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2 October 1994

Online at https://mpra.ub.uni-muenchen.de/15007/
MPRA Paper No. 15007, posted 06 May 2009 14:13 UTC
The Potential of Multi-choice Cooperative Games

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Abstract. We defined the potential for multi-choice cooperative games, and found the relationship between the potential and the multi-choice Shapley value. Moreover, we show that the multi-choice Shapley is consistent.

Introduction. In [1](1991), we extended the traditional cooperative game to the multi-choice cooperative game and extended the traditional Shapley value to the Shapley value for multi-choice cooperative games. In short, we call the shapley value for multi-choice cooperative games the multi-choice Shapley value. In 1990, Shapley asked “what is the potential for multi-choice games?”. In this article, we would answer Shapley’s question.

In [1], [2], [3], we assumed that the players in a multi-choice cooperative game have the same number of options, or say actions. As a matter of fact, from the point of view of the multi-choice Shapley value, it makes no difference whether the players have the same number of options or not. Therefore, by just rewriting the definitions and the proofs in [1], [2], [3], we may define the multi-choice Shapley value for a game where the players have different numbers of actions.

In this article, We would first rewrite the definition of the multi-choice Shapley value and define the potential of multi-choice cooperative games, then, show the relationship between the multi-choice Shapley value and the potential, and prove that the multi-choice Shapley value is consistent.
Finally, we would define “$w$-proportional for two-working-player games”, and show that a solution for multi-choice cooperative game is the multi-choice Shapley value if and only if it satisfies consistency and “$w$-proportional for two-working-player games”.

Definitions and Notations.

Let $I_+ = \{1, 2, ..., n\}$ be the set of players. We allow player $j$ to have $(m_j + 1)$ actions, say $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_{m_j}$, where $\sigma_0$ is the action to do nothing, while $\sigma_k$ is the option to work at level $k$, which is better than $\sigma_{k-1}$. In this article, we assume that all the players have finitely many choices.

For convenience, we will use non-negative integers to denote the players’ actions. Given $\mathbf{m} = (m_1, m_2, ..., m_n) \in I^n_+$, the action space of $N$ is defined by $\Gamma(\mathbf{m}) = \{(x_1, ..., x_n) \mid x_i < m_i \text{ and } x_i \in I_+, \text{ for all } i \in N\}$. Thus $(x_1, ..., x_n)$ is called an action vector of $N$, and $x_i = k$ if and only if player $i$ takes action $\sigma_k$.

Given $\mathbf{z} = (z_1, z_2, ..., z_n)$, $\mathbf{m} = (m_1, m_2, ..., m_n) \in I^n_+$, we define $\mathbf{z} \leq \mathbf{m}$ if and only if $z_i \leq m_i$ for all $i \in N$. It is clear that $\Gamma(\mathbf{z}) \subseteq \Gamma(\mathbf{m})$ whenever $\mathbf{z} \leq \mathbf{m}$.

Definition 1. A multi-choice cooperative game in characteristic function from is the pair $(\mathbf{m}, v)$ defined by, $v : \Gamma(\mathbf{m}) \rightarrow R$, such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, 0, ..., 0)$.

Player $j$ is called a useless player if and only if $m_j = 0$. Moreover, player $j$ is called an working player if and only if $m_j > 0$. We may consider $v(x)$ as the payoff or the cost for the players whenever the players take action vector $\mathbf{x}$. Sometimes, we will denote $v(\mathbf{x})$ by $(\mathbf{m}, v)(\mathbf{x})$ in order to emphasis that the domain of $v$ is $\Gamma(\mathbf{m})$.

Given $\mathbf{z} \in I^n_+$ with $\mathbf{z} \leq \mathbf{m}$, a sub-game of $(\mathbf{m}, v)$ is obtained by restricting the domain of $v$ to $\Gamma(\mathbf{z})$. We denote the sub-game by $(\mathbf{z}, v)$. In other words, Let $\mathbf{z} \in I^n_+$ with $\mathbf{z} \leq \mathbf{m}$, we call $(\mathbf{z}, v)$ a sub-game of $(\mathbf{m}, v)$, if and only if $(\mathbf{z}, v)(\mathbf{x}) - (\mathbf{z}, v)(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{z})$.

We can identify the set of all multi-choice cooperative games defined on $\Gamma(\mathbf{m})$ by $G \simeq R^{\Pi_{-1}(m_j+1)-1}$.

Since we do not assume that action $\sigma_2$ is say, twice as powerful as action $\sigma_1$, and since we do not assume that the difference between $\sigma_{k-1}$ and $\sigma_k$ is the same as the difference between $\sigma_k$ and $\sigma_{k+1}$, etc., giving weights (discrimination) to actions is necessary.

Let $m = \max_{j \in N} \{m_j\}$, and let $w : \{0, 1, ..., m\} \rightarrow R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \leq w(2) \leq ... \leq w(m)$, then $w$ is called a weight function and $w(i)$ is said to be a weight of $\sigma_i$. 2
Given a weight function \( w \) for the actions, we define a value, or say a solution of a multi-choice cooperative game \((m, v)\) by a \( \sum_{j=1}^{n} m_j \) dimensional vector \( \phi^w : G \to R^{\sum_{j=1}^{n} m_j} \) be such that

\[
\phi^w(v) = (\phi^w_{11}(v), \ldots, \phi^w_{m_11}(v), \phi^w_{12}(v), \ldots, \phi^w_{m_22}(v), \ldots, \phi^w_{1n}(v), \ldots, \phi^w_{m_nn}(v))
\]

Here \( \phi^w_{ij}(v) \) is the power index or the value of player \( j \) when he takes action \( \sigma_i \) in game \( v \).

Rewrite [3], we can show that when \( w \) is given, given, there exists a unique \( \phi^w \) satisfying the following four axioms.

**Axiom 1.** Suppose \( w(0), w(1), \ldots, w(m) \) are given. If \( v \) is of the form

\[
v(y) = \begin{cases} 
& c > 0 \quad \text{if } y \geq x \\
& 0 \quad \text{otherwise},
\end{cases}
\]

then \( \phi^w_{x_i,v}(v) \) is proportional to \( w(x_i) \).

Axiom 1 states that for binary valued (0 or c) games that stipulate a minimal exertion from players, the reward, for players using the minimal exertion level is proportional to the weight of his minimal level action.

We denote \( (x \mid x_i = k) \) as an action vector with \( x_i = k \).

Given \( x, y \in \Gamma(m) \), we define \( x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n) \) where \( x_i \vee y_i = \max\{x_i, y_i\} \) for each \( i \). Similarly, we define \( x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n) \) where \( x_i \wedge y_i = \min\{x_i, y_i\} \) for each \( i \).

**Definition 2.** A vector \( x^* \in \Gamma(m) \) is called a carrier of \( v \), if \( v(x^* \wedge x) = v(x) \) for all \( x \in \Gamma(m) \). We call \( x^0 \) a minimal carrier of \( v \) if \( \sum x_i^0 - \min\{\sum x_i \mid x \text{ is a carrier of } v\} \).

**Definition 3.** Player \( i \) is said to be a dummy player if \( v((x \mid x_i = k)) = v((x \mid x_i = 0)) \) for all \( x \in \Gamma(m) \) and for all \( k = 0, 1, 2, \ldots, m_i \).

A useless player is of course a dummy player. The following is a version of the usual efficiency axiom that combines the carrier and the notions of dummy player.

**Axiom 2.** If \( x^* \) is a carrier of \( v \) then, for \( m = (m_1, m_2, \ldots, m_n) \) we have

\[
\sum_{x_i^* \neq 0} \phi^w_{x_i^*,i}(v) = v(m).
\]

By \( x_i^* \in x^* \) we mean \( x_i^* \) is the \( i \)-th component of \( x^* \).
Axiom 3. \( \phi_w(v^1 + v^2) - \phi_w(v^1) + \phi_w(v^2) \), where \((v^1 + v^2)(x) - v^1(x) + v^2(x)\).

Axiom 4. Given \( x^0 \in \Gamma(m) \) if \( v(x) = 0 \), whenever \( x \not\sim x^0 \), then for each \( i \in N \phi_{k,i}^w(v) = 0 \), for all \( k < x^0_i \).

Axiom 4 states that in games that stipulate a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded.

Definition 4. Given \( x \in \Gamma(m) \), let \( S(x) = \{ i \mid x_i \neq 0, x_i \text{ is a component of } x \} \). Given \( S \subseteq N \), let \( e(S) \) be the binary vector with components \( e_i(S) \) satisfying

\[
e_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}
\]

For brevity, we let the standard unit vectors \( e(i) = e_i \), for all \( i \in N \), and let \( |S| \) be the number of elements of \( S \).

Definition 5. Given \( \Gamma(m) \) and \( w(0) = 0, w(1), \ldots, w(m) \), for any \( x \in \Gamma(m) \), we define

\[
\| x \|_w = \sum_{i=1}^{n} w(x_i).
\]

Definition 6. Given \( x \in \Gamma(m) \) and \( j \in N - \{1, 2, \ldots, n\} \), we define \( M_j(x; m) = \{ i \mid x_i \neq m_i, i \neq j \} \).

From Theorem 2 in [3], we have

\[
\phi_{ij}^w(v) = \sum_{k=1}^{i} \sum_{\substack{x_j=k \ x \neq 0 \ x \in \Gamma(m)}} \left[ \sum_{T \subseteq M_j(x; m)} (-1)^{|T|} \frac{w(x_j)}{\|x\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \\
\times [v(x) - v(x - e_j)],
\]

\((*)\)

The Potential.

Given \( m = (m_1, m_2, \ldots, m_n) \in I^n_+ \) and an \( n \)-person multi-choice cooperative game \((m, v)\). We denote the set of all the sub games of \((m, v)\) by

\[
G^* = \{ (z, v) \mid z \in I^n_+ \text{ and } z \leq m \}.
\]

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Given a weight function $w$ for $\{0, 1, \ldots, m\}$, we define a function $P_w : G^* \rightarrow \mathbb{R}$ which associates a real number $P_w((x, v))$.

Given $P_w((x, v))$, we define the following operators.

$$D_{i, j}P_w((x, v)) = w(i) \cdot \left[ P_w( ((x|x_j = i), v)) - P_w( ((x|x_j = i - 1), v)) \right],$$

and

$$H_{x_j, j} = \sum_{\ell=1}^{\ell=x_j} D_{\ell, j}.$$

**Definition 8.** A function $P_w : G \rightarrow \mathbb{R}$ with $P_w((0, v)) = 0$ is called a $w$-potential function if it satisfies the following condition: for each fixed $x \in \Gamma(m)$

$$\sum_{j \in S(x)} H_{x_j, j}P_w((x, v)) = (x, v)(x) \quad (**)$$

Given $j \in N$ and $v(x)$, we define

$$d_jv(x) = v(x) - v(x - e_j),$$

then $d_j$ is associative, i.e. $d_k(d_jv(x)) = d_j(d_kv(x))$. For convenience, we denote $d_id_j = d_{ij}$, $d_{ijk} = d_id_jd_k$, ..., etc. We also denote $d_{i_1, i_2, \ldots, i_t} = d_T$ whenever $\{i_1, i_2, \ldots, i_t\} = T$. Furthermore, for brevity, we denote $d_{S(x)}$ by $d_x$.

**Theorem 1.** The Potential of multi-choice cooperative games is unique, and

$$P_m((x, v)) = \sum_{0 \neq y \leq x} \frac{1}{||y||_w} d_y(x, v)(y) \quad (1)$$

**Proof.** Consider $(m, v)$ and all its sub-games $(x, v)$. It is easy to see that $P_w((0, v)) = 0$. Let $|x| = \sum_{i \in N} x_i$, by mathematical induction on $|x|$, by equation $(**)$, we can easily see that the potential is unique. Now, we show the following claim in order to prove our theorem.

**Claim:** Given any multi-choice cooperative game $(x, v)$, let

$$\psi_w((x, v)) = \sum_{0 \neq y \leq x} \frac{1}{||y||_w} d_y(x, v)(y),$$

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\[ H_{ij} \psi_w((\mathbf{x}, v)) = \phi_{ij}^v((\mathbf{x}, v)). \]

\[
H_{i,j} \psi_w((\mathbf{x}, v)) - \sum_{k=1}^{i} D_{k,j} \psi_w((\mathbf{x}, v)) \\
= \sum_{k=1}^{i} w(k) \cdot \left[ \psi_w((\mathbf{x}|x_i = k), v)) - \psi_w((\mathbf{x}|x_i = k - 1), v)) \right] \\
= \sum_{k=1}^{i} w(k) \cdot \left[ \sum_{y \leq (x|x_j = k)} \frac{1}{|y|} d_y v(y) - \sum_{y \neq 0 \leq (x|x_j = k - 1)} \frac{1}{|y|} d_y v(y) \right] \\
= \sum_{k=1}^{i} w(k) \cdot \left[ \sum_{y \in \Gamma(x)} \frac{1}{|y|} d_y v(y) \right] \\
= \sum_{k=1}^{i} w(k) \cdot \left[ \sum_{y_j = k} \sum_{y \in \Gamma(x)} \frac{1}{|y|} \sum_{T \subseteq S(y)} (-1)^{|T|} v(y - \sum_{r \in T} \mathbf{e}_r) \right] \\
= \sum_{\mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \in \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) + \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|+1} v(y - \mathbf{e}(\mathcal{T} \cup \{j\})) + \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} \left[ v(y - \mathbf{e}(\mathcal{T} \cup \{j\})) \right] \\
= \sum_{T \subseteq S(y)} (-1)^{|T|} v(y - \mathbf{e}(T)).
\]

Consider \( \sum_{\mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \), where \( \mathbf{e}(\mathcal{T}) = \sum_{r \in \mathcal{T}} \mathbf{e}_r \), we have

\[
\sum_{\mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) + \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|+1} v(y - \mathbf{e}(\mathcal{T} \cup \{j\})) + \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|\mathcal{T}|} v(y - \mathbf{e}(\mathcal{T})) \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} \left[ v(y - \mathbf{e}(\mathcal{T} \cup \{j\})) \right] \\
= \sum_{T \subseteq S(y)} (-1)^{|T|} v(y - \mathbf{e}(T)).
\]

Let \( \mathbf{z} = \mathbf{y} - \mathbf{e}(\mathcal{T}) \), then we have

\[
\sum_{T \subseteq S(y)} (-1)^{|T|} v(y - \mathbf{e}(T)) = \sum_{j \notin \mathcal{T} \subseteq S(y)} \left[ v(y - \mathbf{e}(T \cup \{j\})) \right] \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} \left[ v(z) - v(z - \mathbf{e}_j) \right] \\
= \sum_{j \notin \mathcal{T} \subseteq S(y)} (-1)^{|T|} \left[ v(z) - v(z - \mathbf{e}_j) \right].
\]

Since \( \mathbf{y} = \mathbf{z} + \mathbf{e}(\mathcal{T}), \mathbf{y} \neq \mathbf{0} \) and \( j \notin T \),
we have \( \{ T \subseteq S(y) \} = \{ T \subseteq M_j(z; x) \} \).

Hence (1.2) can be written as

\[
\sum_{T \subseteq M_j(z; x)} (-1)^{|T|} \left[ n(z) - n(z - e_j) \right]
\]  

(1.3)

From (1.3 ), we know that (1.1) can be written as

\[
\sum_{z_j = k} \sum_{z \neq 0} \sum_{z \in \Gamma(x)} \left[ \sum_{T \subseteq M_j(z; x)} (-1)^{|T|} \frac{w(k)}{||z + e(T)||_w} \right] \cdot [v(z) - v(z - e_j)]
\]

\[
= \sum_{i} \sum_{k=1} \sum_{z_j = k} \sum_{z \neq 0} \sum_{z \in \Gamma(x)} \left[ \sum_{T \subseteq M_j(z; x)} (-1)^{|T|} \frac{w(k)}{||z||_w + \sum_{r \in T} [w(z_r + 1) - w(z_r)]} \right] \cdot [v(z) - v(z - e_j)]
\]

\[
= \phi^w_{i,j}((x, v)).
\]

Since \((x, v)\) is arbitrarily given, then, by the above claim and (*), we have,

\[
\sum_{j \in S(x)} H_{x, j}^w((x, v)) = \sum_{j \in S(x)} \phi^w_{x, j}((x, v)) = (x, v)(x),
\]

for each fixed \(x \in \Gamma(m)\).

Since the potential of \((x, v)\) is unique and \(\psi^w((x, v))\) satisfies (**) , then \(\psi^w((x, v))\) is the potential.

The proof is complete \(\diamond\)

From the above proof, we can easily see the following theorem.

**Theorem 2.** Given a multi-choice cooperative game \((m, v)\) then the Shapley value and the Potential of \((m, v)\) have the following relationship.

\[
\phi^w_{i,j}((m, v)) = H_{i,j}^w((m, v)).
\]

(2)

**Consistency Property of the Multi-choice Shapley value.**
Given a multi-choice cooperative game \((\mathbf{m}, v)\) and its solution,

\[
(\psi_{m_1}^w(v), \ldots, \psi_{m_1}^w(v), \psi_{m_2}^w(v), \ldots, \psi_{m_2}^w(v), \ldots, \psi_{m_n}^w(v), \ldots, \psi_{m_n}^w(v)),
\]

for each \(z \in \Gamma(\mathbf{m})\), we define an action vector \(z^* = (z_1^*, z_2^*, \ldots, z_n^*)\) where

\[
\begin{align*}
    z_j^* &= m_j \text{ if } z_j < m_j \\
    z_j^* &= 0 \text{ if } z_j = m_j.
\end{align*}
\]

Furthermore, we define a new game \(v^w_z : \Gamma(z) \to R\) such that

\[
v^w_z(y) = v(y \lor z^*) - \sum_{z_j^* \neq 0} \psi_{m_j,j}((b_{y} \lor z^*), v).
\]

We call \(v^w_z\) a reduced game of \(v\) with respect to \(z\) and the solution \(\psi\). Furthermore, we say that the solution \(\psi\) is consistent if \(\psi_{i,j}(v) = \psi_{i,j}(v^w_z)\) for all \(i \leq z_j\) and all \(j \in N - S(z^*)\).

**Theorem 3.** The multi-choice Shapley value \(\phi^w\) is consistent.

**Proof.** Given a multi-choice cooperative game \((\mathbf{m}, v)\) and its Shapley value \(\phi^w\). Given \(z \in \Gamma(\mathbf{m})\), the reduced game of \(v\) with respect to \(z\) and the Shapley is :

\[
v^w_z : \Gamma(\mathbf{m}) \to R \text{ such that }
\]

\[
v^w_z(y) = v(y \lor z^*) - \sum_{z_j^* \neq 0} \phi_{m_j,j}((y \lor z^*), v).
\]

Let \(b = (b_1, \ldots, b_n) = y \lor z^*\), since the Shapley value satisfies Axiom 2, we have

\[
v^w_z(y) = v(y \lor z^*) - \sum_{z_j^* \neq 0} \phi_{m_j,j}((y \lor z^*), v)
= \sum_{b_j \neq 0} \phi_{b_{y,j}}((y \lor z^*), v) - \sum_{z_j^* \neq 0} \phi_{m_j,j}((y \lor z^*), v).
\]

Now, since \(z_j^*\) is either 0 or \(m_j\), for any \(y \leq z\), by Theorem 2, we have

\[
v^w_z(y) = \sum_{b_j \neq 0} \phi_{b_{y,j}}((y \lor z^*), v)
= \sum_{b_j \neq 0} H_{b_{y,j}} P_n((y \lor z^*), v)). \quad (3.1)
\]
By Theorem 1, the potential of the game \((z, v^\omega_z)\) is uniquely determined by formula (\(\ast\ast\)) applied to the game and all its subgames. Comparing this with (3.1), we know that

\[
P_w((y, v^\omega_z)) = P_w((y \lor z^*, v)) + c
\]

for all \(y \leq z\), where \(c\) is a suitable constant so as to make \(P_w((0, v^\omega_z)) = 0\). It is clear that \(z \lor z^* = m\), hence,

\[
\phi_{i,j}^w((z, v^\omega_z)) = H_{i,j}P_w((z, v^\omega_z)) = H_{i,j}P_w((m, v))
\]

for all \(i \leq s_j\) and all \(j \in N - S(z^*)\).

The proof is complete \(\diamondsuit\)

**w-proportional for two-working-player games.**

In the beginning of this article, we define \(\sigma_0\) as the action to do nothing.

Since the solution concept of the multi-choice Shapley value, is dummy free the useless player does not affect the value no matter if he is regard as a player or not, see [2] for detail. However, not all solution concepts are dummy free.

W.L.O.G. suppose \(i < j\), given \(N_1 = \{i, j\}\) and a two person cooperative game \((m_1, v_1)\) with \(m_1 = (m_i, m_j)\), furthermore, given \(N_2 = \{1, \ldots, i, \ldots, j, \ldots n\}\) and an \(n\)-person cooperative game \((m_2, v_2)\) with \(m_2 = (0, 0, \ldots, 0, m_i, 0, \ldots, 0, m_j, 0, \ldots, 0)\). Suppose \(v_1((x_i, x_j)) = v_2((0, 0, \ldots, 0, x_i, 0, \ldots, 0, x_j, 0, \ldots, 0))\) for all \(x_i = 0, 1, \ldots, m_i\) and \(x_j = 0, 1, \ldots, m_j\), we can easily make up some solution concept where the solution for \(v_1\) is different from the solution for \(v_2\) i.e. we can easily make up a solution concept where the value for player \(i\) in \(v_1\) is different from the value for player \(i\) in \(v_2\). Therefore, we can not regard \(v_1\) and \(v_2\) as the same game. To avoid ambiguity, we call \(v_1\) a two person game and call \(v_2\) a two-working-player game. When there is only one \(m_j > 0\), we called \((m, v)\) an one-working-player game.

Given an \(n\)-person multi-choice cooperative game \((m, v)\), and its sub-game \((x, v)\) where \(x \leq m\), as usual, we let \(m = \max\{m_1, \ldots, m_n\}\) Let \((0, x_i = k, x_j = \ell)\) be an action vector where player \(i\) takes action \(\sigma_k\), player takes action \(\sigma_\ell\) and all the other players take action \(\sigma_0\), we have the following definition.

**Definition 9.** Given \(w(0) = 0, w(1), \ldots, w(m)\), a solution function \(\psi\) is said to satisfies “w-proportional for two-working-player games” if for any two-working-player game \((m, v)\) with \(m = (0, m_i, 0, \ldots, m_j, 0, \ldots, 0)\), \(\psi\) satisfies the following.
\[
\psi_{k,i}((m, v)) = \sum_{t=1}^{k} v((0|x_i = t, x_j = 0) + \left[ \frac{w(t)}{w(t) + w(1)} \right] \cdot [v((0|x_i = t, x_j = 1)) - v((0|x_i = t - 1, x_j = 1)) - v((0|x_i = t, x_j = 0))]
\]
\[
+ \left[ \frac{w(t)}{w(t) + w(2)} \right] \cdot [v((0|x_i = t, x_j = 2)) - v((0|x_i = t - 1, x_j = 2)) - v((0|x_i = t, x_j = 1))]
\]
\[
\vdots
\]
\[
+ \left[ \frac{w(t)}{w(t) + w(m_j)} \right] \cdot [v((0|x_i = t, x_j = m_j)) - v((0|x_i = t - 1, x_j = m_j)) - v((0|x_i = t, x_j = m_j - 1))]
\]
\]
\[
(4.1)
\]

For player \( j \), we have a formula of \( \psi_{k,j}((m, v)) \) similar to (4.1) which is omitted.

It is easy to see that (4.1) is an extension of the definition of standard for two-person games in [1]. For convenience, we reformulate (4.1) as follows.

\[
\psi_{k,i}((m, v)) = \sum_{t=1}^{k} \sum_{\substack{z_i = t \in T(m) \setminus \{m_j\} \setminus \{z_j\} \neq z_j \neq m_j}} \left[ \frac{w(z_i)}{w(z_i) + w(z_j)} \right] \cdot [v(z - e_i)] + \sum_{t=1}^{k} \sum_{\substack{z_i = t \in T(m) \setminus \{m_j\} \setminus \{z_j\} \neq z_j \neq m_j}} \left[ \frac{w(z_i)}{w(z_i) + w(z_j + 1)} \right] \cdot [v(z - e_i)]
\]
\]
\[
(4.2)
\]

We have the following conjecture:

**Conjecture.** Given a \( n \)-person multi-choice cooperative game \((m, v)\) with all its sub-games and a weight function for \( \{0, 1, ..., m\} \), let \( \phi^w \) be a solution function. Then:

(i) \( \phi^w \) is consistence; and

(ii) \( \phi^w \) is "\( w \)-proportional for two-working-player cooperative games";

if and only if \( \phi^w \) is the multi-choice Shapley value.
REFERENCES


