Rating philosophy and dynamic properties of internal rating systems: A general framework and an application to backtesting

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RATING PHILOSOPHY AND DYNAMIC PROPERTIES OF INTERNAL RATING SYSTEMS: A GENERAL FRAMEWORK AND AN APPLICATION TO BACKTESTING [1]

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EXECUTIVE SUMMARY

The paper draws a general framework for asset and default dynamics, separating the influence of the economic cycle into a component which is embedded in the rating system and an unobservable risk factor that determines the movements of defaults around the ex ante estimated PDs.
The two components – the sensitivity of ratings to credit cycle and conditional asset correlation - can be quantified through a Maximum Likelihood approach, giving a measure of the cyclicality of the rating system, and allowing for a number of applications: among those the modified binomial test proposed here.

KEYWORDS: **rating philosophy, rating dynamics, cyclicality, asset correlation, migration matrices, ML estimation, backtesting, binomial test.**

1. Introduction

A rating system should be able both to distinguish risk in relative order, that is to discriminate among credit quality of risky borrowers, and to quantify risk in absolute terms.

How these tasks are fulfilled depends on the rating system's underlying philosophy, which in practice cannot be either pure Point in Time or pure Through the Cycle but is somewhere in between these two extremes. Understanding the ratings dynamics is thus crucial in order to assess if the system is well functioning and if it is well suited to encompass for stressed economic conditions.

Aim of this paper is to draw a general framework of asset dynamics: asset value is in fact influenced by systemic factors and idiosyncratic risk, the last one being specific for each firm. As far as the systemic part, which mostly interests us, is concerned, we can ideally separate it into a component which is taken into account, explicitly or implicitly, by the rating model, and an unobservable set of risk factors. Both components contribute to explain the default dynamics, but the first one also determines ratings dynamics, while the second one causes, together with idiosyncratic risk, the movements of defaults around the ex ante estimated PDs. This framework allows us to describe cyclicality in a rating system as the ratio between the sensitivity to macroeconomic conditions which is embedded in the rating and the total sensitivity of defaults: if an appropriate tool is found to quantify the ratio, a rating system underlying philosophy can thus be exactly identified.

Let's in fact think about a pure Through the Cycle rating: no systemic component will be taken into account by the estimated PDs, while conditional asset correlations[2] are maximized, as they should account for the whole systemic risk. On the opposite, a pure Point in Time system captures into the rating all economic conditions, so that conditional asset correlation are zero: realized defaults do not match exactly the default probabilities only because of idiosyncratic factors.

The two components of default sensitivity can both be quantified through a Maximum Likelihood approach: the estimation technique is in fact similar, but as far as the part of cyclicality which is embedded into the rating is concerned we used a multinomial model, while for the component which determines the difference between forecasted and realized defaults we estimated a binomial one-factor model.

In order to quantify the internal rating sensitivity to the economic cycle we used in fact a transition matrix approach: migrations among internal risk buckets are observed through time and used to estimate an underlying single risk factor whose volatility can be interpreted as the named

[2] "Conditional" to rating score at time t. Conditional asset correlations represent the portion of asset volatility not explained by the score.

sensitivity. It is in fact well known that counterparts under Point in Time ratings system tend to migrate frequently, in response to changing economic conditions (while default rates for each rating bucket will remain stable over time): the estimated sensitivity is thus bigger the more PIT the system is. This exercise is useful to understand where the internal rating system is positioned, in particular if compared to other systems and to agencies' ratings. The same quantification was in fact replicated on Standard and Poor's migration matrices, revealing a lower sensitivity of ratings to the economic cycle, which is consistent with the common perception that agencies' ratings approach is Through the Cycle.

The second component needed to quantify defaults' sensitivity to economic conditions is the one which is not embedded into the ratings, or the conditional (to rating) asset correlations. We estimated them through a Bernoulli mixture model, maximizing the probabilities to observe historical default data for each rating class. The assumption underlying this conditional independence technique is that a single systemic risk factor influences, with different sensitivities for each rating group, all counterparts: realized defaults are thus independent from each other. Even in this case, rating philosophy matters: as we said before, in a pure PIT model asset correlation tends to zero, being all economic information useful to predict defaults already captured by score variables; conversely, a significantly different from zero value of asset correlation indicates that realized defaults will be correlated, as it happens in Through the Cycle rating systems, and that default rates per rating bucket will thus vary over time. An important application of asset correlation measurement concerns PDs backtesting against realized Default Rates: statistical tests generally used suffer in fact from an implicit independence assumption, or that PDs are able to assess the current state of the economy so that default events among borrowers may be considered stochastically independent. In the present document we propose a modified binomial test built on a distributional form for default rate which accounts for asset correlation: the test is less strict than the standard one, which would be appropriate only for a pure PIT rating system.

As the two ingredients of default response to economic cycle (the part that is already taken into account by the rating and the one which determines default movement around rating) can be consistently quantified, an omni-comprehensive measure of the rating cyclicality which characterize a specific rating system can be calculated as the amount of the first on the total of the two. This has the advantage to range between 0 (pure TTC models) and 1 (pure PIT models), so that the positioning of a rating system can be immediately perceived: in the present application we found for instance a cyclicality of the internal rating of about 60%, far much higher than the value of around 20% calculated for S&P's ratings.

The proposed framework, which is detailed in the next paragraph, describes cyclicality in sufficiently general terms. Paragraph 3 then explains the methodology underlying the transition matrix approach, which is useful to quantify the sensitivity of a rating system to the credit cycle, while paragraph 4 enters into the details of the Bernoulli mixture model. The following paragraph presents the results of an application of the proposed methodology to an internal rating and Standard and Poor's sample, quantifying and comparing the level of rating response to economic cycle of both systems. Paragraph 6 describes one possible empirical application of the asset correlation values, which is dependent on rating philosophy: we suggest in fact a modified binomial test which can be more realistically used for validating a non pure PIT system. Finally, paragraph 7 draws some conclusions and indicates some next steps.

2. A general framework for cyclicality

Our analysis starts from the description of the dynamic of one-year asset value A_{t+1}^{i}, which derives from Basel II IRB framework and can be defined as *one factor model*[3].

The following equations explain the main features of asset growth value, conditional on information set at time t:

$$S_t^i = \beta_W \cdot W_t^i + \beta_Y \cdot Y_t \qquad [2.1]$$

$$A_{t+1}^i \big| t = \alpha_i + S_t^i \big| w_t^i, y_t + \sqrt{\rho_c} \cdot X_{t+1} + \sqrt{1-\rho_c} \cdot \varepsilon_{t+1}^i \qquad [2.2]$$

$$E\left(A_{t+1}^i \big| t\right) = \alpha_i + \beta_W \cdot w_t^i + \beta_Y \cdot y_t$$

$$\sigma\left(A_{t+1}^i \big| t\right) = 1$$

S_t^i is the normalized credit score of the i-th firm, which depends on obligor specific characteristics W_t^i and on the macroeconomic factor Y_t: this is in fact the full set of information available to a bank for assessing the credit quality of obligor i at date t[4]. We assume here that W_t^i and Y_t are uncorrelated standard normal variables, and that, because of its idiosyncratic nature, W_t^i is also uncorrelated across obligors.

Equation [2.2] represents the asset process conditional on rating model information, where $-\alpha_i$ is the long run debt threshold removed in order to set the default status when $A_{t+1}^i \big| t < 0$.

[3] Capitol letters indicate the risk variables and lower-case letters the realizations.

[4] Even if banks do not usually include macroeconomic variables in their rating models, we argue that they are reflected in balance sheet and behavioral indicators; moreover, for the sake of simplicity, the stylized model splits up the effect of the obligor specific part W_t^i from the common economic factor Y_t, so that they become independent.

X_{t+1} is the "state of the economy" variable, or the single systemic factor common to all firms, that cannot be observed by a bank at date t. Conversely, ε_{t+1}^{i} is the idiosyncratic risk, specific for each firm. X_{t+1} and ε_{t+1}^{i} are orthogonal, time independent and both follow a standard Normal distribution.

$A_{t+1}^{i}\big|t$ expected value is $\alpha_i + S_t^i\big|w_t^i, y_t$: the information set underlying the score, that is the linear combination of W_t^i and Y_t, can be thought as being independent of both X_{t+1} and ε_{t+1}^{i} because we assume that the rating model maximizes all available information at time t to predict default events in t+1 (or future asset values).

The factor loading $\sqrt{\rho_c}$, assumed for now fixed for all firms[5], can be interpreted either as the sensitivity to systematic risk or as the (square root of) conditional asset correlation: the term "conditional" refers here to the portion of asset volatility explained given all available information about the score S_t^i in t.

$$\rho_{con}(i,j) = \frac{E(A_{t+1}^{i}\big|t \cdot A_{t+1}^{j}\big|t) - E(A_{t+1}^{i}\big|t)\cdot E(A_{t+1}^{j}\big|t)}{\sigma(A_{t+1}^{i}\big|t)\cdot\sigma(A_{t+1}^{j}\big|t)} = \rho_c$$

The above formula is used to calculate conditional asset correlation between firm i and firm j.

Finally, it's worthwhile pointing out that all the coefficients β_W, β_Y and ρ_c involved in the framework are greater than zero, being each risk factor positively related to credit quality; furthermore, we set $\beta_W = \sqrt{1-\beta_Y^2}$ in order to grant S_t^i standard deviation equals to one.

A value of ρ_c near zero implies that, conditional on information at time t, defaults at time t+1 are independent because all the economic information useful to predict the default event is captured by the score variables. This is consistent with a *Point In Time* philosophy, that considers the rating model (S_t^i) to be the best estimate of obligors default likelihood, reflecting all cyclical and systematic information.

Conversely, a value of ρ_c significantly different from zero means that defaults are correlated among firms; rating score is thus not able to catch all systemic information at time t, whose residual and not predictable part will anyway arise in t+1 through X_{t+1}. Two situations are consistent with a non-zero ρ_c : a "pure" *Through The Cycle* model with $\beta_Y = 0$, where S_t^i reflects only non-cyclical indicators, or a *Hybrid* one, which incorporates features of both TTC and PIT model.

[5] It could depend, as will be shown in the following, on rating grade.

In order to evaluate the degree of "pitness" for the stylized model described above, we need to introduce the concept of unconditional asset correlations: these can be interpreted as the correlations between firms when rating model does not include the systemic contribution Y_t, as it happens under TTC philosophy. Unconditional correlations should be larger than the conditional ones because in this case the rating model would be stable over the cycle, implying a greater volatility of defaults around unconditional default probabilities.

$$A_{t+1}^i \big| w_t^i = \alpha_i + \beta_W \cdot w_t^i + \beta_Y \cdot Y_t + \sqrt{\rho_c} \cdot X_{t+1} + \sqrt{1-\rho_c} \cdot \varepsilon_{t+1}^i \qquad [2.3]$$

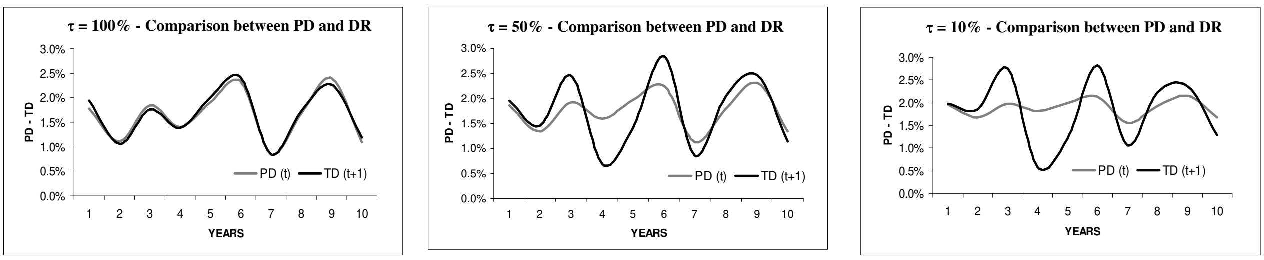
$$\rho_{un}(i,j) = \frac{E(A_{t+1}^i \big| w_t^i \cdot A_{t+1}^j \big| w_t^j) - E(A_{t+1}^i \big| w_t^i) \cdot E(A_{t+1}^j \big| w_t^j)}{\sigma(A_{t+1}^i \big| w_t^i) \cdot \sigma(A_{t+1}^j \big| w_t^j)} = \frac{\beta_Y^2 + \rho_c}{\left(\beta_Y^2 + 1\right)} \qquad [2.4]$$

Equation [2.4] expresses unconditional asset correlation (ρ_{un}) in relation to ρ_c: it becomes thus very easy to prove the inequality $\rho_{un} \geq \rho_c$. The level of cyclicality embedded in the rating model (τ), which we have defined as degree of "pitness", is the contribution of the variable Y_t to the total systemic asset variance:

$$\tau = \frac{\beta_Y^2}{\beta_Y^2 + \rho_c} \qquad [2.5]$$

In a PIT model, where ρ_c tends to zero, τ is near one, while in a TTC one, where $\sqrt{\rho_c}$ is much bigger than β_Y, it approaches zero.

FIGURE 1 Examples of rating system with different τ values. Each graph come from a 10 years simulation comparing average PD (calculated at the beginning of the year using the rating score) and DR (at the end of the year) for 20'000 counterparties.



Until now, we outlined a general framework for cyclicality. Our next task will be to quantify τ, through a building block approach: it considers separately the internal rating sensitivity β_Y and the conditional asset correlations $\sqrt{\rho_c}$ using for both a maximum likelihood estimation technique:

- In the first step, rating sensitivity calculation is based on historical transition matrix: we maximize the migration rates likelihood among internal performing risk classes, relating them to a single risk factor whose volatility is β_Y;

- in the second step, conditional asset correlations $\sqrt{\rho_c}$ and long run PD $\Phi(-\alpha_i)$ (grouped by rating grades) are estimated using a *Bernoulli mixture model*, where the probability of observing the sample default rates for each rating class and sample year are maximized.

3. Internal rating sensitivity through a transition matrix approach

In the proposed framework, we assume that the bank's internal rating model produces a score based on financial and behavioral ratios and grouped into homogeneous risk classes[6].

Let S_t^i be the score of each borrower which is, as it was defined before, made up by an uncorrelated and time independent specific part W_t^i and a systematic part Y_t; at the beginning of year t, scores are grouped in G performing rating (score) grades. Following the *CreditMetrics* approach described by Gupton, Finger, and Bhatia (1997), we assume that one-year transitions between grades reflect an underlying, continuous credit-change indicator (asset) explained in this case by Y_t, a normally distributed "credit rating cycle" variable:

FIGURE 2 Illustration of Y_t impact on migration rates for rating grade 4.

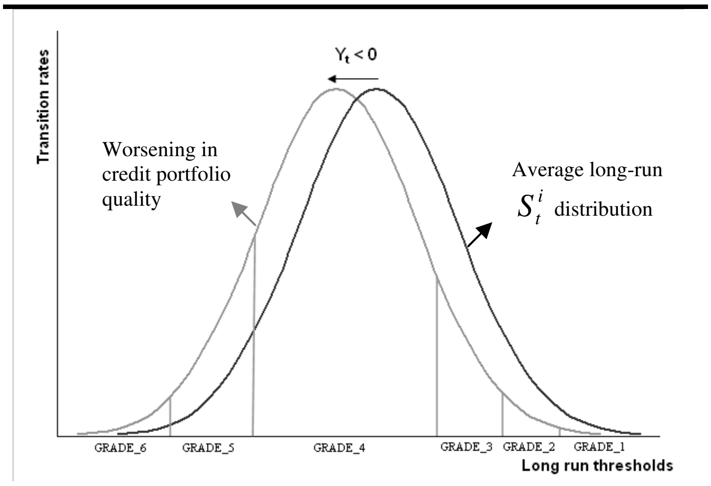


Figure 2 illustrates the distribution of a score initially in grade *g* (in the example grade 4). The score movement is caused by the common cyclical variable Y_t. On the x-axis, long run bins are defined so that the probability (assumed to be normal) that S_t^i falls within a given interval equals

[6] The Basel II IRB approach requires in fact that the score values are mapped on a relatively small number of rating grades (at least seven non-default grades), but leaves their exact number at the institution's discretion. This number will thus depend on the methodology the bank chooses for aggregating, such as cluster analysis (e.g. minimizing and maximizing within and between variance of potential buckets) or kernel density evaluation (in this case one could analyze the non-parametric score distribution and use the observed discontinuity points to assign firms to different buckets). However, the task is in any case to build a mapping function based on similarity rules, which classifies the score in risk classes.

the corresponding historical average transition rate observed for grade g. Theoretical migration rate $\hat{P}\left(S^{g,k}, S^{g,k-1}, Y_t\right)$ (from class g to class k) can thus be calculated in the following way:

$$\hat{P}_t^{g,k} = \hat{P}\left(S^{g,k}, S^{g,k-1}, Y_t\right) = \Pr\left(S^{g,k} < S_t^i < S^{g,k-1}\right)$$

$$\hat{P}\left(S^{g,k}, S^{g,k-1}, Y_t\right) = \Pr\left(S^{g,k} < \beta_Y \cdot Y_t + \sqrt{1-\beta_Y^2} \cdot W_t^i < S^{g,k-1}\right) \qquad [3.1]$$

This relation can be transformed into the following one:

$$\hat{P}\left(S^{g,k}, S^{g,k-1}, Y_t\right) = \Pr\left(\frac{S^{g,k} - \beta_Y \cdot Y}{\sqrt{1-\beta_Y^2}} < W_t^i < \frac{S^{g,k-1} - \beta_Y \cdot Y_t}{\sqrt{1-\beta_Y^2}}\right) = \Phi\left(\frac{S^{g,k-1} - \beta_Y \cdot Y_t}{\sqrt{1-\beta_Y^2}}\right) - \Phi\left(\frac{S^{g,k} - \beta_Y \cdot Y_t}{\sqrt{1-\beta_Y^2}}\right)$$

where $S^{g,k}$ and $S^{g,k-1}$ are the long run thresholds which delimit the grade k range when starting from the initial rating g. Φ is the normal cumulative density function.

β_Y is estimated maximizing the probability of observing historical migration rates, which are, conditional to migration probability as in [3.1], independent and multinomial distributed. The following equation indicates the unconditional likelihood of transition matrix at time t:

$$L\left(\beta_y\right)_t = P\left(N_t^{g,1} = n_t^{g,1}, \ldots, N_t^{g,G} = n_t^{G,G}\right) = \int_{\Re} \prod_{g=1}^{G} \frac{n_t^g !}{n_t^{g,1}! \cdots n_t^{g,G}!} \cdot \left(\hat{P}_t^{g,1}\right)^{n_t^{g,1}} \cdots \left(\hat{P}_t^{G,G}\right)^{n_t^{g,1}} \cdot dF\left(Y_t\right) \qquad [3.2]$$

where

$$n_t^g = \sum_{i=1}^{G} n_t^{g,i}$$

is the total number of observations in rating state g and F is the cumulative normal density function of Y_t factor common to all transitions. The first term of the product is called multinomial coefficient and explains all possible combination of n_t^g firms across all G rating classes, each one containing $n_t^{g,i}$ (i from 1 to G) counterparties. From a statistical point of view, [3.2] quantifies the probability of an experiment repeated n_t^g times where $n_t^{g,i}$ is the number of times (migrations) the different outcomes occurred with probability $\hat{P}_t^{g,i}$. The integral operator is used to generate all possible $\hat{P}_t^{g,i}$ scenarios over Y_t.

Due to the time independency assumption of Y_t, the probabilities of jointly observing all historical migration rates are calculated by the product

$$L\left(\beta_y\right) = \prod_{t=1}^{T} L\left(\beta_y\right)_t$$

which through logarithmic transformation leads to the *Log-Likelihood* (*LL*) function:

$$LL(\beta_y) = \sum_{t=1}^{T} Ln\left[L(\beta_y)_t\right]$$

This expression, made of one-dimensional integrals sum, should be maximized over β_Y.

Unfortunately, there is no analytical solution to this problem because the usual procedure – setting the first derivatives of the likelihood to zero – is not feasible; this expression is in fact tractable only through a numerical approach such as the *gaussian-quadrature* we choose[7].

To sum up, this method allows us to give an estimate of how much cyclical the rating model is: β_Y in fact quantifies the sensitivity of rating scores to the common factor, usually identified with or explained by macroeconomic variables.

4. Asset correlation estimation: a Bernoulli mixture model with rating effect

Estimating (conditional) asset correlations is difficult in practice because of the historical data scarcity and the large number of parameters to be found. A natural solution is to impose some restrictions on parameters: in this case we used an exponential functional form for long run PDs and correlations, which in some way provides for the data span shortness. The method adopted here, called *Bernoulli mixture model with rating effect*, follows a maximum likelihood estimation technique similar to the one described in the previous paragraph: it determines long run PDs and asset correlations such that the probability of observing historical default data for each rating class is maximized.

The main hypothesis here is that, once the S_t^i score has been assigned and grouped in G rating grade, it exists an unobservable systemic risk factor X_{t+1}, shared by all firms and rating groups with different sensitivities, which allows for independence among all realized defaults (conditional independence technique). In addition, the G risk classes are homogeneous enough to assign the same long run PD $\Phi(-\alpha_g)$ and correlations ρ_c^g to all firms within a given risk grade.

In the remainder of the paragraph, we assume the historical performance data for the bank's rating system to be available. For each one of the T years and G rating grades, we observe the number of obligors at the beginning of the year (N_t^g), classified using a mapping function based on score S_t^i value, and the number of those obligors that default by year-end (D_{t+1}^g).

Conditional on systemic risk X_{t+1}, firm's defaults are independent in grade g and can be described as the outcome of a Bernoulli trial with success (default) probability

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7 This approach, like other numerical ones, is normally solved by standard statistical software (e.g. MATLAB or SAS).

$$PD^g \left| X_{t+1} = \Phi\left[\frac{\alpha_g - \sqrt{\rho_c^g} \cdot X_{t+1}}{\sqrt{1-\rho_c^g}}\right]\right. \qquad [4.1]$$

simply recovered analyzing and solving g-grade asset process in the following way:

$$PD^g \left| X_{t+1} = P\left(A_{t+1}^g < 0\right) = P\left(\alpha_g + \sqrt{\rho_c^g} \cdot X_{t+1} + \sqrt{1-\rho_c^g} \cdot \varepsilon_{t+1}^i < 0\right)\right.$$

where

$$\alpha_g = \Phi^{-1}\left(PD^g\right)$$

PD^g and ρ_c^g are respectively the long run (unconditional) default probability and conditional asset correlation to be estimated. At time t, when S_t^i scores are properly classified, they determine the distribution of obligors across rating grades: consequently, the score variables do not appear any more in the process A_{t+1}^g, but influence N_t^g in describing the distribution of defaults number and thus also ρ_c^g estimate. The g-grade number of defaults follows in fact a binomial distribution $L\left(D_{t+1}^g \middle| X_{t+1}\right)$, with probability stated in [4.1]:

$$L\left(D_{t+1}^g \middle| X_{t+1}\right) = \binom{N_t^g}{D_{t+1}^g} \cdot PD^g \left|X_{t+1}\right.^{D_{t+1}^g} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - D_{t+1}^g} \qquad [4.2]$$

Since defaults are also conditionally independent across grades thanks to the uniqueness of systematic risk X_{t+1}, the joint likelihood $D_{t+1} = \sum_g D_{t+1}^g$ is just the product of the G conditional ones (*Bernoulli* distributions) defined in [4.2]:

$$L\left(D_{t+1} \middle| X_{t+1}\right) = \prod_{g=1}^{G} \binom{N_t^g}{D_{t+1}^g} \cdot PD^g \left|X_{t+1}\right.^{D_{t+1}^g} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - D_{t+1}^g} \qquad [4.3]$$

The unconditional likelihood is thus calculated integrating equation [4.3] over all possible outcome of X_{t+1}

$$L\left(D_{t+1}\right) = \int_{\Re} \prod_{g=1}^{G} \binom{N_t^g}{D_{t+1}^g} \cdot PD^g \left|X_{t+1}\right.^{D_{t+1}^g} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - D_{t+1}^g} \cdot dF\left(X_{t+1}\right)$$

where $F\left(X_{t+1}\right)$ is the normal cumulative density function. If we maintain the hypothesis that X is time independent, we can represent the probability of total sample default as in the following equations:

$$L(D) = \prod_{t=1}^{T} \int_{\Re} \prod_{g=1}^{G} \binom{N_t^g}{D_{t+1}^g} \cdot PD^g \left|X_{t+1}\right.^{D_{t+1}^g} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - D_{t+1}^g} \cdot dF\left(X_{t+1}\right) \qquad [4.4]$$

$$LL(D) = \sum_{t=1}^{T} \log\left(\int_{\Re} \prod_{g=1}^{G} \binom{N_t^g}{D_{t+1}^g} \cdot PD^g \left|X_{t+1}\right.^{D_{t+1}^g} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - D_{t+1}^g} \cdot dF\left(X_{t+1}\right)\right) \qquad [4.5]$$

[4.5] indicates the Maximum Log-Likelihood (*LL*) function we have to maximize over PD^g and ρ_c^g parameters, given the observed values of N_t^g and D_t^g [8].

Rather than directly estimate PD^g and ρ_c^g, we can express these parameters in a more parsimonious way, through a monotonous function such as the exponential. We propose in fact two alternative ways for estimating:

- hp1: $PD^g = e^{\alpha_1 + \beta_1 \cdot g}$ and ρ_c constant across grades as in the [2.2] asset equation;
- hp2: $PD^g = e^{\alpha_1 + \beta_1 \cdot g}$ and $\rho_c^g = e^{\alpha_2 + \beta_2 \cdot PD^g}$ depending on credit quality and thus allowing for a "rating effect".

In the first case, only three parameters need to be estimated: the intercept α_1, the slope β_1 defining long run PD and the conditional asset correlation ρ_c. In the second, α_2 defines the level and β_2 the relationship between asset correlations and long run PDs. β_2 is expected to be negative, as suggested by both empirical evidence and economic consistency: to a higher borrower's risk is associated a stronger idiosyncratic component, meaning that default probability depends less on the overall state of the economy and more on individual risk drivers.

In order to explore the reliability of the estimated parameters, we simulated their sample distribution through a Monte Carlo technique. The main purpose of the simulation is to check the robustness and significativity of parameters and in particular to test the hypothesis of $\rho_c > 0$ and $\beta_2 < 0$. A second issue to be analyzed is the entity of asset correlation (downward) bias, which typically occurs in small-sample estimation[9]. Assuming that the model is correctly specified, *LL* estimators will in fact be asymptotically consistent in the sense that the estimated parameters will approach the true ones as the number of *T* years of performance data gets increasingly large: unfortunately, in real-world applications, we have to deal with data span shortness, as it is very infrequent to observe a default dataset covering a sufficient number of years, particularly when referring to internal rating models.

Thus, as there is no guarantee that *LL* will produce unbiased parameter estimates, it was decided to check the magnitude of the bias and verify if it can be considered as negligible.

So as to perform the Monte Carlo simulation, we drew several historical default paths and maximized the Log-Likelihood function for each year, over α_1, β_1, ρ_c in hp1 and over α_1, β_1, α_2, β_2 in hp2. It was thus necessary to:

1. specify a probability distribution apt to describe empirical default data: in this case, equations [4.1] – [4.4] with *LL* parameters;

[8] Also in this case we solved the integral numerically, as explained in the previous paragraph.
[9] This phenomenon was studied in many empirical works.

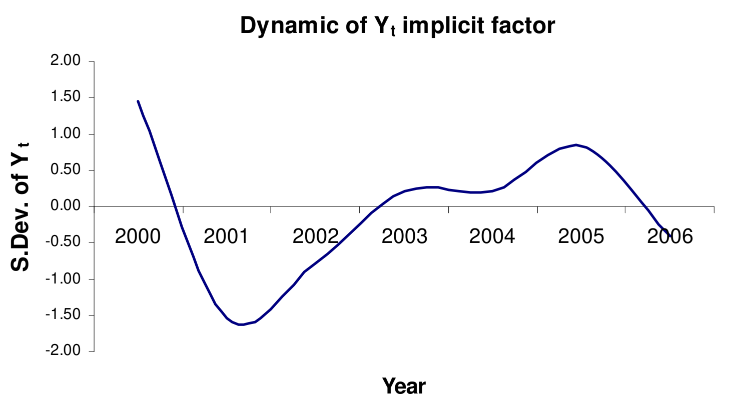
2. randomly draw a hypothetical dataset from the distribution specified in step 1: for each year, draw the time independent X_{t+1} systemic factor, calculate g-grade conditional probability $PD^g \left| X_{t+1} \right.$ and finally extract the number of defaults from a binomial distribution where N_t^g is the fixed number of firms at the beginning of year t;
3. determine the *LL* estimators (α_1, β_1, ρ_c or α_1, β_1, α_2, β_2) on the basis of the simulated data from step 2;
4. repeat steps 2 and 3 several times to trace the parameters sample distribution.

5. Empirical evidence: application of the methodology on an internal rating model and comparison with S&P ratings

The dataset used to estimate contains about 61'000 Italian firms belonging to the corporate segment and covers seven years of defaults data, from 2000 to 2006. Each firm is evaluated through an internal rating model, based on balancesheet and behavioral ratios combined with a logistic approach: the output is a credit score, finally grouped in 15 homogeneous classes of increasing risk level built by cluster analisys.

First of all we estimated the rating sensitivity β_Y as described before, obtaining a value of 1.99%. The implied systemic factor Y_t ("credit cycle"), calculated through the minimization of the quadratic distance between theoretical and observed one-year transition rates, shows the following trend:

FIGURE 3 Trend over time of Y_t



What emerges from the graph is a negative fluctuation in 2001-2002 (twin towers, financial crisis) and a positive economic growth in 2003-2004-2005 followed, in 2006, by a downturn which is likely to continue in the following years.

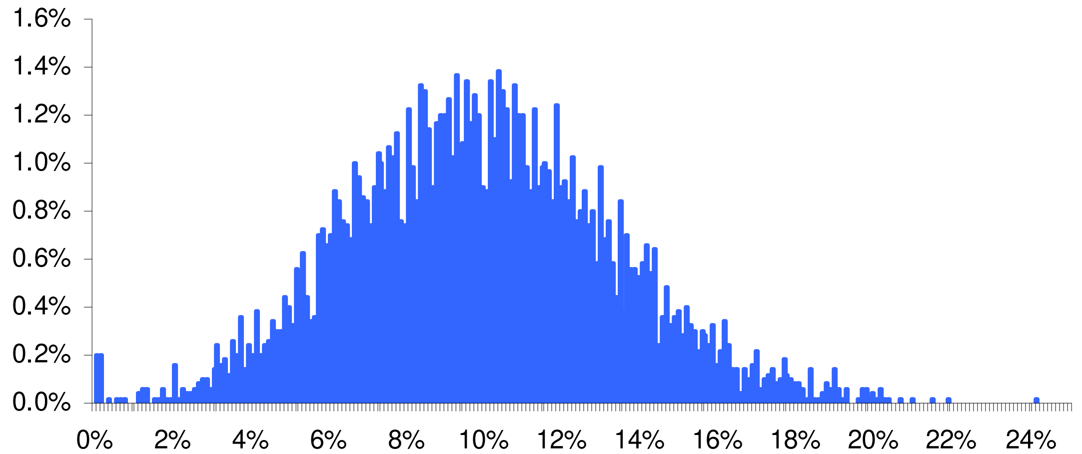
TABLE 1 Hp1: *Bernoulli mixture model* in the case of asset correlations not differentiated per rating class. Statistics are generated simulating 5000 sample of 7 years default paths.

	Estimates	Mean	Median	σ	Δ%Bias	P2.5%	P5%	P95%	P97.5%
α_1	-8.121	-8.134	-8.127	0.223	0.16%	-8.587	-8.512	-7.768	-7.712
β_1	0.433	0.434	0.434	0.013	0.20%	0.409	0.413	0.457	0.461
ρ_c	1.299%	1.092%	0.957%	0.707%	-15.97%	0.13%	0.21%	2.44%	2.86%
$\sqrt{\rho_c}$	11.397%	9.884%	9.784%	3.385%	-13.27%	3.57%	4.56%	15.61%	16.91%

As far as conditional asset correlations are concerned, we present the results of *LL* optimization under Hp1, where a single ρ_c is estimated. Table 1 summarizes *LL* estimates in the first column, and the statistics deriving from simulation in the following ones: mean, median, standard error, bias (defined as percentage ratio between mean and *LL* estimates) and some percentiles of the bootstrapped samples.

Long run PD parameters α_1 and β_1 show a low standard error and are significantly different from zero: in particular, the slope β_1 indicates that the rating model discriminates quite well among rating grades; moreover, the upward bias we found seems to be rather small and probably would disappear when increasing the number of simulations. ρ_c assumes a low value and presents a huge standard deviation, in relation to the average, which anyway becomes lower considering $\sqrt{\rho_c}$, or the sensitivity, as represented in figure 4.

FIGURE 4 Empirical distribution of conditional asset sensitivity $\sqrt{\rho_c}$, derived from 5000 trials of 7-years default samples



As it is shown in the graph, the hypothesis of zero asset correlation can be refused, since the probability of observing a null value is about 0.4%. Furthermore, 2.5th and 5th percentiles are 0.13% and 0.21% for ρ_c, 3.57% and 4.56% for $\sqrt{\rho_c}$; in other words, the independence assumption which would support a pure PIT philosophy seems not to be justified.

The skewed shape of the distribution in figure 4 suggests the existence of a downward bias, mainly due to the historical data series shortness (T)[10]. It is anyway in line with the evidences presented in previous studies[11], and could be taken into account through a prudential (e.g. 13.27%) add-on on estimated sensitivity[12], in order to get a simulated mean roughly corresponding to the value we think to be the "true" one.

Combining the results for rating sensitivity (β_Y) and for conditional asset correlation (ρ_c) we obtain a value for τ, or the level of cyclicality embedded in the rating model, equal to 60% (see table 3).

The same type of analysis was also applied on Standard & Poor's data[13], in order to compare the level of cyclicality of the two rating systems.

The comparison is anyway not completely fair because of some differences in the dataset, as for instance: the data span, which, being for Standard & Poor's much longer (from 1981 to 2003) and thus covering more than one credit cycle, is probably linked to a less stable default rate; the number of rating classes, as transition matrices were calculated for S&Ps on coarse rating grades (7 performing risk buckets). Furthermore, the internal portfolio is the result of customers selection for credit quality and of diversification strategies, which leads to a lower default volatility.

The analysis on S&Ps data leads to a rating sensitivity β_Y of 1.34%, while the following table summarizes the estimates for ρ_c and the related statistics. We notice that the ρ_c's downward bias between simulated mean and estimate is lower than for the internal model, due to the longer time series.

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[10] This phenomenon is in fact negatively related to long run PD level and tends to disappear when the number of years T increases.

[11] See for instance Gordy & Heitfied (2002), Dullman & Scheule (2003), Demey et al. (2005)

[12] This is consistent with what Loffler & Posch (2007) suggested.

[13] "Special report, rating performance 2003", Standard & Poor's 02/2004.

TABLE 2 *Bernoulli mixture model* applied on S&P data from 1981 to 2003. Asset correlations are not differentiated among rating class and statistics are generated simulating 5000 sample of 23 years default paths

	Estimates	Mean	Median	σ	Δ%Bias	P2.5%	P5%	P95%	P97.5%
α_1	-12.175	-12.204	-12.193	0.389	0.24%	-12.987	-12.862	-11.589	-11.468
β_1	1.570	1.574	1.573	0.052	0.27%	1.477	1.492	1.661	1.681
ρ_c	5.205%	4.951%	4.788%	1.684%	-4.89%	2.114%	2.472%	8.002%	8.750%
$\sqrt{\rho_c}$	22.814%	21.923%	21.881%	3.802%	-3.91%	14.540%	15.724%	28.287%	28.287%

Table 3 compares the parameters for the two rating models: consistently with expectations, correlations are higher and rating sensitivity is lower for S&P. τ - the level of cyclicality embedded in the rating model - is thus much lower for S&Ps data, with a value of about 20%.

TABLE 3 Estimated parameters for equation [2.4] – [2.6] to assess the degree of cyclicality τ. Comparison between internal and agency models.

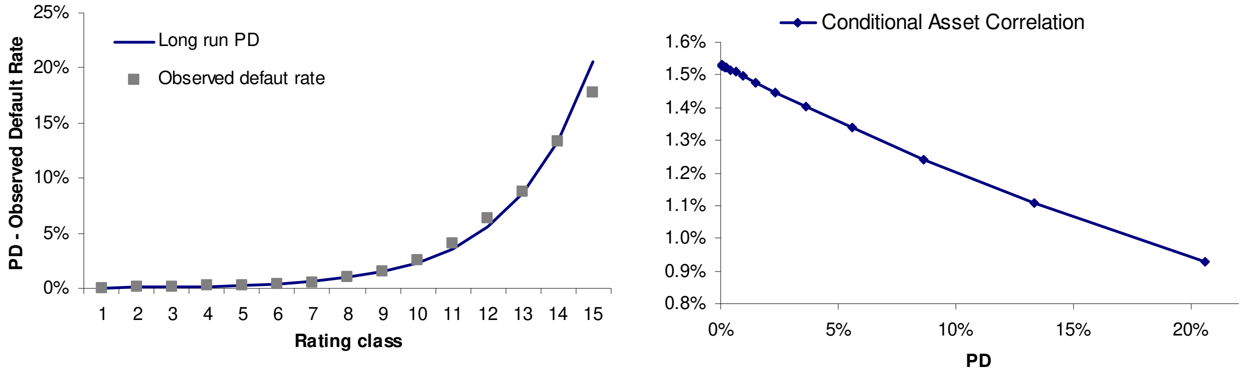
	ρ_c	β_Y	ρ_{un}	$\sqrt{\rho_{un}}$	τ
Internal model (2000-2006)	1.29%	1.99%	3.23%	17.97%	60.51%
S&P (1981-2003)	5.21%	1.34%	6.46%	25.41%	20.52%

As far as the component of asset correlation is concerned, we estimated the parameters also for the second hypothesis referred to in the previous pages, or the one which considers asset correlation as negatively dependent on PD. This was done only on internal data, in order to use the results for the binomial test application. Table 4 summarizes the parameters values and the related statistics, while figure 5 plots the results for PDs and asset correlations.

TABLE 4 Hp2: *Bernoulli mixture model* where asset correlations depend on rating class. Statistics are generated simulating 5000 sample of 7 years default paths

	Estimates	Mean	Median	σ	Δ%Bias	P2.5%	P5%	P95%	P97.5%
α_1	-8.172	-8.188	-8.127	0.246	0.82%	-8.690	-8.601	-7.797	-7.730
β_1	0.436	0.437	0.434	0.015	0.27%	0.410	0.414	0.462	0.468
α_2	-4.179	-4.608	0.010	0.981	10.27%	-6.652	-6.104	-3.479	-3.319
β_2	-2.433	-2.802	0.098	5.522	15.15%	-13.620	-12.850	6.357	7.695

FIGURE 5 Hp2 estimation results. Long run PD compared with sample default rates using α_1 and β_1 parameters on the left, conditional asset correlation using α_2 and β_2 on the right side.



Applying α_2 and β_2 coefficients, we found asset correlation ranging from 1.53% to 0.95% (the related sensitivity goes from 12.36% to 9.75%); β_2 slope is negative as expected but not significantly greater than zero (2.5th and 97.5th percentiles are in fact -13.62 and 7.6 including zero value).

Finally, if we compare the level of the asset correlation with those settled for corporate risk-weight supervisory formula[14], we find that our estimates are considerably lower. Basel II corporate sensitivities, which depends negatively on PD and firm size, lie in fact within a range of about 35%-45%, compared to the internal ones that range from 9.75% to 12.36%. This strong difference is of course influenced by the fact that Basel II correlations are unconditional: however, even if we had used the internal sensitivity derived from unconditional asset correlation presented in table 3, we would not have joined the supervisory lower bound. The main reasons that could explain this gap are that:

- Basel II correlations incorporate a certain degree of conservatism because they are derived for capital purposes and thus calculated at a stressed level;
- the historical period for internal estimation might be too short (2000- 2006) so that default rates appear to be more stable than they would have been over a longer time window;
- as already said, the internal portfolio is selected, thus showing better credit quality, higher diversification and lower default volatility than average.

6. Backtesting hybrid PD through a correlated binomial distribution

If up to now the effort to define and quantify the degree of cyclicality of a rating system may seem to be a pure theoretical theme, some practical applications of this exercise can be found in a

[14] "An Explanatory Note on the Basel II IRB Risk Weight Functions", BIS, July 2005.

number of fields, like backtesting, benchmarking, stress testing. In the following we will just explore one among the different issues, i.e. the backtesting as a tool for validation.

As stated in the WP14, supervisors and risk managers: "will need to understand how a bank assigns risk ratings and how it calculates default probabilities in order to accurately evaluate the accuracy of reported PDs"; "will not be able to apply a single formulaic approach to PD validation because dynamic properties of pooled PDs depend on each bank's particular approach to rating obligors. …. will have to exercise considerable skill to verify that a bank's approach to PD quantification is consistent with its rating philosophy"; "to effectively validate pooled PD's, …. will need to understand the rating philosophy applied by a bank in assigning obligors to risk buckets".

The same idea that validation techniques should take into account the underlying rating philosophy turns up also in the Capital Adequacy Directive, where it is said that "credit institutions shall have sound internal standards for situations where deviations in realised PDs, LGDs […] from expectations become significant enough to call the validity of the estimates into question. These standards shall take account of business cycles and similar systematic variability in default experience."

Statistical tests generally used for backtesting, or to assess the distance between PD and DR (binomial, Hosmer-Lemeshow, and Mean Square Error), suffer from the independence assumption. They are in fact implicitly assuming that PDs are able to reflect the current state of the economy, so that default events among borrowers may be considered stochastically independent and so driven by orthogonal specific factors. From the point of view of the regulator (as it is for instance expressed in the Working Paper 14) this kind of tests go in the desired prudential direction: e. g. the binomial test is a one-side test, apt to detect if the ex ante PDs underestimate the realized defaults, but not a mis-calibration in terms of overestimation of PDs. Furthermore, from a statistical point of view this approach is very conservative in stating the distance between PDs and DRs. This framework can in fact only reasonably be used with PIT rating when conditional asset/default correlations are zero, while in all other cases the probability of rejecting the correct calibration hypothesis is higher than the "true" one. At the other extreme, there is the stylized TTC model, where unconditional correlations reach their highest level, thus maximizing the bias of the standard binomial confidence intervals with respect to the "true" ones, or those that would be calculated if correlations were taken into account.

In the following paragraphs, we will illustrate an example of how the conditional asset correlation we calculated for the internal rating system (classified as hybrid) can be used to modify the standard binomial test: the aim is to get the best-suited confidence intervals according to the cyclicality degree, even if we still apply a one-side approach.

Generally, in the standard binomial test used for backtesting model calibration, we test the null hypothesis (Hp0) that stand-alone PD of a rating category is correct against the alternative (Hp1) of a default rate underestimation. This is a one-side test and can be represented, given a confidence level α (e.g. 95%), as in the following:

$$P\left(D_{t+1}^{g} \le k_{ind}^{g}\right) = \sum_{i=k_{ind}^{g}}^{N_t^g} \binom{N_t^g}{i} \cdot \left(PD^g\right)^i \cdot \left(1 - PD^g\right)^{N_t^g - i} \qquad [6.1]$$

$$k_{ind}^{g*}(\alpha) = \min\left\{k_{ind}^{g} \,\middle|\, P\left(D_{t+1}^{g} \le k_{ind}^{g}\right) \le 1 - \alpha\right\} \qquad [6.2]$$

$P\left(D_{t+1}^{g} \le k_{ind}^{g}\right)$ is the cumulative binomial distribution of D_{t+1}^{g} future theoretical default, N_t^g is the number of firms in g-grade at the beginning of period t, $k_{ind}^{g*}(\alpha)$ is the maximum number of default we observe for α confidence level, under the assumption of independence. In this case, the null hypothesis is rejected if the observed number of default is greater than or equal to $k_{ind}^{g*}(\alpha)$.

Once we introduce asset dependency according to the parameterization shown in table 4, [6.1] becomes

$$P\left(D_{t+1}^{g} \le k_{cor}^{g}\right) = \int_{\Re} \sum_{i=0}^{k_{cor}^{g}} \binom{N_t^g}{i} \cdot PD^g \middle| X_{t+1}{}^{i} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - i} \cdot dF\left(X_{t+1}\right) \qquad [6.3]$$

where $k_{cor}^{g*}(\alpha)$ is calculated as in [6.2] but through a numerical integration method or Monte Carlo simulation[15].

A further interesting method for backtesting is the validation of total default rate, also viewed as a joint test on rating class PDs. In this case, a copula approach is needed (usually called *factor gaussian copula model*), in fact, resorting to conditional independence assumption:

$$P\left(D_{t+1} \le k_{cor} = \sum_{g=1}^{G} k_{cor}^{g}\right) = E\left(\prod_{g=1}^{G} \sum_{i=0}^{k_{cor}^{g}} \binom{N_t^g}{i} \cdot PD^g \middle| X_{t+1}{}^{i} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - i}\right)$$

$$P\left(D_{t+1} \le k_{cor} = \sum_{g=1}^{G} k_{cor}^{g}\right) = \int_{\Re} \left(\prod_{g=1}^{G} \sum_{i=0}^{k_{cor}^{g}} \binom{N_t^g}{i} \cdot PD^g \middle| X_{t+1}{}^{i} \cdot \left(1 - PD^g \middle| X_{t+1}\right)^{N_t^g - i}\right) \cdot dF\left(X_{t+1}\right) \qquad [6.4]$$

At this stage, the aim is to calculate the observed total portfolio number of defaults and then compare it with the theoretical $k^*(\alpha)$ at a given a confidence level. Under the assumptions of

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[15] The latter consists in generating the X_{t+1} variable contained in $PD^g \middle| X_{t+1}$ and then randomly inverting the binomial cumulative function to recover the defaults number (i). Through iteration of the process, it's possible to trace the stand alone class g defaults distribution and thus determine $k_{cor}^{g*}(\alpha)$.

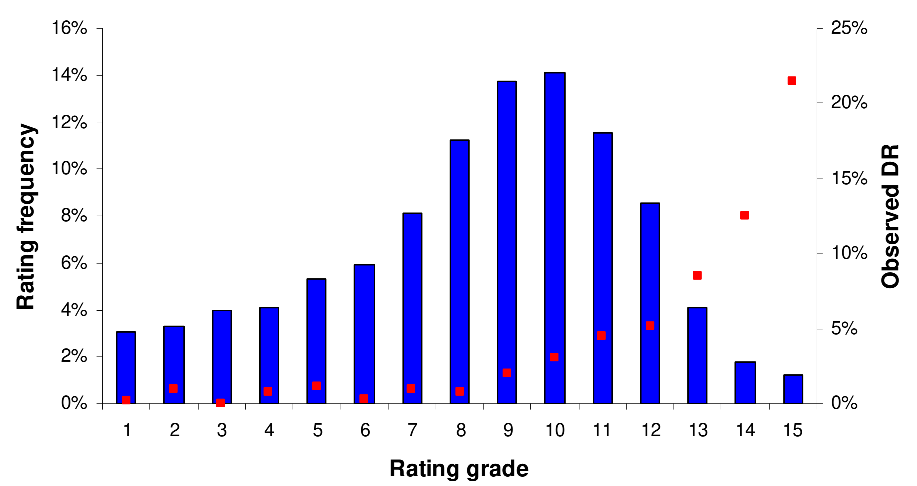
independence or correlation we'll call it respectively $k^*_{ind}(\alpha)$ and $k^*_{cor}(\alpha)$. For the latter we sketch the algorithm below:

1) generate a realization x_{t+1} of X_{t+1};
2) for each g grade, substitute x_{t+1} into $PD^g \big| X_{t+1}$ where $PD^g = e^{\alpha_1+\beta_1 \cdot g}$ and $\rho^g_c = e^{\alpha_2+\beta_2 \cdot PD^g}$ (table 4 parameters);
3) generate g-grade independent k^g_{cor} defaults from the binomial distributions (inside the [6.3]) and sum up the portfolio default number $k_{cor} = \sum_{g=1}^{G} k^g_{cor}$;
4) repeat step 1 to 3 many times;
5) compute the whole distribution and calculate $k_{cor}(\alpha) = \min\{k_{cor} \big| P(D_{t+1} \le k_{cor}) \le 1-\alpha\}$.

Under the independence assumption, we adopt the same methodology starting from point 3 but with a consistent estimation of long run PD, using [4.5] without asset correlation parameters and thus removing integral treatment[16]. This slightly different PD calibration is also applied to stand-alone test.

Next step is to build a real case study in order to compare standard binomial with binomial test accounting for estimated correlations. For the purpose of illustration, we propose a realistic corporate portfolio at year t composed by 16'000 firms, with the following rating and t+1 defaults distribution:

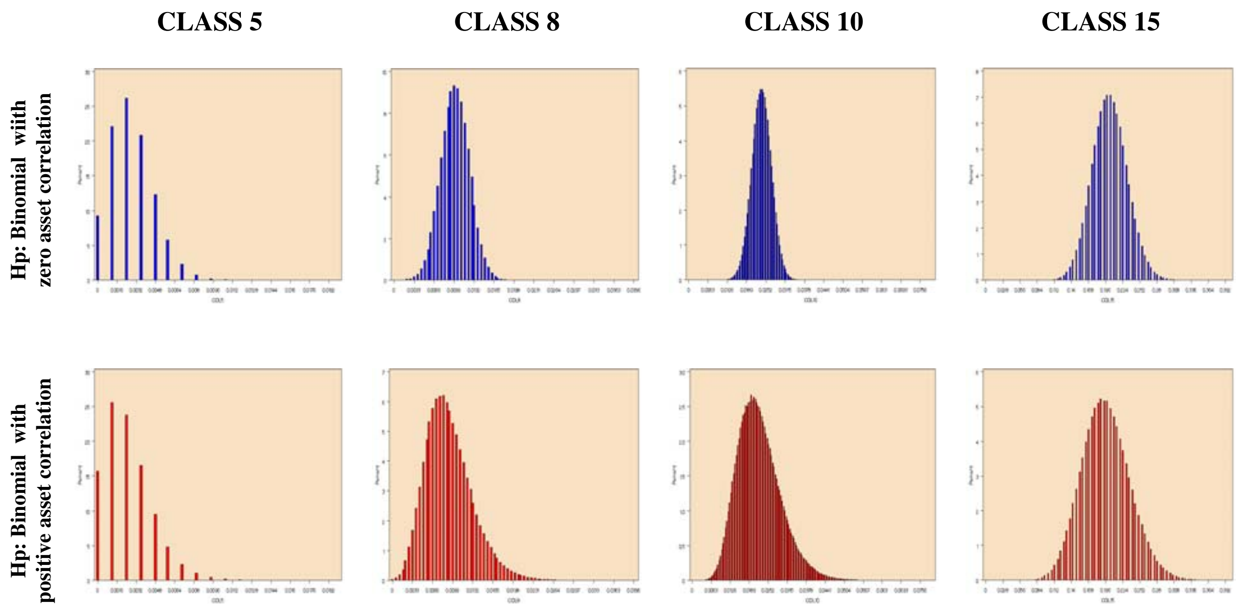
FIGURE 6 Corporate rating distribution at the beginning of year t (left y axis) and observed defaults (right axis).



[16] This estimation leads to α_1=-8.025, β_1=0.429 and *LL*(D)=-258.41 while in table 4 we found α_1=-8.172, β_1=0.436 and *LL*(D)=-228.43. As we expected, the performance expressed by *log-likelihood* is lower although we observe a slight increase in long run PD; this is essentially due to the fatter tail of default distribution when estimation is conducted under asset correlation assumption, implying a decrease in the mean value.

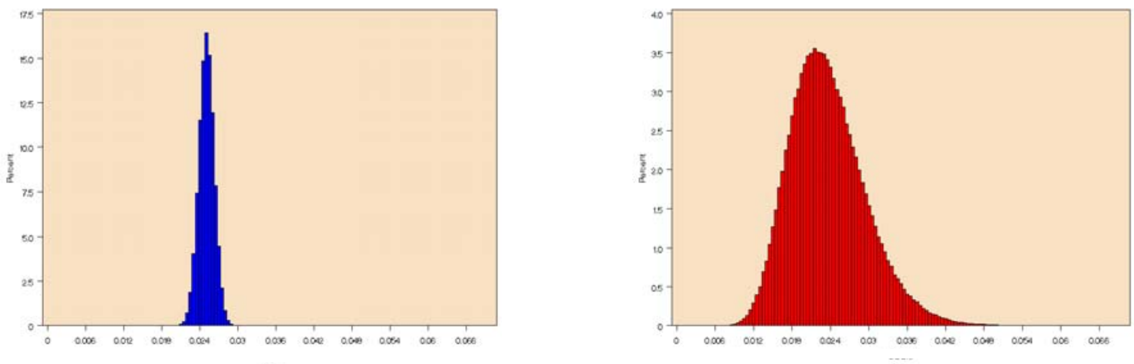
Average PD calculated under correlation assumption (table 4 parameters) is 2.36%, whereas the average PD in the case of asset independency is 2.50% (the two values are different as PDs are endogenously estimated according to different calibrations). Figure 7 outlines the difference in shape between simulated stand-alone default rate distribution for some rating classes (classes 5-8-10-15), according to the two assumptions (thus of the using [6.1] and [6.3] with 500'000 trials):

FIGURE 7 Comparison between independent (blue) and correlated (red) binomial default rate distribution.



Furthermore, next figure compares in the same way the portfolio default distribution we traced according to the above explained algorithm for copula implementation:

FIGURE 8 Comparison between independent (blue) and correlated (red) portfolio default rate distribution.



When the whole default distribution is calculated, the granularity-effect, related to the independency assumption and due to the compensation of specific risk among risk grades, is stronger: this can be noticed in the shape of the blue distribution, which is more compressed around its mean than in the single class cases.

Since our intention is to evaluate the reasonability of PD forecast, we build table 5, where observed default rates (DR) are compared to the 95th and the 99th percentiles of the theoretical distribution: in the right part of the table, statistics are based on the estimated coefficients shown in table 4 for Bernoulli mixture model, while in the left one binomial distributions without correlation assumption are computed for comparison. In the last row, figures refer to the whole portfolio distribution.

TABLE 5 Summary statistics on defaults rates distributions under different correlation assumptions. To make comparison homogeneous, all numbers are computed by Monte Carlo simulation with 500'000 draws.

		Binomial distribution with no correlations					Binomial distribution with correlations				
Rating	**DR**	**Mean**	**Median**	$\frac{k_{ind}^{g*}(95\%)}{N_t^g}$	$\frac{k_{ind}^{g*}(99\%)}{N_t^g}$	**p-value (DR)**	**Mean**	**Median**	$\frac{k_{cor}^{g*}(95\%)}{N_t^g}$	$\frac{k_{cor}^{g*}(99\%)}{N_t^g}$	**p-value (DR)**
1	0.206%	0.050%	0.000%	0.206%	0.412%	2.55%	0.044%	0.000%	0.206%	0.412%	2.25%
2	0.943%	0.077%	0.000%	0.377%	0.377%	0.00%	0.068%	0.000%	0.377%	0.377%	0.00%
3	0.000%	0.118%	0.157%	0.314%	0.472%	52.90%	0.104%	0.000%	0.314%	0.472%	46.60%
4	0.769%	0.181%	0.154%	0.462%	0.615%	0.14%	0.162%	0.154%	0.462%	0.615%	0.26%
5	1.176%	0.279%	0.235%	0.588%	0.824%	68.60%	0.250%	0.235%	0.588%	0.824%	58.76%
6	0.316%	0.428%	0.421%	0.842%	0.947%	57.98%	0.388%	0.316%	0.842%	1.158%	46.97%
7	0.923%	0.657%	0.692%	1.000%	1.154%	8.70%	0.599%	0.538%	1.154%	1.462%	11.69%
8	0.778%	1.008%	1.000%	1.389%	1.556%	80.42%	0.926%	0.889%	1.611%	2.056%	58.25%
9	2.045%	1.548%	1.545%	2.000%	2.182%	2.42%	1.432%	1.364%	2.364%	2.909%	11.31%
10	3.017%	2.377%	2.396%	2.884%	3.106%	1.97%	2.215%	2.130%	3.505%	4.259%	12.64%
11	4.494%	3.648%	3.628%	4.385%	4.656%	2.28%	3.427%	3.303%	5.252%	6.226%	13.75%
12	5.109%	5.600%	5.620%	6.642%	7.080%	76.75%	5.298%	5.182%	7.810%	9.124%	50.39%
13	8.449%	8.596%	8.602%	10.445%	11.214%	52.50%	8.195%	7.988%	11.674%	13.518%	40.79%
14	12.500%	13.190%	13.214%	16.429%	18.214%	59.11%	12.674%	12.500%	17.857%	20.357%	47.72%
15	21.429%	20.256%	20.408%	25.000%	27.041%	30.46%	19.603%	19.388%	26.531%	29.592%	28.77%
TOT	2.800%	2.500%	2.500%	2.700%	2.781%	0.62%	2.357%	2.300%	3.425%	4.006%	20.92%

Looking at stand alone rating class, the standard binomial test would reject the hypothesis of correct calibration for seven grades (1,2,4,5,9,10,11) at 95% confidence level and for three grades (2,4,5) at 99%. We are facing a situation slightly less conservative when we introduce correlation parameters, because the test rejects the null for four grades (1,2,4,5) at 95% and for two grades (2,4) at 99%. However, the most relevant difference concerns the test performed on the entire portfolio: here, as far as default independency is concerned, we would not accept the bank's forecast as adequate because the probability to observe a default rate greater than 2.8% is only 0.62% ("p-value DR" in table 5). This probability becomes much less extreme in the case of asset dependency (20.92%), suggesting that the model and the calibration are not yet to be revised. This remark is deemed convincing only if a bank can explain somehow the dependency structure of its portfolio, for example by statistical evidence based on historical defaults, as we did. If in such a situation the standard binomial test was applied, the proper size of type I errors (rejection of the null hypothesis when it is true) would be higher than the α-level of confidence indicated by the test.

7. Conclusion

The paper presents a general framework of asset and default dynamics, which separates the cyclicality effect into a component which is embedded into the rating system and another one which explains the fluctuation of realized defaults around the ex-ante calculated probability of default. This framework allows us to detect the point where the rating system is situated in between the two purely theoretical extremes of Point In Time and Through The Cycle.

Understanding and quantifying the philosophy which characterizes a system, and the implied rating dynamics, is crucial for a number of issues, like validation, pricing, stress testing, economic and regulatory capital. In this paper some results were presented regarding validation, and specifically a method to estimate asset correlations was suggested which can be usefully applied by banks to modify the standard tests that suffer from independence assumption.

Still there are applications to other fields that can benefit from the cyclicality framework we sketched and that are still to be explored. As far as migration analysis is concerned, it is directly applicable to stress testing: in the proposed framework the cyclical (systemic) variable is not identified, so that scenarios can only be expressed in terms of percentiles. Further work could anyway go in the direction of explaining this factor, at least partially, by macroeconomic variables, in order to better understand its contribution and to describe expected scenarios. A foreseeable problem in this case could be the shortness of the historical series of internal migrations, which doesn't guarantee the necessary robustness of the estimated relationship between macro variables and the implicit cyclical factor.

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