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Abstract

Amid the controversies around the optimisation criteria and the objective functions when applying mathematical methods in economics, we proposed a method of quantifying a multi-criteria optimum, called *critical distance method***.** The demonstration of this method is exemplified by assessing the investment optimum at microeconomic level (project or company portfolio choice). A hyperbolic paraboloid function of three variables (the recovery time, the investment value and the unit cost) representing a surface of the second degree has been defined. The intersection of the hyperbolic parabola planes identifies the point where the three considered variables have the same value, signifying an equal importance attached to them and revealing the optimum level of their interaction. The distance from this critical point to the origin represents, in fact, the criterion according to which one could choose the most efficient investment alternative. In our opinion, the proposed method could be extended to the study of any economic process.

Key words: microeconomic optimum, critical distance method, portfolio choice, investment alternatives, multi-criteria optimum.

JEL classification: B21, B23, C02, C61, G11

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A QUANTIFYING METHOD OF MICROINVESTMENT OPTIMUM

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As a rule, when we speak about the issue of the optimum of a certain process, we start by setting a criterion on the basis of which we are establishing the objective function; usually, this choice has a definite subjective character and one could not deny that, under certain circumstances, the present state of affairs itself imposes the restrictions and the factor which is to be optimised. But starting from a normal situation, when we do not need to fix a specific criterion, we shall usually try, after a selective filtering of possibilities, to choose the most convenient one, from our own point of view. Being fully aware of the importance of the volitional element, which lies at the basis of this consideration, no one could be satisfied because the maximisation or minimisation of a certain aspect of the respective process represents only a partial optimum. What we are interested in is optimising the process as a whole, not only one of its elements from a particular viewpoint.

Considering all these elements and focusing on the economic field, all controversies that arise around the optimisation criteria and the objective functions, which have been chosen, are generated, in our opinion, by the one-sided way of approaching the essence of the economic optimum. This can be explained by the deficiencies of the mathematical methods applied in the economic field, which identify the optimum with the extreme value of a function, without taking into account that the economic processes are extremely complex. Besides, the influencing factors are so numerous that, giving priority to one of them is risky. We emphasise the interdependent character of the economic processes in order to point out that, mainly at a microeconomic level, the optimum essence consists in ensuring the system functionality of the factors interdependence at the optimum level and not in the maximisation or minimisation of the action or the level of only one of them. Attaching importance to all the influencing factors, at least to the most relevant ones is required, so that a multi-criteria optimum of the economic process and not an economic optimum may come out from it. It is essential to find that method which is able to quantify the factors' interaction simultaneously as well as its optimum level, considering the equality between the assignments of the factors' importance in order to eliminate almost entirely the subjectivism when the priority is chosen. In this paper, starting from the example of the investment process, we propose a method, we have called the *critical distance method* by means of which one can establish the distance up to the point where the interaction of three investment factors is situated at an optimum level; the method as such can be generalised when approaching any kind of economic process, and the factors which are considered can be more numerous, only the mathematical calculus being a little bit more complicated. Being applied to the calculus of investment economic efficiency, to the level of the investment project or to the level of the economic unit, the optimum that is assessed by the critical distance method is a multi-criteria one.

As regards the investment recovery time – the number of years in which the fixed assets value is recovered from the benefits at the level of the investment project or of the enterprise – it is calculated as follows:

$$d = I / B \qquad (1)$$

where:

d – the investment recovery time

I – the total value of the investment

B – the annual benefit.

The annual benefit resulting from subtracting the cost of the forecasted annual production (C) from the value of the commodity output (P) expected to be achieved, can be written:

$$d = I / (P - C) \tag{2}$$

The annual production cost being obtained as a multiplication of the annual capacity of production expressed in physical units (Q) by the unit cost (c), it results that:

$$d = I / (P - Q\,c) \tag{3}$$

Relation (3) is a basic prerequisite in our analysis because it has three main indicators that are used especially for choosing one or another of the investment alternatives. The correlation between the recovery time, the investment value and the unit cost, whose values are different from one alternative to another, is decisive in choosing the optimal one, taking into consideration that the implementation of a new investment's projects needs, is in most cases, a given value of the physical production, and under the circumstances of a specific selling price, the ensuring of a certain value of the annual production, the same for all alternatives included in the economic and technical feasibility studies.

As a consequence, relation (3) is equivalent with:

$$y = z / (b - a x) \tag{4}$$

where:

y – the recovery time

z – the total value of the investment

a – the annual production capacity expressed in physical units

b – the annual value of the commodity output.

Analysing mathematically the relation (4) which will be explained in detail, we could choose the most efficient alternative of the investment, which will be suitable for an optimal correlation of the recovery time with the investment value and with the unit cost respectively, specifying that the critical distance method applied in this case, is valid for the same annual production capacities and for the annual values of the commodity output considered constant, as mentioned before.

It is obvious that this method can be applied in other cases, too, for example, when one or both constants are meant to be considered variables due to the context and we have to choose other parameters.

Writing relation (4) differently, we have:

$$f (x, y, z) = a x y - b y + z = 0 \tag{5}$$

This is a function of three variables representing a surface of the second degree, namely, a quadric. The invariant elements of the equation (5) (we call them invariant as they remain unaltered when the coordinate system is transformed), are:

- the linear invariant $I = 0$
- the square invariant $J = [(a^2) / 4] < 0$
- the cubic invariant $\delta = 0$

- the biquadratic invariant $\Delta = [(a^2) / 16] > 0$

So, Δ being different from zero, it results that the function is a quadric proper, without a central point at a finite distance and because $J<0$ and $\Delta>0$ it comes out a *hyperbolic* paraboloid function (the quadric or the surface of the second degree).

In a general formula, a hyperbolic paraboloid function written in a canonical form represents the following equation:

$$[(x^2) / m] - [(y^2) / n] = 2 z \qquad (6)$$

where m, n are parameters.

In the canonical form (6), a hyperbolic paraboloid has as symmetrical axes the standard coordinates, Ox, Oy, Oz, and as a central point the origin of these axes, namely the point O (0; 0; 0). The graphical representation of the canonical form is given in Fig. 1.

In case of the hyperbolic paraboloid function defined by relation (5), from that we start in the mathematical demonstration of the critical distance method, we operated a translation of symmetrical axes of coordinates, as it is shown in Fig. 2. Also the origin point was translated to O' (b/a; 0; 0). The translation of coordinates was made under the evidence that only the hyperbolic paraboloid surface for x, y, z > 0 is significant from an economic viewpoint; the negative values of these variables have no economic meaning.

It is worth emphasising that the axis O'y' represents the asymptote of the hyperbolic paraboloid function, namely the maximum value just on the line of x, that is b/a, which, from an economic viewpoint stands for the proportion between the value of the annual commodity output and the physical production, being in fact the unit selling price.

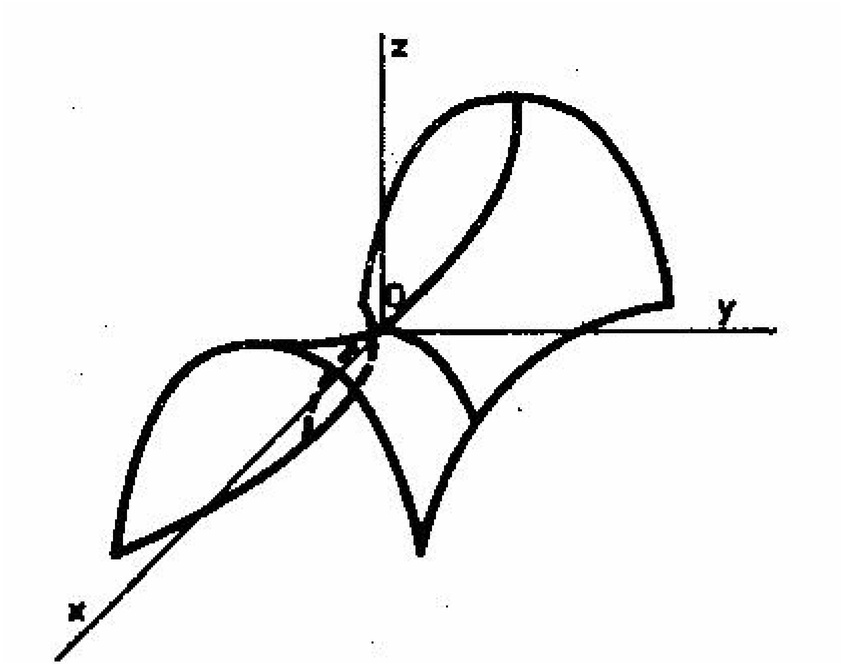

Fig. 1

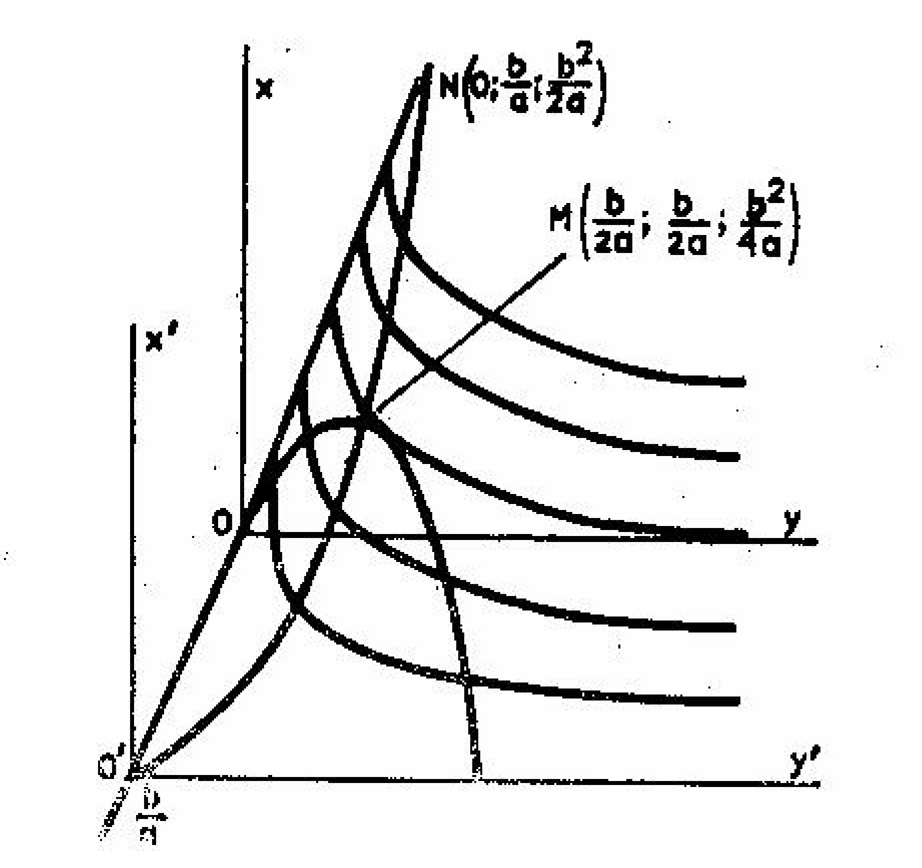

Fig. 2

Applying the mathematical analysis to the function (5) it comes out that this one does not have extremes and we have to find a point that belongs to the hyperbolic paraboloid and could constitute a condition for efficiency.

For this purpose, we consider each variable as constant, in turn.

a) Considering x as constant, x = I, relation (5) becomes:

$$z = (b - a\,I)\,y \tag{7}$$

This represents the equation of a straight line which passes through the origin point and whose slope is m = b - a I.

We can see that for:

$m = 1,$	$I = (b - 1) / a$	for $y = z$
$m < 1,$	$I > (b - 1) / a$	for $y > z$
$m > 1,$	$I > (b - 1) / a$	for $y < z$

So, for I = (b-1)/a, z = y for any value of y and any value of z. This is a special point on Ox because $(\forall)y$ and $(\forall)z$, x = (b-1)/a. For any other x different from this value, $z \neq y$ for any value of y and z.

The intersection of the plane x = (b-1)/a (which is parallel to the plane xOy) with the hyperbolic paraboloid will be a straight line parallel to the bisection line of the angle zOy. The nearer it is to zero (slowdown), being continuously moving away from (b-1)/a, the bigger the value of z becomes, as compared to y, and the nearer x comes to b/a, then y is bigger than z. For x = b/a, the intersection of the plane with the hyperbolic paraboloid is the straight line O'y'.

b) Considering y as constant, namely y = t, the relation (5) becomes:

$$z = bt - a\,t\,x \qquad (8)$$

This represents the equation of a straight line, having the ordinate n=bt and the slope m=at. We can see that for:

$$t = x / (b - a\,x) \quad \text{or} \quad t = z / (b - a\,z), \quad x = z$$

This is a special point on Oy as for any value of x and any value of z, the point where

$$x = z \quad \text{is} \quad y = x / (b - a\,x) = z / (b - a\,z)$$

For any other $y \neq x / (b-ax) \neq z / (b-az)$, x is different from z for $(\forall)x$ and $(\forall)z$. Intersecting plane x = y (bisection line to planes zOy and xOy) with the hyperbolic paraboloid, we obtain a straight line which intersects the hyperbolic paraboloid in points of the value x = z. The nearer x is to the plane xOy, the bigger its value will be in comparison with z.

c) Considering z as being constant (z = k) the relation (5) becomes:

$$b\,y - a\,x\,y = k \qquad (9)$$

This is the equation of a hyperbola which has as asymptotes, the axes Ox and O'y' and as asymptotical axis, the bisecting line of the OO'y angle. This is an equilateral hyperbola with the central point in 0' and the vertex in M with the following coordinates:

$$[(b/a) - k/a; k/a; k]$$

The bisection plane of the angle formed by planes xOy and yOz intersected with the hyperbolic paraboloid represents a parabola and from relations $x = y$ and $y = z/(b-ax)$ we have the following parabolic equation:

$$a x^2 + b x - z = 0 \tag{10}$$

$$a y^2 + b y - z = 0 \tag{11}$$

The parabola defined by relations (10) and (11) has the maximum of coordinates:

$$[b/2a; b/2a; (b^2)/4a]$$

For any $z < (b^2)/4a$, any x and any y, there will exist two points in which $x = y$, which are the roots of the parabola of values $x = y$. For any $z > (b^2)/4a$, any x and any y, there will exist no point in which $x = y$ but $x < y$.

For each level of z, there results, one by one, a hyperbole whose vertex M draws a central parabolic curve whose equation is:

$$a x^2 - 2 b x + (b^2 / a) - z = 0 \tag{12}$$

The parabola defined by the relation (12) has as a minimum, the point 0' [(b/a); 0; 0] and as a final point from the hyperbolic paraboloid, the point N, having the coordinates $[0; b/a; (b^2)/2a]$.

The parabolas (10) and (12) mark the limits of four zones on the hyperbolic paraboloid:

- *zone 1*, between the plane z0v and the branch of the parabola (10) which is situated towards the origin. Starting from the value x =0 and y > 0, x grows up faster than y, but x < y for the whole zone;

- *zone 2*, between the branch of the parabola (10) situated towards the origin and parabola (12). Starting from the value x = y, x grows up faster than y, and x > y for the whole zone;

- *zone 3*, between parabola (12) and the other branch of the parabola (10). Starting from the central parabola, x grows up more slowly than y, but x > y for the whole zone;

- *zone 4*, between the branch situated on the right side of the parabola (10) and the straight line O'y'. Starting from x = y, x grows up more slowly than y (x tends to b/a, y tends to the infinite), and x < y for the whole zone.

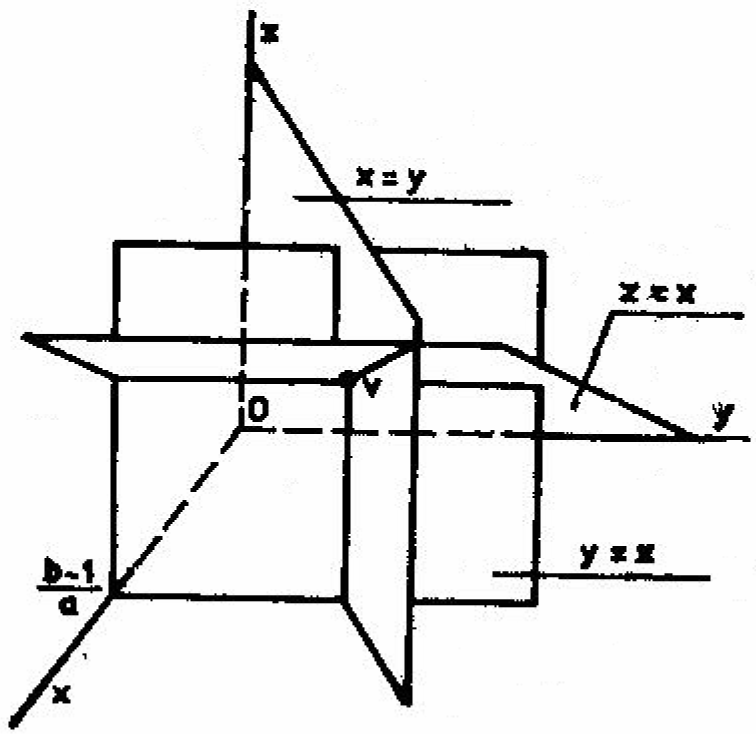

Fig. 3

Intersecting the resulting planes at points a, b, and c (Fig. 3) there results the point V that will be called "critical point", having the coordinates:

$$[x_{cr} = (b-1)/a;\ y_{cr} = (b-1)/a;\ z_{cr} = (b-1)/a\ ]$$

This is the only point on the hyperbolic paraboloid in which the three variables considered, i.e. the recovery time, the investment value and the unit cost have the same value, signifying that equal an importance is attached to them. The coordinates of the critical point V are fundamental for establishing the unit of measure for the three variables. We mention that there exists a point on the hyperbolic paraboloid in which x, y, and z have the same coordinates, namely the origin O (0; 0; 0) which, however, has not an economic meaning.

The importance of the existence and determination of the critical point comes out from the fact that this is the maximum limit of the efficiency domain given by the distance from this point to the origin which is in fact, the criterion according to which we choose the efficient alternatives.

The position of the critical point as to the origin is expressed by the critical distance d* which is calculated by the following formula:

$$d^* = (\ x_{cr}^2 + y_{cr}^2 + z_{cr}^2\ )^{1/2} = [\ (\ b - 1\ )\ /\ a\ ]\ (\ 3^{1/2}\ ) \qquad (13)$$

The efficiency condition of the investment alternatives will be given by respecting the inequality:

$$d = (\ x^2 + y^2 + z^2\ )^{1/2} < d^* \qquad (14)$$

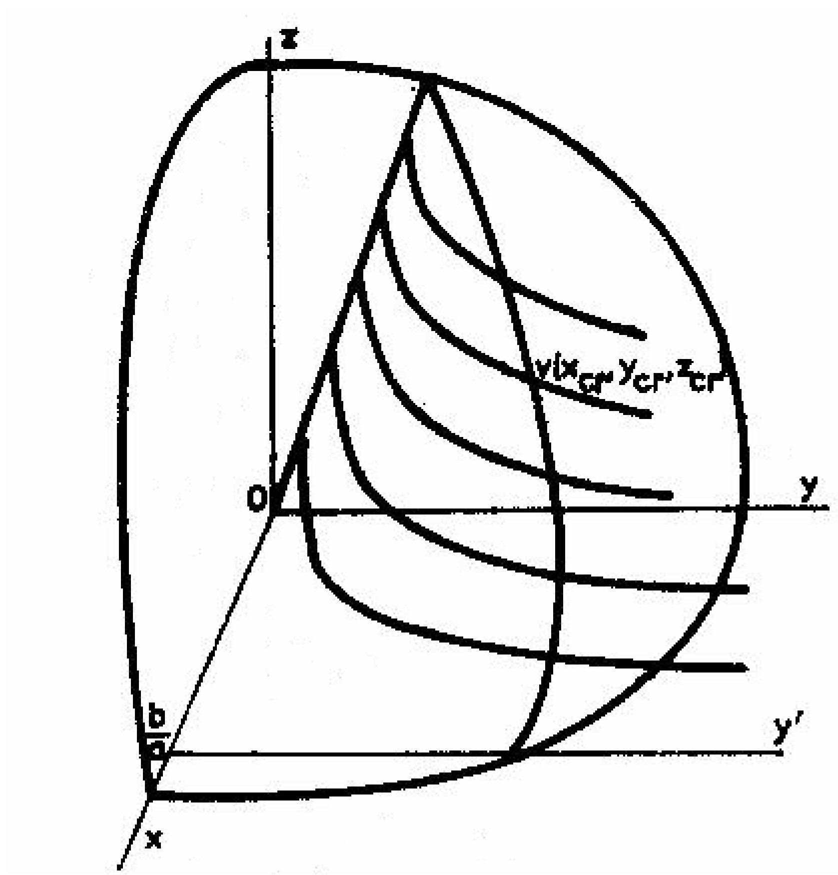

Fig. 4

Relation (14), is, in fact, a sphere with a radius $d^* = [(b-1)/a](3^{1/2})$ with the centre in the origin whose intersection with the hyperbolic paraboloid will give birth to the efficiency domain:

$$\begin{cases} x^2 + y^2 + z^2 - 3\,[\,(b-1)/a\,]^2 = 0 \\ y = z/(b - a\,x) \end{cases} \tag{15}$$

The graphical representation of the investment alternatives efficiency domain is given in Fig. 4.

From a mathematical viewpoint, the interpretation of the correlation between the recovery time, the investment value and the unit cost, did not take into account the units of measurement in which these indicators are expressed, but only their mathematical relation, the variation of one in comparison with others, reflected by the spatial representation of their interdependence.

The recovery time of investments is generally measured in years, i.e. in units of time, because the total value of the investments is related to the benefit obtained in one unit of time: the year.

On the other hand, the recovery time can be interpreted as a proportion between value units of measures, representing how many monetary units have to be invested to obtain one monetary unit of the return in a year. This means that, being interpreted in this way, the measurement units of these three indicators studied from their mathematical relation viewpoint are comparable, and that their spatial representation on the three axes has significance, and can be economically interpreted.

The application of the critical distance method raises some issues related to setting a unitary criterion according to which one could choose the measurement units both for expressing the variables and for the parameters. It is worth mentioning that the right choice of the conventional measurement units with a view to determining correctly both the critical distance and the distances as to the critical point of each alternative depends on the concrete situation of the investment alternatives. We also point out that because of the diversity and complexity in practice of the investment process, the application of a unitary criterion according to which we could choose the measurement units is not possible.

Synthesising the above elements, the algorithm for the investment optimum assessment according to the critical distance method is the following:

1. One establishes the conventional units of measure of the parameters a and b and, on their basis, those of the variables x and z, y being expressed a priori in years;

2. One calculates the values reduced to conventional units for all investment variants, which will be taken into consideration later on;

3. One calculates $d^* = [(b - 1) / a] (3^{1/2})$

4. One calculates $d_i = (x^2 + y^2 + z^2)^{1/2}$ for each investment alternative

5. One chooses the efficient alternatives on the basis of the criterion:

$$d_i < d^*$$

6. One assesses the optimum on the basis of the criterion:

$$\min (di)$$

The mathematical approach for assessing the economic efficiency of the investment alternatives by the critical distance method, points out general characteristics of the investment process, analysing the recovery time in correlation with the investment value and the unit cost. This means that:

- for a low level of the unit cost, the investment value increases more in comparison with the increase of the recovery time, and the nearer the unit cost is to the selling price, the bigger the recovery time is in comparison with the increase of the investment value;

- for a low level of the investment, the unit cost increases more in comparison with the increase of the recovery time and the bigger the level of the investment is, the bigger the increase of the recovery time, tending to the infinite, in comparison with the increase of the unit cost which tends to the value of the selling price;

- for a low level of the recovery time, the increase of the unit cost is bigger in comparison with the increase of the investment value, and the bigger the recovery time, the higher the increase of the investment value in comparison with the increase of the unit cost.

The critical distance method proposed in order to quantify the microinvestment optimum has to be applied only according to specific circumstances. Its general principles being valuable for all cases, it can be extended, in our opinion, to the study of any economic process.

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