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Project selection and equivalent CAPM-based investment criteria

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Abstract. This paper shows that the CAPM-based capital budgeting criteria proposed by Tuttle and Litzenberger (1968), Mossin (1969), Hamada (1969), Stapleton (1971), Rubinstein (1973), Bierman and Hass (1973), Bogue and Roll (1974) are equivalent: They all state that a project is profitable if its internal rate of return is greater than the risk-adjusted cost of capital, where the latter is given by the sum of the risk-free rate and a risk-premium which is a function of the systematic risk of the project, itself a function of the project cost.

Keywords and phrases. Capital budgeting, investment decisions, capital asset pricing model, equivalence.

Introduction

This paper deals with a world where a security market exists satisfying the assumptions of the Capital Asset Pricing Model (CAPM) and where a firm has the opportunity of undertaking a one-period project (say project *Z*). The question is: Under what conditions the project is profitable? In the literature we have several seemingly different answers. Using simple algebraic manipulations this paper shows that the answers provided by Tuttle and Litzenberger (1968), Mossin (1969), Hamada (1969), Stapleton (1971), Rubinstein (1973), Bierman and Hass (1973), Bogue and Roll (1974) are all equivalent.[1] To this end, Rubinstein's criterion is introduced first, and then the equivalence of each criterion with Rubinstein's is shown. (Notational conventions are collected at the end of the paper. Expectation will be indicated by an overbar.)

1. Rubinstein

Let us suppose a project is available to a firm and decision must be taken about undertaking it or not. Rubinstein (1973, pp. 171-172 and footnote 10) proves that if the security market is in equilibrium (so that the fundamental CAPM equation holds), and if shareholders' wealth is to be maximized, then the project is worth undertaking if

$$\bar{r}_Z > r_f + \lambda \frac{\operatorname{cov}(F_Z, r_m)}{I} \qquad \text{(1a)}$$

where $\bar{r}_Z := \frac{\bar{F}_Z - I}{I}$ and $\lambda := \frac{\bar{r}_m - r_f}{\sigma_m^2}$. The right-hand side of (1a) is the *risk-adjusted cost of capital*, and the covariance term (which is a function of the project cost I)[2] is the *systematic risk*. Criterion (1a) is equivalent to

$$\mathrm{NPV}_Z = -I + \frac{\bar{F}_Z}{1 + r_f + \lambda \frac{\operatorname{cov}(F_Z, r_m)}{I}} > 0, \qquad \text{(1b)}$$

which means that the project is worth undertaking if its NPV is positive. Equivalently, we may say that the project is worth undertaking if the project value $V := \frac{\bar{F}_Z}{1 + r_f + \lambda \frac{\operatorname{cov}(F_Z, r_m)}{I}}$ is greater than the cost.

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[1] For an alternative proof, see Senbet and Thompson (1978). The authors provide the proof of equivalence of the criteria proposed by Hamada (1969), Rubinstein (1973), Bierman and Hass (1973), Bogue and Roll (1974). They do not deal with Tuttle and Litzenberger's criterion nor with Mossin's (they only devote to the latter some verbal consideration at p. 399 of their paper).

[2] The fact that the covariance term is a *disequilibrium* covariance (depending on cost, not on project's equilibrium value) bears striking relations to the principle of arbitrage (see Magni, 2006).

2. Mossin

Mossin (1969, p. 755, left column) shows that, assuming the market is in equilibrium, an investment Z will be undertaken by a firm l if and only if

$$\frac{1}{1+r_f}(\overline{F}_Z - \mathrm{R}\,\mathrm{cov}(F_Z, V_1^m)) > I \tag{2}$$

where $\mathrm{R} := \dfrac{\overline{F}_l - (1+r_f)V_l}{\mathrm{cov}(F_l, V_1^m)}$ with $\overline{F}_l$:=free cash flow of firm l. We have

The term $\dfrac{\overline{r}_l - r_f}{\mathrm{cov}(r_l, r_m)}$ "is the same for all companies" (Mossin, 1969, p. 755, right column), so that $\dfrac{\overline{r}_l - r_f}{\mathrm{cov}(r_l, r_m)} = \dfrac{\overline{r}_m - r_f}{\sigma_m^2}$. As a result, eq. (2) becomes

$$\overline{r}_Z > r_f + \lambda\,\mathrm{cov}(\frac{F_Z}{I}, r_m)$$

which is Rubinstein's criterion.

3. Hamada

Hamada (1969, section V) shows that the project is profitable for a firm l if and only if

$$\frac{E(\Delta\,\text{Net Earnings})}{\Delta\,\text{Equity}} > r_f + \frac{\lambda}{V_0^m}\left[\frac{\mathrm{cov}(\Delta\,\text{Net Earnings}, V_1^m)}{\Delta\,\text{Equity}}\right] \tag{3}$$

(see also his eq. (24)). Using the fact that the left-hand side of (3) is just the expected rate of return of the project $\overline{r}_Z = \dfrac{\overline{F}_Z - I}{I}$ and that $\Delta\,\text{Equity} = I$, the equivalence of Hamada's criterion and Rubinstein's is easily found:

$$\frac{\overline{F}_Z - I}{I} > r_f + \frac{\lambda}{V_0^m}\left[\frac{\mathrm{cov}(F_Z - I, V_1^m)}{I}\right] \Leftrightarrow \overline{r}_Z > r_f + \lambda\,\mathrm{cov}\left[\frac{F_Z - I}{I}, \frac{V_1^m}{V_0^m}\right]$$

$$\Leftrightarrow \overline{r}_Z > r_f + \lambda\,\mathrm{cov}(\frac{F_Z}{I}, r_m).$$

4. Bierman and Hass

Bierman and Hass (1973) show that the shareholders' wealth increases if and only if

$$\bar{k}_I > \lambda^* \operatorname{cov}(k_I, k_m) \tag{4}$$

where

$$k_I := \frac{F_Z}{(1+r_f)I} - 1$$

$$k_m := \frac{V_1^m}{(1+r_f)V_0^m} - 1$$

$$\lambda^* := \frac{\bar{k}_m}{\sigma^2\left(\frac{V_1^m}{(1+r_f)V_0^m} - 1\right)}$$

(see their eq. (10) and their proof in section II). Using the above definitions and substituting in (4), we have

$$\frac{\bar{F}_Z}{(1+r_f)I} - 1 > \frac{\frac{\bar{V}_1^m}{(1+r_f)V_0^m} - 1}{\sigma^2\left(\frac{V_1^m}{(1+r_f)V_0^m} - 1\right)} \operatorname{cov}\left(\frac{F_Z}{(1+r_f)I} - 1, \frac{V_1^m}{(1+r_f)V_0^m} - 1\right)$$

$$\Leftrightarrow \frac{\bar{F}_Z}{(1+r_f)I} - 1 > \left(\frac{\bar{V}_1^m}{(1+r_f)V_0^m} - 1\right) \operatorname{cov}\left(\frac{F_Z}{(1+r_f)I}, \frac{V_1^m}{(1+r_f)V_0^m}\right) \frac{1}{\sigma^2\left(\frac{V_1^m}{(1+r_f)V_0^m} - 1\right)}$$

$$\Leftrightarrow \frac{\bar{F}_Z}{(1+r_f)I} - 1 > \left(\frac{\bar{V}_1^m}{(1+r_f)V_0^m} - 1\right) \frac{\operatorname{cov}(r_Z, r_m)}{(1+r_f)^2} \frac{(1+r_f)^2}{\sigma_m^2}$$

$$\Leftrightarrow \frac{\bar{F}_Z}{(1+r_f)I} - 1 > \left(\frac{\bar{V}_1^m - (1+r_f)V_0^m}{(1+r_f)V_0^m}\right) \frac{\operatorname{cov}(r_Z, r_m)}{\sigma_m^2}$$

$$\Leftrightarrow \bar{r}_Z - r_f > (r_m - r_f) \frac{\operatorname{cov}(r_Z, r_m)}{\sigma_m^2}$$

which is Rubinstein's criterion.

5. Bogue and Roll

Bogue and Roll's (1974, p.606) criterion is

$$\frac{\overline{F}_Z - \lambda^\circ \operatorname{cov}(F_Z, V_1^m)}{1+r_f} > I$$

with $\lambda^\circ := \dfrac{\overline{V}_1^m - (1+r_f)V_0^m}{\sigma^2(V_1^m)}$. Manipulating algebraically we have

$$\frac{\overline{F}_Z - \dfrac{\overline{V}_1^m - (1+r_f)V_0^m}{\sigma^2(V_1^m)} \operatorname{cov}(F_Z, V_1^m)}{1+r_f} > I$$

$$\Leftrightarrow \quad \frac{\overline{F}_Z - \dfrac{\dfrac{\overline{V}_1^m}{V_0^m} - (1+r_f)}{\dfrac{1}{V_0^m}\sigma^2(V_1^m)} V_0^m \operatorname{cov}\left(F_Z, \dfrac{V_1^m}{V_0^m}\right)}{1+r_f} > I$$

$$\Leftrightarrow \quad \frac{\overline{F}_Z - \dfrac{\overline{r}_m - r_f}{\sigma_m^2} \operatorname{cov}\left(F_Z, r_m\right)}{1+r_f} > I$$

$$\Leftrightarrow \quad \frac{\overline{F}_Z - I\dfrac{\overline{r}_m - r_f}{\sigma_m^2} \operatorname{cov}\left(r_Z, r_m\right)}{1+r_f} > I$$

$$\Leftrightarrow \quad \frac{\overline{F}_Z}{I} - \frac{\overline{r}_m - r_f}{\sigma_m^2} \operatorname{cov}\left(r_Z, r_m\right) > 1 + r_f$$

$$\Leftrightarrow \quad \overline{r}_Z - \lambda \operatorname{cov}\left(r_Z, r_m\right) > r_f$$

which is again the standard criterion described in eq. (1).

6. Tuttle and Litzenberger

Tuttle and Litzenberger's (1968) cost of capital is given by

$$\text{cost of capital}= r_f + b^* \rho(Z,m)\sigma_Z$$

where $b^* := \dfrac{\bar{r}_l - r_f}{\rho(i,m)\sigma_l}$ (see their eq. (18)). But "in order for the equilibrium to occur, b^* must be the same for all firms" (Tuttle and Litzenberger, 1968, p. 437). This implies $b^* = \dfrac{\bar{r}_m - r_f}{\sigma_m}$ so that

$$\text{cost of capital}= r_f + \frac{\bar{r}_m - r_f}{\sigma_m}\rho(Z,m)\sigma_Z = r_f + \frac{\bar{r}_m - r_f}{\sigma_m^2}\operatorname{cov}(r_Z, r_m)$$

which is Rubinstein's cost of capital in eq. (1).

7. Stapleton

Stapleton (1971) introduces a criterion which he himself recognizes as equivalent to the criterion provided by Bierman and Hass (1973). He introduces the random variable $k'_I = E(k_I / k_m)$, and after noticing that Bierman and Hass's condition may be written as

$$\bar{k}_I > \rho(k_I, k_m)\sigma_I \frac{\bar{k}_m}{\sigma_m},$$

he claims that his criterion is "simply a short cut to calculating the amount of non-diversifiable risk … It is shown in my paper that the absolute value of this amount is simply the standard deviation of the conditional expected value variable k'_I defined above" (Stapleton, 1974, p. 1584). In other terms, his criterion is just Bierman and Hass's criterion, where $\rho(k_I, k_m)\sigma_I$ is replaced by $\sigma_{k'_I}$.[3] As a result, his criterion is equivalent to Rubinstein's criterion (and therefore to all the other ones).

[3] This replacement is legitimate as far as Stapleton assumes that k'_I is perfectly linearly correlated with k_m (see Bierman and Hass, 1974).

References

Bierman, H. and Hass, J. E. (1973), Capital budgeting under uncertainty: a reformulation. *Journal of Finance*, 28(1), 119–129.

Bierman, H. and Hass, J. E. (1974), Reply, *Journal of Finance*, 29(5), 1585.

Bogue, M. C. and Roll, R. (1974), Capital budgeting of risky projects with 'imperfect markets' for physical capital, *Journal of Finance*, 29(2), 601–613, May.

Hamada, R. S. (1969), Portfolio analysis, market equilibrium and corporation finance, *Journal of Finance*, 24(1), 13–31, March.

Litzenberger, R. H. and Budd, A. P. (1970), Corporate investment criteria and the valuation of risk assets, *Journal of Financial and Quantitative Analysis*, 5(4), 395–418, December.

Magni, C. A. (2006). Project valuation and investment decisions: CAPM versus arbitrage. *Applied Financial Economics Letters*, 3, 1–4.

Mossin, J. (1969), Security pricing and investment criteria in competitive markets, *American Economic Review*, 59(5), 749–756, December.

Rubinstein, M. (1973), A mean-variance synthesis of corporate financial theory, *Journal of Finance*, 28, 167–182, March.

Senbet, L. W. and Thompson, H. E. (1978), The equivalence of mean-variance capital budgeting models, *Journal of Finance*, 23(29), 395–401, May.

Stapleton, R. C. (1971), Portfolio analysis, stock valuation and capital budgeting decision rule for risky projects, *Journal of Finance*, 26(1), 95–117, March.

Stapleton, R. C. (1974), Capital Budgeting under Uncertainty: A Reformation: Comment, *Journal of Finance*, 29(5), 1583–1584.

Tuttle, D. L. and Litzenberger, R. H. (1968), Leverage, diversification and capital market effects on a risk-adjusted capital budgeting framework, *Journal of Finance,* 23(3), 427–443.

Notational conventions used in the paper

F_Z = End-of-period cash flow of project Z	I = Cost of project Z
$\overline{F}_l$ = Free cash flow of firm l	r_Z = Rate of return of project Z
r_l = rate of return of firm l	r_m = Market rate of return
r_f = Risk-free rate of return	V_0^m =Market value at time 0
V_1^m = End-of-period market value	V_l = value of firm l
σ_X =Standard deviation of X	$\sigma^2(X)$ = Variance of X
$\sigma_m^2 = \sigma^2(r_m)$	$\mathrm{cov}(X,Y)$ = Covariance of X and Y
$\rho(X,Y)$ =Correlation between X and Y	NPV_Z = NPV of project Z

X and Y are generic random variables. The Δ *symbol indicates variation, and a bar over a symbol means that expectation is taken.*