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Lecture Notes on The Optimal Growth Problem

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LECTURE NOTES ON ECONOMIC GROWTH

The Optimal Growth Problem

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These notes provide an introduction to the study of optimal growth in the one-sector neoclassical growth model in continuous time. The model is developed using the analogy of Robinson Crusoe living on a deserted island. Both the Hamiltonian method and the phase diagram are presented and explained on an intuitive level. Some familiarity with optimization theory and differential equations, as well as a thorough understanding of intermediate-level microeconomics, is assumed.

Rough draft. Comments and corrections would be would be greatly appreciated. These notes may be distributed freely, provided that they are distributed in their entirety.

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1 A “Robinson Crusoe” Model

We are going to start with what is in many ways a rather simple model. It is going to look nothing like a model of a modern economy, for instance. Nevertheless, we are going to work through this model completely, going over all the details of how to solve it. Some of this may already be familiar to you, but I want you to go over it carefully because it is the foundation of everything we will do in this course. In addition, we will see later that, by reinterpreting the variables in the model and making a few minor changes, we *can* interpret the model as a representation of a modern, market-based economy.

I am going to set up the model by telling a story about a family of people who live alone in a very primitive environment. The goal of this analogy is to help you see very clearly what the variables and the equations in the model mean, and to gain solid intuition for the solution. Once we understand the model very well, we will see that different interpretations can be given to these same equations. For now, we will talk about a person named Robinson Crusoe,¹ who has been shipwrecked with his family on a small, deserted island. The family will live in isolation on this island forever.

- Crusoe and his descendants (the “Crusoe household” or “dynasty”) live alone on an island, and will stay there forever
- Time is continuous, indexed by $t \in [0, \infty)$

Think of the variable t as representing the date on a calendar. We will label the day that the family arrives on the island as “date 0.” Measuring time as a continuous variable [using $t \in [0, \infty)$] rather than a discrete variable [using $t = 0, 1, 2, \dots$] is not very important. Writing the model in discrete time leads to nearly identical results (but is somewhat messier and less elegant).

We are going to assume that the size of the Crusoe household grows over time, and that it grows at a constant rate. To keep the story simple, we will often imagine that Crusoe arrives on the island alone, so that the initial size of the household is one person. You might wonder how he could have descendants in this case, but biology is not our concern here. This is a stylized representation of how a group of people grows over time.

- Let N_0 = number of people who arrive on the island (can imagine $N_0 = 1$, ignoring biological concerns)
- Let $N(t)$ = number of people in the household at time t
 - Assume N grows at rate n , so that

$$\frac{\dot{N}(t)}{N(t)} = n$$

¹ Robinson Crusoe was the main character in a novel by Daniel Defoe, which was published in 1719. This book is considered by many people to be the first true novel published in the English language. In the novel, Crusoe is shipwrecked on an (almost) deserted island and lives there alone for 27 years. Our story will differ from the details in the novel, of course. The novel is available online at <http://www.bibliomania.com/0/0/17/31/frameset.html>.

or

$$N(t) = N_0 e^{nt}$$

A comment about notation: the variable $N(t)$ is a single number; it denotes the number of people who are alive on the island at time t . If we write N by itself, however, it represents an entire function

$$N : [0, \infty) \rightarrow \mathbb{R}_+,$$

that gives the number of people on the island at every point in time. We should always be careful to distinguish between the number $N(t)$ and the function N . This same notational convention will apply to all of the variables in our model.

1.1 Technology

There is only one source of food on the island: coconuts. These coconuts grow on trees, and when Crusoe arrives on the island, there are already some coconut trees there.

- The only source of food is coconuts
- When Crusoe arrives on the island, there are some coconut trees there
- A coconut can either be consumed or planted. If a coconut is planted “today”, it becomes a new tree “tomorrow”

For intuition purposes, I will often talk about “today” and “tomorrow” as discrete events. This is because the fundamental tradeoff in the model is between the present (consuming coconuts today) and the future (planting trees that will yield coconuts at later dates). Because time is a continuous variable, however, this use of the word “tomorrow” is not exactly correct. In the model, a coconut that is planted instantly becomes a new (fully-grown) tree. Again, our goal here is not to deal with biology (of people or of plants), but to set up an analogy that will help us understand and interpret the mathematics of the model.

- Let $K(t)$ = number of coconut trees on the island at time t
- Let K_0 denote the number of trees when Crusoe arrives on the island

Throughout the course, it will be important to distinguish an upper-case letter $K(t)$ from the lower-case $k(t)$. The number of trees on the island at time t is represented by an upper-case letter $K(t)$. As with N , the letter K by itself will represent the function that tells us an entire path of the variable $K(t)$ over time.

The Crusoe household must harvest the coconuts from the trees before they can be either eaten or planted. The number of coconuts that are harvested depends on both how many trees there are on the island and how many people there are in the household to do the harvesting.

- The number of coconuts harvested when there are $K(t)$ trees is

$$Y(t) = F(K(t), N(t))$$

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We will call F the “harvest function” or the “production function.” We are assuming that all people work a fixed number of hours per day; otherwise the number of hours worked would also be an argument in the harvest function. We assume this function has certain properties (these should be familiar from a microeconomics course).

- Assumptions on harvesting technology

(1) Constant Returns to Scale (homogeneous of degree one):

$$F(\lambda K(t), \lambda N(t)) = \lambda F(K(t), N(t)) \text{ for any } \lambda > 0$$

(2) Positive but diminishing marginal products:

$$\begin{aligned} \frac{\partial F}{\partial K} &> 0 & \frac{\partial^2 F}{\partial K^2} &< 0 \\ \frac{\partial F}{\partial N} &> 0 & \frac{\partial^2 F}{\partial N^2} &< 0 \end{aligned} \quad \text{for all } (K(t), N(t))$$

(3) Boundary (Inada) Conditions:

$$\begin{aligned} \lim_{K(t) \rightarrow 0} \frac{\partial F}{\partial K} &= \infty = \lim_{N(t) \rightarrow 0} \frac{\partial F}{\partial N} \\ \lim_{K(t) \rightarrow \infty} \frac{\partial F}{\partial K} &= 0 = \lim_{N(t) \rightarrow \infty} \frac{\partial F}{\partial N} \end{aligned}$$

Condition (1) comes from a replication argument: If there were twice as many trees and twice as many people collecting coconuts, it seems natural to think that twice as many coconuts would be collected. Condition (2) says (a) having either more trees or more people will always lead to a larger harvest and (b) there are diminishing returns to the harvesting process. If you think about a group of people collecting coconuts, these assumptions should seem very reasonable. Condition (3) is technical in nature. As we will see later on, it guarantees that certain optimization problems always have an interior solution.

To keep the model simple, we assume that all coconut trees are equally productive while they are alive. However, at every point in time, some of the trees suddenly die. This implies that if Crusoe and his family do not plant at least some trees, eventually the population of trees would completely die out.

- In each period of time, a fraction δ of the existing trees dies

The interesting part of this story is how the decision Crusoe makes today – how many coconuts to eat and how many to plant – affects the number of trees on the island and hence the number of coconuts available to the family in the future. We already have all of the information about this tradeoff; now I want to write it down in a compact way.

- Let $C(t)$ = number of coconuts consumed at time t

- Then we have

$$\underbrace{Y(t) - C(t)}_1 = \underbrace{\delta K(t)}_2 + \underbrace{\frac{dK(t)}{dt}}_3$$

where

- 1 = number of coconuts planted
- 2 = number of existing trees that die
- 3 = change in number of trees

The left-hand side of this equation is the number of coconuts that are harvested but not consumed; this must be equal to the number of coconuts planted. The right-hand side says that each planted coconut either replaces a tree that just died or becomes an increase in the total stock of trees. If the number of coconuts planted is larger than the number of trees that died, the last term is positive and the stock of trees will be growing over time. However, if the number of coconuts planted is smaller than the number of trees that died, the last term is negative and the stock of trees is becoming smaller over time.

- Using $\dot{K}(t)$ to represent the derivative $dK(t)/dt$, we have

$$\dot{K}(t) = F(K(t), N(t)) - C(t) - \delta K(t) \quad (\text{RC})$$

This equation tells us how the stock of trees on the island evolves over time, given the consumption decisions of Crusoe and his descendants at each point in time. We will refer to it as the “resource constraint”, or (RC).

Let’s take a step back for a minute. Many of you are probably looking at this equation and seeing something familiar. $K(t)$ stands for capital, $N(t)$ for labor, and this is a standard capital-accumulation equation. Why am I telling you a silly story about Robinson Crusoe and coconuts?

The reason is that this simple story (hopefully) gives you an easy way to think about the model. “Capital”, “investment”, and “output” are somewhat vague terms: What units are these measured in? What exactly is “aggregate consumption”? We will get to these issues soon enough. In our simple story, however, the units of all of the variables are perfectly clear. We can picture a coconut very well, and we know that when you plant one coconut it will grow into one coconut tree. As we work through the math and think about the intuition, it will be helpful to literally think about a family living on an island full of coconut trees.

1.2 Intensive Form

It will also be helpful to talk about the harvest per person.

- Define

$$y(t) = \frac{Y(t)}{N(t)} \quad \text{harvest per person}$$

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- Similarly

$$c(t) = \frac{C(t)}{N(t)} \quad \text{consumption per person}$$

$$k(t) = \frac{K(t)}{N(t)} \quad \text{trees per person}$$

In this last case, don't think about each individual owning some trees. All of the trees on the island belong to the entire family. We are just looking at the ratio of the total number of trees to the total number of people collecting coconuts from them.

- We know

$$F(\lambda K(t), \lambda N(t)) = \lambda F(K(t), N(t)) \quad \text{for any } \lambda > 0$$

- Let $\lambda = \frac{1}{N(t)}$. Then

$$F(k(t), 1) = \underbrace{\frac{F(K(t), N(t))}{N(t)}}_{\text{harvest/person}} = y(t)$$

This equation tells us something of fundamental importance: the variable $y(t)$ depends only on the variable $k(t)$, and not on the variables $K(t)$ and $N(t)$ independently. In other words:

- The harvest per worker depends only on the number of trees per worker.

I want to give this property a name so that we can refer to it later.

- Scale independence: When large and small economies are equally productive in per-capita terms

Suppose we look at two separate island economies that are identical except for size: one island has twice as many trees and twice as many people on it. If scale independence holds, the harvest will be exactly twice as large on the larger island, so that the harvest per person is the same on both islands. What we see here is that our Robinson Crusoe economy has the scale independence property. This property will hold in some, but not all, of the models we see later in the course.

- Define the function

$$f(k(t)) \equiv F(k(t), 1)$$

- This is the intensive harvest function.

The word 'intensive' means 'per capita' or 'per person'. We could also call f the 'per capita harvest function', but the term 'intensive' is more commonly used.

It is important to keep in mind that the function F is something primitive – the relationship between inputs and outputs in the harvesting process. It has two arguments: input of trees and input of people. The function f , on the other hand, is just short-hand notation. Whenever we have arranged things so that the level of the second input in F is equal to 1 (which will be very often) we will use f just to avoid having to write “1” all of the time.

Using the properties of the total harvest function F given above, we can derive the properties of the intensive harvest function f . I will leave it to you to verify the following

- Verify: The function f is continuous, strictly increasing, and strictly concave with

$$\lim_{k(t) \rightarrow 0} f'(k(t)) = \infty \quad \text{and} \quad \lim_{k(t) \rightarrow \infty} f'(k(t)) = 0$$

The scale independence property tells us that if we are interested in how many coconuts are collected per person, we need to focus on the number of trees per person (rather than on the total number of trees). Therefore, we would like to have some information about how the number of trees per person evolves over time.

- Want: an equation for $\dot{k}(t)$

How can we get this information? Consider the following approach.

- Differentiate the definition of $k(t)$ with respect to t

$$\begin{aligned} \dot{k}(t) &= \frac{d}{dt} \left(\frac{K(t)}{N(t)} \right) = \frac{\dot{K}(t) N(t) - K(t) \dot{N}(t)}{N(t)^2} \\ &= \frac{\dot{K}(t)}{N(t)} - nk(t) \end{aligned}$$

- Substitute our equation for $\dot{K}(t)$ into this expression

$$\begin{aligned} \dot{k}(t) &= \frac{F(K(t), N(t)) - C(t) - \delta K(t)}{N(t)} - nk(t) \\ &= \frac{F(K(t), N(t))}{N(t)} - c(t) - (\delta + n)k(t) \end{aligned}$$

or

$$\boxed{\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t)} \quad (\text{rc})$$

This equation is the intensive form of the resource constraint (RC) above. It tells us how the stock of trees per person evolves over time, depending on how many coconuts are consumed per person at each point in time.

Notice that n enters this equation in exactly the same way as δ . Why? Recall that the variable we are working with is $k(t) = K(t)/N(t)$, the ratio of trees to people. The parameter δ measures the decrease in the numerator of this ratio due to the deaths of trees. The parameter n measure the increase in the denominator of this ratio due to births in the family. Increasing the denominator

has the same effect on a ratio as does decreasing the numerator (that is, they both make the ratio smaller). For this reason, δ and n enter the equation that governs the evolution of trees per person in the same way.

- Note: n (increases in the number of people) is “like” δ (decreases in number of trees).
- $(\delta + n)$ is the “effective depreciation rate” of the trees-per-person ratio

These steps we are going through now are very important. We are going to repeat them (in different contexts, with different variables) many, many times during the course. If you have any doubts about what we are doing, you should try to clear them up now.

1.3 Preferences

Figuring out how many coconuts the Crusoe household should consume and how many they should plant at every point in time requires us to know their preferences over different consumption streams. In a micro class, the way you typically construct preferences is by looking at different pairs of consumption bundles and asking the consumer which one she prefers. This allows you to draw indifference curves and to find a utility function to represent the preferences.

Here we will do the same thing. The first question is: What is a “consumption bundle” in our setting? There is only one good (coconuts). We will always assume that people prefer eating more coconuts to eating fewer. What else do we need to know?

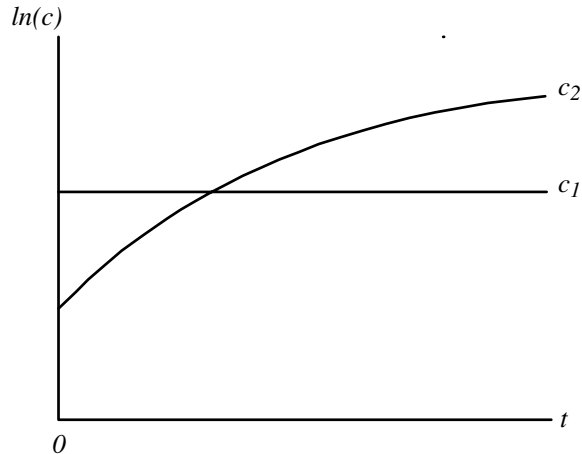
We need to know how the household feels about present consumption versus future consumption. The standard trick for dealing with dynamic models is to think of a good at different points in time as different commodities. A coconut at $t = 0$ is one commodity, a coconut at $t = 0.5$ is a different commodity, and so on. In other words, there are infinitely many commodities in our model, and a consumption bundle is a function that specifies a quantity of each of these commodities. We need to know the household’s preferences over these consumption bundles.

We are going to make an important assumption here: everyone within the household agrees about what is best for the whole group. That is, this is not a group of greedy people, each of whom is trying to get more for him or herself. Rather, this is a big, happy family where people like to share things equally. In other words, I want you to literally think of this household as a family and not, say, as the population of an entire country.

- Assume: consumption is shared equally
 - each person consumes $c(t) = \frac{C(t)}{N(t)}$ at time t

What the members of the household care about, therefore, is the variable $c(t)$ and how it behaves over time. In other words, they care about the function $c : [0, \infty) \rightarrow \mathbb{R}_+$.

- Consider



- c_1 and c_2 are consumption plans
- we need to know which one the family prefers

Think again about the exercise you would do in a micro class; you would ask a person if she prefers to consume one orange and two apples or two oranges and one apple. The person would tell you which bundle she prefers, and this answer gives you information about her preferences. The same reasoning applies here. We are going to ask Crusoe if he would prefer the consumption plan c_1 or the plan c_2 for his family. In fact, we want to imagine asking this question about all possible pairs of all conceivable consumption plans. These answers would give us Crusoe's complete preferences, from which we could construct his utility function. More precisely, if we let \mathbb{C} denote the set of all possible consumption plans,

- we want a functional

$$U : \mathbb{C} \rightarrow \mathbb{R}$$

U is like a standard utility function; it takes each possible consumption “bundle” and assigns a number to it. The only difference from the standard micro situation is that a consumption bundle is now a function c (a level of consumption at every point in time) instead of a vector. Because U operates on functions (instead of vectors) is called a “functional” (instead of a “function”). We will put a lot of structure on this U functional by constructing it in the following way.

- Let $u[c(t)] =$ utility one person gets from consuming $c(t)$ coconuts
- assume

$$u'(c(t)) > 0, \quad u''(c(t)) < 0 \quad \text{for all } c(t), \text{ and}$$

$$\lim_{c(t) \rightarrow 0} u'(c(t)) = \infty, \quad \lim_{c(t) \rightarrow \infty} u'(c(t)) = 0$$

The first two assumptions are standard: consuming more is better, but there is diminishing marginal utility of consumption. The last two are technical conditions to ensure that the solution to the utility maximization problem is always interior.

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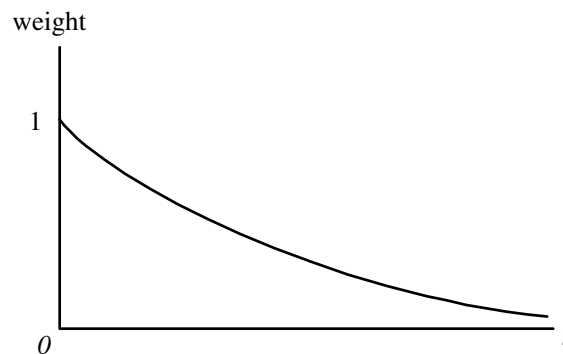
The important thing is how the Crusoe household compares, or “adds up”, these utilities across time. That is, how much weight is given to consumption today and how much weight is given to consumption in the future?

Imagine that Crusoe has just arrived on the island and collected his first batch of coconuts. He is thinking about how many of these coconuts should be planted and how many should be consumed. He does some calculations and figures out that the plans c_1 and c_2 above are both feasible. He needs to decide whether he wants to consume more today (and therefore plant fewer trees and consume less in the future – plan c_1) or consume less today (and therefore plant more trees and consume more in the future – plan c_2).

We will assume that there are two effects here. First, he is impatient. Crusoe (and all of his descendants) care more about the present than about the future. We will assume that this impatience takes a particular form:

(1) Exponential discounting (impatience)

- utility at time t is weighted by $e^{-\rho t}$, where $\rho > 0$ is the discount rate



Second, Crusoe realizes that if $n > 0$, future generations will be larger than the present generation, and he gives future generations more weight because of that. In other words, when he looks at $c(0)$, he realizes that this is the consumption of one person – himself. However, when he looks at $c(t)$ in the distant future, he realizes that a lot of people will be receiving that number of coconuts each (remember that $c(t)$ measures consumption per person), and as a result he cares more about the value of $c(t)$.

(2) Size-based weights

- utility at time t is also weighted by the number of people receiving it, $N(t)$
- recall $N(t) = N_0 e^{nt}$

The number $u[c(t)]$ tells us how happy each person in the household will be at time t if a certain consumption plan is followed. When Crusoe is laying out his plan on the first day, he gives this level of happiness a certain weight (or importance) based on (i) how far away it is, and (ii) how many people will receive it.

⇒ Weighted utility at time t is

$$\begin{aligned} & u [c(t)] e^{-\rho t} N_0 e^{nt} \\ = & N_0 u [c(t)] e^{-(\rho-n)t} \end{aligned}$$

- Assume: $\rho > n$

For the objective function to be well-defined, we need for Crusoe to place more weight on current consumption than on future consumption. Future generations automatically get more weight because they will be larger, so we need the discount rate to be large enough to more than offset this effect. One piece of evidence in favor of this assumption is that interest rates in the real world are almost always positive: future consumption is cheaper than current consumption. This seems to imply that decision makers do indeed place more weight on current consumption than on future consumption.

- Total utility of the household is

$$N_0 \int_0^\infty u [c(t)] e^{-(\rho-n)t} dt \equiv U \quad \{\text{just adding up all of the weighted utilities}\}$$

or

$$\boxed{\int_0^\infty u [c(t)] e^{-(\rho-n)t} dt}$$

This is our functional U . Given any possible consumption plan c , we plug it into this expression, do the integration, and we get a number telling us how well the Crusoe household likes that plan.

Keep in mind that we are making a lot of assumptions here. There is no reason that Crusoe's preferences must be of this form. Any functional $U : \mathbb{C} \rightarrow \mathbb{R}$ satisfying certain properties could represent his preferences. However, we will see that this particular function has some nice properties that will be very useful for us. In addition, it seems to be “reasonable”.

1.4 Feasible consumption plans

Now that we know Crusoe's preferences over consumption plans, we need to determine what consumption plans are actually possible, or feasible, for him and his family to follow. The set of feasible plans will obviously depend on the initial stock of trees on the island and on the harvesting technology described above.

- Definition: A consumption plan c is feasible from k_0 if there exists a function k such that

$$\begin{aligned} (i) \quad k(0) &= k_0 \\ (ii) \quad k(t) &\geq 0 \text{ for all } t, \quad \text{and} \\ (iii) \quad \dot{k}(t) &= f(k(t)) - c(t) - (\delta + n)k(t) \text{ for all } t \end{aligned}$$

In other words, imagine trying to follow the plan c , starting at $t = 0$. We know the initial number of trees per person is equal to k_0 . We also know that a fraction δ of the trees will die and that the number of people in the family will increase by a fraction n . Therefore, there will be a “loss” in

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the number of trees per person equal to $(\delta + n)k_0$. The number of new trees planted per person will be $f(k(0)) - c(0)$; this is just the number of coconuts per person that are harvested but not eaten. The resource constraint (iii) then allows us to calculate $\dot{k}(0)$, the net change in the number of trees at $t = 0$, which tells us how the stock of trees is changing at that time.

From this point, we can simply follow the plan forward, using the consumption decision at each point in time to determine how the stock of trees is evolving. If we can follow the plan c and the stock of trees remains positive at all points in time, then c is feasible. However, if following c would lead to a negative number of trees at any point in time, the plan is clearly not feasible. If we imagine doing this exercise for every conceivable plan $c \in \mathbb{C}$, we will find the set of consumption plans that are actually feasible for the Crusoe household to follow.

One might want to include another constraint on the set of feasible consumption plans: the number of coconuts consumed at time t cannot be larger than the harvest, or

$$c(t) \leq f(k(t)) \quad \text{for all } t.$$

If a consumption plan violates this constraint for some t , it would be telling Crusoe to eat more coconuts than are available, which certainly seems infeasible. However, we are going to ignore this constraint. We will see below that the solution we derive will typically satisfy the constraint, and hence it does not matter whether or not we include it in the analysis.²

Now, the exercise we want to do is the following. Suppose Crusoe has just arrived on the island and he wants to know what he should do. He tells us his preferences over consumption plans, and our goal is to tell him which plan is the best one for his family to follow. That is, we want to set up a utility maximization problem and find the solution.

1.5 The optimal growth problem

We now have the necessary ingredients to write down the problem we want to solve. We simply want to find the consumption plan that maximizes Crusoe's utility function, subject to the constraint that the consumption plan be feasible. We are going to call this utility-maximization problem the "optimal growth problem," because its solution will tell Crusoe the optimal way for his stock of trees to grow (or possibly shrink) over time. The problem is also called the "social planner's problem"; it is as if Crusoe has asked us to write down the best plan for his island society. The maximization problem is as follows.

² In general, whether or not this constraint should be included in a model depends on whether or not investment is viewed as being reversible. Investment is said to be reversible if the capital stock can be converted back into consumption goods and irreversible if it cannot. In our analogy of trees and coconuts, investment is clearly irreversible: once a coconut is planted and becomes into a tree, it is impossible to turn the tree back into a coconut and eat it. In other environments, however, investment is more reversible (perhaps some machinery can be used for consumption purposes). For studying issues related to economic growth, whether investment is assumed to be reversible or irreversible typically makes no difference in the predictions of the model. For other issues, such as business cycle analysis, it is much more important.

- Optimal growth problem :

$$\begin{aligned} & \max_{\{c\}} \int_0^{\infty} u [c(t)] e^{-(p-n)t} dt && \text{(OGP)} \\ & \text{subject to} \end{aligned}$$

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - (\delta + n)k(t) \\ k(0) &= k_0 \quad \text{and} \\ k(t), c(t) &\geq 0 \quad \text{for all } t. \end{aligned}$$

This is the problem Crusoe faces the moment he arrives on the island. Notice that solving the problem involves working out a plan for the entire future. In other words, deciding how many coconuts should be consumed and how many should be planted on the very first day requires us to know how many should be consumed/planted at every point in the future as well. Fortunately, we only need to solve the problem once. Since all of Crusoe’s descendants have the same preferences as he does, the solution to (OGP) will also be the best plan for them to follow when the time comes.

This is a constrained maximization problem, where we need to choose a value $c(t)$ for every point in time. In other words, there is an infinite number of choices to be made here, which is what makes this problem difficult. The important question now is how we can solve a problem like this.

2 The Hamiltonian Method:

There are different approaches one can take to this problem. I am going to present the Hamiltonian method. We can think of this as a dynamic version of the standard Lagrangian method that you use to solve static optimization problems with equality constraints. First I will present the steps needed to solve a general class of problems that includes (OGP) as a special case. You can think of these steps as a “recipe” that can be followed to solve problems. Then I will spend a fair amount of time deriving some exact intuition for these steps, so that we understand fully how this recipe works.

2.1 A 3-step recipe

- Consider the (more general) problem:

$$\begin{aligned} & \max_{\{c\}} \int_0^{\infty} v [k(t), c(t), t] dt \\ & \text{subject to} \\ & \dot{k}(t) = g [k(t), c(t), t] \\ & k(0) = k_0 \quad \text{and} \\ & k(t), c(t) \geq 0 \quad \text{for all } t \end{aligned}$$

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Notice that the optimal growth problem does indeed fit into this general form, by setting³

$$v[k(t), c(t), t] = u[c(t)] e^{-(p-n)t}$$

and

$$g[k(t), c(t), t] = f(k(t)) - c(t) - (\delta + n)k(t).$$

There are two types of variables in this problem, called “state” variables and “control” (or “choice”) variables. It is always important to know which of these categories each variable falls into.

- **Definition:** The value of a state variable at time t is completely determined by decisions made before t . A variable that is not a state variable is a control (or choice) variable.

At any point in time, the number of trees on the island is not something that Crusoe can choose. This number is determined entirely by how many trees he planted in the past (and how many trees were on the island when he arrived). Therefore $k(t)$ is a state variable. On the other hand, once the coconuts have been harvested, the number of coconuts that are consumed *is* a choice that can be made today. Hence $c(t)$ is a control variable.

- Here:
 - state variable: $k(t)$
 - control variable: $c(t)$

We are now ready to go through the 3 steps of the recipe.

- Step 1: Construct the Hamiltonian function

$$H = \underbrace{v[k(t), c(t), t]}_1 + \underbrace{\mu(t)}_2 \underbrace{g[k(t), c(t), t]}_3$$

1: objective function at time t

2: multiplier

3: “constraint” at time t (transition function)

- Step 2: Take the first-order conditions (FOC)

$$(a) \frac{\partial H(t)}{\partial c(t)} = 0, \quad (b) \frac{\partial H(t)}{\partial k(t)} = -\dot{\mu}(t), \quad (c) \frac{\partial H(t)}{\partial \mu(t)} = \dot{k}(t)$$

The first-order conditions are some derivatives of the Hamiltonian function. We will see why these are the relevant derivatives soon. First, though, we will need one other piece of information. Notice that the second FOC is a differential equation for μ . To solve this, we will need a boundary condition.

³ The general form has some features that are not present in the optimal growth problem. In particular, the variable $k(t)$ may appear in the objective function, and the variable t may appear independently in the constraint. These features are not needed now, but will be useful later in the course.

- Step 3: Write the transversality condition (TVC)

$$\lim_{t \rightarrow \infty} [\mu(t) k(t)] = 0$$

These are necessary conditions for a solution of our problem. We will not worry about sufficient conditions. We will see later that only one consumption plans satisfies these conditions, and hence this plan must be the solution to the problem.

If we apply these formulas to our optimal growth problem, we get

$$H = u[c(t)] e^{-(p-n)t} + \mu(t) [f(k(t)) - c(t) - (\delta + n)k(t)]$$

FOC:

$$u'(c(t)) e^{-(p-n)t} - \mu(t) = 0 \tag{a}$$

$$\mu(t) [f'(k(t)) - (\delta + n)] = -\dot{\mu}(t) \tag{b}$$

$$f(k(t)) - c(t) - (\delta + n)k(t) = \dot{k}(t) \tag{resource constraint} \tag{c}$$

TVC:

$$\lim_{t \rightarrow \infty} \mu(t) k(t) = 0$$

This recipe is fairly easy to remember and straightforward to apply. Now we are going to dig a little deeper and see that the equations (a) – (c) also have very interesting economic interpretations.

2.2 Intuition for the Hamiltonian approach

I do not want the equations above to be just some formulas that you memorize; I want to make sure that we really understand the economics behind these equations.

- Want the economic intuition for:
 - the multiplier $\mu(t)$
 - condition (a)
 - the function $H(t)$
 - condition (b)

Condition (c) is just the resource constraint, which we already understand. We will look at the intuition for the transversality condition later.

2.2.1 Understanding the multiplier $\mu(t)$

The economic interpretation multiplier $\mu(t)$ is critical for understanding all of the other conditions. This multiplier equals the marginal value of a tree at time t to Crusoe, measured from the point of view of $t = 0$. To see exactly what this means, we need to set up some new notation.

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- Let

$$V(k_0) = \max_{\{c\}} \int_0^{\infty} u[c(t)] e^{-(p-n)t} dt$$

subject to

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - (\delta + n)k(t) \\ k(0) &= k_0 \\ k(t), c(t) &\geq 0 \quad \text{for all } t \end{aligned}$$

That is, $V(k_0)$ is the utility value of the solution to our maximization problem. Of course, we don't know what this solution is yet. All we are saying here is that there is a solution, and let's call the value of this solution V .

- More generally, for any $t \geq 0$, let

$$V(k(t), t) = \max_{\{c\}} \int_t^{\infty} u[c(s)] e^{-(p-s)t} ds$$

subject to

$$\begin{aligned} \dot{k}(s) &= f(k(s)) - c(s) - (\delta + n)k(s) \\ k(t) &= k_t \\ k(s), c(s) &\geq 0 \quad \text{for all } s \geq t \end{aligned}$$

Here we are choosing some point in time t and ignoring everything that happens before t . Suppose Crusoe finds himself at time t with $k(t)$ trees. What is the best he can do from that point onwards? Call the utility value of this best plan $V(k(t), t)$.

Now let's focus on a very short interval of time.

- Consider $[t, t + \Delta]$ for Δ small

Suppose that on this entire interval, a fixed consumption level \bar{c} must be chosen. This is only an approximation, of course. The number of coconuts consumed can be different at each point in time. Our approach is going to be to make this approximation, and then to take the limit as Δ goes to zero, so that the approximation becomes exactly correct.

- The following equation must then hold :

$$\begin{aligned} \text{maximal value from } &= \text{benefit from} &+ & \text{value of starting tomorrow} \\ k(t) \text{ trees today} &: & \text{consumption today} & \text{with } k(t + \Delta) \\ \overbrace{V(k(t), t)} &= \max_{\{\bar{c}\}} \left\{ \overbrace{u[\bar{c}] e^{-(p-n)t} \Delta} + \overbrace{V(k(t + \Delta), t + \Delta)} \right\} \\ &: & & \text{depends implicitly on } \bar{c} \end{aligned}$$

This expression says that the way to get the highest possible utility starting at time t is to make the

best consumption decision today (\bar{c}) and then to get the highest possible utility starting tomorrow (at time $t + \Delta$). This is true by definition.

Let's step back for a minute. What are we trying to do here? We have a problem in which we need to make an infinite number of choices – how much to consume at each point in time. What we are trying to do is make these choices one at a time, by just looking at how much should be consumed today. We do this by grouping all of the points in time after today together and call them “the future” (or “tomorrow”). Then in making today's decision, we only need to look at (1) the benefit of consuming today, and (2) the effect consumption today has on the future. The optimal choice of \bar{c} is the one that properly balances these two concerns.

- Take the FOC for this problem

$$\underbrace{u'[\bar{c}] e^{-(p-n)t} \Delta}_1 + \underbrace{\frac{\partial}{\partial k} V(k(t+\Delta), t+\Delta)}_2 \underbrace{\frac{\partial k(t+\Delta)}{\partial \bar{c}}}_3 = 0 \quad (\star)$$

1 = marginal benefit of consuming a coconut today

2 = loss in future utility if there is one less tree

3 = decrease in future trees caused by consuming one more coconut today

2 · 3 = loss in future utility caused by consuming one more coconut today

Q: What is $\frac{\partial k(t+\Delta)}{\partial \bar{c}}$?

In other words, how are future values of $k(t)$ affected by today's consumption choice? The fact that we are working with a very simple environment (with only trees and coconuts) should help us understand the answer. First, let's go through some mathematics.

A: Begin with the following linear approximation

$$k(t+\Delta) \approx k(t) + \dot{k}(t) \Delta \quad (\text{for small } \Delta)$$

$$\Rightarrow \frac{\partial k(t+\Delta)}{\partial \bar{c}} \approx \Delta \frac{\partial \dot{k}(t)}{\partial \bar{c}}$$

Keep in mind that we are using \bar{c} to denote the level of consumption on the whole interval $[t, t + \Delta]$. Therefore $c(t)$ is equal to \bar{c} , and we can replace $c(t)$ in the resource constraint by \bar{c} .

- Using the resource constraint

$$\dot{k}(t) = f(k(t)) - \bar{c} - (\delta + n)k(t)$$

we have

$$\frac{\partial \dot{k}(t)}{\partial \bar{c}} = -1$$

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- Therefore

$$\frac{\partial k(t + \Delta)}{\partial \bar{c}} = -\Delta.$$

The intuition behind these equations is very simple. If you consume one more coconut today, you will have one less tree tomorrow. If you continue this extra consumption for a period of time with length Δ , the total decline in the number of trees is one multiplied by Δ .

Next, substitute this result into the first-order condition (★) that we derived above.

- FOC is now:

$$u'[\bar{c}] e^{-(p-n)t} \Delta - \frac{\partial}{\partial k} V(k(t + \Delta), t + \delta) \Delta = 0$$

- Cancel out the Δ term, and then let $\Delta \rightarrow 0$. With $\Delta = 0$, we can replace \bar{c} with $c(t)$

$$u'[c(t)] e^{-(p-n)t} - \frac{\partial}{\partial k} V(k(t), t) = 0 \quad \text{for any } t \geq 0$$

- Comparing this with condition (a), we see that

$$\boxed{\mu(t) = \frac{\partial}{\partial k} V(k(t), t)}$$

This gives us an interpretation of $\mu(t)$. We defined the variable $V(k(t), t)$ to be the utility value of having $k(t)$ trees at time t . Our calculations have shown that the multiplier $\mu(t)$ is exactly equal to the derivative of this V function with respect to its first argument. In other words, $\mu(t)$ measures the amount by which total utility could be increased if Crusoe were given one more tree at time t .

- We have shown

$$\begin{aligned} \mu(t) &= \text{marginal value of a tree at time } t \\ &= \text{amount by which total utility} \\ &\quad \int_0^{\infty} u[c(t)] e^{-(p-n)t} dt \\ &\quad \text{could be increased if Crusoe had one more tree at } t \end{aligned}$$

Notice this is just like in a static optimization problem: the multiplier is the shadow value of the resource constraint. We are going to use this interpretation repeatedly.

This was our first goal: understanding the meaning of the multiplier $\mu(t)$. Next we want to understand the meaning of the first-order condition (a).

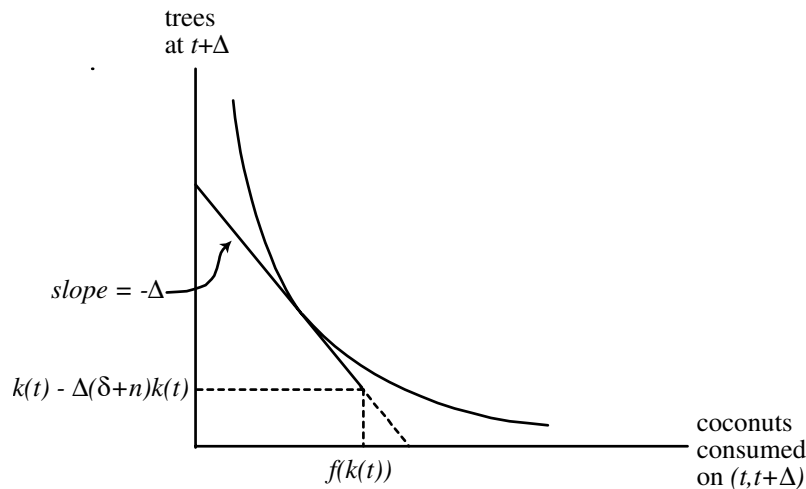
2.2.2 Understanding condition (a)

Our approach above was to turn a dynamic optimization problem into a series of static problems by using the V function to measure the value of trees. Let's look at what we did a little more carefully.

- Recall

$$V(k(t), t) = \max_{\{\bar{c}\}} \{u[\bar{c}] e^{-(p-n)t} \Delta + V(k(t + \Delta), t + \Delta)\}$$

The right-hand side of this equation is an easy maximization problem involving only one variable (\bar{c}). Once we know the utility value of a tree, the problem of choosing how many coconuts to consume and how many to plant becomes a standard, static problem from microeconomics. Utility is defined over two goods (consumption today and trees tomorrow), and the decision maker faces a tradeoff between these goods. In other words, the function V allows us to draw indifference curves between coconuts consumed today and future trees, as in the following picture.⁴



Condition (a) is simply the first-order condition for this (easy) optimization problem.

- Condition (a) says

$$\underbrace{u'(c(t)) e^{-(p-n)t}}_{\text{marginal utility of } c(t)} = \underbrace{\mu(t)}_{\text{marginal utility of } k(t)}$$

- notice that both terms are discounted to $t = 0$
- Optimality requires that the consumption–planting decision be such that, at the margin, Crusoe is indifferent between eating a coconut and planting it

2.2.3 Economic interpretation of the Hamiltonian function

The value $H(t)$ also has an important economic interpretation. To see it, we need to focus on what I will call the “net harvest” at time t .

- Define the net harvest (per person) at time t to be $y(t) - (\delta + n)k(t)$

⁴ When looking at this picture, it is important to keep in mind that we are not saying that Crusoe gets utility from the trees directly. The utility of a tree at time t comes indirectly, from the coconuts that will be consumed from the tree in the future. This indirect utility is what the function V captures.

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From the total harvest, we subtract enough coconuts to replace the trees that have just died and to have new trees for the new members of the household. This remaining “net” harvest can either be consumed or used to increase the stock of trees per person.

- Then

$$\text{net harvest} = \text{consumption} + \text{increase in total stock of trees}$$

or

$$f(k(t)) - (\delta + n)k(t) = c(t) + \dot{k}(t)$$

This last equation is just the resource constraint (rc) written in a different order. Once Crusoe has decided how much of the net harvest will be consumed and how much will become new trees, we can ask:

Q: What is the utility the Crusoe household gets from the net harvest? (discounted to $t = 0$)

To answer this question, we need to add together the utility from current consumption **and** the utility from the future consumption made possible by increasing the number of trees. We know from above that, at the margin, a new tree is worth $\mu(t)$ in terms of utility at $t = 0$. Therefore, the utility from having $\dot{k}(t)$ more trees is equal to the product $\mu(t)\dot{k}(t)$.⁵ The answer to the question is then:

A: Add the two benefits together

$$\begin{aligned} & : u[c(t)]e^{-(\rho-n)t} + \mu(t)\dot{k}(t) \\ & = u[c(t)]e^{-(\rho-n)t} + \mu(t)[f(k(t)) - c(t) - (\delta + n)k(t)] \\ & = H(t) \quad \quad \quad \{\text{The Hamiltonian function}\} \end{aligned}$$

In words, we have shown that the Hamiltonian function measures the utility that the household receives from the current net harvest. It combines the current benefit from consuming with the future benefit from planting new trees.

This interpretation gives us another way of looking at condition (a). When we decide how to divide today’s net output into consumption and planting, we should clearly do it in a way that maximizes the total utility from this output. In other words, $c(t)$ should be chosen to maximize the value of $H(t)$. This is exactly what condition (a) says to do.

- Another view of condition (a) : choose $c(t)$ to maximize $H(t)$

$$\Rightarrow \frac{\partial H(t)}{\partial c(t)} = 0$$

It almost seems like we should be done. The only choice we need to make is how many coconuts to consume at each point in time. We have an equation telling us how to do that. Why do we need anything else?

⁵ More precisely, this product is a linear approximation of the value, because $\mu(t)$ is the derivative of the indirect utility function V . Because we are looking at a single instant of time, the linear approximation will be exactly correct.

The problem is this: we have introduced the new variable $\mu(t)$. That is, we said that *if* we knew the utility value of a tree at every point in time, we would know how to make the consumption-planting decision. However, we do not yet know anything about this variable $\mu(t)$. Condition (b) gives us this information.

2.2.4 Interpreting Condition (b)

Condition (b) is a differential equation for the variable μ ; it tells us how the marginal value of a tree changes over time. To see the intuition behind this condition, we need to go back to our equation relating the value of a tree today to the value of a tree tomorrow.

- Recall

$$V(k(t), t) = \max_{\{\bar{c}\}} \{u[\bar{c}] e^{-(p-n)t} \Delta + V(k(t + \Delta), t + \Delta)\}$$

Suppose we hold the level of consumption \bar{c} fixed and just focus on the variables $k(t)$ and $k(t + \Delta)$. The equation above tells us about the relationship between the utility value of the stock of trees today and the utility value of the stock of trees tomorrow. What we want to know is the relationship between the marginal utility of trees today and the marginal utility of trees tomorrow. Recall that $\mu(t)$ is a derivative of the V function, so we can

- Differentiate both sides with respect to $k(t)$, holding \bar{c} fixed

$$\frac{\partial}{\partial k} V(k(t), t) = 0 + \frac{\partial}{\partial k} V(k(t + \Delta), t + \Delta) \frac{dk(t + \Delta)}{dk(t)}$$

or

$$\mu(t) = \mu(t + \Delta) \frac{dk(t + \Delta)}{dk(t)} \quad (\blacklozenge)$$

We can interpret this equation as saying

$$\text{marginal utility today} = \text{marginal utility tomorrow} \times \text{a rate of transformation.}$$

We can also rearrange the terms as follows.

- Or

$$\frac{\mu(t)}{\mu(t + \Delta)} = \frac{dk(t + \Delta)}{dk(t)}$$

What does this condition say? The left-hand side is a ratio of marginal utilities, which is a marginal rate of substitution (MRS). The right-hand side is a marginal rate of transformation (MRT). In other words, this condition tells us that when we consider the two commodities “trees today” and “trees tomorrow”, the standard MRS=MRT condition must hold.

It might seem a little bit odd to talk about a marginal rate of substitution for trees (instead of for consumption), but remember what the function V and the variable μ are. They measure the (indirect) utility value of trees. Using their values, we could draw an indifference curve for trees today and trees tomorrow. The equation above just tells us that if we are doing things correctly,

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the slope of this indifference curve should be equal to the rate at which a tree today can be used to create a tree tomorrow.

It turns out that this equation is equivalent to condition (b). Seeing this requires a little bit of algebra. First, we need to determine what the marginal rate of substitution between $k(t)$ and $k(t + \Delta)$ is.

- Take the linear approximation

$$k(t + \Delta) \approx k(t) + \dot{k}(t) \Delta \quad \{\text{for } \Delta \text{ small}\}$$

$$\Rightarrow \frac{\partial k(t + \Delta)}{\partial k(t)} = 1 + \Delta [f'(k(t)) - (\delta + n)]$$

Notice what this last equation tells us: the marginal rate of transformation is not equal to one (as we might have naively expected). In other words, one tree today does not create one tree tomorrow. For one thing, trees sometimes die, so having one tree today would give you less than one tree tomorrow. There is also the issue of the new members of the household. However, the tree today also produces coconuts, which can be planted to yield trees tomorrow. How many coconuts does a tree today give? Well, for the marginal rate of transformation what matters is the last (marginal) tree, which gives $f'(k(t))$ coconuts. Therefore one tree (per capita) today generates one tree (per capita) tomorrow **minus** $(\delta + n)$ “depreciated” trees **plus** $f'(k(t))$ new trees.

- Also take the linear approximation

$$\mu(t + \Delta) \approx \mu(t) + \dot{\mu}(t) \Delta$$

Substituting these conditions into equation (◆) from above,

- Then we have:

$$\begin{aligned} \mu(t) &= [\mu(t) + \Delta \dot{\mu}(t)] [1 + \Delta (f'(k(t)) - (\delta + n))] \\ \mu(t) &= \mu(t) + \mu(t) \Delta [f'(k(t)) - (\delta + n)] + \Delta \dot{\mu}(t) + \Delta^2 (\dots) \end{aligned}$$

Since we are considering small values of Δ , the last term (which has Δ^2 in it) will disappear. For this reason I did not write the term out, leaving just (...) instead. Now, cancel out the first $\mu(t)$ term on each side, and then divide through by Δ

$$0 = \mu(t) [f'(k(t)) - (\delta + n)] + \dot{\mu}(t) + \Delta (\dots)$$

let $\Delta \rightarrow 0$

$$\begin{aligned} \mu(t) [f'(k(t)) - \delta] &= -\dot{\mu}(t) \\ \Rightarrow &\text{condition (b)} \end{aligned}$$

These calculations therefore show what I claimed above: condition (b) can be thought of as simply an intertemporal MRS=MRT condition for trees. We can also write the condition as follows.

- Or

$$\underbrace{\frac{\dot{\mu}(t)}{\mu(t)}}_{\text{growth rate of the value of a tree}} = \underbrace{-[f'(k(t)) - (\delta + n)]}_{\text{(net) marginal rate of transformation}}$$

growth rate of the value of a tree = (net) marginal rate of transformation

We can think of the growth rate of μ as the “instantaneous” marginal rate of substitution between trees today and trees tomorrow (as $\Delta \rightarrow 0$).

- Summary so far:

$H(t)$ = total benefit from the net harvest at t

$$(a) : \frac{\partial H(t)}{\partial c(t)} = 0 \Rightarrow \text{static MRS=MRT for consumption-planting decision}$$

$$(b) : \frac{\partial H(t)}{\partial k(t)} = -\dot{\mu}(t) \Rightarrow \text{dynamic MRS=MRT for trees}$$

$$(c) : \frac{\partial H(t)}{\partial \mu(t)} = \dot{k}(t) \Rightarrow \text{constraint (transition function).}$$

One way to think of what we have done so far is the following. We said that if we knew how to assign a value to trees, the consumption-planting decision at each point in time is an easy, static optimization problem whose first-order condition is (a). The question then becomes how we should assign the value of trees. Condition (b) says that we need to assign these values so that the intertemporal MRS=MRT condition holds at every point in time. Now let’s take these equations and try to solve them to see what the complete functions k and c will look like.

2.3 Working with the FOC

I am going to use a specific utility function here to simplify the algebra. I will also assume that the Crusoe household does not increase in size over time.

- Assume:

$$\begin{aligned} u(c) &= \ln(c) \\ n &= 0 \end{aligned}$$

In the first Problem Set, you will solve these equations with $n > 0$ and using a different utility function. This problem set is very important, because we are going to use that utility function in the remainder of the course.

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- Then the FOC are given by

$$\frac{1}{c(t)}e^{-\rho t} = \mu(t) \quad (a)$$

$$-\frac{\dot{\mu}(t)}{\mu(t)} = f'(k(t)) - \delta \quad (b)$$

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t) \quad (c)$$

- Thus we have

$$(a) \Rightarrow \text{relationship between } c(t) \text{ and } \mu(t)$$

$$(b) \Rightarrow \text{differential equation for } \mu$$

$$(c) \Rightarrow \text{differential equation for } k$$

So this is a 2-dimensional system of differential equations, and we need to find the solution. We have 3 variables here, but we can use condition (a) to eliminate one of them. One option would be to solve (a) for $c(t)$, and plug this solution into equation (c). It turns out to be more useful to solve (a) for $\mu(t)$ and plug the result into equation (b). Either of these approaches would lead to the same answer, of course, but we will always follow the latter.

$$(a) \Rightarrow \mu(t) = \frac{1}{c(t)}e^{-\rho t}$$

- Take logs and differentiate with respect to t

$$\ln(\mu(t)) = -\ln(c(t)) - \rho t$$

or

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\frac{\dot{c}(t)}{c(t)} - \rho$$

or

$$-\frac{\dot{\mu}(t)}{\mu(t)} = \frac{\dot{c}(t)}{c(t)} + \rho$$

Here we see that the instantaneous marginal rate of substitution for consumption is linked to the marginal rate of substitution for trees. They differ in the discount factor. We will see some intuition for this relationship later.

We are going to follow the procedure of taking logs and differentiating throughout the course. If you are a little rusty on the rules associated with the natural logarithm (for example, $\ln(xy) = ??$), I recommend that you brush up on them now.

- Substitute this into (b) :

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho$$

or

$$\dot{c}(t) = [f'(k(t)) - \delta - \rho] c(t) \quad (1)$$

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t) \quad (2)$$

This is a system of ordinary differential equations. The word ‘ordinary’ here just means that all of the derivatives are with respect to the same variable: time.

Remember our goal: we want to give Crusoe a plan that tells him how many coconuts should be consumed at each point in time. That plan is a function c . The first-order conditions for Hamiltonian method did not give us the function, but it did give us some information about it. In particular, we have a differential equation that tells us how this level of consumption should evolve over time. However, this equation depends on the number of trees and how that number evolves over time. Therefore, in order to get our answer (the function c), we need to study how the optimal consumption of coconuts and the optimal number of trees evolve together over time. That is what these two differential equations tell us.

3 Finding the Solution

The next question is how we can analyze this system of equations. In general, the analysis of differential equations can be done using graphical, analytical, or numerical methods. We will use primarily graphical methods in this course.

3.1 The phase diagram

Our approach will be based on the phase diagram. When I draw a phase diagram, I always go through the following steps. First, we draw the phase plane, the set of all possible values for the variables k and c . Since we know the the number of coconuts consumed and the number of trees on the island are both non-negative numbers, the relevant part of the phase plane for our problem is the non-negative orthant.

For each point in the phase plane, the differential equations specify a direction of movement (either k and c are both increasing, or both decreasing, or one is increasing and the other decreasing). We want to know what the direction of movement is at each point. To do this, we will begin by looking at special sets of points, where one of the variables is neither increasing nor decreasing.

- Definition: An isocline is a set of points in the phase plane where one of the variables is unchanging. In our system, the isoclines are where $\dot{c} = 0$ and where $\dot{k} = 0$ hold.

Our method for drawing the phase diagram consists of four steps, two each for the variables c and k . We begin with the variable c .

- Step (1): Find the isocline(s) for c . That is, where is $\dot{c} = 0$?
 - For \dot{c} to be zero we need either $c = 0$ or

$$f'(k(t)) = \delta + \rho \quad \rightarrow \quad \text{Solution : } k^*$$

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Because the intensive harvest function f is strictly concave, there is only one value of k such that the marginal productivity of trees is equal to $(\delta + \rho)$. Let's call that value k^* . The isoclines for \dot{c} are therefore the horizontal axis and a vertical line at the value k^* .⁶

- Step (2): Classify the dynamics of c , drawing arrows (up and down)

The vertical isocline has divided the phase plane into two regions. The variable c is rising in one of these regions and falling in the other. We need to figure out which region is which. We do this by picking any one point and then asking: if we are at that point, would c be increasing or decreasing? Start with a point to the left of k^* in the figure below. We know that at k^* ,

$$f'(k^*) - \delta - \rho = 0$$

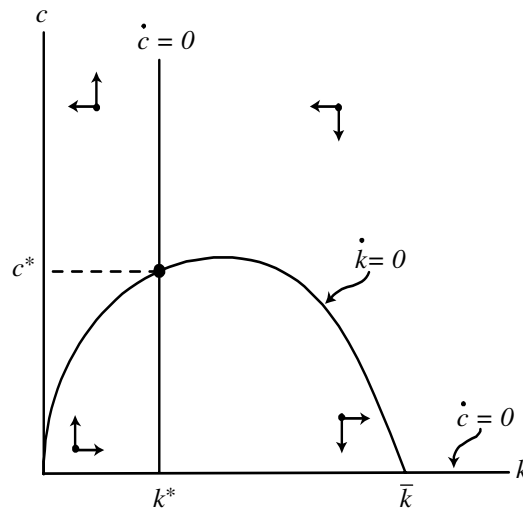
holds. Since f is concave, decreasing k will raise $f'(k)$. For any value of k below k^* , therefore, we must have

$$f'(k(t)) - \delta - \rho > 0$$

and thus $\dot{c} > 0$ holds and c will be growing. At a couple of points to the left of k^* , we draw an arrow pointing upwards, as in the figure. For values of k greater than k^* , on the other hand,

$$f'(k(t)) - \delta - \rho < 0$$

holds, and therefore at these points the level of c will be falling. For a couple of such points, we draw an arrow pointing downward, as in the figure.



Now we repeat these two steps, but focusing on the variable k .

⁶ With log utility, $c(t)$ cannot be zero (because the log of zero is undefined). To be precise, therefore, our our phase diagram should not contain the $c = 0$ axis. However, the utility function that you will use in the first problem set (and that we will use in the remainder of the course) is defined for $c(t) = 0$, at least for some parameter values. For this reason, I have chosen to include the axis in the diagram here. We shall see that including or excluding this axis makes no difference in the analysis.

- Step (3): Find the isocline(s) for k . That is, where is $\dot{k} = 0$?

$$c = f(k) - \delta k$$

This equation is easy to understand: \dot{k} is zero when consumption is equal to all of the net harvest (recall that we have set $n = 0$). We need to draw this curve on our phase diagram. When k is zero, the value of c corresponding to this curve is also zero, and hence the curve starts at the origin. The strict concavity of the function f implies that this curve is strictly concave. The curve reaches a maximum where $f'(k) = \delta$ holds and then decreases until it reaches the horizontal axis. Let's call the point where it hits the axis \bar{k} . Note that the maximum of this curve (where $f'(k) = \delta$ holds) is necessarily to the right of k^* (where $f'(k) = \delta + \rho$ holds)

- Step (4): Classify the dynamics of k , drawing arrows (left and right)

This isocline has also divided the phase plane into two regions, one below the curve and the other above it. We need to figure out in which of these two regions k is rising and in which it is falling. Pick some point below the curve, like the point near the origin in the figure above. Compare this point to the one directly above it on the curve. By construction the value of k is the same, but c is lower at the point near the origin. Since \dot{k} is zero along the isocline, it must be positive at any point below the curve. This should make intuitive sense. If consumption is relatively small compared to the number of trees and hence the size of the harvest, then a lot of trees are being planted and the number of trees must be growing. Similarly, if we look at a point above the curve, \dot{k} must be negative and hence k is falling. In other words, if the level of consumption is very high, not enough new trees are being planted to replace the trees that are dying, and the total number of trees on the island will be decreasing. Adding these arrows to the diagram completes the figure above.

Now we have this diagram, and we need to figure out what it is telling us. Let's start by looking at the special points where the isoclines cross.

- **Definition:** A point (k^*, c^*) such that both \dot{k} and \dot{c} are equal to zero is called a steady state. (**not** an "equilibrium")

I am going to insist on using this terminology throughout the course. I know that physicists and mathematicians often use the word 'equilibrium' to refer to a point like (k^*, c^*) , but to an economist an equilibrium is something entirely different. We are going to use the word 'equilibrium' very soon, and it is critical that we not confuse an (economic) equilibrium with a steady state. Therefore the point (k^*, c^*) is, for us, a steady state.

Steady states are important points because (i) they are easy to find and (ii) they may tell us something useful about the long-run behavior of the system.

- Steady states in our diagram:

$$\begin{aligned} (k, c) &= (0, 0) \\ &= (\bar{k}, 0) \\ &= (k^*, c^*) \end{aligned}$$

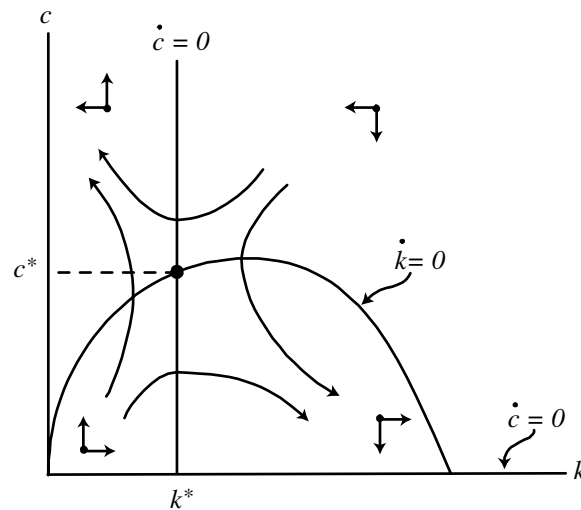
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- (k^*, c^*) is the unique interior steady state.

By definition, if our island economy happens to start at a point that is a steady state, it will stay at that point forever. Next we want to know what happens if the island economy starts somewhere else in the diagram.

- Pick any point on the phase plane. There is a unique trajectory from that point that satisfies the differential equations.

Given any starting point in the phase plane, the differential equations (1) and (2) tell us exactly how the variables k and c will evolve over time. The word ‘trajectory’ refers to this path in the phase plane. The figure below shows some trajectories from different starting points.



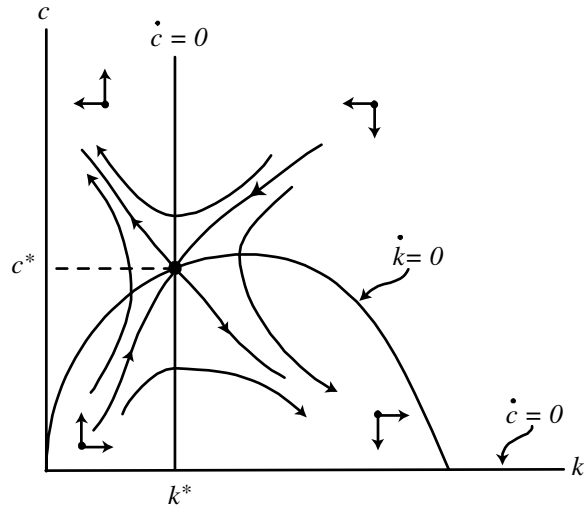
Notice that the trajectories must “respect” the isoclines. That is, when a trajectory crosses the isocline for k , it must be vertical (because k is neither rising nor falling at this point). Similarly, when a trajectory crosses the isocline for c , it must be horizontal at that point.

- An interior steady state usually has two special trajectories called arms.

If the differential equations are linear, the arms are straight lines. If the equations are nonlinear, like the ones we are studying here, the arms are curves. In our diagram, the arms are the only trajectories that are connected to the steady state.

- Each arm can be either stable or unstable.

We are just going to use the arrows in our phase diagram to determine stability. If you wanted to verify this rigorously, you would linearize the differential equations around the steady state, and then find the eigenvalues of the Jacobian matrix. Each eigenvalue tells you the stability along one of the arms: a negative number means stable and a positive number means unstable. If we did this for our system of equations, we would find one negative and one positive eigenvalue.



- When one arm is stable and the other is unstable, the steady state is called a saddle point.⁷

If both arms are stable, the steady state is **locally stable**, and if both arms are unstable the steady state is **locally unstable**. We will see some locally stable steady states later in the course, but for now we will see a lot of saddle points. It is not a coincidence that we ended up with a saddle point here; there are important economic reasons for this, which we will see very soon.

So far we have a diagram with many trajectories on it, all of which are solutions to our system of differential equations. Our maximization problem only has one solution, though. We need to know which trajectory on this diagram is actually the solution to our problem. Another way of saying this is that we need to know the “initial conditions”; where does the optimal trajectory start?

Q: What do we know about the starting point (initial condition)?

A: Only $k(0) = k_0$

We are given an initial value for $k(t)$, but not for $c(t)$. The initial level of consumption is something Crusoe can choose. We need to find a way of determining the correct value for $c(0)$. This is the role of the transversality condition.

3.2 Transversality and feasibility

Recall that our system of differential equations is given by

$$\begin{aligned}\dot{c}(t) &= [f'(k(t)) - \delta - \rho] c(t) \\ \dot{k}(t) &= f(k(t)) - c(t) - \delta k(t)\end{aligned}$$

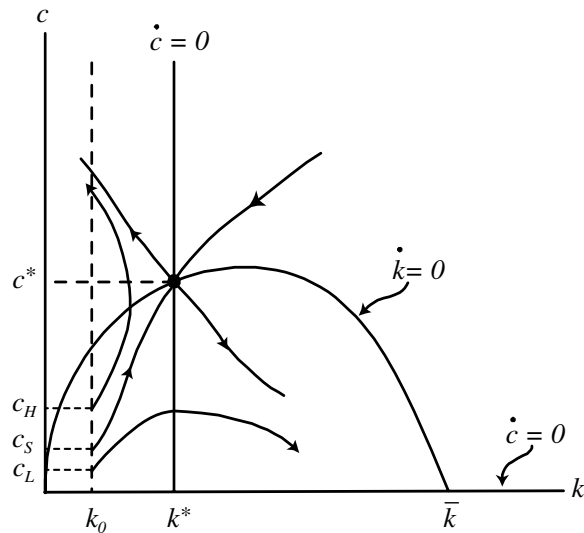
with the initial condition

$$k(0) = k_0.$$

⁷ The name “saddle point” comes from comparing our figure with the dynamics that would be generated by placing a marble at various places on a horse saddle.

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The figure below is a cleaned-up version of the phase diagram, with some of the trajectories removed to make things easier to see. We know the starting value for k , and we need to find the starting value for c .



- Initial condition: We know that $k(0)$ must start at k_0 . What should $c(0)$ be?

The one piece of information we haven't used is the Transversality condition. It is going to take some work, but eventually this condition will tell us the optimal level of consumption at $t = 0$.

- TVC:

$$\lim_{t \rightarrow \infty} \mu(t) k(t) = 0$$

- Intuition:

$$\begin{aligned} \mu(t) &= \text{value of one tree} \\ k(t) &= \text{number of trees} \end{aligned}$$

\Rightarrow

$$\mu(t) k(t) = \text{value of entire time } t \text{ stock of trees}$$

So the transversality condition says that the present value (measured at $t = 0$) of the stock of trees must go to zero as t goes to infinity. To understand this condition better, let's look at a finite-horizon problem.

3.2.1 A finite-horizon problem

Suppose that instead of living on the island forever, Crusoe knows that he will be rescued on date T , where T is some large number. In making his plans for the island, therefore, he will only care

about what happens between dates zero and T . The optimal growth problem becomes:

$$\begin{aligned} & \max \int_0^T u[c(t)] e^{-(\rho-n)t} dt \\ & \text{subject to} \\ & \dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t) \\ & k(0) = k_0 \quad \text{and} \\ & k(t), c(t) \geq 0 \quad \text{for all } t \in [0, T] \end{aligned}$$

The Hamiltonian function and first-order conditions are exactly the same as those given above. The only difference is in the transversality condition, which is now given by

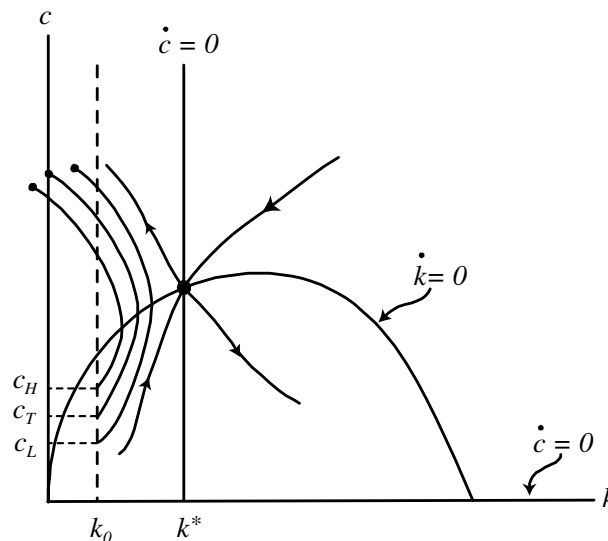
$$\mu(T)k(T) = 0.$$

This is very much like a complementary slackness condition in a static optimization problem. It says that at time T , either there should be no trees left [$k(T) = 0$] or trees should have no value [$\mu(T) = 0$]. From first-order condition (a) we have

$$\mu(T) = u'[c(T)] e^{-\rho T} > 0.$$

That is, the value of a tree at time T is positive, and therefore $k(T) = 0$ must hold.⁸ In other words, for the finite horizon case the transversality condition says that when Crusoe departs the island, there should be no trees left. If there were trees left, it would have been possible for him to consume more (and plant less) while he was on the island, which would have made him better off.

Looking at the phase diagram, then, we need to find the trajectory that satisfies $k(0) = k_0$ and $k(T) = 0$. In the figure below, the trajectory from (k_0, c_T) satisfies these two conditions.



⁸ We are assuming here, as in the analysis above, that investment is reversible. This means that when the date gets very close to T , Crusoe is able to turn some of the existing trees back into coconuts and eat them. If we impose the irreversibility constraint $c(t) \leq f(k(t))$, matters become more complicated. Because we are only trying to get some intuition here, we will stick to the simpler case where investment is perfectly reversible.

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Notice that setting $c(0) = c_T$ is the only way for Crusoe to satisfy both the differential equations and the transversality condition. If he initially chose a slightly higher level of consumption (like c_H in the figure), the trajectory dictated by the differential equations would cross the vertical axis before time T , and hence would not be feasible. In other words, if he consumes too much at $t = 0$, he will run out of trees too early. Conversely, if Crusoe chose an initial consumption level slightly lower than c_T (like c_L in the figure), the trajectory will not reach the axis by time T and the transversality condition will not be satisfied. In other words, if he consumes too little at $t = 0$, he will have trees left over at time T , which is clearly not optimal. Only by choosing $c(0) = c_T$ will he hit the “target” of $k(T) = 0$. Therefore, the trajectory starting at (k_0, c_T) is the solution to the finite-horizon optimal growth problem.

3.2.2 Back to the infinite horizon

In our original problem, the Crusoe household will live on the island forever and, therefore, there is no point in time when the stock of trees should be zero. Nevertheless, we still need to make sure that they do not accumulate “too many” trees. As discussed above, the transversality condition tells us that the period-zero value of the stock of trees should go to zero in the limit:

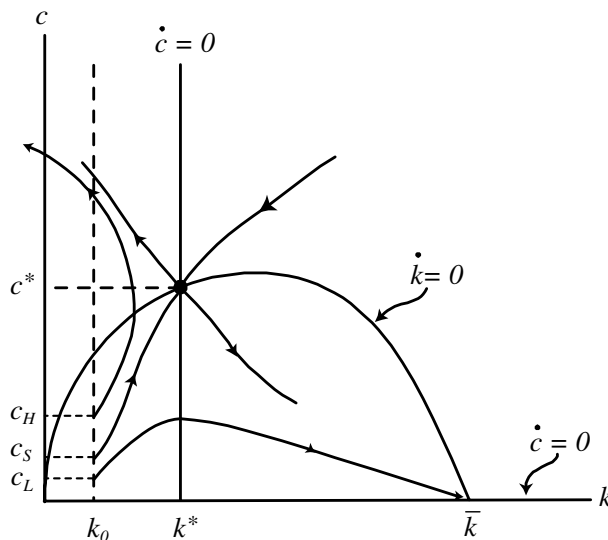
$$\lim_{t \rightarrow \infty} \mu(t) k(t) = 0.$$

If this condition is not satisfied, Crusoe is consuming too little and planting too much (just like in the finite case). Let’s use the first-order condition (a) to replace the variable $\mu(t)$.

- For our problem (with $n = 0$):

$$\begin{aligned} \text{(a):} \quad \mu(t) &= \frac{1}{c(t)} e^{-\rho t} \\ \Rightarrow \text{TVC:} \quad \lim_{t \rightarrow \infty} \frac{k(t)}{c(t)} e^{-\rho t} &= 0 \end{aligned}$$

Let’s pick some possible initial values for c and see if the resulting trajectories satisfy this condition. Consider the phase diagram below.



- Suppose $c(0) = c_L$ Does this trajectory look efficient?
- Then:

$$\begin{aligned} k(t) &\rightarrow \text{constant } (\bar{k}) \\ c(t) &\rightarrow 0 \\ \frac{k(t)}{c(t)} e^{-\rho t} &\rightarrow \frac{\bar{k}}{0} \quad \{\text{this is a problem!}\} \end{aligned}$$

Intuitively, this trajectory looks inefficient. Consumption is small (and goes to zero in the long run) while the stock of trees is growing very large (up to \bar{k}). However, when we evaluate the transversality condition, we get a zero-over-zero problem because of the discounting term. So, we have more work to do. We need to figure out whether $c(t)$ or the discounting terms is going to zero faster. To do this, we are going to look at a linear approximation of the function $c(t)$ that is valid in the limit.

- Note: near \bar{k} we have

$$\frac{\dot{c}(t)}{c(t)} \approx f'(\bar{k}) - \delta - \rho \quad \{\text{a constant}\}$$

We don't know the function $c(t)$. However, all we need to know is what happens in the limit as t goes to infinity. So we can take a linear approximation to the function $c(t)$ that is valid in this limit.

- Therefore, near \bar{k} we can say

$$c(t) \approx b e^{[f'(\bar{k}) - \delta - \rho]t} \quad \{\text{where } b \text{ is a constant of integration}\}$$

- Plug this into the TVC:

$$\begin{aligned} \lim_{t \rightarrow \infty} k(t) \frac{e^{-\rho t}}{c(t)} &= \bar{k} \lim_{t \rightarrow \infty} \frac{e^{-\rho t}}{b e^{[f'(\bar{k}) - \delta - \rho]t}} \\ &= \frac{\bar{k}}{b} \lim_{t \rightarrow \infty} e^{[-\rho - f'(\bar{k}) + \delta + \rho]t} \\ &= \frac{\bar{k}}{b} \lim_{t \rightarrow \infty} e^{[\delta - f'(\bar{k})]t} \end{aligned}$$

This expression shows that the value of the limit depends crucially on the sign of $(\delta - f'(\bar{k}))$.

Q: What is the sign of $(\delta - f'(\bar{k}))$?

We know the answer to this question. Where is the marginal product of trees equal to δ ? Recall

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that the equation for the isocline for k is

$$c = f(k) - \delta k.$$

The maximum of this isocline occurs where $f'(k)$ is equal to δ . This maximum point obviously lies to the left of \bar{k} , and hence by diminishing returns we know that $f'(\bar{k})$ is less than δ .

A: Positive. Therefore,

$$\lim_{t \rightarrow \infty} e^{[\delta - f'(\bar{k})]t} = \infty$$

and the TVC is *violated*.

This means that the trajectory starting at (c_L, k_0) is not a solution to our dynamic optimization problem. This was pretty hard to show. In the diagram the intuition is easier to see. For example, the time path of consumption from c_s looks clearly better than that from c_L .

Result 1: The trajectory from any $c(0) < c_s$ violates the transversality condition.

- Now suppose $c(0) = c_H$

This looks like a pretty good path: you get a lot of consumption. But something is wrong with it. (What?)

- Recall the feasibility constraint:

$$k(t), c(t) \geq 0 \quad \text{for all } t$$

From the phase diagram, we can see that the trajectory starting at (k_0, c_H) will eventually have $k(t) < 0$ and therefore will violate the feasibility constraint.⁹

Result 2: The trajectory from any $c(0) > c_s$ violates the feasibility constraint.

There is now only one possibility left. Suppose $c(0) = c_s$

- Transversality:

$$\frac{k(t)}{c(t)} \rightarrow \text{constant} \quad \text{and} \quad e^{-\rho t} \rightarrow 0$$

therefore

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{k(t)}{c(t)} e^{-\rho t} &= 0 \\ &\Rightarrow \text{TVC is satisfied.} \end{aligned}$$

⁹ To be more precise, we should verify that the picture is drawn correctly and that the trajectory does eventually cross the axis. If the trajectory did not cross the axis, it must be the case that it is slowing down and asymptotically approaching the axis. By continuity, this would imply that $\dot{k} = 0$ holds along the axis. However,

$$\dot{k}|_{k=0} = f(0) - c - \delta \cdot 0 = -c \neq 0.$$

Therefore, the trajectory does not approach the axis; it crosses over the axis and hence is infeasible.

Along this trajectory, the stock of trees on the island does not go to zero; it approaches the value k^* in the long run. However, because of discounting, the utility value of this stock (measured at $t = 0$) does go to zero as t goes to infinity, and hence the trajectory satisfies the transversality condition.

- Feasibility:

$$\begin{aligned} \text{clearly } k(t) &\geq 0 \text{ for all } t \\ &\Rightarrow \text{feasibility is satisfied} \end{aligned}$$

Result 3: The trajectory from (c_s, k_0) satisfies both transversality and feasibility.

Finally, we have our answer:

\Rightarrow The solution to the optimal growth problem is the trajectory starting at (c_s, k_0) .

I want to emphasize that this last statement is the solution to our problem. The phase diagram is not the solution, because there are many trajectories on that diagram. Just drawing a phase diagram does not tell Crusoe what to do. Completely solving the problem requires us to identify the one trajectory that Crusoe and his family should follow.

3.3 Drawing time paths

We now have the solution in the phase diagram, but sometimes it will be more useful to draw the solution in another way. Think back to the pictures I showed in the first class: we looked at the graph of real output over time for a variety of economies. I want to draw the same types of pictures for the Crusoe economy.

- Suppose $k_0 < k^*$
- Want: time paths of k , y , and c

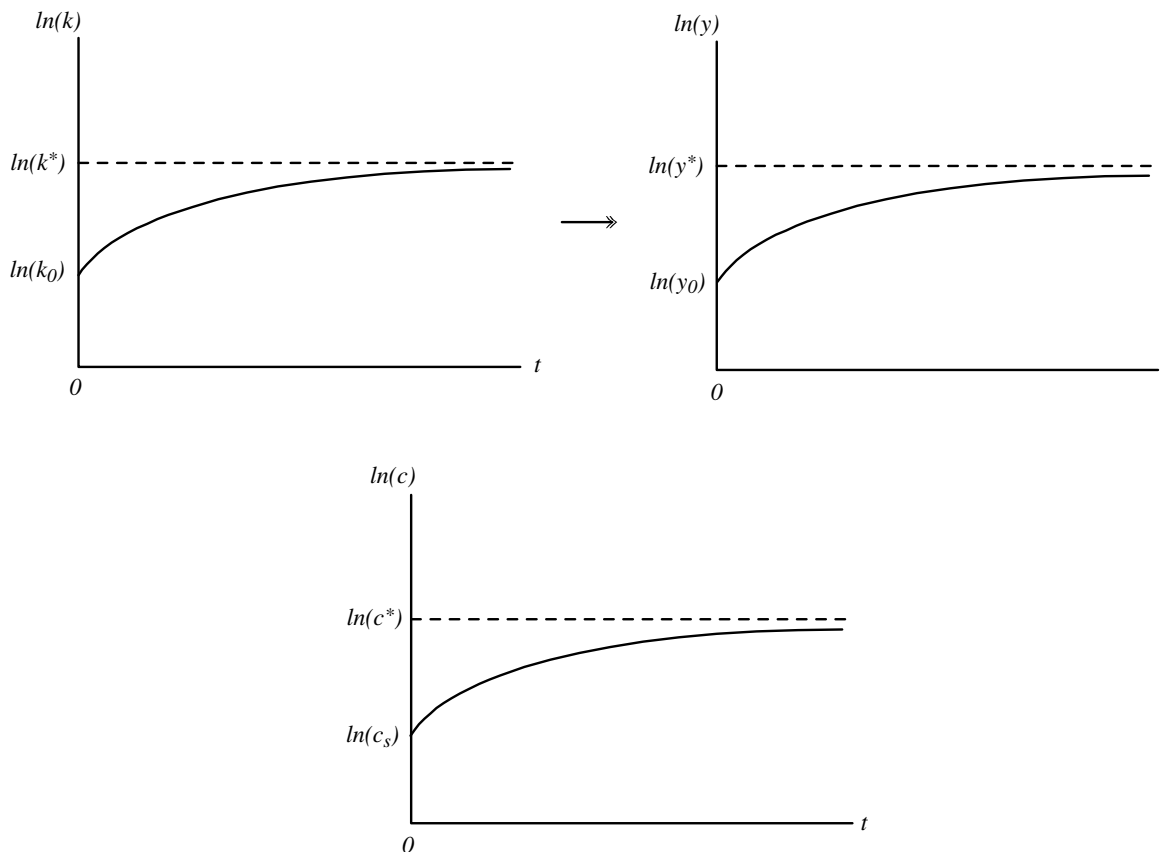
In other words, we want to draw the functions k , y , and c that solve the optimal growth problem. In the phase diagram, we see that k starts at the level k_0 and increases monotonically through time. As t goes to infinity, the value of k approaches k^* . The time path of k therefore looks like the first picture below.

To draw the time path of y , we only need to remember the intensive production function $y(t) = f(k(t))$. The function y must therefore start at $y_0 = f(k_0)$. As k grows over time, so does y . In the long run, the value of y approaches $y^* = f(k^*)$.

Finally, we can draw the time path of c in the same way. From the phase diagram, we see that $c(t)$ starts at c_s and grows over time, approaching the value c^* in the long run.

Notice that I have put the natural logarithm of each variable on the axis in these pictures, rather than the variable itself. If I had not done this, and instead put k , y , and c on these axes, the pictures would look qualitatively the same. However, this will not always be true, and for consistency I will always use the logs of variables when we draw time paths.

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These time paths give us the *solution* of the model. The time path for c is the plan that we should write down and give to Crusoe.

As the last part of this study of the Robinson Crusoe model, I want to do some exercises.

4 Comparative Dynamics Exercises

Now that we have the solution to our problem, we might want to ask how the solution would change if something about the environment the Crusoe household lives in were different. For example, suppose Crusoe was less patient. How would the optimal path of consumption change?

This is a standard type of exercise in economics. For example, if you have supply and demand curves for some good, you might ask how much the equilibrium price would change if the demand curve shifted up by $x\%$. This is called a comparative statics exercise. What we want to do is very much in the same spirit, but we want to trace out the change in the solution to our problem over the whole time period; this is why we call it comparative dynamics.

We will do a couple of exercises. The way an exercise works is the following. We want to compare two different worlds, one in which (say) the Crusoe household is fairly patient and one in which they are more impatient. We do this by drawing the phase diagrams and the time paths for the two different solutions, and seeing how they compare.

- Begin with a baseline case (the model we just solved)

- Draw the phase diagram
- Suppose that $k_0 = k^*$ (that is, we start at the steady state of the baseline case)

This is just to make it easier to “see” the answer. If we start at the steady state, we know that we will stay there. So the time paths for the baseline case will all be flat lines.

- The modified case differs in one (or more) parameters. Example: $\rho' > \rho$
- Draw the modified phase diagram, indicating clearly what has changed
- Find the appropriate initial condition (the starting level for k will be the same, but the starting level for c may be different)

I want to think about this exercise in the following way. We are comparing two separate worlds, one where the household is fairly patient and one where the household is less patient. Everything else is the same. We want to see how the optimal consumption plans in these two different worlds compare.

- Goal: Draw the time paths of k and c for both cases

4.1 Different levels of impatience

- Modified case has $\rho' > \rho$

The baseline phase diagram is on the left in the figure below. What does the modified phase diagram (with a larger value of ρ) look like?

- Isocline for k :

$$\dot{k} = 0 \quad \Rightarrow \quad c = f(k) - \delta k$$

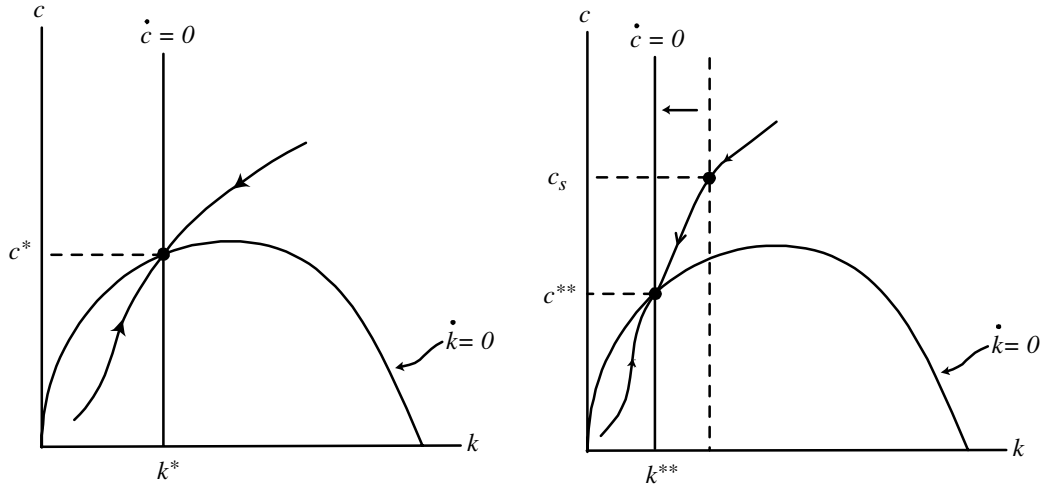
- no change
- Isocline for c :

$$\begin{aligned} \dot{c} = 0 &\quad \Rightarrow \quad f'(k^*) = \delta + \rho' \\ &\Rightarrow \quad f'(k^*) \text{ is larger} \\ &\Rightarrow \quad k^* \text{ is smaller} \quad \{ \text{moves to } k^{**} \} \end{aligned}$$

- isocline for c shifts to the left

The panel on the right side of the figure below shows the phase diagram for the modified case, where the vertical isocline for c is shifted to the left. The modified steady state is at (k^{**}, c^{**}) , where both consumption and the stock of trees is smaller than in the baseline steady state.

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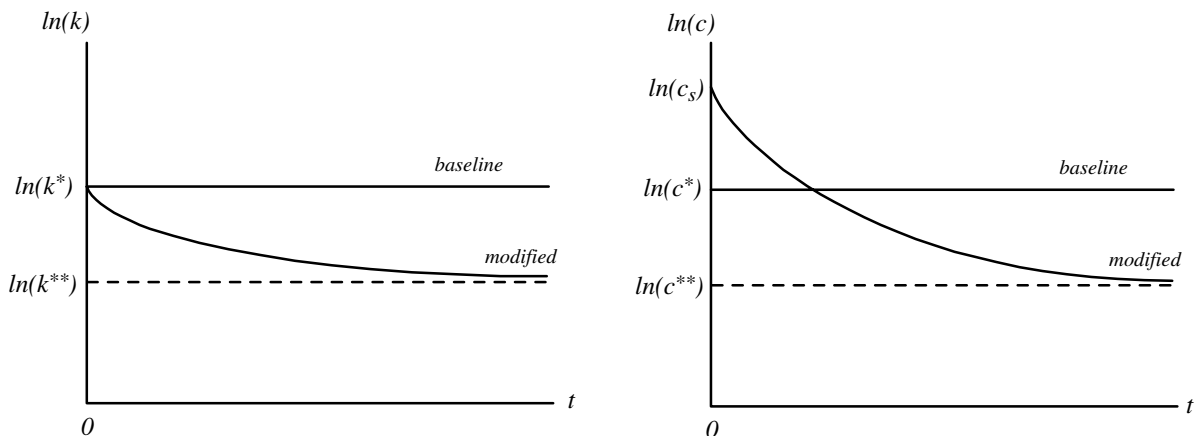


What is the optimal trajectory in the modified case? The initial stock of trees is equal to the baseline steady-state level k^* . What about the initial level of consumption? Is it equal to c^* ?

NO! The level of c must adjust to put the economy on the stable arm of the saddle point for the modified case. The optimal level of $c(0)$ in this case is therefore the value c_S shown in the figure, which is bigger than c^* . If Crusoe instead set $c(0) = c^*$, he would end up violating the transversality condition. The solution to the (modified) optimal growth problem starts at c_S .

We can use these phase diagrams to draw the time paths of k and c .

- Time paths of k and c



The results here are not surprising. If the household is less patient, they should consume more today, which implies planting fewer trees and consuming less in the future.

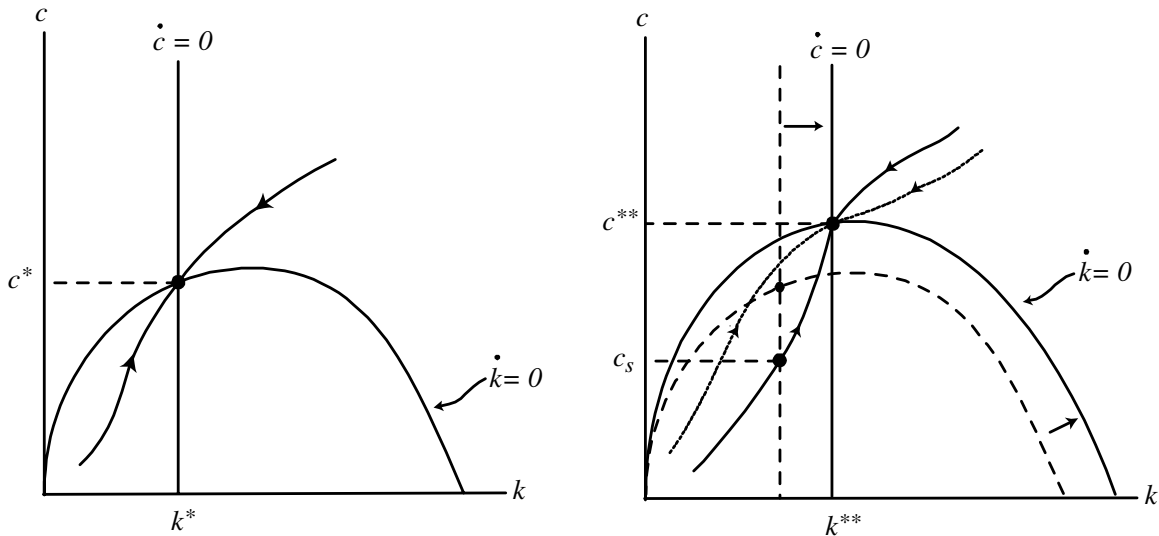
4.2 Different depreciation rates

- Another exercise: $\delta' < \delta$

Comparative dynamics exercises are not always so easy. Suppose we compare our baseline

environment with one where trees die less often. Before we draw anything, let me ask a question: How do you expect the initial level of consumption to be different between the baseline and the modified cases? If trees live longer, should the household initially consume more or plant more? Think about it for a minute.

Now let's go through the same steps as before. First, we draw the baseline phase diagram, and then the modified phase diagram, showing what has changed.



- Isocline for k :

$$\dot{k} = 0 \quad \Rightarrow \quad c = f(k) - \delta'k$$

- the isocline for k rotates upward
- the distance between the baseline and modified isoclines is $(\delta - \delta')k$
- Isocline for c :

$$\begin{aligned} \dot{c} = 0 &\quad \Rightarrow \quad f'(k^*) = \delta' + \rho \\ &\Rightarrow f'(k^*) \text{ is smaller} \\ &\Rightarrow k^* \text{ is larger} \quad \{ \text{moves to } k^{**} \} \end{aligned}$$

- isocline for c shifts to the right

Here it is possible to draw the stable arm of the saddle point for the modified case so that it passes either above or below the baseline steady state at (k^*, c^*) . In other words, consumption may initially be either higher or lower than in the baseline case. Why? How do we know which of the two stable arms in the figure above is the correct one?

The fact that the modified stable arm can be drawn on either side of c^* indicates that there are two opposing effects at work.

- Substitution effect: lower δ makes future consumption less expensive (relative to current consumption) \Rightarrow consume less today (and more in the future)

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- Income effect: lower δ makes household “richer” \Rightarrow consume more today (and more in the future)

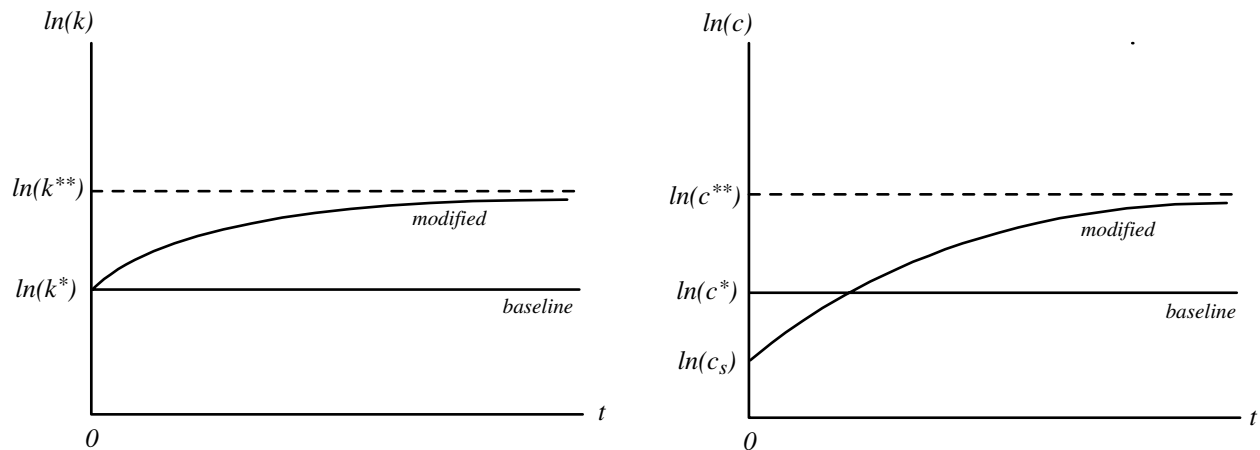
In fact, we can see these two effects in the shifts of the isoclines discussed above. The isocline for k reflects the resource constraint. A lower value of δ implies that net output is higher at $t = 0$ in the modified case, which is what rotates the isocline for k upward. Having more net output tends to increase both consumption and planting; this is the income effect.

The isocline for c reflects the optimal consumption-planting decision. A lower value of δ makes trees more valuable (because they die less often), and this shifts the isocline for c to the right. Making trees more valuable tends to make Crusoe want to consume less and plant more; this is the income effect.

Depending on the specific parameter values, either of these effects may dominate. For this exercise, we will

- Assume substitution effect dominates.

Then the time paths look like the following:



We will do comparative dynamics exercises like these throughout the course, because they allow us to clearly trace out the predictions of the model. Working through such exercises also forces you to think about all of the elements of the model and how they fit together; for this reason your problem sets (and exams) will contain many exercises of this type.

At this point, we (should) understand very well the optimal way for a family living on a deserted island eating only coconuts to behave. The next question is how this analysis relates to modern, market-based economies and the growth of nations. In the next few classes, we will write down a simple model of a market-based economy, and we will investigate the relationship between that model and the results we have derived here.