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Athanasios Andrikopoulos

University of Ioannina

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CHARACTERIZATION OF THE GENERALIZED TOP-CHOICE ASSUMPTION (SMITH) SET

ATHANASIOS ANDRIKOPOULOS

ABSTRACT. In this paper, I give a characterization of the Generalized Top-Choice Assumption set of a binary relation in terms of choice from minimal negative consistent superrelations. This result provides a characterization of Schwart's set in tournaments.

JEL classification: D60, D71.

Key words: Negative Consistency, Generalized Top-Choice Assumption (Smith) set, Generalized Optimal-Choice Axiom (Schwartz) set.

1. INTRODUCTION

A fundamental problem of rational choice theory is to determine whether a choice is optimal relative to some preference relation or not. So a rational agent who knows his preference relation, chooses the maximal elements according to this relation, in every feasible set presented for choice. In other words, the optimal choice set of a choice process consists of the maximal alternatives in the feasible set according to the viewpoint of a binary relation. However, when will the set of optimal choices be non-empty? If the feasible set is finite and the binary relation is acyclic, then the set of optimal choices is always non-empty; When the optimal choice set is empty\(^1\), the crucial question which has been arisen, is what to count as a choice. That is, what sets of alternatives may be considered as reasonable solutions? To answer this question, several methods (solution theories) for constructing non-empty choice sets have been proposed. Such a solution is the Generalized Top-Choice Assumption set (GETCHA set), introduced by Schwartz in [4], which is a generalization of the optimal choice set. Smith in [5] introduces a generalization of Condorcet Criterion that is satisfied when pairwise election are based on simple majority choices. He uses the notion of dominant set, that is, any candidate in this set is collectively preferred to any candidate not into this set. But Smith does not discuss the idea of a smallest dominant set. Fishburn in [3] narrows Smith’s generalization of the Condorcet Criterion to the smallest dominant set and calls it Smith’s Condorcet Principle. Schwartz in [4] discusses the Smith’s Condorcet Principle as a possible standard for optimal collective choice and he call it GETCHA.

\(^1\)This problem is common in the analysis of pairwise majority voting, in the choice of a winning sport team, in the aggregation of multiple choice criteria, in committee selection, in the choice under uncertainty, etc.
The $\text{GOCHA}$ set ($\text{Smith set}$)\textsuperscript{2} is the choice set from a given set specified by the $\text{GOCHA}$ condition. To address the absence of maximal elements, Schwartz [4] gives another general solution for constructing non-empty choice sets which is called Generalized Optimal-Choice Axiom ($\text{GOCHA}$). The choice set from a given set specified by the $\text{GOCHA}$ condition is the union of minimal sets which each one of them has the following property: no alternative outside this set is preferable to an alternative inside it.

In this paper, I show that an alternative belongs to the $\text{GOCHA}$ set of a binary relation if and only if it is maximal for a minimal negative consistent superrelation. This result provides a characterization of the $\text{GOCHA}$ set ($\text{Schwartz set}$)\textsuperscript{3} in tournaments.

2. Notation and definitions

Let $X$ be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary binary relation on $X$. We will say that $R$ is a subrelation of $R'$, and $R'$ a superrelation of $R$, denoted $R \subseteq R'$, when for all $x,y \in X$, $xRy$ implies $xR'y$. The complement of $R$ is denoted by $R^c$, that is for all $x,y \in X$, $R^c = \{(x,y)| (x,y) \notin R\}$. We sometimes abbreviate $(x,y) \in R$ as $xRy$. For any $x \in X$, $Rx = \{y \in X | yRx\}$ and $xR = \{y \in X | xRy\}$ denote respectively the upper contour set and lower contour set of $R$ at $x$. The asymmetric part $P(R)$ of $R$ is given by: $(x,y) \in P(R)$ if and only if $(x,y) \in R$ and $(y,x) \notin R$. $\mathcal{M}(R)$ denote the elements of $X$ that are $R$-maximal in $X$, i.e., $\mathcal{M}(R) = \{x \in X | \text{for all } y \in X, yRx \text{ implies } xRy\}$. We say that $R$ is transitive if for all $x,y,z \in X$, $(x,z) \in R$ and $(z,y) \in R$ implies that $(x,y) \in R$. The transitive closure of $R$ is denoted by $\overline{R}$, that is for all $x,y \in X$, $(x,y) \in \overline{R}$ if there exist $k \in \mathbb{N}$ and $x_0,...,x_k \in X$ such that $x = x_0$, $(x_{k-1},x_k) \in R$ for all $k \in \{1,...,K\}$ and $x_K = y$. A subset $Y \subseteq X$ is an $R$-cycle if, for all $x,y \in Y$, we have $(x,y) \in \overline{R}$ and $(y,x) \in \overline{R}$. We say that $R$ is acyclic if there does not exist an $R$-cycle.

Suzumura [6] provides the following definition, which generalize the notions of transitivity and acyclicity: The binary relation $R$ is consistent, if for all $x,y \in X$, for all $k \in \mathbb{N}$, and for all $x_0,x_1,...,x_K \in X$, if $x = x_0$, $(x_{k-1},x_k) \in R$ for all $k \in \{1,...,K\}$ and $x_K = y$, then $(y,x) \notin \overline{P(R)}$. If in the definition of consistent binary relation above we replace $R$ with $R^c$, we get the notion of a negative consistent binary relation. As binary relations are subsets of $X \times X$, they are naturally partially ordered by set-inclusion. A chain, denoted $\mathcal{C}$, is a class of relations such that $B,B' \in \mathcal{C}$ implies $B \subseteq B'$ or $B' \subseteq B$. A class $\mathcal{B}$ of relations is closed downward if, for all chains $\mathcal{C}$ in $\mathcal{B}$, $\bigcap \{B|B \in \mathcal{C}\} \in \mathcal{B}$.

Let $\Omega$ be a family of non-empty subsets of $X$ that represents the different feasible sets presented for choice. A choice function is a mapping that assigns to each choice situation a subset of it:

\textsuperscript{2}The Smith set also appears in the literature as weak top cycle.

\textsuperscript{3}The Smith set is also sometimes confused with the Schwartz set because in tournaments (asymmetric and complete binary relations) both sets coincide.
The traditional choice-theoretic approach takes behavior as rational if it can be explained as the outcome of maximizing a binary relation $R$. In this direction, best choices can be expressed as the maximization of the individuals’s preferences over a set of alternatives. That is, for every $A \in \Omega$, $C(A) = \mathcal{M}(R/A)$ ($\mathcal{M}(R/A)$ denote the elements of $X$ that are $R$-maximal in $A$). An $A \in \Omega$ is $R$-undominated iff for no $x \in A$ there is a $y \in X \setminus A$ such that $yRx$. An $R$-undominated set is minimal if none of its proper subsets has this property. The set $A$ is $R$-dominant if and only if $xRy$ for each $x \in A$ and each $y \in X \setminus A$. An $R$-dominant set is minimal if none of its proper subsets is an $R$-dominant subset of $X$. To deal with the case where the set of maximal choices $C(A)$ is empty, Schwartz [4, Definition in page 141] has been proposed the following general solutions:

**Generalized Top Optimal-Choice Axiom (GETCHA):** For each $A \in \Omega$, $C(A)$ is equivalent to the minimum $R$-dominated subsets of $A$.

**Generalized Optimal-Choice Axiom (GOCHA):** For each $A \in \Omega$, $C(A)$ is equivalent to the union of minimum $R$-undominated subsets of $A$. The $\text{GETCHA}(R)$ set (resp. $\text{GOCHA}(R)$ set) is the choice set from a given set specified by the $\text{GETCHA}$ (resp. $\text{GOCHA}$) condition according to $R$.

### 3. Main Result

The main result in this paper establishes a binary characterization of the choices generated by negative consistent superrelations. Let $\mathcal{R}_N$ denote the negative consistent superrelations of $R$, i.e., $\mathcal{R}_N = \{R \subseteq R | R \text{ is negative consistent}\}$, and let $R_{N^*}$ denote the elements of $\mathcal{R}_N$ that are minimal with respect to set inclusion.

**Zorn’s Lemma:** If every chain of a partially ordered set has a lower bound, then $E$ has a minimal element.

In order to prove the main result of this paper, we need the following proposition which is a simplification of the dual version of the definition of Duggan [1, Definition 4] for consistent binary relations.

**Proposition 1.** A binary relation $R$ is negative consistent if and only if $P(R) \subseteq P(\overline{R^c})$.

**Proof.** Suppose that $R$ fulfills the definition of negative transitivity and for $x, y \in X$, $(x, y) \in P(R)$. By way of contradiction, we assume that $(x, y) \notin P((\overline{R^c})^c)$. We have two cases: either $(x, y) \notin (\overline{R^c})^c$, or $(x, y) \in (\overline{R^c})^c$ and $(y, x) \in (\overline{R^c})^c$. In the first case we have $(x, y) \in \overline{R^c}$, that is, there exists a natural number $n$ and alternatives $x_1, x_2, \ldots, x_n \in X$ such that

$$x = x_1R^c x_2 \ldots x_{n-1}R^c x_n = y.$$ 

Thus, negative consistency yields $(y, x) \notin P(R^c)$ which contradicts the hypothesis that $(x, y) \in P(R)$. For the second case, where $(x, y) \notin \overline{R^c}$ and $(y, x) \notin \overline{R^c}$, it follows that $(x, y) \in I(R)$ which leads to a contradiction too.
To see the converse, first suppose $P(R) \subseteq P((\overline{R^c})^c)$ and take $n$ and $x_1, x_2, \ldots, x_n \in X$ such that

$$x = x_1 R^c x_2 \ldots x_{n-1} R^c x_n = y.$$ 

Thus, $(x, y) \in \overline{R^c}$, implying $(x, y) \notin (\overline{R^c})^c$, and by supposition $(x, y) \notin P(R)$. Therefore, $(y, x) \notin P(R^c)$, as required.

The following proposition is the dual of the Proposition 5 in [1].

**Proposition 2.** The class of all negative consistent binary relations is closed downward.

**Proof.** Let $\mathcal{B}$ be the class of all negative consistent binary relations. To prove that $\mathcal{B}$ is closed downward, take a chain $\mathcal{C}$ in $\mathcal{B}$, let $C = \bigcap_{B \in \mathcal{C}} B$, and take $(x, y) \in P(C)$. We prove that $(x, y) \in P((\overline{C^c})^c)$. We proceed by the way of contradiction, suppose that $(x, y) \notin P((\overline{C^c})^c)$, then there are two cases to consider: (i) $(x, y) \notin (\overline{C^c})^c$; (ii) $(x, y) \in (\overline{C^c})^c$ and $(y, x) \in (\overline{C^c})^c$. In the first case, if $(x, y) \notin (\overline{C^c})^c$, then, $(x, y) \in \overline{C^c}$. Thus, there exist $x_0, x_1, \ldots, x_K \in X$ such that

$$x = x_0, (x_{k-1}, x_k) \in C^c \text{ for all } k \in \{0, \ldots, K\} \text{ and } x_K = y.$$ 

But then, for each $k \in \{1, \ldots, K\}$, there is a $B_k \in \mathcal{C}$ such that $(x_{k-1}, x_k) \in B_k$. Since $\mathcal{C}$ is a chain, $\mathcal{B} = \{B_k|k = 1, 2, \ldots, K\}$ contains a relation, $B_\lambda$, minimum with respect to set-inclusion. Hence,

$$x = x_0 B_\lambda^c x_1 \ldots x_{K-1} B_\lambda^c x_K = y.$$ 

On the other hand, since $(x, y) \in P(C)$, there is $B_\mu \in \mathcal{C}$ such that $(x, y) \in P(B_\mu)$ for each $B_\nu \subseteq B_\mu$. We have the following two subcases to consider: (i$_{\mu}$) $B_\lambda \subseteq B_\mu$; (ii$_{\mu}$) $B_\mu \subseteq B_\lambda$. For (i$_{\mu}$), we have

$$x = x_0 B_\mu^c x_1 \ldots x_{K-1} B_\mu^c x_K = y \text{ and } (x, y) \in P(B_\lambda).$$ 

Since $B_\lambda$ is negative consistent, it must be that $(y, x) \notin P(B_\lambda^c)$. Hence, $(y, x) \in B_\lambda$ or $(y, x) \in I(B_\lambda)$, contradicting $(x, y) \in P(B_\lambda)$. Now consider the subcase (i$_{\mu}$). Since $B_\mu \subseteq B_\lambda$, we have

$$x = x_0 B_\mu^c x_1 \ldots x_{K-1} B_\mu^c x_K = y \text{ and } (x, y) \in P(B_\mu).$$ 

This is a contradiction as well.

We come now to the second case, that of $(x, y) \in (\overline{C^c})^c$ and $(y, x) \in (\overline{C^c})^c$. In this case, we have $(x, y) \in I(C)$ which contradicts that $(x, y) \in P(B_\mu)$. □

The next two propositions are used in the proof of Theorem 5 below. The proof of the Proposition 3 uses the technique of Lemma 1 in [2]

**Proposition 3.** Let $R$ be a binary relation on $X$. For each $x \in X$, there exists a negative consistent superrelation $R_{c(x)} \supseteq R$ such that $\overline{R_{c(x)}} \ x = \overline{R^c x \ \{x\}}$. 
Proof. Let us define \( Y = \overline{R} \cup \{ x \} \). Denote by \( \mathcal{R} \) be the set of negative consistent superrelations \( R_N \subseteq X \times X \) of \( R \) which satisfies the following property (c):

\[
(\text{c}) \quad \text{For each } z, y \in X, \text{ if } (z, y) \notin R_N, \text{ then } x = y \text{ or } (y, x) \notin \overline{R}_N.
\]

Since \( X \times X \) lies in \( \mathcal{R} \), this set is non-empty. Let \( \mathcal{C} \) be a chain in \( \mathcal{R} \), and let \( \mathcal{D} = \bigcap \mathcal{C} \). Since the class of negative consistent relations is closed downward (Proposition 2), \( \mathcal{D} \) is negative consistent. Moreover, \( \mathcal{D} \) satisfies the condition (c). To see that, take any \( s, t \in X \) such that \( (t, s) \notin \mathcal{D} \), so there exists \( R_N \in \mathcal{C} \) such that \( (t, s) \notin R_N \). Hence, \( s = x \) or \( (s, x) \in \overline{R}_N \subseteq \overline{\mathcal{D}} \). Therefore, by Zorn’s lemma, \( \mathcal{R} \) has an element, say \( R_{c(e)} \), that is minimal with respect to set inclusion. To prove that \( \overline{R}_{c(e)} x = Y \setminus \{ x \} \), it suffices to show that \( \Lambda = (Y \setminus \{ x \}) \setminus \overline{R}_{c(e)} x = \emptyset \). Now suppose to the contrary that there exists a point \( y \in \Lambda \). Then, there exists a natural number \( n \) and alternatives \( y_1, y_2, \ldots, y_n \in X \) such that

\[
y = y_1 R^c y_2 \cdots y_{n-1} R^c y_n = x \quad \text{and} \quad (y, x) \notin \overline{R}_{c(e)}
\]

Since \( x \neq y \), we may assume that the elements \( y_1, y_2, \ldots, y_n \) are distinct. Now define

\[
Q_N = R_{c(e)} \setminus \{(y_1, y_2), \ldots, (y_{n-1}, y_n)\}.
\]

Then, we have \( R \subseteq Q_N \subseteq R_{c(e)} \). The first inclusion is easy: For each \( k \in \{1, \ldots, n-1\} \), \((y_k, y_{k+1}) \notin R \). For the second inclusion, it suffices to show that there is at most one \( k \in \{1, \ldots, n-1\} \) such that \((y_k, y_{k+1}) \in R_{c(e)} \). Indeed, if for each \( k \in \{1, \ldots, n-1\} \) we let \((y_k, y_{k+1}) \notin R_{c(e)} \), we obtain \((y, x) \in \overline{R}_{c(e)} \) a contradiction. Furthermore, \( Q_N \) satisfies the condition (c). Indeed, assume that \( s, t \in X \) are such that \((t, s) \notin Q_N \). There are two cases to consider: (i) \((t, s) \notin R_{c(e)} \); (ii) \((t, s) = (y_k, y_{k+1}) \) for some \( i \in \{1, \ldots, n-1\} \). In the first case, by construction we have \( x = s \) or \((s, x) \in \overline{R}_{c(e)} \) \( \subseteq Q_N \). In the second case, there must exist \( k \in \{1, \ldots, n-1\} \) such that \( s = y_{k+1} \). If \( k = n-1 \), then \( s = y_n = x \). Otherwise, \( s = y_{k+1} \) for some \( k^* \in \{1, \ldots, n-2\} \). Since \((y_k, y_{k^*+1}) \in Q_N^c \), \((y_{k^*+1}, y_k) = (y_{k^*+1}, x) \in Q_N^c \), we conclude that \((s, x) \in \overline{Q_N} \). Therefore, by minimality of \( R_{c(e)} \), it is clear that \( Q_N \) is not negative consistent. Thus, there exists a natural number \( m \) and alternatives \( z_0, z_1, \ldots, z_m \in X \) such that

\[
\mu = z_0 Q_N^c z_1 \cdots z_{m-1} Q_N^c z_m = \nu \quad \text{and} \quad (\nu, \mu) \in P(Q_N^c).
\]

Since \( R_{c(e)} \) is negative consistent and \( Q_N^c = R_{c(e)} \setminus \{(y_1, y_2), \ldots, (y_{n-1}, y_n)\} \), there must exist \( \kappa = 1, \ldots, n-1 \) and \( \lambda = 0, 1, \ldots, m-1 \) such that \((y_\kappa, y_{\kappa+1}) = (z_\lambda, z_{\lambda+1}) \). Consider the smallest \( \kappa \) for which there exist such \( \mu, \nu, m, z_0, \ldots, z_m \), and \( \lambda \). We show that there is no \( j \in \{1, \ldots, n-1\} \) such that \((z_{\lambda+1} \in R_N^c (z_{\lambda}),(y_\kappa, y_{\kappa+1})) \). We proceed by the way of contradiction. Suppose that \( y_{j+1} = z_\lambda R^c z_{\lambda+1} = y_{k+1} \). Since the elements \( y_1, \ldots, y_n \) are distinct, it follows that \( \kappa \neq j \) and so \( \kappa < j \). But then, from \( y_\kappa = z_\lambda = y_{j+1} \) we conclude
that \( \kappa = j + 1 \) which is impossible. Thus, from \( (z_{\text{mod}(\lambda m + 1) + m + \lambda, m + 1}, z_\lambda) \in Q_N^c \) we deduce that \( (z_{\text{mod}(\lambda m + 1) + m + \lambda, m + 1}, z_\lambda) \in R_{C(x)}^c \). Since \( R_{C(x)}^c \in R \) and \( (z_{\text{mod}(\lambda m + 1) + m + \lambda, m + 1}, z_\lambda) \notin R_{C(x)}^c \), we have \( x = z_\lambda \) or \( (z_\lambda, x) \in R_{C(x)}^c \). Using \( z_\lambda = y_k \neq x \), we exclude the first case. Hence, 

\[
y = y_1 R^c y_2 ... R^c y_k R_{C(x)}^c x.
\]

Now define

\[
\Gamma_N = R_{C(x)} \setminus \{(y_1, y_2), ... (y_{k-1}, y_k)\}.
\]

As in the proof of \( Q_N \), we conclude that \( R \subseteq \Gamma_N \subseteq R_{C(x)} \). Furthermore, for all \( s, t \in X \), if \( (t, s) \notin \Gamma_N \), then similarly to the case of the relation \( Q_N \) we can prove that \( x = s \) or \( (s, x) \in \Gamma_N \). Thus, \( \Gamma_N \) satisfies the condition (c). Finally, because of choice of \( \kappa \) we conclude that \( \Gamma_N \) is negative consistent. Hence, \( \Gamma_N \in R \), contradicting the minimality of \( R_{C(x)} \). This contradiction establish that \( \Lambda = \emptyset \) and completes the proof.

\[\Box\]

**Proposition 4.** Let \( X \) be a nonempty set of alternatives and let \( R \) be a binary relation over \( X \). Then, the \( \text{GETCHA}(R) \) set is equivalent to \( \mathcal{M}([\mathcal{P}(R)^c]^{-1}) \).

**Proof.** Let \( x \in \text{GETCHA}(R) \). We have two cases to consider: (i) For each \( y \in Y \) there holds \( (x, y) \in R \); (ii) There exists \( y_0 \in Y \) such that \( (x, y_0) \notin R \). In the first case, we have \( (y, x) \notin P(R) \) which implies that \( (x, y) \in ([P(R)]^{-1} \subseteq \mathcal{M}([\mathcal{P}(R)^c]^{-1}) \). Hence, \( x \in \mathcal{M}([\mathcal{P}(R)^c]^{-1}) \). In the second case, since \( (x, y_0) \notin R \), it follows that \( y_0 \notin \text{GETCHA} \), for otherwise \( (x, y_0) \in R \) which is impossible.

Let \( A_x = \{t \in \text{GETCHA}(R) \mid (x, t) \in R^c \} \). We have that \( A_x \neq \emptyset \), because otherwise, for each \( t \in \text{GETCHA}(R) \), \( (x, t) \notin R^c \). But then, \( (x, t) \in R \), which implies that \( \{x\} \subset \text{GETCHA} \) is an \( R \)-dominant subset of \( X \), a contradiction because of the minimal character of \( \text{GETCHA}(R) \). Let \( G = \text{GETCHA}(R) \setminus A_x \). We prove that \( G = \emptyset \). We proceed by the way of contradiction. Suppose that \( G \neq \emptyset \). Then, for each \( t \in A_x \) and each \( s \in G \) we have \( (t, s) \in R \) for suppose otherwise, \( (t, s) \in R^c \) which implies that \( (x, s) \in R^c \). Thus, \( s \in G \). Therefore, \( A_x \subset \text{GETCHA}(R) \) is an \( R \)-dominant subset of \( X \), again a contradiction. Hence, \( A_x = \text{GETCHA}(R) \). Since, \( y_0 \in \text{GETCHA}(R) \) we conclude that \( (x, y_0) \in R^c \). Similarly, we can prove that \( (y_0, x) \in R^c \). Hence, since \( R^c \subseteq [\mathcal{P}(R)^c]^{-1} \) we conclude that \( x \) and \( y_0 \) belong to a \( ([P(R)]^{-1}) \)-cycle. On the other hand, for each \( y \in Y \setminus \text{GETCHA}(R) \), as in the case (i), we deduce that \( (x, y) \in ([P(R)]^{-1}) \). Hence in any case we have \( (y, x) \notin P([\mathcal{P}(R)]^{-1}) \) which implies that \( x \in \mathcal{M}([\mathcal{P}(R)]^{-1}) \).

To prove the converse, take any \( x \in \mathcal{M}([\mathcal{P}(R)]^{-1}) \). We show that \( x \in \text{GETCHA}(R) \). We will consider two cases:

**Case 1:** For each \( y \in X \) there holds \((y, x) \notin ([P(R)]^{-1}) \). In this case we have \( (y, x) \notin P(R) \subseteq R \). Hence, \( x \) is an \( R \)-dominant element of \( X \) which implies that \( \text{GETCHA}(R) = \{x\} \).
Case 2. There exists \( y \in X \) such that \((x, y) \in (\mathbb{P}(R)^c)^{-1}\) and \((y, x) \in (\mathbb{P}(R)^c)^{-1}\). In this case, \(x\) belongs to a \([P(R)]^c\)-cycle. Let \(C(x)\) be a \([P(R)]^c\)-cycle containing \(x\) that is maximal in the sense that it is not a proper subset of any other \([P(R)]^c\)-cycle. We prove that \(C(x) = \mathcal{GETCHA}\). Suppose on the contrary, that \((t, z) \notin R\) for some \(t \in C(x)\) and \(z \in X \setminus C(x)\); to deduce a contradiction. It follows that \((t, z) \in [P(R)]^c\) which implies that \((x, z) \in [P(R)]^c\). Hence, \((z, x) \in (\mathbb{P}(R)^c)^{-1}\). Since \((z, x) \notin P((\mathbb{P}(R)^c)^{-1})\) we conclude that \((x, z) \in (\mathbb{P}(R)^c)^{-1}\). Hence, \(C(x) \cup \{z\}\) is a \([P(R)]^c\)-cycle, a contradiction. \(\square\)

The next result shows the connection between the \(\mathcal{GETCHA}(R)\) set and the choice sets generated from negative consistent superrelations.

**Theorem 5.** Let \(X\) be a nonempty set of alternatives and let \(R\) be a binary relation over \(X\). Then, the \(\mathcal{GETCHA}(R)\) set is equivalent to the union of maximal elements of all minimal negative consistent superrelations of \(R\).

**Proof.** Let \(R_{n^*} \in R_{n^*}\) be minimal, take any \(x \in M(R_{n^*})\). We prove that \(x \in \mathcal{GETCHA}(R)\). Suppose to the contrary that \(x \notin \mathcal{GETCHA}(R)\), then by Proposition 4 there exists \(y \in X\) such that \((y, x) \in \mathbb{P}(\mathbb{P}(R)^c)^{-1}\). It follows that \((x, y) \notin \mathbb{P}(\mathbb{P}(R)^c)^{-1}\) which implies that \((y, x) \notin R\). Hence, \((y, x) \in R \subseteq R_{n^*}\). Therefore by \((y, x) \notin \mathbb{P}(R_{n^*})\) we conclude that \((y, x) \in R_{n^*}\). Let us define \(R_{n^*} = R_{n^*} \setminus \{x, y\}\). Since \((x, y) \notin R\), we conclude that \(R \subseteq R_{n^*} \subseteq R_{n^*}\), and \(R_{n^*}\) is non negative consistent (the assumption that \(R_{n^*}\) is negative consistent contradicts to the fact that \(R_{n^*}\) is minimal with respect to set-inclusion). Hence, there exist \(s, t \in X\), \(\lambda \in \mathbb{N}^\ast\) and \(z_0, z_1, ..., z_\lambda \in X\) such that \(s = z_0, (z_{\lambda - 1}, z_\lambda) \in R_{n^*}\) for all \(\lambda \in \{1, ..., \Lambda\}\), \(z_\lambda = t\) and \((t, s) \in \mathbb{P}(R_{n^*}) \subseteq R_{n^*}\). Since \(R_{n^*}\) is negative consistent and \(R_{n^*} = R_{n^*}\) \(\cup \{x, y\}\), there must exists \(\lambda_0 \in \{1, ..., \Lambda\}\) such that \((z_{\lambda_0 - 1}, z_{\lambda_0}) = (x, y)\) and for all \(\lambda \in \{1, ..., \Lambda\}\) with \(\lambda \neq \lambda_0\), \((z_{\lambda - 1}, z_\lambda) \in R_{n^*}\) if and only if \((z_{\lambda - 1}, z_\lambda) \in R_{n^*}\). It then follows that \((z_{\lambda_0 - 1}, z_{\lambda_0}) \in R_{n^*}\). Therefore, \((y, x) \in \mathbb{P}(R_{n^*}) \subseteq \mathbb{P}(R)\). But then, \((x, y) \in (\mathbb{P}(R)^c)^{-1}\) contradicting \((y, x) \in \mathbb{P}(\mathbb{P}(R)^c)^{-1}\). This contradiction confirms the claim.

To prove the converse, take any \(x \in \mathcal{GETCHA}(R)\). We show that there exists \(R_{n^*} \in R_{n^*}\) such that \(x \in M(R_{n^*})\). Let \(R_{C(x)}\) be as in Proposition 3. First, observe that \(x\) is \(R_{C(x)}\)-maximal in \(X\). Indeed, suppose to the contrary that there exists \(y \in X\) such that \((y, x) \in \mathbb{P}(R_{C(x)}) \subseteq \mathbb{P}(\mathbb{P}(R_{C(x)})^c)\). Since \((x, y) \notin R_{C(x)} \supseteq R\), it follows that \(y \in \mathcal{GETCHA}(R)\), for otherwise \((x, y) \notin R \subseteq R_{C(x)}\) which is impossible. From \(x \in \mathcal{GETCHA}(R)\) by using the proof of Proposition 4 we conclude that \((y, x) \in \mathbb{P}(R_{C(x)})^c\). Therefore, \((y, x) \notin \mathbb{P}(R_{C(x)})\), contradicting \((y, x) \in \mathbb{P}(R_{C(x)})^c\) \(\subseteq (R_{C(x)})^c\). Hence, \(x\) is \(R_{C(x)}\)-maximal in \(X\). If \(R_{C(x)}\) is minimal with respect to set-inclusion in \(X\), then the proof is over. Otherwise, there exists at least one negative consistent superrelation \(Q\) such that \(R \subseteq Q \subseteq R_{C(x)}\). Let \(Q\) be the set of negative consistent superrelations \(Q\) satisfying the latter condition. Let \(C\) be a chain in \(Q\), and let \(D = \bigcap C\). Evidently, \(R \subseteq \)}
$\mathcal{D} \subset R_{C(x)}$. Since the class of negative consistent relations is closed downward (proposition 2), $\mathcal{D}$ is negative consistent. Therefore, by Zorn’s lemma, $Q$ has an element, say $\tilde{Q}$, that is minimal with respect to set inclusion. We prove that $x$ is $\tilde{Q}$-maximal. We proceed by way of contradiction. Let $y \in X$ such that $(y, x) \in P(\tilde{Q})$. Since $x \in \text{GETCHA}(R)$ and $(x, y) \notin R$, as above, we conclude that $y \in \overline{R^c x \setminus \{x\}} = \overline{R_{C(x)}^c x} \subset \overline{\tilde{Q}^c x}$, contradicting $(y, x) \in P(\tilde{Q}) \subseteq P((\tilde{Q}^c)^c) \subseteq (Q^c)^c$. The proof is over. □

References