-flow vector is

$$\vec{l} = (cr, cr, \ldots, cr, cr + c).$$

It is a fixed-income asset and its profit-sibling is given, as seen, by a variable-income asset

$$\vec{s} = \big(\alpha_1, \alpha_1(1+i), \ldots, \alpha_1(1+i)^{n-1} + c(1+i)^n\big).$$

This means that it is possible to offer different financial contracts by declaring the same growth rate for residual income, depending on the notion of residual income employed. Symmetrically, this $\mathcal{L}$-induced asset coincides with an $\mathcal{S}$-induced asset where $g = 0$ (see above). This means that the same asset may then be interpreted as giving either constant $\mathcal{S}$ residual incomes or $\mathcal{L}$ residual incomes growing at the market rate of return: it is thus possible to offer the same financial contract by declaring different growing rates for residual income, depending on the notion of excess profit employed.

Case 2. $g = 0$.

This case is described by eq. (30). Its profit-sibling is the fixed-income asset

$$\vec{s} = (cr, cr, \ldots, cr, cr + c).$$

One may reframe l_k in eq. (30) as $l_k = i(c - s_k) + (1 - ik)s_k$, $k \in \mathbb{N}_{n-1}$, which tells us that the l_k is equal to a coupon calculated by applying the market rate to $c - s_k$, plus a variable term which equals the present value, at linear discount, of the $\mathcal{S}$-induced asset's cash flows.

Case 3. $g = r$.

The cash-flow vector is

$$\vec{l} = \left(ci, ci, \ldots, ci, ci + c + c(r-i)\frac{(1+r)^n - 1}{r}\right)$$

and, as seen, its profit-sibling is a zero-coupon bond $\vec{s} = (\,0, 0, \ldots, c(1+r)^n)$. Such an asset is interpretable as a fixed-income asset differing from a coupon bond traded in the market by an additional amount, which may be interpreted as a redemption premium to the asset's holder. Such a premium is just the net final value $F_n(\vec{l})$; therefore, this asset is very informative because it self-contains its own net final value, incorporated in the last cash flow:

$$\vec{l} = (ci, ci, \ldots, ci, ci + c + F_n(\vec{l})).$$

Equivalently, the asset is a portfolio of a coupon bond traded at par and a security that pays off $F_n(\vec{l})$ at the final date:

$$\vec{l} = (ci, ci, \ldots, ci, ci + c) + (0, 0, \ldots, 0, F_n(\vec{l})).$$

To hold this asset is therefore financially equivalent to holding a coupon bond traded in the market at par plus a money amount equal to the asset's NPV.

While we have found zero-coupon assets from $\mathcal{S}$-induced assets, this is not possible for $\mathcal{L}$-induced assets, as long as a constant increase rate $g \neq 0$ (and a constant internal rate of return) is maintained: indeed, in this case, the system

$$l_k = l_k(g) = 0, \quad k \in \mathbb{N}_{n-1}$$

may not be solved for g in the relevant case $r \neq 0$. However, it is possible to generate zero-coupon assets from $\mathcal{L}$-induced assets by allowing the growth rate to vary over time.

Proposition 9. *Let $\vec{s}(\vec{\alpha})$ and $\vec{l}(\vec{\alpha})$ be profit-siblings. Suppose that $\vec{\alpha}$ is governed by the following dynamic recurrence equation:*

$$\alpha_{k+1} = (1 + g_k)\alpha_k, \qquad \text{for all } k \in \mathbb{N}_{n-1} \tag{34}$$

and let the growth factor $\nu_{0,k}(\vec{g})$ be recursively defined as

$$\nu_{0,k}(\vec{g}) = (1 + i)\nu_{0,k-1}(\vec{g}) + (g_1 - i)\nu_{0,k-1}(r), \quad \text{for all } k = 2, \ldots, n-1, \tag{35}$$

with the boundary condition

$$g_1 \in \mathbb{R} \setminus \{\mu_k\}_{k \leq n-1}, \tag{36}$$

where $\{\mu_k\}_{k\in\mathbb{N}}$ is given by

$$\mu_k = -1 + (1 + i)(1 + r) \cdot \frac{(1 + i)^{k-1} - (1 + r)^{k-1}}{(1 + i)^k - (1 + r)^k}, \ k \in \mathbb{N}.$$

Then

$$\begin{cases} l_k = c(i + r - g_1) \qquad \text{for all } k \in \mathbb{N}_{n-1} \\ l_n = c(i + r - g_1) + c + c(g_1 - i)\dfrac{(1 + r)^n - 1}{r}. \end{cases} \tag{37}$$

Proof. We claim that

$$\nu_{0,k}(\vec{g}) = \nu_{0,1}(\vec{g})\left(\sum_{j=0}^{k-1}\nu_{0,k-1-j}(i)\cdot\nu_{0,j}(r)\right) - \left(\sum_{j=0}^{k-2}\nu_{0,k-1-j}(i)\cdot\nu_{0,j+1}(r)\right) \tag{38}$$

for all $k = 2,\dots,n-1$. In fact, for $k = 2$, eq. (38) is easily shown to be equivalent to eq. (35). Set $E := \{p \in \{2,\dots,n-1\} : \text{eq. (38) holds for all } k \in \{2,\dots,p\}\}$: we know that E is not void because it contains at least the index 2, so we denote with M the maximum of E. Suppose *ab absurdo* that $M < n-1$: then, by (35) and through a simple algebraic manipulation, we get $\nu_{0,M+1}(\vec{g}) = \nu_{0,1}(i)\nu_{0,M}(\vec{g}) + \nu_{0,1}(\vec{g})\nu_{0,M}(r) - \nu_{0,1}(i)\nu_{0,M}(r)$. Applying (38) for $k = M$ to $\nu_{0,M}(\vec{g})$ in the above equation one easily gets to

$$\nu_{0,M+1}(\vec{g}) = \nu_{0,1}(\vec{g})\left(\sum_{j=0}^{M}\nu_{0,M-j}(i)\cdot\nu_{0,j}(r)\right) - \left(\sum_{j=0}^{M-1}\nu_{0,M-j}(i)\cdot\nu_{0,j+1}(r)\right)$$

so contradicting $M = \max E$ and proving the claim.

We now assert that $\nu_{0,k}(\vec{g}) \neq 0$ for all $k \in \mathbb{N}_{n-1}$. If $k = 1$, the assertion directly derives from (36); if $k > 1$, let k^* be the first index such that $\nu_{0,k^*}(\vec{g}) = 0$. Then, by the claim, $\nu_{0,k^*}(\vec{g})$ is equivalent to $g_1 = \mu_{k^*}$, which is excluded by (36), so showing the assertion. By the assertion, we can rewrite eq. (35) as

$$1 + g_k = (1+i) + (g_1 - i)\cdot\frac{\nu_{0,k-1}(r)}{\nu_{0,k-1}(\vec{g})}. \tag{39}$$

Hence, equations (34) and (39) imply

$$\alpha_{k+1} = (1+i)\alpha_k + (1+r)^{k-1}(g_1 - i)\alpha_1,$$

which, by induction, becomes

$$\alpha_k = c(g_1 - i)(1+r)^{k-1} + c(r - g_1)(1+i)^{k-1} \tag{40}$$

for all $k \in \mathbb{N}_n$. Replacing (40) into (19), one finds

$$s_k = c(r - g_1)(1+i)^{k-1} \tag{41}$$

for all $k \in \mathbb{N}_{n-1}$. From eqs. (40) and (41) one gets $s_k - \alpha_k = c(i - g_1)(1+r)^{k-1}$ for all $k \in \mathbb{N}_{n-1}$, which in turn gives, alongside (21), $l_k = c(i + r - g_1)$ for all $k \in \mathbb{N}_{n-1}$. The latter and the fact that $\vec{l} \in A_n(r,c)$ lead to $l_n = c + c(i + r - g_1) + c(g_1 - i)\dfrac{(1+r)^n - 1}{r}$. $\square$

Corollary 1. *Let $\vec{s}(\vec{\alpha})$ and $\vec{l}(\vec{\alpha})$ be profit-siblings. Suppose that $\vec{\alpha}$ is governed by the following dynamic recurrence equation:*

$$\alpha_{k+1} = (1 + g_k)\alpha_k, \qquad \textit{for all } k \in \mathbb{N}_{n-1} \tag{42}$$

where

$$g_k = i + r \cdot \frac{\nu_{0,k-1}(r)}{\nu_{0,k-1}(\vec{g})} \quad \textit{for all } k \in \mathbb{N}_{n-1} \tag{43}$$

Then

$$\begin{cases} l_k = 0 & \textit{for all } k \in \mathbb{N}_{n-1} \\ l_n = c(1+r)^n. \end{cases} \tag{44}$$

Proof. Note that eq. (43) is a particular case of (35) for $g_1 = i + r$, which implies $g_1 \geq \min\{i, r\}$ (remind that $i > 0$). It is not difficult to show that $\{\mu_k\}$ is a strictly increasing sequence converging to $\min\{i, r\}$. Hence, $g_1 > \mu_k$ for all $k \in \mathbb{N}$, so satisfying the boundary condition (36); consequently, (43) is well defined. Finally, (37) goes to (44). □

The above proposition tells us that zero coupon assets are generated by $\mathcal{L}$-induced assets if the growth rate for residual income is equal to the sum of the market rate of return and a multiple of the internal rate of return (in particular, the first growth rate is exactly equal to internal rate plus market rate). Such a multiple is greater or smaller than 1 depending exclusively upon the internal and the market rate (besides the index k).

5 Generation of zero-coupon and coupon bonds from residual-income-based assets

The results shown in the previous section enables one to construct financial contracts where residual incomes are fixed first, and cash flows are univocally determined. Whenever any such financial contract is introduced in the market, market tends to arbitrage away disequilibrium: the internal rate of return tends toward the market rate of return, until equilibrium is reached again. Zero-coupon and coupon bonds traded in the market may then be seen as the result of equilibrium forces acting on residual income-based contracts. In other words, we may easily define such bonds as financial securities generated by either $\mathcal{S}$-induced or $\mathcal{L}$-induced financial contracts. To this end, we first need to prove that bonds traded in the market are obtained when residual-income-based contracts with appropriate growing rates are introduced in the market. We start with the following

Definition 9. A financial contract is said to be $\mathcal{S}$-induced (respectively, $\mathcal{L}$-induced) if its cash-flow vector is $\mathcal{S}$-induced (respectively, $\mathcal{L}$-induced).

Proposition 10. *A coupon bond* $(ci, ci, \ldots, ci, ci + c)$ *traded at par in an equilibrium market is generated by:*

- *an* $\mathcal{S}$*-induced contract with constant residual incomes*
- *an* $\mathcal{L}$*-induced contract with residual incomes growing at the market rate* i

Proof. Pick $g = 0$ in (25): the resulting cash-flow vector $\vec{s}(r) = (cr, cr, \ldots, cr, cr + c)$. Pick $g = i$ in (29): the resulting cash-flow is $\vec{l}(r) = (cr, cr, \ldots, cr, cr + c)$ as well. Whenever any such asset is introduced in the market, disequilibrium is gradually arbitraged away, which means that r tends to i, so that the asset boils down to a coupon bond:

$$\lim_{r \to i} \vec{l}(r) = \lim_{r \to i} \vec{s}(r) = (ci, ci, \ldots, ci, ci + c).$$

□

Proposition 11. *A zero-coupon bond* $(0, 0, \ldots, c(1 + i)^n)$ *traded at* c *in an equilibrium market is generated by:*

- *an* $\mathcal{S}$*-induced contract with residual incomes increasing at the internal rate of return* r
- *an* $\mathcal{L}$*-induced contract with residual incomes growing at the rate* $g_k = i + r \cdot \nu_{0,k-1}(r)/\nu_{0,k-1}(\vec{g})$

Proof. Picking $g = r$ in (25) one finds $\vec{s}(r) = (0, 0, \ldots, c(1 + r)^n)$. Owing to Corollary 1, picking $g_k = i + r \cdot \nu_{0,k-1}(r)/\nu_{0,k-1}(\vec{g})$ one finds $\vec{l}(r) = (0, 0, \ldots, c(1 + r)^n)$. Whenever any such asset is introduced in the market, the internal rate of return tends to the market rate of return so as to arbitrage away disequilibrium. Hence,

$$\lim_{r \to i} \vec{l}(r) = \lim_{r \to i} \vec{s}(r) = (0, 0, \ldots, c(1 + i)^n).$$

□

Remark 15. These securities are generated starting from nonzero-NPV assets which are introduced in the market. After they are immersed into the market and the equilibrium process is over, not only the internal rate of return coincides with the market rate of return, but residual income in each period becomes zero, so that the resulting NPV of the bonds is zero. It is worth noting that the limiting process regards cash flows, not c. It is true that, practically, when a fixed cash-flow vector is introduced in the market, it is the cost c that tends to its equilibrium value. However, in Propositions 10 and 11, c is, *by assumption*, the value of the bond, which implies that cash flows (rather than cost) adjust to obtain the equilibrium.

Remark 16. Zero-coupon and coupon bonds may not be found as solutions of the linear systems (16) or (17) if one assumes $i_k = i$ for all k and $i = r$. In this case, neither system has a unique solution: for example, system (16) may be rewritten as $\phi_k^{\mathcal{S}}(\vec{a}) = 0$, $k \in \mathbb{N}_n$; hence, its solution is described by a vector space with dimension $n-1$ (only a_n is univocally determined, for $\vec{a} \in A_n(c, r)$). By contrast, we have solved the systems for $r \neq i$, finding a unique asset. Afterwards, the asset is introduced in the market, which tends to sweep away disequilibrium. This means that r gradually aligns to i, and, keeping c independent of r, we have found the *unique* equilibrium cash-flow vector whose value is exactly c.

Propositions 10 and 11 imply that the well-known notions of zero-coupon and coupon bonds may be defined starting from a prefixed pattern of residual incomes of assets which are induced by either the standard paradigm or the lost-capital paradigm of excess profit. We are now allowed to state the following definitions.

Definition 10. A coupon bond $(ci, ci, \dots, ci, ci{+}c)$ selling at par is the *equilibrium* vector generated by a financial contract having one of the following characteristics:

- $\mathcal{S}$ residual incomes are constant
- $\mathcal{L}$ residual incomes grow at the market rate i

Definition 11. A zero-coupon bond $(0, 0, \dots, c(1 + i)^n)$ selling at c is the *equilibrium* vector generated by a financial contract having one of the following characteristics:

- $\mathcal{S}$ residual incomes grow at the internal rate of return r.
- $\mathcal{L}$ residual incomes grow at increasing rates $g_k = i + r \cdot \frac{\nu_{0,k-1}(r)}{\nu_{0,k-1}(\vec{g})}$

Conclusion

This paper addresses several issues related to the notion of *excess profit* (*residual income*), a ubiquitous concept in economic theory, finance, and accounting. As a first concern, it provides an axiomatization: depending on the relations among the investor's excess wealth and the asset's excess profit, different notions of residual income are generated, in particular the *standard* residual income and the *lost-capital* residual income. The latter employs a comprehensive cost of capital, which is a multiple of the market rate, and takes account of the capital lost by the investor. This implies a nice property of additive coherence: the grand total of residual incomes, regardless of when they are received, leads to the asset's Net Future Value (and thus to the Net Present Value). This notion is naturally embraced in an arbitrage theory perspective: the lost-capital residual income is the arbitrage gain derived from the investment undertaking and is obtained as the difference between interest revenues on a long position and interest expenses on a short position. Conversely, the standard paradigm employs the market rate as the cost of capital, which implies that additive coherence is satisfied only if one correctly ascribes residual incomes to the periods where they are generated.

While axiomatization is addressed starting from a given vector of cash flows, the analysis in the second part of the paper is reversed: a given sequence of residual incomes is fixed *a priori*, assuming that the internal rate of return of the asset differs from the market rate (i.e. the asset is a nonzero-NPV investment). By making use of different linear systems corresponding to either paradigm, it is shown that different vectors of cash flows are induced depending on the pre-determined residual income's growth rate. Fixed-income or variable-income assets are generated and their relations with zero-coupon and coupon bonds traded in the capital market are stressed.

Finally, the paper shows how the assets, being *unique* solutions of the linear systems, unambiguously converge to either zero-coupon or coupon bonds if they are traded in a capital market. This implies that a definition of bonds may be given in terms of residual incomes rather than in terms of cash flows: zero-coupon and coupon bonds are the equilibrium vectors of specific residual-income based assets.

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