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Corporate debt pricing I.

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Abstract.

In this article we discuss fundamentals of the debt securities pricing. We begin with a generalization of the present value concept. Though the present value is the base valuation method in the modern finance we will illustrate that this concept does not sufficiently accurate in producing instrument pricing. The incompleteness of the unique present value approach stems from variability of the interest rates. Admitting variability of the interest rates we define two present values one for buyer other for seller. Therefore future buyer and seller cash payments can be described by the correspondent present values. Usually used assumption that future interest on investment over a specified time period would be the same as before specified period is a theoretical simplification that might be admitted or not. Admitting such assumption leads to eliminating an important component of the market risk. Recall that the assumption that a future payment can be invested with the same constant interest rate equal to the one used in the past is a component of the group conditions that specify frictionless of the market. We use this new concept that splits present value within two counterparties to outline details of the new valuation method of the fixed income securities.

The primary goal of this paper is a credit derivative pricing method of the risky debt instruments. First we introduce a formal definition of the default. It somewhat close but does not coincide with the reduced form of the default setting.

The first distinction is the “risk neutral” valuation used for credit derivatives. It is not difficult to note that the risk neutral valuation originated by mathematical interpretation of the option pricing. There are two different ways in the modern finance to introduce risk neutrality. These ways are specified by binomial scheme or the continuous time. In continuous time setting the “risk neutral” world came up from measure change technique used in stochastic calculus. Taking into account that Girsanov theorem it is possible to change real expected rate of return of an option’s underlying security on risk free interest rate. This adjustment can be provided by the transition from original probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ to $\{\Omega, \mathcal{F}, \rho\mathbb{P}\}$ where ρ is the exponent Girsanov density

that provides a change a drift coefficient in stochastic Ito equations. This technique was intended to apply for Black Scholes equation (BSE) to present its solution in the stochastic form. On the probability space $\{\Omega, F, \rho P\}$ with the appropriate density ρ the real expected return (the drift coefficient) of the security will be replaced by the risk free rate of return. Nevertheless as far as the solution of the BSE is the expected value of the functional along the security price then its value does not change with the transition from original probability space to risk-neutral world $\{\Omega, F, \rho P\}$. Thus the risk-neutral world is incorrect interpretation of the Cauchy problem for the parabolic equation.

In binomial scheme the “risk neutral” probability distribution does not relate to original probability measure whether it exists or not. This is the essence of the option valuation. As far as the real world distribution does not involve in option pricing either for discrete or continuous time then for example two securities having expecting returns say 10% or – 20% with the equal volatility have the same option price over the same time period and equal strike prices. The last remark shows that either binomial scheme or BSE present a wrong understanding of the option price.

For credit derivatives the risk neutral world has been used as an original probability space. It is common tradition does not specify the structure of the probability measure on the risk neutral probability space. This implies that risky securities and credit derivatives dynamics are represented with respect to measure $Q = \rho P$ where ρ is the Girsanov density. This density depends on risk free rate as well as the equity market parameters. It also depends on the exact number of stocks on the equity market their expected returns and volatilities. One should use measure ρP for calculation local characteristics of the stochastic differential equations (SDEs) which govern risky securities dynamics. For instance if we apply the measure P to estimate the expected value of a random variable it can be the arithmetic average value. On the other hand calculation of the expected value of the random variable with respect to $Q = \rho P$ leads to the different value. It would be important to take into account peculiarities of the risk neutral world statistical calculations. Theoretically it is possible to transform SDEs given on $\{\Omega, F, \rho P\}$ to the SDEs on $\{\Omega, F, P\}$ or inverse but it does not make any sense because we involve for calculation equity market parameters that actually should not play any role. Either risky securities or credit derivatives valuation methods could be apply regardless on whether equities exist or not. It looks like that risk neutrality is a needless complex construction used for the credit derivatives pricing.

There is other distinction of our approach. We relax a condition related to the market frictionless. This condition implies that payments received in the future moments of time could immediately be invested with the same interest rates. It is not difficult to check that real market data does not follow this assumption. In contrast to commonly accepted cash flow modeling dealing with expected values our model deals with the real stochastic flows. Thus given stochastic setting any market price accepted by a market participant anticipates a risk. The value of the risk is determined by the event when the value of the rate of return of a particular instrument implied by the market is lower than implied by the original market price.

The paper is structured as follows. In section 1 we assume that all parameters of the model are available. We present a formal definition of the default and develop pricing formulas for zero and nonzero coupon corporate bonds. Then in the second section we focus on the practical problem how using the theoretical model and given market data to study risk characteristics of the corporate debt instruments.

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1. A risky bond pricing model.

In this section we perform a model of the risky bond pricing. We present a formal definition of the default using a simple model. Then applying this definition we establish valuation formulas for main types of the bonds.

In this paper we follow a primary type of credit events defined by the ISDA as the “Failure to pay”. This by definition is an event when an entity fails to make scheduled full payment either a coupon or a principal of the risky debt. Within other important types of credit events defined by ISDA are bankruptcy, obligation default, moratorium, restructuring. We do not study these credit events here.

Risky 0-coupon debt valuation.

We begin with the pricing formulas of a 0-coupon bond. The bond price $B(t, T)$ by definition is a function of two variables t and T . The t is a current time and T denotes the bond maturity. In many practical situations it seems more convenient to use as independent variables t and time to maturity $T - t$ which would replace the variable T . The value $B(t, T)$ represents the value at date t of the \$1 at date T . Thus $B(t, T)$ is the present value at date t of receiving one dollar at T with no risk of default. The standard form used for pricing the Treasury security is

$$B(t, T) = 1 - i_d \frac{T - t}{360} = \frac{1}{1 + i_s \frac{T - t}{365}}$$

Here $B(t, T)$ is might be a notation for the zero-coupon T-bill, note, or bond price at date t , and $B(T, T) = 1$. Parameters i_d and i_s are assumed here to be constants but they also can depend on time. In a more complex environment they might be assumed to be stochastic too. They are known as discount rate and simple interest rate respectively. Given $B(t, T)$ the values of interest and discount rates can be easily calculated and

vice versa. The financial tables usually provide investors information about the bond prices. In continuous compounding we assume that

$$\frac{dB(t,T)}{dt} = r B(t,T)$$

$t < T$, with a boundary condition $B(T, T) = 1$.

A 0-coupon debt-security price with no risk of default is sometimes referred to as a present value or a discount factor.

Now we consider a valuation method of the default debt securities. Default is a class of the wider notion such as the credit event. We formalize now the event “failure to pay” which we consider as a definition of the default. Though this method is close to the reduced form of the risky bond valuation it has also some significant distinctions. We begin with a formal definition of the default in a discrete time setting. In a simple example it will be clear the difference between the standard reduced form models introduced in [4], [5] and the approach that we will develop below. Let us consider in details the model example the zero coupon risky bond. A risky corporate bond like a government zero default bond promises \$1 at maturity T . As far as there is no cash flow prior to maturity it seems quite reasonable to assume that the only time when corporate bond can represent its risk is a maturity date. Assume that the value of the risky corporate bond at the maturity T is defined as

$$\begin{aligned} &1, && \text{if no default at the date } T \\ &\Delta, && \text{if default occurred on the date } T \end{aligned} \tag{1.1}$$

where Δ is a known constant $\Delta \in [0, 1)$. The case $\Delta = 1$ is a limit situation when the risky bond coincides with the Treasury bond having by definition 0 chance of default. From (1.1) it follows that the risky bond pricing implies stochastic setting. Denote $R(t, T) = R(t, T; \omega)$ the risky bond value at date t , $t \leq T$, where T is the bond maturity. The variable ω is associated with two elementary events or scenarios if one prefers financial terminology. Let the value ω_0 signifies the ‘no default’ scenario and ω_d denotes the default event of the risky bond. The random function $R(t, T)$ is a stochastic process for which the face value $R(T, T)$ is defined by (1.1). The risky bond value at any time prior to maturity can be easily established by expressing its value through the risk-free bond values with the same maturity. Instead of the using informal no arbitrage argument we prefer to give a formal definition of the equality for two investments. Thus two investments are called equal over a given time interval $[t, T]$ if for equal notional values their rates of return are also equal. Using this definition the price of the risky bond should promise the same rates of return as the risk free bond regardless a scenario. This ‘perfect replication’ can be achieved by presenting the risky bond price in the form

$$\begin{aligned}
R(t, T; \omega) &= B(t, T) \{ [1 - \chi(\omega, D)] + \Delta \chi(\omega, D) \} = \\
(1.2) \quad &= B(t, T) \{ 1 - (1 - \Delta) \chi(\omega, D) \}
\end{aligned}$$

where the function

$$\chi(\omega, D) = \chi(\omega, D, T) = \begin{cases} 1, & \text{if } \omega \in D \\ 0, & \text{if } \omega \notin D \end{cases}$$

is the indicator of the default event D at the date T . Indeed (1.2) follows from the equation

$$\frac{R(T, T; \omega)}{R(t, T; \omega)} = \frac{B(T, T)}{B(t, T)}$$

that uniquely represents equality of the two investments in risky and government bonds. This equation is a short form of the equal rates of return relationship. Thus the perfect replication calculates the risky bond price based on other three given values. Note that it promises the same rate of return for each ω in contrast to the expected values as it accepted in derivatives pricing. Here $B(T, T) = 1$, and corporate bond face value $R(T, T; \omega)$, $\omega \in \Omega = \{ \omega_o, \omega_d \}$ is given by (1.1). The value

$$s(\omega) = B(t, T) (1 - \Delta) \chi(\omega, D)$$

is the deference between 0-default Government bond and the corporate bond at the date t . This differential can be called a stochastic spread in contrast with commonly used spread notion that is the defined as the expected value of the stochastic spread we introduced above. The stochastic spread shows the price difference of the risky and 0-default Treasury bond. The expectation and standard deviation of the risky bond are

$$E R(t, T; \omega) = B(t, T) [1 - (1 - \Delta) P(D)]$$

$$STDV R(t, T; \omega) = B(t, T) (1 - \Delta) \sqrt{P(D)(1 - P(D))}$$

These formulas represent primary risk characteristics of the risky bond given recovery rate (RR) and probability of default (PD).

Remark 1. The formula of the expected value of the risky bond first was presented by P. Jarrow and S. Turnbull first in [4]. This formula was used as a definition of the risky bond price. In this interpretation the spot price was interpreted as the expected value of the undefined random function that in our interpretation is the risky bond price. Reducing

the random price definition of the risky bond to its expectation eliminates market risk and related risk information. Thus stochastic setting helps us to price a corporate debt in more realistic environment.

The difference between the benchmark Treasury bond and the corporate bond with the same maturity called spread and is defined as

$$s = [B (t , T) - E R (t , T ; \omega)] = B (t , T) (1 - \Delta) P (D) \quad (1.3)$$

Here $E R (t , T ; \omega)$ is expected value of the corporate bond. This equality is similar to the credit triangle relationship [6]. Indeed admitting the expected value of the bond as the market price automatically implies risk. The risk is connected to the possible losses when the market price is interpreted as the expected value of the risky bond. The value of the risk then is the measure of the chance that investors pay higher price than the ‘perfect’ price. This risk can be expressed with the help of probability

$$P \{ R (t , T ; \omega) < E R (t , T ; \omega) \}$$

Note that this definition of the bond price theoretically implied but unfortunately have not been explicitly established by the reduced form approach. In reduced form the risky bond price corresponds to the expected value or market price. If the recovery ratio Δ is equal to 0 then the probability of default could be expressed by a simple formula

$$P (D) = B (t , T) - E R (t , T)$$

Now we extend the pricing method of the corporate bond by letting the bond to default at a discrete set of dates. Assume that 0-coupon risky bond might default only at a known sequence of the dates $t = t_0 < t_1 < t_2 < \dots < t_N = T$. As far as there is no up front a payment to counterparty it might be reasonable to exclude the chance of default at the initiation date t . Otherwise some insignificant adjustments can be made to cover this possibility. Let ω_j ,

$j = 0, 1, \dots, N$ denote the default event at the date t_j and let $\tau(\omega)$ denote the random time of default. Putting by definition $\omega_0 = \{ \omega : \tau(\omega) > T \}$ we note that ω_0 denotes the event “there is no default during the lifetime of the risky bond”. Then $\omega_j = \{ \omega : \tau(\omega) = t_j \}$, denotes the event that default occurred at the date $t_j, j > 0$. Then the union of the mutually exclusive events $\omega_j, j = 0, 1, \dots, N$ provides a decomposition of the probability space Ω . Assume that RR at the date t_j is a known constant $\Delta_j < 1$. The equality $\Delta_j = 1$ implies that there is no default at the date t_j . Then the present value of the risky 0-coupon bond can be written in the form

$$R (t , T ; \omega) = \sum_{j=1}^N \Delta_j B (t , t_j) \chi(\tau = t_j) + B (t , T) \chi(\tau > T) \quad (1.4)$$

where time of default $\tau = \tau(\omega)$ is defined as

$$\tau = \sum_{k=1}^N t_k \chi(\tau = t_k)$$

The formula (1.4) is a generalization of the formula (1.2). It represents the value at t of the seller's credit commitments over the lifetime of the bond. The only technical difference is that the lifetime of the bond is now a random variable with recovery rate that might or might not depend on time. There are several ways to establish RR. For example when recovery rate does not depend on time might be expressed as a fixed portion of the face value of the bond. In more complex setting RR can depend upon a benchmark fixed income security.

The distribution of the random time $\tau(\omega)$ in the pricing model (1.4) is assumed to be given. Note that for an arbitrary real world scenario ω the only one term on the right hand side (1.4) does not equal to 0 and therefore the bond pricing for every scenario ω is identical to the model (1.2) studied above. This follows from the fact that ω_j decompose the probability space Ω . The proof of the formula (1.4) is straightforward. Indeed for the scenarios belonging to the ω_j , $1 \leq j \leq N$ the price of the risky bond is simply the present value of the Δ_j . This follows from the case studied above when default might occur at maturity only. For the scenario ω_0 the risky bond price coincides with the risk free bond and its present value is the well-known benchmark price. Then the pricing formula for any scenario from $\omega_0 \cup \omega_N$ coincides with the risky present value of the bond when default might occurred at maturity only. Summing up over all possible scenarios leads to the formula (1.4).

Adding and subtracting appropriate terms in (1.4) we note that it could be rewritten in the form

$$\begin{aligned} R(t, T; \omega) &= \sum_{j=1}^N \Delta_j B(t, t_j) \chi(\tau = t_j) + B(t, T) \chi(\tau > T) = \\ &= \sum_{j=1}^N \bar{R}_j(t, t_j; \omega) - \sum_{j=1}^{N-1} B(t, t_j) \chi(\tau > t_j) \end{aligned}$$

Here the upper line over risky bonds signifies that bonds could default at maturity only. The low index 'j' points on dependence of the bond value on recovery rate Δ_j . Note that if default occurred at t_j then the terms through 1 to j in the first sum and from 1 to $j-1$ in the second sum in above formula will not be equal to 0.

Continuous time zero coupon bond valuation.

Continuous time valuation formula can be obtained from (1.3) when $\max (t_{j+1} - t_j)$ tends to zero. Then the valuation formula can be rewritten then in the form

$$R (t, T; \omega) = \int_t^T \Delta(s)B(t, s) \chi \{ \tau (\omega) \in ds \} + B(t, T) \chi (\tau > T)$$

where $\Delta(s)$ is a given nonrandom recovery rate. Assume that the probability of default over a small time interval can be presented as follows

$$P \{ \tau (\omega) \in (s, s + \Delta s] \} = \mu (s) \Delta s + o (\Delta s)$$

where $\mu (s)$ is a given function. Then for the first moment of the risky bond price can be written in the form

$$ER (t, T; \omega) = \int_t^T \Delta(s)B(t, s) \mu(s)ds + B(t, T)M(t,T) \quad (1.5)$$

Here $M (t, T) = P \{ \tau (\omega) > T \}$. The case when

$$\mu (s) = \mu / [T - t]^{-1}$$

corresponds to uniform distribution of default on $[t, T]$. The uniform model is an important simple example. Under the assumption default probability does not increases over the time in contrast to Poisson default time model. In many situations the uniform default time distribution looks more realistic than alternatives. Note that risk characteristics in continuous time can be approximated by the corresponding risk characteristics in discrete time. The functions $\Delta (*)$ and $\mu (*)$ in the discrete and continuous time settings could be construct follow next steps. First we need to present an estimate of the function $M (t, s)$. It will help then in construction the default density μ . Given μ , M observing $ER (t, T; \omega)$ we see that the function $\Delta (s)$ is a solution of the integral equation (1.5).

In above we assumed that recovery rate is a given portion of the face value. Now let us assume that recovery rate is a random variable taking a finite number of different values

$$\Delta(\omega) = \sum_{j=1}^k \Delta_j \chi(D_j)$$

Thus in a simple model when a risky bond can default at maturity T only the face value of the bond is defined as

$$\bar{R}(T, T; \omega) = \begin{cases} 1, & \text{if } \omega \notin D \\ \Delta_j, & \text{if } \omega \in D_j, \Delta_j < \Delta_{j+1}, j = 1, 2, \dots, k \text{ and } \bigcup_{j=1}^k D_j = D \end{cases}$$

and therefore

$$R(t, T; \omega) = \sum_{j=1}^k \Delta_j \chi(D_j) + \chi(\Omega \setminus D) = 1 - \sum_{j=1}^k (1 - \Delta_j) \chi(D_j)$$

Here the event $\Omega \setminus D$ is no default at maturity. It is not difficult to study random recovery rate model when default might occur at a discrete or in continuous time setting.

Risky coupon-bond valuation.

The benchmark formula

$$PVB(t, T, c) = PV(t; t_1, c, \dots, t_{N-1}, c, t_N, c+F) = \sum_{j=1}^N cB(t, t_j) + FB(t, T) \quad (1.6)$$

represents the well known present value (PV) of the cash flow associated with a coupon bearing bond. Here

*) c is the coupon paid by issuer of the bond to the bond holders on predetermined dates $t = t_0 < t_1 < t_2 < \dots < t_N = T$

**) B (t , T) is the value of strips (0-coupon bond) with no risk of default.

***) F is the face (par) value of the bond.

Thus, the benchmark price of the coupon bond is by definition the present value of the all the payments attached to the bond over its lifetime. The formula (1.6) uses the value of

0-coupon bonds with various maturities. By definition the function B (t , T) when t is fixed and T is considered as a variable argument is called the term structure of the bond. Thus term structure depends on a current moment t as a parameter. This is the financial standard to use PV as a price of the coupon bearing bonds. We will show bellow that this pricing method should be completed in order to represent real market more accurately. With this interpretation the coupon bond pricing formula (1.6) represents the bond

seller's price while the bond buyer's bond price can be represented by a different formula.

To highlight a motivation for the distinctive buyer's price one should remark that bond seller price over lifetime of the bond is the present value of the cash flow (t_1, c) , (t_2, c) , ..., $(t_N, c + F)$ paid by the seller to the buyer. Note that the coupon value of $\$c$ at a date $s \in (t, T)$ can be generated by $\$c B(t, s)$, $B(t, s) \leq 1$ invested at the date t . Note here that the issuer of the bond is the owner of the $\$c$ over the time $[t, s)$. On the other hand a bond buyer who is also the investor receives the coupon payment at the date s owns $\$c$ sum over the future adjacent period $(s, T]$ until the bond expiration. Recall that we have assumed that there is no chance of default of the government bond. The market observation shows that the discount rates over different time intervals are not equal. That proves the necessity to distinguish seller and buyer prices generated by the same coupon bond. Hence it is clear that the bondholder return should be estimated based on forward discounted cash flow. The bond investor's price can be constructed as follows. Note that the value at time T of the amount of $\$c$ paid at t_j is by definition equal to $c \times B(t_j, T)$. Thus the bond buyer accumulated capital at the date T is

$$FV(t; t_1, c, \dots, t_{N-1}, c, t_N, c+F) = \sum_{j=1}^N c B^{-1}(t_j, T) + F$$

Recall that by definition $B(T, T) = 1$. The pricing problem is to establish the price of the coupon-bearing bond at any moment t prior to the bond expiration T . Thus, the bond buyer has two alternatives: the investment in 0-coupon or in c -coupon bond. Note that if 0-coupon does not exist then it can be considered hypothetically. To avoid arbitrage opportunity market should promise equal rate of return on any type of the government bonds with the same expiration date regardless of the coupon value. Thus if the PV amount invested in the risk free bond with expiration date at T yields strictly smaller or larger then the FV represented above then there exist an arbitrage opportunity. Therefore the equation

$$\frac{FV(t; t_1, c, \dots, t_{N-1}, c, t_N, c+F)}{B_c(t, T)} = \frac{B(T, T)}{B(t, T)} \quad (1.7)$$

for $B_c(t, T)$ is the unique way to avoid arbitrage. The solution of the equation (1.7) is the investor price at the time t of the coupon-bearing bond. From (1.7) it follows that

$$B_c(t, T) = \left[\sum_{j=1}^N c B^{-1}(t_j, T) + F \right] B(t, T) \quad (1.8)$$

The equality (1.8) states that the bond buyer price of the coupon bond is the present value of the total cash amount accumulated at maturity. Note that the value $B_c(t, T)$ is not equal to the price on the right-hand side (1.6). The formula (1.8) contains value of the bonds related to the future moments of time that are not known at t . In stochastic setting these values can be modeled by the random functions. Therefore the bond buyer faces a risk accepting the date t market price. The real rate of return can be either lower or higher implied by the market. The value of the bid-ask spread could be considered as an indicator of the stability of the market.

Remark 2. The idea that the present value methodology does not perfectly fitted to the real market has been highlighted in some papers. In [7] it was highlighted the assumptions behind definition of the yield to maturity. These are

1. An investor who buys bond can only achieve a return equal to the yield if the bond is held to maturity and if all coupons can be reinvested at the same rate as the yield
 2. The yield curve is flat. That means equal reinvestment rates for different maturities.
- Also it was noted in [7] that either of these assumptions do not take place in practice.

The idea to differentiate buyer and seller transactions in bond trading was presented in [Fixed Income Pricing]. As far as the values $B(t_j, T)$ for $t_j > t, j = 1, 2, \dots, N$ are unknown at the date t the one commonly excepted way to proceed is a randomization of the problem setting. Admitting stochastic of the bond price one can apply statistical hypotheses testing along with statistical estimates theory to draw conclusions from population presented by the market data. The forward contract historical data can be applied for developing hypothetical distribution. Other way that often used in finance is an analytic assumption in a form of a stochastic differential equation about a bond price dynamics.

Let s be the difference between (1.6) and (1.8). Then the value

$$s = s(\omega) = PV B(t, T; c) - B_c(t, T; \omega)$$

is a random depending on parameters t, T . This interpretation of the market price implies a risky settlement between two counterparties. Let for example the present value $PV(t, T; c)$ is a market settlement price. Then the counterparties risk can be expressed with the help of the cumulative distribution function $F(x)$ of the random variable $s(\omega)$. Indeed

$$F(x) - F(0) = P\{0 \leq B_c(t, T; \omega) - PV(t, c, T) < x\}$$

$$F(-x) = P\{B_c(t, T; \omega) - PV(t, c, T) < -x\}$$

The first equality above represents the probability of the chance that investor price $PV(*)$ is bellow of the real price $B_c(*) ; \omega$. This is the value of the bond seller risk. The second equality represents the probability of the complimentary events i.e. the probability that the market price $PV(*)$ of the bond is above than the bond value $B_c(*) ; \omega$. That is the

bond buyer risk value. An assumption on the bond price dynamics is needed in order to present the distribution function $F(x)$ in an analytic form. Implied approach is the common way in the modern finance sciences to avoid statistical inference regarding the explicit form of the function $F(x)$. Thus the contemporary implied approach admits a hypothetical distribution without statistical testing the model.

The probability that bond price exceeds the present value is $F(x) - F(0)$. This value specifies the chance that bond price is higher than was admitted at the moment t . Indeed, from seller's point of view the cost of the coupon bond at time t is given by (1.6). The bond value given by (1.8) is what the bond buyer assumes to be the bond price at the date t . The present value of the bond in the neutral market would have a symmetric distribution with respect to expected return. In this case PV could be a good unbiased estimate of the bond price. Below we will illustrate such a situation in details.

We prove the formula (1.8) by using the method of mathematical induction. We begin with the last interval $(t_{N-1}, T]$. Over this interval the value of the coupon bond can be received from the 0-coupon bond curve by multiplying it by the factor $(F + c)$. Indeed bonds with 0 or $c > 0$ coupon issued by a financial institution having 0 chance of default should promise equal rate of return for any moment t from $(t_{N-1}, T]$. Otherwise, there exist an arbitrage opportunity. From the equation

$$\frac{c + F}{B_c(t, T)} = \frac{1}{B(t, T)}$$

it follows that for any t from $(t_{N-1}, T]$

$$B_c(t, T) = (c + F) B(t, T)$$

At the date t_{N-1} the bondholders receive a coupon of $\$c$ and from the formula above we see that

$$B_c(t_{N-1}, T) = B_c(t_{N-1} + 0, T) + c$$

where $B_c(t_{N-1} + 0, T) = \lim_{h \rightarrow 0} B_c(t_{N-1} + h, T)$ when the variable $h > 0$ tends to 0. Next let us repeat the pricing method over the next interval $(t_{N-2}, t_{N-1}]$. Then from the equation

$$\frac{B_c(t_{N-1}, T)}{B_c(t, T)} = \frac{B(t_{N-1}, T)}{B(t, T)}$$

follows that

$$B_c(t, T) = \frac{[B_c(t_{N-1} + 0, T) + c] B(t, T)}{B(t_{N-1}, T)} = [(c + F) + c B^{-1}(t_{N-1}, T)] B(t, T)$$

for arbitrary t from the interval $(t_{N-2}, t_{N-1}]$. Note that $B(t_{N-1} + 0, T) = B(t_{N-1}, T)$ as far as the 0 coupon bond price is assumed to be a continuous function. Since there is a finite number subintervals $(t_{j-1}, t_j]$, $j = 1, 2, \dots, N$ on $[t, T]$ then the construction can be completed for the finite number of steps. Indeed let this formula is true on $(t_j, T]$.

Then

$$B_c(t, T) = [\sum_{k=j+1}^N c B^{-1}(t_k, T) + F] B(t, T)$$

for any t from $(t_j, t_{j+1}]$. In particular

$$B_c(t_j + 0, T) = [\sum_{k=j+1}^N c B^{-1}(t_k, T) + F] B(t_j, T)$$

Hence

$$B_c(t_j, T) = B_c(t_j + 0, T) + c$$

Then

$$\frac{B_c(t_j, T)}{B_c(t, T)} = \frac{B(t_j, T)}{B(t, T)}$$

for any t from $(t_{j-1}, t_j]$. Therefore

$$\begin{aligned} B_c(t, T) &= \frac{[B_c(t_j + 0, T) + c] B(t, T)}{B(t_j, T)} = \\ &= \frac{[B(t_j, T) (\sum_{k=j+1}^N c B^{-1}(t_k, T) + F) + c] B(t, T)}{B(t_j, T)} = \end{aligned}$$

$$= B(t, T) \left[\sum_{k=j}^N c B^{-1}(t_k, T) + F \right]$$

That justifies formula (1.8).

We highlighted the difference in bond pricing given by (1.8) and (1.6).

Statement. In order that the values (1.6) and (1.8) were equal for an arbitrary date t it is necessary and sufficient that the 0-coupon bond prices satisfy the equality

$$B(t, s) B(s, T) = B(t, T)$$

for arbitrary $0 \leq t \leq s \leq T$. Historical data shows that this equality does not take place in the real world.

Putting in (1.8) $c = 0$ we arrive at the 0 coupon bond with face value F and therefore its present value is $B(t, T)$. This price represents an instrument referred to as a 'strip'. In this case we see that the bond price is equal to its present value. If $F = 0$, then the price of this component of the Treasury bond can be obtained from the formula (1.6). This financial instrument is a claim on pure coupon payments over the lifetime of the bond. The Wall Street Journal uses abbreviations: "np" and "bp" for the Treasury note and bond respectively, and "ci" for the claim on pure coupon payments. Using historical data one can easily figure out that the values (1.6) and (1.8) are different. A source of such discrepancy is a variability of the interest rates.

Assume now that the bond value at the date t_j is estimated at the moment t . Denote this estimate $B(t_j, T | t; \omega)$, $j = 1, 2, \dots, N$. We interpret this value as a random variable and therefore the spread $s(\omega) = PVB(t, T; c) - B_c(t, T)$ is also a random variable. The buyer's risk is then associated with the probability of the event $\{\omega : s < 0\}$ and the bond seller's risk is associated with the scenario $\{\omega : s > 0\}$. The distribution of the spread is a random function $s = s(t, T; \omega)$ depending on the unobservable at date t random variables $B(t_j, T)$, $t_j > t$. To derive statistical characteristics of the spread the unobserved random variables $B(t_j, T)$ would be replaced by their statistical estimates. The data related to the bond forward contracts could be used to construct reliable estimates. The mathematical statistics usually uses the conditional expectation $E\{B(t_j, T) | F_t\}$ as such estimates. Here F_t is the σ -algebra generated by the bond price values prior to the date t .

Let us return to the risky coupon bond valuation. Let t_j be coupon payment dates and let the recovery rate at t_j be a known constant Δ_j , $j = 1, 2, \dots, N$. Let $B_c(t, T; F)$ be the value of the risk free coupon bond at date t with coupon c with maturity T and face value F . That is a holder of the bond will receive a coupon payment of c at the dates t_j and $(F + c)$ at the date T . From (1.8) it follows that the buyer's price of the bond of the risky coupon bond is a random variable depending on scenario ω and equal to

$$B_c(t, T; F + c), \text{ if } \omega \in \{\tau(\omega) > T\}$$

$$B_c(t, t_j; rr(j)), \text{ if } \omega \in \{ \tau(\omega) = t_j \}, t < t_j < T$$

$j = 1, 2, \dots, N$. Here the recovery rate $rr(j)$ is a claim paid by the bond issuer to the bond holders if default occurred at the date t_j . There are several reasonable possibilities to establish the value of the $rr(j)$ claim. For example it can be chosen equal to $c \Delta_j$, or $(c + F) \Delta_j$, or the portion of the value of the government bond at the date of default t_j covered the total loss remained to paid after default by the bond seller to the bond holders, i.e. the sum

$$\Delta_j \left\{ \sum_{k=j}^N c B(t_j, t_k) + F B(t_j, T) \right\}$$

Obviously that different recovery rates would lead to the different bond values. Recall that the coupon bond is an example of the path dependent security. Taking into account equalities (1.6) and (1.8) the price of the risky coupon bond can be written now in one of the next forms

$$R_s^{(c)}(t, T, \omega) = \sum_{j=1}^N B_c(t, t_j; rr(j)) \chi(\tau = t_j) + B_c(t, T; F + c) \chi(\tau > T) \quad (1.9)$$

$$R_b^{(c)}(t, T, \omega) = \sum_{j=1}^N \chi(\tau = t_j) B(t, t_j) \left[\sum_{k=1}^{j-1} c B^{-1}(t_k, t_j) + rr(j) \right] + \chi(\tau > T) B(t, T) \left[\sum_{k=1}^N c B^{-1}(t_k, T) + F \right] \quad (1.10)$$

The lower indexes 's' and 'b' here stand for buyer and seller pricing. In the second formula the expression in the first brackets represents the future value of the c-coupon bond with maturity t_j with the face value equal to the claim amount $rr(j)$, $j = 1, 2, \dots, N - 1$. The expression in the second brackets represents the future value of the no default bond.

Let us consider an example. Suppose that a credit event could occur only on coupon payment dates and the recovery rate is a known constant Δ times default date payment. Let F and c be a face value and coupon value correspondingly. Then the risky cash flow can be presented in the form

$$c \sum_{j=1}^N [\Delta \chi(\tau = t_j) + \chi(\tau > t_j)] +$$

$$+ F [\Delta \chi(\tau = t_N) + \chi(\tau > t_N)]$$

Therefore at the date t the spot price of the short or long positions of the risky coupon bearing bond are the standard present value or the discounted future value of the above cash flow. Thus

$$R_s^{(c)}(t, T; \omega) = \sum_{j=1}^N B(t, t_j) c [\Delta \chi(\tau = t_j) + \chi(\tau > t_j)] \quad (1.11)$$

$$R_b^{(c)}(t, T, \omega) = B(t, T) \sum_{j=1}^N c B^{-1}(t_j, T) [\Delta \chi(\tau = t_j) + \chi(\tau > t_j)] +$$

$$+ F [\Delta \chi(\tau = t_N) + \chi(\tau > t_N)] \quad (1.12)$$

Thus the buyer and seller prices are random processes depending on the distribution of the time of default. Note that the buyer price also depends on random forward interest rates. Given a distribution of the default time and forward rate statistics one can easily calculate mean, variance and other risk characteristics of the corporate bond price. If the expression (1.8) or (1.11) or any other is the market price of the risky bond then counterparties take a risk. Applying a particular model distribution for the default time it is easy to present statistical characteristics of the corporate bond price in a compact closed form.

Let us consider a more general corporate pricing problem that involves tri-party transactions. In this problem we assume that the bond buyer might decide to buy protection against a possible default. Here we assume that a protection seller and the bond seller are the same party. In the event of default the bondholder would receive a protection payment. In return protection buyer pays a constant premium until default or the bond maturity whichever one comes first. In the discrete time setting this type of insurance problem can be easily resolved. Note that this problem contains two types of transactions. The first one is the bond-pricing problem studied above. Other type of transaction taken separately is known as a credit default swap (CDS). We briefly describe here the CDS pricing model. If the protection buyer does not a risky bondholder then CDS is rather game in insurance where one party pays fixed payments and other party pays a claim value at the event of on default.

Let us first consider a case $N = 1$. In this case default can occur at maturity only. If the bond defaults at the date T the payment from protection seller to protection buyer

would be a claim equal to $(1 - \Delta) \chi\{\tau = T\}$. Assume then that the first payment d paid by bondholder who also might be the buyer of protection takes place at initiation at the date t and the second payment d takes place at maturity if there is no default at this date. That is

$d[\chi\{\tau > t\} + \chi\{\tau > T\}]$. The two cash flows are equal if and only if when the premium payment d is equal to

$$d(\omega) = \frac{(1 - \Delta) \chi\{\tau = T\}}{1 + \chi\{\tau > T\}} = \frac{(1 - \Delta) \chi\{\tau = T\}}{2 - \chi\{\tau = T\}}$$

If the premium will be paid at maturity only then the number 2 in the denominator on the right hand side would be replaced by 1. There does not exist an universal constant that solves pricing problem. Then every market choice of the premium implies the risk. In the modern applications it is common to assume that counterparties use expected value of the cash flows to define the premium value. Note that this value does not coincide with the expected value of the exact premium value $d(\omega)$ presented above. If we follow this way then it is not difficult to see that the premium value could be written as

$$[d] = \frac{(1 - \Delta) P(D)}{2 - P(D)}$$

where $D = \{\tau = T\}$ denotes default event at the date T . Note that the volatility of the protection seller is

$$(1 - \Delta)^2 \text{Var} \chi\{\tau = T\} = (1 - \Delta)^2 P(D) [1 - P(D)]$$

and the protection buyer volatility

$$\langle d \rangle \text{Var} \chi\{\tau > T\} = \langle d \rangle \text{Var} [1 - \chi\{\tau = T\}]$$

These equalities demonstrate that the risk exposures of the protection buyer and protection seller are not equal. The expected value of the exact premium is

$$\langle d \rangle = E d(\omega) = (1 - \Delta) P(D)$$

If the premium payment does not take place at t then the expected value of the exact solution of the equation

$$(1 - \Delta) \chi\{\tau = T\} = d \chi\{\tau > T\}$$

$$E d(\omega) = E \frac{(1 - \Delta) \chi\{\tau = T\}}{\chi\{\tau > T\}} = 0 P\{\Omega \setminus D\} + \frac{0}{\infty} P(D)$$

is undefined. That is it makes sense do not use this setting in the market practice. In contrast one can use the estimate [d]. This value is defined as

$$[d] = \frac{(1 - \Delta) P(D)}{1 - P(D)}$$

Let us return now to the more general case when default might occur at a finite number of dates $t_j, j = 1, 2, \dots, N$. The cash flow from protection seller to protection buyer is

$$s_b = \sum_{j=1}^N (1 - rr(j)) \chi\{\tau = t_j\} + 0 \chi\{\tau > t_N\}$$

The cash flow from protection buyer to protection seller is

$$b_s = \sum_{j=1}^N \left(\sum_{i=1}^j d \right) \chi\{\tau = t_j\} + \chi\{\tau > T\} \sum_{i=1}^N d$$

For every scenario there is only one term does not equal to 0 and therefore there is no need to reduce cash flows to its present value. These two cash flows are equal when

$$d = d(\omega) = \frac{\sum_{j=1}^N (1 - rr(j)) \chi\{\tau = t_j\}}{\sum_{j=1}^N j \chi\{\tau = t_j\} + N \chi\{\tau > T\}}$$

This is a definition of the premium representing its value as a random variable that is also depends on t and T . This implies that a market settlement value implies a market risk for either counterparty. Note that

$$\langle d \rangle = E d(\omega) = \sum_{j=1}^N \frac{(1 - rr(j))}{j} P\{\tau = t_j\}$$

Let us assume that counterparties used expected cash flows to derive the equilibrium premium value. Then the value of premium [d] that equates the expected cash flows is equal to

$$[d] = \frac{\sum_{j=1}^N (1 - rr(j)) P\{\tau = t_j\}}{\sum_{j=1}^N j P\{\tau = t_j\} + N P\{\tau > T\}}$$

These two premium are different and therefore the estimate [d] is biased. The risk management specifies the risk value of each leg of the deal. The bondholder risk combines two unfavorable events. The first is the bond buyer risk. The value of the bond buyer risk is measured by the probability of the event that the buyer price (1.12) for the realized scenario is bellow than the market price. The second bondholder's risk is the protection buyer risk measured by the probability of the event that cumulative premium over the lifetime of the bond for the realized scenario exceeds the default payoff. Let the bond buyer purchases the risky bond for \$A. Then the probability

$$P\{\omega : R_b^{(c)}(t, T; \omega) < A\}$$

is the measure of market risk.

Let d^* be a settlement premium value paid by protection buyer to protection seller. Then protection buyer risk is $P\{\Sigma d^* > s_b\}$. It represents the probability of all events when the cumulative protection premium Σd^* is larger than default payoff. Thus overall bondholder risk is associated with the union of the next unfavorable events

$$\{\omega : R_b^{(c)}(t, T; \omega) < [R]\} \bigcup_{k=1}^N \{[\omega : kd > (1 - rr(k))]\} \cap (\tau = t_k) \bigcup \{\omega : \tau > T\}$$

Continuous time risky bond pricing.

Continuous time study we begin with pricing of the 0-coupon risky bond. Assume that default might occur at any time prior to maturity. Recall that a high yield risky bond known also as junk bond has a significant lower price and therefore promises significantly higher rate of return than the 0-default government bond with the same face value and maturity. Assume as usually that it is possible to interpret the risky bond price as a random process. The risk-free bond price is also can be a stochastic process though this particular case will not be studied here. Thus the time of default τ is now a random variable having a continuous probability distribution depending on parameters t and T .

Assume that the default is defined as a moment when stochastically continuous bond price process breaches a fixed barrier H . We note that in this case τ admits exposure

$$\tau = \lim_{\lambda \rightarrow 0} \sum t_i \{ R(t_j, T; \omega) \leq H \} \prod_{j=0}^{i-1} \chi \{ R(t_j, T; \omega) > H \}$$

where $R(*)$ in the above formula denotes the risky bond price or more accurately its separable modification. For writing simplicity and without loss of generality we also assumed that the partition leg $\lambda = t_i - t_{i-1}$ does not depend on i .

The bond payoff at maturity T is equal to \$1 for any scenario ω for which $\tau(\omega) > T$. If $\tau(\omega) \leq T$ then the value of the recovery rate δ should be specified. For instance the recovery rate $\delta(\omega)$ in continuous time can be one from follows

- 1) $\delta_1(\omega) = \Delta$, 2) $\delta_2(\omega) = \Delta B(\tau(\omega), T)$, or
- 3) $\delta_3(\omega) = \Delta [B(t, \tau(\omega)) - R(t, \tau(\omega); \omega)]$

where the given constant $\Delta < 1$. It is also possible that the ratio Δ might depend on time. The third version of the RR is interpreted as a known fraction of the risk free and risky bonds spread at the time of default. We primarily assume that given $\{ \tau(\omega) \leq T \}$ the corporate bond price at the moment of default is defined as $R(t, \tau(\omega); \omega) = \Delta B(t, \tau(\omega))$.

Let $D = \{ \omega : \tau(\omega) \leq T \}$ denote default event. Putting the face value of the corporate bond equal to \$1 we note that payoff on risky bond is either \$1 at date T for the scenarios ω for which $\tau(\omega) > T$ or the value $\delta_i(\omega)$, $i = 1, 2, 3$ at date $\tau(\omega)$ if $\tau(\omega) \leq T$. Here the values $\delta_i(\omega)$ are defined above recovery rates. If the risky bond price is a separable random process then we can apply the discrete time approximation to present risky bond valuation

Remark. Note that recovery rate classification used for example in [8] deals with the market data statistics rather than with its theoretical counterparts. There are two different approaches to risky valuation problems. One approach has been discussed above. In this approach we assumed that complete information about parameters and distributions are given. The other approach follows to mathematical statistics. This approach deals with historical data and its randomization. The accurate randomization includes definition of a probability space Ω and a hypothesis regarding a sample probability distribution of the risky bond. Recall that the spot price is commonly interpreted as expected value of the risky bond. For example the stochastic recovery rate $\delta_1(\omega)$ above corresponds to Fractional Recovery of Par (FRP). This FRP recovery rate is a given fraction of its face value. Thus this recovery rate implies that

$$\delta_2(\omega) = \Delta B(\tau(\omega), T; \omega) \chi(\tau(\omega) < T)$$

Recall that the risk free bond price is assumed here to be deterministic continuous function.

Let the time of the default time be a continuously distributed random variable and assume that the random process $R(t, T; \omega)$ is continuous in t with probability 1. The structural model associates the default event with the event that company's equity price falls below a certain level $\{t : S(t, T; \omega) \leq d\}$ for the some particular value d . In this case default time $\tau_d(\omega) = \inf\{t : S(t, T; \omega) \leq d\}$. This assumption can help to determine a distribution of the default time.

We now present a risky bond valuation. Recall that the 0-coupon risky bond does not a path-depended security and therefore there is a unique valuation formula for either buyer or seller. This pricing formula is the present value benchmark formula

$$R_i(t, T, \omega) = B(t, T) \chi(\tau > T) + B(t, \tau) \delta_i(\omega) \chi(\tau \leq T)$$

where variables $\delta_i(\omega)$, $i = 1, 2, 3$ are given above. There are several models are commonly have been used in different continuous time applications to approximate the random time $\tau(\omega)$. The most popular model is when default is associated with the first jump of the Poisson process. Taking into account this interpretation the well-developed mathematical theory with adjusted for Finance terminology can cover the credit events study. Let us briefly recall this well-known mathematical construction. Denote $N(t)$ the standard Poisson process. The function $Q(t) = P\{\tau(\omega) < t\}$ is the probability that the credit event occurred until t . Assume that the function $Q(t)$ is continuously differentiable function that is the derivative $q(t) = Q'(t)$ is a continuous function on $[0, T]$. The function $P(t) = 1 - Q(t)$ presents the probability that there is no default before the moment t . Let us introduce the condition probability

$$P(t, T) = P\{\tau(\omega) > T \mid \tau(\omega) > t\}$$

The function $P(t, T)$ represents the probability that there is no default until T given that there is no default before the date t . By definition of the condition probability we have

$$P(t, T) = \frac{P\{\tau(\omega) > T \cap \tau(\omega) > t\}}{P\{\tau(\omega) > t\}} = \frac{P\{\tau(\omega) > T\}}{P\{\tau(\omega) > t\}} = \frac{P(T)}{P(t)}$$

Then the probability of default over the period $[t, T]$ is $Q(t, T) = 1 - P(t, T)$. Putting

$T = t + \Delta t$ we have

$$Q(t, t + \Delta t) = \frac{P(t) - P(t + \Delta t)}{P(t)} = - \frac{P'(t)}{P(t)} \Delta t + o(\Delta t)$$

Denote

$$\lambda(t) = - \frac{P'(t)}{P(t)} \quad (1.13)$$

The function $\lambda(t)$ is called the hazard rate and

$$Q(t, t + \Delta t) \approx \lambda(t) \Delta t$$

The hazard rate at time t is the probability that the default occurred at next to t unit of time if no default until the moment t . The solution of the differential equation (1.13) leads to

$$P(s) = \exp - \int_t^s \lambda(l) dl$$

From this formula follows that the probability that there is no default that on $[t, T]$ is

$$P(t, T) = \exp - \int_t^T \lambda(l) dl \quad (1.14)$$

The function $\lambda(t)$ can be estimated empirically. Hence the probability of default over $[t, T]$ is

$$Q(t, T) = P\{t \leq \tau(\omega) < T\} = 1 - P(t, T)$$

where $P(t, T)$ is given by (1.14).

Remark. In modern credit derivatives research field a popular assumption is that hazard rate follows a stochastic differential equation. For example

$$d\lambda(t) = (a - b\lambda(t)) dt + g\sqrt{\lambda(t)} dw(t)$$

Note that this assumption is implied. Similar model equations are used for short-term interest rates models for interest rates derivatives pricing. The term “implied” in finance sciences means that constants a, b, g are calculated from historical data though there is no statistical test has been used to justify the hypothetical distribution assumed for the random process $\lambda(t)$. It is somewhat uncommon in mathematical statistics. From statistical point of view the implied distribution of the short term process $\lambda(t)$ is an

assumption about hypothetical distribution with unknown parameters and according to the mathematical statistics observed data must be tested in order to accept or reject the hypothetical distribution. Note that the ‘implied’ technique is equivalent to accepting the statistical hypothesis H_0 without making even attempt to test it.

The second remark we want to highlight here is an assumption that hazard rate $\lambda(t)$ is governed by the stochastic differential equation (SDE). It is not common in applications such as for example operating research. Indeed the relationship (1.14) was used as a definition of the hazard rate $\lambda(t)$. The SDE implies other than (1.14) definition of the probability $P(t, T)$. This probability is now a random function and therefore can not be considered as a standard probability. It might be make sense to interpret it as a conditional probability. Then the problem is needed to be more accurate outlined. For instance from the very beginning it should be explicitly formulate parameters of the model that effect probability $P(t)$ (or $Q(t)$). Then it might become possible to introduce conditional probabilities and consider the evidence in a favor of using the special form of stochastic differential equation for hazard rate. Let us recall a O.A.Vasicek result that widely used generalized in the modern theory of the derivatives pricing. It might be used for stochastic interpretation of the hazard rate. Let companies stocks are governed by the system

$$dA_i(t) = r A_i(t) dt + \sigma_i A_i(t) dW_i(t)$$

$i = 1, 2, \dots, n$, and $E W_i(t) W_j(t) = \rho t$ for $i \neq j$. Then Wiener processes $W_i(t)$ admit representation

$$W_i(t) = \sqrt{\rho} x(t) + \sqrt{1 - \rho} \varepsilon_i(t)$$

where $x(t)$ and $\varepsilon_i(t)$, $i = 1, 2, \dots, n$ are independent Wiener processes. Indeed putting

$$x(t) = a \sum_{i=1}^n W_i(t) + bU, \quad \varepsilon_i(t) = \frac{1}{\sqrt{1 - \rho}} (W_i(t) - x(t) \sqrt{\rho})$$

where the random process U is a Wiener process independent upon W_i , $i = 1, 2, \dots, n$ and

$$a = \frac{\sqrt{\rho}}{1 + (n - 1) \rho} \quad \text{and} \quad b = \frac{\sqrt{1 - \rho}}{\sqrt{1 + (n - 1) \rho}}$$

One can easy check that $x(t)$ and $\varepsilon_i(t)$ are independent Wiener processes. In many later publications the Wiener process $x(t)$ has been interpreted as a common risk term

and then one can study the conditional market dynamics with respect to the common risk factor $x(t)$. We can remark here that by construction this common factor is a weighted sum of the market risk factors $W_i(t)$ and an independent factor $U(t)$. From this independence it follows that the factor $U(t)$ does not relate to the market associated with stocks $A_i(t)$, $i = 1, 2, \dots, n$. If other components of the market exist it does not make any effect and we can ignore their existence. Recall that U are usually attempted to explained as a common economic factor though as we can see it does not relate to the market at all. In this case it really independent factor as it was assumed above. Therefore it should be defined explicitly similar to the Wiener process $W_i(t)$ that could be constructed from stock prices. If we use other orthogonalization method of the Wiener system $W_i(t)$ presented in [3e] then there is no common risk factor exist for the system of stocks and it is not clear what type of conditional distribution could lead us to stochastic hazard rate.

Now let us return to the risky bond valuation. There are several types of recovery rates were presented above. In the case when recovery rate is a fixed portion of the face value of the bond it is easy to display statistical characteristics of the risky bond. For example assuming that default time is govern by the Poisson process we see that the first two moments of the risky bond price are equal to

$$E R_1(t, T; \omega) = B(t, T) P(\tau > T) + \Delta E B(t, \tau) \chi(\tau \leq T) =$$

$$= B(t, T) \exp - \int_t^T \lambda(s) ds + \Delta \int_t^T B(t, s) \lambda(s) ds$$

$$E R_1^2(t, T, \omega) = B^2(t, T) \exp - \int_t^T \lambda(s) ds + \Delta^2 \int_t^T B^2(t, s) \lambda(s) ds$$

The second case is in general similar to the presented above. In order to estimate the default time distribution we need a realistic assumption regarding the bond price evolution. The Poisson approximation of the default time implies that the corporate bond spread suggests that the probability of default should increase over time. Note that if the spread volatility does not increase over time the Poisson distribution of the default time might be fail. If recovery rates are chosen in the form of 3) then risk and other statistical characteristics can also be easy calculated.

A risky coupon bond pricing model in continuous time.

Assume that the default can occur at any time within time interval $(t, T]$. Denote t_i ,

$i = 0, 1, \dots, N$ the dates of the coupon payments. As it was emphasize the long and short positions are not perfectly symmetric. Assume that the bond issuer pays coupon $\$c$ to the bond buyer at the dates t_i . Then the bond seller remains a holder of the $\$c$ during the period $[0, t_i)$. On the other hand bond buyer would own this amount starting from the date t_i until maturity or default which one comes first. The present value at the date t of the cash flow that represents seller's commitment can be written in the form

$$R_s^{(c)}(t, T; \omega) = \sum_{i=1}^N \chi(t_{i-1} \leq \tau < t_i) \left\{ \sum_{j=1}^{i-1} cB(t, t_j) + B(t, \tau) \Delta B^{(c)}(\tau, T) \right\} + \\ + \chi(\tau = T) \left[\sum_{j=1}^{N-1} cB(t, t_j) + B(t, T) \Delta(F + c) \right] + \quad (1.15)$$

$$+ \chi(\tau > T) \left[\sum_{i=1}^N cB(t, t_j) + B(t, T)F \right] = \Delta B_s^{(c)}(t, \tau) \chi(\tau \leq T) + B_s^{(c)}(t, T) \chi(\tau > T)$$

Note that the pricing in the formula (1.15) depends on Treasury 0-default coupon bond and the random default time. The buyer price at the date t is also a present value of the accumulated cash flow taking at $\tau \wedge T = \min(T, \tau)$. The periodic interest payments at the moments of its delivery are invested immediately at risk free 0-coupon bond with the same maturity at T . Then at the $\tau \wedge T$ the cumulative sum should be discounted by the risk-free bond over the interval $[t, \tau \wedge T]$. Hence

$$R_b^{(c)}(t, T; \omega) = \sum_{i=1}^N \chi(t_{i-1} \leq \tau < t_i) B(t, \tau) \left\{ \left[\sum_{j=1}^{i-1} cB^{-1}(t_j, \tau) \right] + \right. \\ \left. + \Delta B^{(c)}(\tau, T) \right\} + \chi(\tau = T) B(t, T) \left[\sum_{j=1}^{N-1} cB^{-1}(t_j, T) + \Delta(F + c) \right] + \\ + \chi(\tau > T) B(t, T) \left[\sum_{i=1}^N cB^{-1}(t_j, \tau) + F \right]$$

This formula can be rewritten in a more compact form

$$R_b^{(c)}(t, T; \omega) = \Delta B_b^{(c)}(t, \tau) \chi(\tau \leq T) + B_b^{(c)}(t, T) \chi(\tau > T) \quad (1.16)$$

Using (1.15), (1.16) the risky bond can be evaluated for different hypothetical default time and the bond forward price distributions.

The Reliability Theory uses several popular stochastic models that could be adopted for default events. The most popular model of the default time is the exponential model we introduced above. Bearing in mind stochastic interpretation of the event of default we can note that exponential distribution is reasonable when default occurred ‘unexpectedly’. If a credit event results in ‘gradually’ internal deteriorated changes of the debt-equity structure of a firm then it might make sense to use a normal approximation of the default time. Denote

$\theta = E \tau$ and $\sigma = D \tau$. Then we see that conditional probability of the ‘no default prior T’ event given that there is no default until t can be represented in the form

$$\begin{aligned} P(t, T) &= P\{\tau(\omega) > T \mid \tau(\omega) > t\} = 1 - Q(t, T) = \\ &= \frac{1}{\sqrt{2\pi}} \left[1 - \Phi\left(\frac{T-\theta}{\sigma}\right) \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[1 - \Phi\left(\frac{t-\theta}{\sigma}\right) \right] \end{aligned}$$

where Φ is the Gaussian cumulative distribution function.

There are other types of distributions that might be applied in capacity of an approximation of the default time are Gamma and log-normal distributions. Recall that a particular choice of the distribution parameters of the Gamma distribution reduces it to the exponential or chi-squared distributions. It is also a possibility to use approximation of the default time by the Weibull distribution which density is

$$f(x) = \begin{cases} \lambda \alpha x^{\alpha-1} \exp(-\lambda x^\alpha), & \text{when } x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

The mean and variance of the Weibull distribution can be expressed by the simple formulas using the Gamma function.

Floating rate risky bond valuation.

Let us first consider a risk free floating rate bond contract. Beside the popularity of this type of contracts it also can be applied to valuation of the interest rate swap. The valuation method introduced below is based on the scheme presented in [3f].

Let $t = t_0 < t_1 < \dots < t_N = T$ be interest rate reset dates and assume that value $\Delta = t_{j-1} - t_j$ does not depend on j . Let $i(t_j, t_j + \delta)$ be the a floating rate at t_j which is applied for floating leg transaction at the time t_{j+1} . For writing simplicity assume that notional principal of the bond is \$1. Otherwise all transactions should be proportionally changed. The floating interest rates that would be applied for payments from buyer to seller are represented in the table

Dates	t_0	t_1	t_2	...	$t_N = T$
Floating flow	-1	$i(t_0, t_0 + \delta)$	$i(t_1, t_1 + \delta)$...	$1 + i(t_{N-1}, t_{N-1} + \delta)$

From the table one can see that one-dollar at date t_{N-1} is equal to

$$\$1(t_{N-1}) = \$1(T) [1 + i(t_{N-1}, T)]$$

Hence in particular

$$\$1(t_{N-1}) i(t_{N-2}, t_{N-1}) + \$1(T) [1 + i(t_{N-1}, T)] = \$1(t_{N-1}) [1 + i(t_{N-2}, t_{N-1})]$$

Therefore cumulative cash flow to the bond buyer over $[t, T]$ is calculated backward in time from T to t yields

$$\begin{aligned} & \$1(t_1) i(t_0, t_1) + \$1(t_2) i(t_1, t_2) + \dots + \$1(T) [1 + i(t_{N-1}, T)] = \\ & = \$1(t_1) i(t_0, t_1) + \$1(t_2) i(t_1, t_2) + \dots + \$1(t_{N-1}) [1 + i(t_{N-2}, t_{N-1})] = \dots \\ & \dots = \$1(t_1) [1 + i(t_0, t_1)] = \$1(t) \end{aligned}$$

These calculations prove that \$1 at date t is the price of the floating rate bond. Thus the bond buyer paying \$1 at t will receive equivalent cash payments over the period $(t, T]$. The variability of the interest rate does not effect on valuation. The issuer of the floating bond will receive equivalent cash flow with opposite sign. That is

Dates	t_0	t_1	t_2	...	$t_N = T$
Floating flow	1	$-i(t_0, t_0 + \delta)$	$-i(t_1, t_1 + \delta)$...	$- [1 + i(t_{N-1}, t_{N-1} + \delta)]$

This floating rate bond valuation uses the present value reduction to justify pricing model. The seller of the floating bond receives \$1 at the date $t_0 = t$. Investing it in exchange of the sequence of $\$i(t_{k-1}, t_k)$ at t_k , $k = 1, 2, \dots, N - 1$ payments and paying

at the bond maturity T the amount of $\$(1 + i(t_{N-1}, t_N))$ would exhausted the up front funding. Note that this construction actually does not depend on N and value δ .

On the other hand buyer estimates the future value of the contract using formula

$$Fl(T) = \sum_{j=1}^N i(t_{j-1}, t_j) B^{-1}(t_j, T) + 1$$

This formula presents the date- T value of the floating bond payments and the Δ -compounding interest rate formula should be applied for the bond value at T . To avoid arbitrage one might expect that the floating bond and the 0-coupon bond issued by the Government should provide the same rate of return. That implies in particular that

$$Fl(T) / Fl(t) = 1 / B(t, T)$$

The solution of this equation is

$$Fl(t) = B(t, T) \left[\sum_{j=1}^N i(t_{j-1}, t_j) B^{-1}(t_j, T) + 1 \right]$$

As far as the rates $i(t_{j-1}, t_j)$, $j > 1$ are unknown at the date $t = t_0$ it could be interpreted as a sequence of random variables. Let

$$FB(s, [t, T]; \lambda) = \sum_{j=1}^N i(t_{j-1}, t_j) \chi\{s = t_j\} + 1 \chi\{s = T\}$$

$\lambda = \{t_j; j = 1, 2, \dots, N\}$ denote the cash flow generated by the sequence of payments $i(t_{j-1}, t_j)$ paid at t_j , $j = 1, 2, \dots, N-2$ and $1 + i(t_{N-1}, t_N)$ paid at T . Then by definition the present value at t of the cash flow $PV\{FB(s, [t, T]; \lambda)\}$ is equal to $\$1(t)$ for any values λ , t and T .

Now let us take a look at a risky floating bond contract. A seller of the risky floating bond pays λ -reset floating interest rate payments until default or maturity which one comes first. At the default date that by an assumption can occur only at a reset date the bond seller would pay to the bond holder a specified ratio $0 \leq \Delta < 1$. In return the bondholder pays $\$s$ at initiation of the contract. The pricing problem is to derive the value of 's' given the nonrandom recovery rate Δ and the distribution of the default time. The bond buyer pays up front $\$s$ and receives from bond seller the cash flow until default or maturity which one comes first. Thus

$$s_s = \sum_{j=1}^N \chi(\tau = t_j) \left\{ \left[\sum_{i=1}^{j-1} B(t, t_i) i(t_{i-1}, t_i) \right] + B(t, t_j) \Delta \right\}$$

This is the price that we used to call “seller” price. As far as values $i(t_{j-1}, t_j)$, $j > 1$ are unknown at t then any market price $\langle s \rangle$ implies the risk associated with the default time distribution and deviation real $i(t_{j-1}, t_j)$ from the model estimates of the corresponding values. On the other hand the “buyer” price is

$$s_b = \sum_{j=1}^N \chi(\tau = t_j) B(t, t_j) \left\{ \left[\sum_{i=1}^{j-1} B^{-1}(t_i, t_j) i(t_i, t_i) \right] + \Delta \right\}$$

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