A Decision Analysis Approach To Solving the Signaling Game

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Abstract

Decision analysis has traditionally been applied to choices under uncertainty involving a single decision maker. Game theory has been applied to solving games of strategic interaction between two or more players. Building upon recent work of van Binsbergen and Marx (2007. Exploring relations between decision analysis and game theory. *Decision Anal.* 4(1) 32–40.), this paper defines a modified decision-theoretic approach to solving games of strategic interaction between two players. Using this method, the choices of the two players are modeled with separate decision trees comprised entirely of chance nodes. Optimal policies are reflected in the probabilities in the decision trees of each player. In many cases, the optimal strategy for each player can be obtained by rolling back the opponent's decision tree. Results are demonstrated for the multi-stage signaling game, which is difficult to model using decision nodes to represent strategies, as in the approach of van Binsbergen and Marx.

Key Words: decision analysis, decision tree, game theory, mixed strategy, signaling game.

*Comments and suggestions for improvement are welcome and will be gratefully appreciated.

1 Introduction

A recent paper by van Binsbergen and Marx (2007)—hereafter VBM—offers an alternative to traditional game theory approaches to solving decision problems that involve strategic interactions among multiple decision makers. As inspiration for their work, these authors cite research by Cavuosoglu and Raghunathan (2004) that gives the details of a game-theoretic approach for configuring detection software. One conclusion made by Cavuosoglu and Raghunathan (2004) is that a decision-theoretic approach cannot be used to appropriately solve for the optimal detection choice, necessitating a game-theoretic solution. VBM outline their decision-theoretic approach to strategic games and apply it to solve the software configuration problem.

The VBM approach is based on a duality between decision analysis and game theory. For consistency, we define game theory and decision theory as in their article, and for completeness provide a brief review. As stated by VBM, who paraphrase definitions formulated by the Decision Analysis Society, decision analysis provides a structured approach to examining how actions taken in a decision environment affect the results of the decision maker. Four aspects of a decision analysis are (i) the set of alternatives available to the decision maker, (ii) the random or chance events that influence the outcomes, (iii) a value model that describes outcomes for the various combinations of alternatives and chance events, and (iv) the solution technique.

Game theory defines a strategic interaction between two or more players where the payoff to each player depends not only on her own choice, but the choices of all other players. The game theory analogs to the four components of decision analysis are (i) the strategy set, (ii) the moves of nature, (iii) the payoff function, and (iv) the equilibrium concept. Two aspects of game theory that differentiate it from decision analysis are (v) the other players and (vi) the dependence of each players payoffs on the other players in the game. VBM state that these additional aspects do not necessarily limit the ability of decision analysis to model situations where the payoffs to the decision maker are affected by the actions of other players. Their approach uses decision analysis techniques to model the actions of each player.

Koller and Milch (2003) also apply a decision-theoretic model to a multi-agent decision making context. Their approach constructs one influence diagram model with separate decision nodes for each player. Their paper describes an algorithm that exploits the notion of strategic relevance to determine global equilibrium strategies using local computation on several interacting smaller games. In contrast to this paper, decision variables are used to represent the strategies of the players, and these variables have finite state spaces. In the spirit of our research and the VBM article, Koller and Milch (1999) state the following regarding the use of a decision-theoretic approach to modeling games of strategic interaction:

> "...the traditional representations of games are primarily designed to be amenable to abstract mathematical formulation and analysis. As a consequence, the standard game representations...obscure certain important structure that is often present in real-world scenarios—the decomposition of the situation into chance and decision *variables*, and the dependence relationships between these variables" (p. 182).

We concur with VBM that decision theory offers models that may be more amenable to developing an intuitive graphical model of a strategic game. In situations involving strategic interaction, decision makers are deemed to be rational if they act using strategies that meet the Nash equilibrium criterion—that no player can change actions unilaterally to earn a better outcome. Traditional game theory approaches to determining Nash equilibrium strategies require decision makers to reason abstractly about how the strategies of each player affect other players in the game. Modeling games of strategic interaction with decision-theoretic models—such as decision trees and influence diagrams—can allow decision makers to use well-known solution algorithms—such as a dynamic programming approach to rolling back a decision tree—to find equilibrium strategies in some cases.

The methodology presented by VBM and extended in our paper can also be used to incorporate game-theoretic thinking into a decision analysis context. In decision analysis, any event not under the control of the decision maker—such as an action taken by a rival firm—is represented by a random variable. Probability distributions are used to describe the uncertainty associated with these random variables. If actions are chosen strategically by the two firms, the actions of the decision maker may affect the probability that the rival firm chooses a certain action. The techniques presented in this paper allow the decision maker more flexibility in modeling the affects of strategic interaction on a decision problem.

To model games of strategic interaction, VBM construct a decision tree for each player where the player's own strategies are represented as decision nodes and the strategies of other players (or moves of nature) are represented by chance nodes. The method presented in this paper also depends on building a decision tree for each player. However, our approach models the choices of all players with chance nodes. The probabilities assigned to these chance nodes represent the chosen strategies of the players. Allowing only chance nodes in the decision tree offers some computational advantages. Principally, the rollback of the decision tree can proceed without any maximization operations. Although the techniques we present can be used to model games with more than two players, we will focus on two-player games because equilibrium strategies in these games can often be determined without the iterative solution process required by the VBM approach.

The goal of this paper is to extend the VBM approach to more easily model more complicated games, including the signaling game (Gibbons 1992, Dixit and Skeath 1999). In the signaling game, a less informed player relies on a signal from a more informed player to understand the potential strength (or type) of its opponent. The remainder of this paper is organized as follows. In §2, game-theoretic, decision analysis, and modified decision analysis approaches for solving a normal-form game are introduced, with the inspection game of Fudenberg and Tirole (1993) used as an example. In §3, the signaling game is introduced and is solved using the modified approach, with a game-theoretic solution offered as a comparison. §4 compares the three approaches, provides a discussion of the implementation of the modified decision analysis approach, and summarizes the paper.

2 Normal Form Games

In this section we focus on static games of complete information. A normal form representation of such a game with two players, $G = \{S_1, S_2; u_1, u_2\}$, is defined by the i-th player's strategy space S_i and payoff function u_i. These games may be solved using either a game-theoretic approach or a decision-theoretic approach. In this section, we illustrate that in

Table 1: The normal form of the inspection game.

	Firm	
Worker	Inspect	No Inspection
Shirk	0, −10	50, −50
Work	30, 40	30, 50

either case the solution does not change. Also, we present a decision analysis approach that is a modification to the VBM approach and obtain the same solution.

Applying the Nash equilibrium concept will lead to a prediction of the strategies that ought to be chosen by the players. By definition, there will be no incentive to deviate from these strategies. In other words the predicted actions will be strategically stable (Gibbons 1992, p. 8). More formally, the strategies s_1^*, s_2^*, where $s_i \in S_i$, are a Nash equilibrium if $u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*)$ and $u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2)$.

2.1 Example: The Inspection Game

The inspection game (Fudenberg and Tirole 1993, p. 17) is an example of a two-player static game of complete information. This game is played between a worker—who can choose to Shirk (S) or Work (W)—and a firm—who can Inspect (I) the worker or make No Inspection (NI). Producing output valued at \$100 costs the worker \$20. The firm pays the worker \$50 and it costs \$10 to inspect. The normal form table for the inspection game is shown in Table 1. The Worker's payoffs are the first entry in each pair of values.

The Worker can choose Shirk, Work, or a mixed strategy where it plays Shirk with some probability $y \in [0, 1]$. Shirk and Work are *pure strategies* and coincide with $y = 1$ and $y = 0$, respectively. The Firm can choose to play Inspect, No Inspection or a mixed strategy where it plays Inspect with some probability $x \in [0, 1]$ ($x = 1$ is a pure Inspect strategy and $x = 0$ is a pure No Inspection strategy). In other words, the Worker's and Firm's strategies are y and x, respectively.

It is clear that there cannot be any pure strategy Nash equilibria in this game. If the Firm performs No Inspection then the Worker will prefer to Shirk, thus giving the Firm an incentive to deviate from No Inspection. Similarly, if the Firm chooses to Inspect then the Worker will prefer to Work, which in turn creates an incentive for the Firm to deviate from Inspect. Thus, both players must play a mixed strategy in equilibrium.

2.2 Game-Theoretic Solution

The Worker's expected value from Shirking is $EV_{W,S} = 0x + (1 - x)50 = 50 - 50x$ and her expected value from Working is $EV_{W,W} = 30x + (1 - x)30 = 30$. The Firm's expected value from Inspecting is $EV_{F,I} = -10y + (1 - y)40 = 40 - 50y$ and its expected value from choosing No Inspection is $EV_{F,NI} = -50y + (1 - y)50 = 50 - 100y$. Each player's strategy is in equilibrium only if it does not provide an incentive to the other player to deviate. Thus the Firm's equilibrium strategy x^* must be such that $EV_{W,S} = EV_{W,W}$ and the Worker's

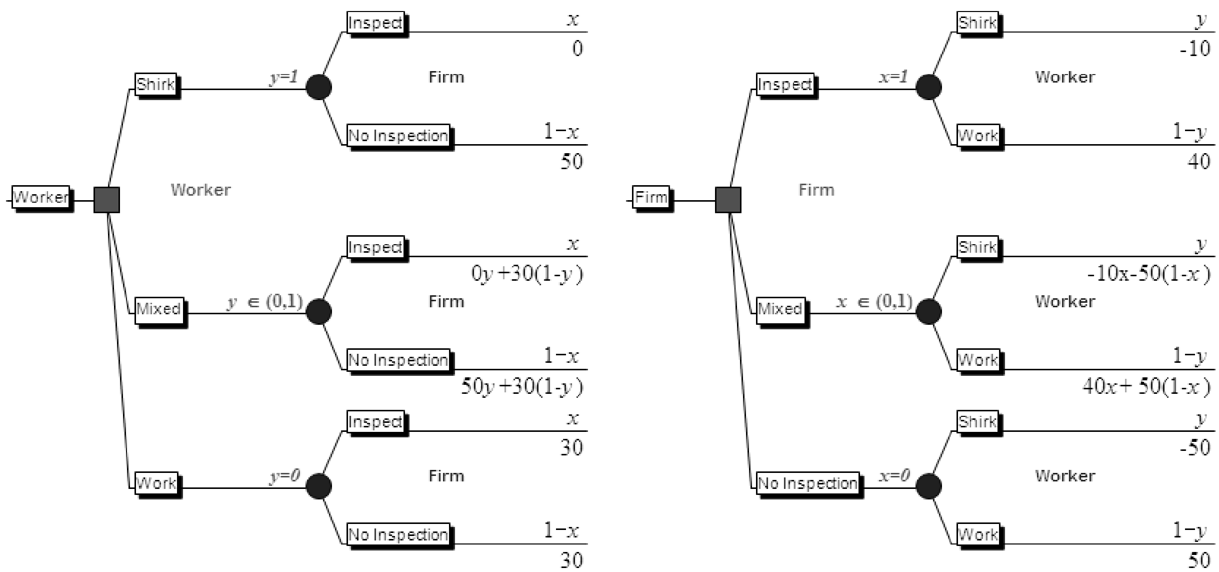

Figure 1: The VBM paired decision analysis representation of the inspection game.

equilibrium strategy y^* must be such that $EV_{F,I} = EV_{F,NI}$. Solving for x^* and y^* gives the Nash equilibrium strategies

$$\begin{array}{ccc} 50 - 50x = 30 & \quad\text{and}\quad & 40 - 50y = 50 - 100y \\ x^* = 0.40 & & y^* = 0.20\ . \end{array}$$

2.3 Decision-Theoretic Solution

In their paper, VBM use an approach called *paired decision analysis* to solve normal-form games similar to the one with payoffs in Table 1. Using this approach requires constructing a decision tree for each player with a decision node representing their own choice, and a chance node representing the opponent's choice. A graphical depiction of the VBM paired decision analysis model for the inspection game with payoffs in Table 1 is shown in Figure 1.

As shown in the decision tree, the Worker chooses either one of two pure strategies (Shirk or Work) or chooses a mixed strategy y. The chance node beyond the initial decision node in the Worker's decision tree represents the Firm's strategy. On the mixed strategy branch, the payoffs at the endpoints of the decision tree are expected values. The setup of the Firm's decision tree is similar, except that the Firm's own strategy is represented as a decision node.

VBM state that to identify the Nash equilibrium in the game "...one could iterate until the two decision trees are consistent..." and that there is "...no single pass approach, such as backward induction or rolling back the tree, that delivers the solution to this problem" (p. 37). To identify the equilibrium, the decision maker must try various strategy pairs (x, y) until a pair provides a solution that does not vary when each player's strategy is subsequently changed. When such an iteration process arrives at the strategies x^* and y^* from the previous section, the Nash equilibrium solution is found.

In the next section, we present a modification to the VBM approach with some computational advantages that can aid in performing such an analysis.

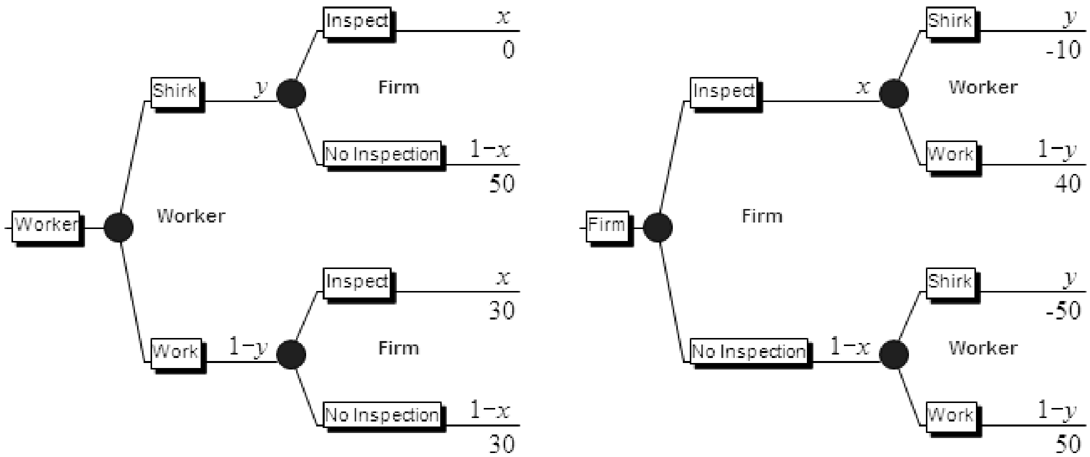

Figure 2: Decision tree models for the two players in the modified decision analysis approach to solving the inspection game.

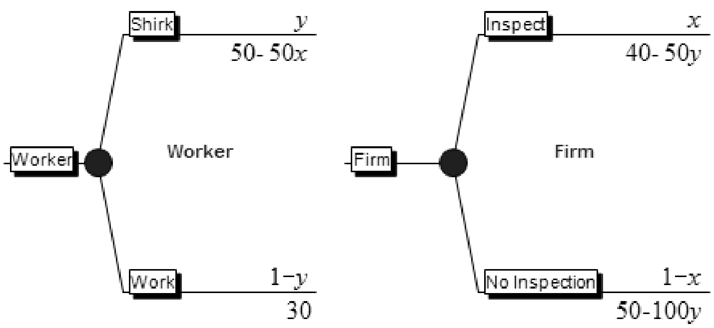

Figure 3: Decision tree models for the two players in the inspection game after rolling back the tree one level.

2.4 Modified Decision-Theoretic Solution

In this paper, we introduce a modification to the *paired decision analysis* concept defined by VBM, which we refer to as a *paired mixed decision analysis*. In a paired mixed decision analysis, the choices of the player and opponent are modeled with chance nodes called *random strategy nodes*, as the players may randomize between available strategies. A pure strategy can be represented in this model by assigning a probability of one to a single strategy and zero to all other strategies in the strategy set.

In contrast to the VBM approach, the expected payoff for each player in single-stage games can be calculated by rolling back the tree using expected value calculations. A modified decision analysis representation of the inspection game is shown in Figure 2. A selection of the probabilities x and y by the two players represents a strategy pair. Rolling back these decision trees one level, as demonstrated in Figure 3, helps reveal the equilibrium strategies.

In the Worker's tree, the expected values are calculated at the two nodes for the Firm's choice in the original decision tree and noted in the model in Figure 3 as

$$EV_{W,S}(x) = 0x + 50(1-x) = 50 - 50x$$
$$EV_{W,W}(x) = 30x + 30(1-x) = 30\ .$$

Because the expected values $EV_{W,S}$ and $EV_{W,W}$ will be weighted by the probabilities y and $1 - y$, respectively, to determine the Worker's overall payoff in the game, it is clear from the decision tree and the rollback procedure used to solve the decision tree that the Firm must choose x to satisfy $EV_{W,S} = EV_{W,W}$. When the Firm does so, the Worker has no incentive to deviate from any specific value for its strategy y. The Firm's optimal strategy x^* is found as

$$\begin{gathered} EV_{W,S}(x) = EV_{W,W}(x) \\ 50 - 50x = 30 \\ x^* = 0.40 \end{gathered}$$

By using the decision tree for the Firm and following the same logic, the Worker's optimal strategy is found as $y^* = 0.20$. The optimal strategy in a mixed equilibrium for the Worker is the probability that makes the Firm indifferent as to whether it follows the Inspection or No Inspection branches from the random strategy node in its decision tree. Similary, he optimal strategy in a mixed equilibrium for the Firm is the probability that makes the Worker indifferent as to whether she follows the Shirk or Work branches from the random strategy node in her decision tree.

In the modified decision-theoretic approach, the equations needed to determine the optimal strategies x and y are found directly through rolling back the decision trees. The payoffs to the players at equilibrium can also be determined by substituting the optimal strategies and rolling back the decision trees. In this example, the expected value in the game at Nash equilibrium is \$30 for each player.

In contrast to the VBM method, the modified decision-theoretic approach provided a solution to this problem using a "single pass" to calculate the expected values of each player in the game. These expected value expressions are then used to solve for the Nash equilibrium strategies. No iterative solution procedure is required. This approach can also provide solutions using a single pass in some more complicated two-player games, including the signaling game presented in the next section.

3 The Signaling Game

Signaling games are games of asymmetric information where the more informed player (the Sender) has a choice about whether to provide information to his opponent (the Receiver). The information contained in the signal may affect the judgment of the less informed player about the first player's true type. Signaling in a way that confounds the understanding of the less informed player may be costly. The less informed player has to decide how to respond, taking into consideration the uncertainty about the opponent's type, recognizing that the signal may be strategically chosen. In this paper we consider the signaling game with two players where the Sender can be one of two types.

3.1 Example: Nova and Oldstar

Dixit and Skeath (1999) present an example of a signaling game. There are two firms in a market entry game: the incumbent firm is Oldstar (the Receiver) and the new firm is Nova (the Sender). Oldstar reasons that Nova is one of two types: weak (Type 1) with probability

Table 2: Descriptions for the parameters in the Nova–Oldstar signaling game.

Parameter	Description
p	Probability that Nova is weak
m_W	Weak Nova's strategy (% of time weak Nova challenges)
m_S	Strong Nova's strategy (% of time strong Nova challenges)
a_C	Oldstar's strategy after observing challenge (% of time Oldstar retreats upon observing challenge)
a_N	Oldstar's strategy after observing no challenge (% of time Oldstar retreats upon observing no challenge)

Table 3: The payoffs to each player in the Nova–Oldstar signaling game.

	Weak Nova with probability p			Strong Nova with probability $1-p$	
Oldstar	Challenge	No Challenge	Oldstar	Challenge	No Challenge
Retreat	$0,\ 2-c$	3, 0	Retreat	0, 4	3, 0
Fight	$1,\ -2-c$	3, 0	Fight	-2, 2	3, 0

p or strong (Type 2) with probability $1-p$. If Nova enters the market, Oldstar can beat a weak Nova and have the market to itself, but a strong Nova can beat Oldstar and capture the entire market.

Now suppose Nova can signal its type by presenting prototypes of advanced products before it can produce and distribute these products on a large scale. Such a signal seems to indicate that Nova is strong, but even a weak Nova can display these prototypes, albeit at a cost c. A weak Nova, therefore, may choose to signal strategically and imitate a strong firm. If it can do so and cause Oldstar to retreat, it can gain the market. Oldstar on the other hand could engage in a price war and fight with Nova. However, Oldstar's payoff from fighting with a strong Nova is the least preferred outcome for Oldstar. On the other hand, Oldstar's payoff from fighting with a weak Nova is preferable to retreating. The task for Oldstar then is to strategize around this risk. A weak Nova has to decide on whether to incur the cost of the challenge c or not. A strong Nova will always challenge. Assume the parameters and payoffs in the game to each player are as shown in Tables 2 and 3. In Table 3, the left (right) panel shows payoffs that occur when Nova is weak (strong). In both cases, Oldstar's payoff is the first value listed in each pair. All of these parameters and payoffs are common knowledge to both firms.

We next present the game-theoretic solutions to the Nova-Oldstar game, then demonstrate the modified decision-theoretic approach to solving the game.

3.1.1 Game-Theoretic Approach

A game-theoretic solution to the Nova-Oldstar example uses a Bayesian-Nash approach that will be demonstrated in more detail later in this section. In this environment there are a number of possible equilibria. We will state rather than derive these equilibria here (see Dixit and Skeath (1999, pp. 416–424) for a detailed solution). Our point is to show that the modified decision-theoretic approach provides the same solution as the game theory approach.

Pooling Equilibrium

When the cost of challenging is small, then even a weak Nova is likely to challenge ($m_W = 1$) and pretend to be a strong Nova. Thus, a pooling equilibrium is likely when $c < 2$. At the same time if p is small enough, then Oldstar might believe that the likelihood of facing a strong Nova is too high for it to take the risk of fighting. It turns out that for $p < 2/3$ and $c < 2$, a weak Nova will always challenge and Oldstar will always retreat. Since neither player has an incentive to deviate from their strategy, we have a Nash equilibrium.

Separating Equilibrium

When $c > 2$, a weak Nova has no incentive to challenge. However, a strong Nova always challenges ($m_S = 1$). In this scenario Oldstar will always believe that a challenger is a strong Nova and will always retreat in the face of a challenge. Conversely, faced with a Nova that does not challenge Oldstar will decide to fight. Again, these are strategies that neither player has an incentive to deviate from and form a separating Bayesian-Nash equilibrium.

Semi-separating Equilibrium

An interesting situation arises when $p > 2/3$ and $c < 2$. The cost of challenging is small enough to provide an incentive for a weak Nova to challenge. At the same time, Oldstar faces enough of a chance of meeting up with a weak Nova to have an incentive to fight when faced with a challenge. In this situation both players have a Bayesian-Nash mixed-strategy equilibrium. The game is structured so that $m_S = 1$. Therefore, Oldstar's decision to fight is based on its belief about Nova's type given it observes the signal—a challenge. Oldstar generates this belief using Bayes' rule. Thus, Oldstar uses Bayes' rule to find $P(\text{Strong} \mid \text{Challenge}) = m_W \cdot p/(m_W \cdot p - p + 1)$. Given this belief Oldstar fights ($a_C = 0$) only if $m_W > (2 - 2p)/p$. Consequently, Oldstar never fights ($a_C = 1$) only if $m_W < (2 - 2p)/p$.

Of course, now Nova's strategy depends on whether Oldstar fights or not—in this game Nova does not have to worry about Oldstar's type. Nova challenges only if it finds that a_C is low enough. Thus $m_W = 1$ only when $a_C < (2 + c)/4$. Consequently $m_W = 0$ only when $a_C > (2 + c)/4$. Thus, a Bayesian-Nash mixed-strategy equilibrium where neither player has an incentive to deviate from occurs when $a_C = (2 + c)/4$ and $m_W = (2 - 2p)/p$. Applying our modified decision-theoretic approach to this game produces exactly this result, as demonstrated in the next section.

3.1.2 Modified Decision-Theoretic Approach

To use the modified decision-theoretic approach to determine the Basyesian-Nash equilibrium strategies, we build a decision tree for each player, as shown in Figure 4.

In Nova's decision tree, nature's selection of a weak or strong is represented as a chance node. Nova learns its type, then chooses its strategy m_W, as modeled with a random strategy

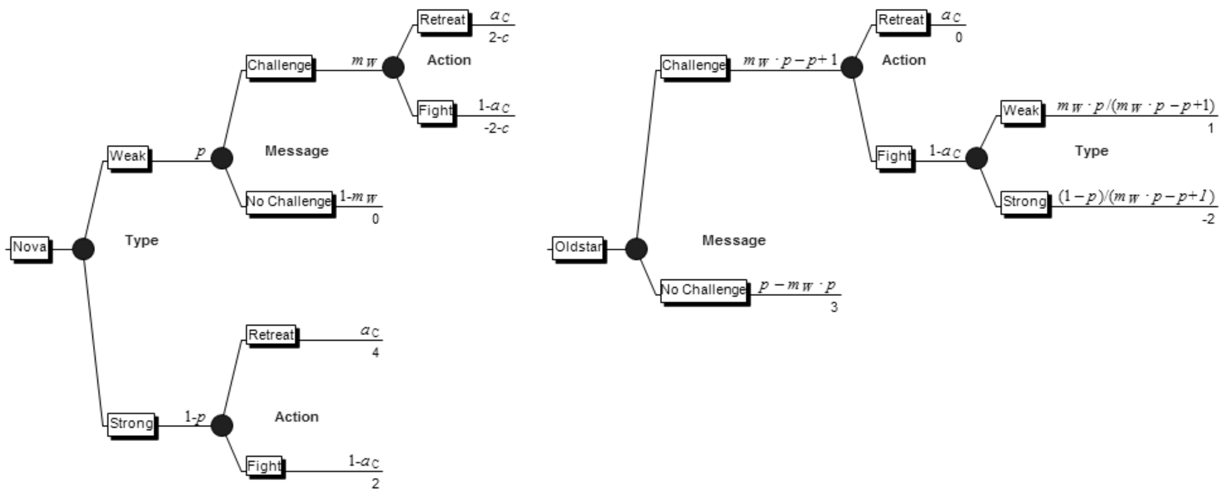

Figure 4: The decision trees in the modified decision-theoretic approach for the Nova–Oldstar signaling game.

node. If Nova does not challenge, it earns a payoff of zero. If Nova challenges, its payoff is determined according to whether Oldstar retreats or fights. Oldstar will retreat with probability a_C, leaving Nova with the market and a payoff of $2 - c$. Otherwise, Oldstar will fight and Nova's payoff will be $-2-c$. If Nova is strong it will always challenge. Oldstar will again retreat with probability a_C leaving Nova with a payoff of four. Otherwise, Oldstar will fight and Nova will earn two. Note that the tree assumes that $a_N = 0$ because observing no challenge, Oldstar has no reason to retreat. Similarly, $m_S = 1$ because a strong Nova should always challenge.

Oldstar will first observe whether or not Nova challenges, as shown by the initial chance node in its decision tree. The probabilities at this chance node are determined as

$$\begin{aligned}
P(\text{Challenge}) &= P(\text{Weak} \cap \text{Challenge}) + P(\text{Strong} \cap \text{Challenge}) \\
P(\text{Challenge}) &= P(\text{Challenge} \mid \text{Weak}) \cdot P(\text{Weak}) + P(\text{Challenge} \mid \text{Strong}) \cdot P(\text{Strong}) \\
P(\text{Challenge}) &= m_W \cdot p + m_S \cdot (1-p) = m_W \cdot p + 1 \cdot (1-p) = m_W \cdot p - p + 1 \\
\\
P(\text{No Challenge}) &= 1 - (m_W \cdot p - p + 1) = p - m_W \cdot p \ .
\end{aligned}$$

If Oldstar observes the challenge, it retreats with probability a_C, earning nothing. If Oldstar fights a weak Nova, it earns a payoff of one, while it loses two by fighting a strong Nova. Its probabilities for Nova's type after observing a challenge are determined according to Bayes' rule as

$$\begin{aligned}
P(\text{Weak} \mid \text{Challenge}) &= P(\text{Weak} \cap \text{Challenge})/P(\text{Challenge}) = m_W \cdot p/(m_W \cdot p - p + 1) \\
\\
P(\text{Strong} \mid \text{Challenge}) &= P(\text{Strong} \cap \text{Challenge})/P(\text{Challenge}) = (1-p)/(m_W \cdot p - p + 1) \ .
\end{aligned}$$

Rolling back the decision trees one level gives the results shown in Figure 5. Continuing rolling back the trees a second level gives the results shown in Figure 6. As a result of rolling back the decision trees, we can find that Nova's expected payoff in the game is

$$EV_N\left(a_C, m_W\right) = 2 - \left(2 + (2+c)m_W\right)p + a_C\left(2 + \left(4m_W - 2\right)p\right) \ . \tag{1}$$

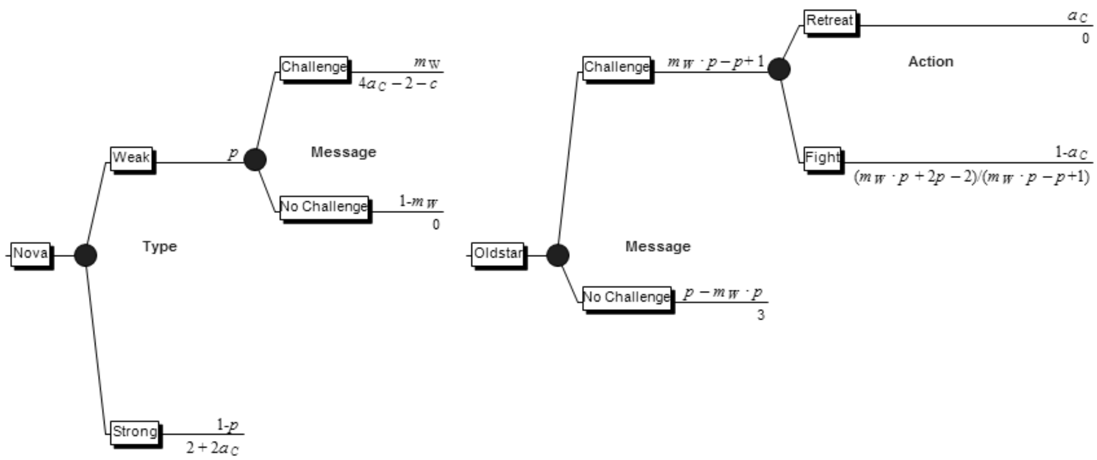

Figure 5: The modified decision-theoretic approach for the Nova–Oldstar signaling game after rolling the decision trees back one level.

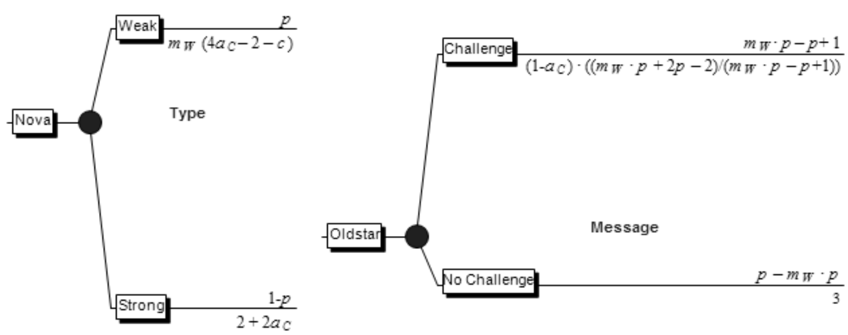

Figure 6: The modified decision-theoretic approach for the Nova–Oldstar signaling game after rolling the decision trees back two levels.

Oldstar's expected payoff in the game is

$$EV_O\left(a_C, m_W\right) = \left(5 - 2m_W\right)p - a_C\left(-2 + \left(2 + m_W\right)p\right) - 2\ . \qquad (2)$$

Pooling Equilibrium

The functions in (1) and (2) can be used to determine the conditions that result in a pooling equilibrium where both a weak Nova and strong Nova will challenge (or $m_W = 1$ and $m_S = 1$). This equilibrium exists with Oldstar always retreating in the face of a challenge ($a_C = 1$) if

$$\left(EV_N(1,1) \geq EV_N(1,0)\right) \cap \left(EV_O(1,1) \geq EV_O(0,1)\right)\ .$$

Since $EV_N(1,1) = 4(1-p) + (2-c)p$ and $EV_N(1,0) = 4(1-p)$, the first condition is met when $c < 2$. Since $EV_O(1,1) = 0$ and $EV_O(0,1) = -2(1-p) + p$, the second condition is met when $p < 2/3$. Thus, we can conclude that if $c < 2$ and $p < 2/3$ a pooling equilibrium exists with $m_W = 1$ and $m_S = 1$.

Separating Equilibrium

Suppose we want to determine if a separating equilibrium exists where weak Novas do not challenge, or $m_W = 0$ and $m_S = 1$. Since $a_N = 0$ (Oldstar never retreats if not challenged), a separating equilibrium exists if

$$(EV_N(0,0) \geq EV_N(0,1)) \cap (EV_O(0,0) \geq EV_O(1,0)) \ ,$$

assuming Oldstar plays $a_C = 1$ and always retreats if challenged. Since $EV_N(0,0) = 4(1-p)$ and $EV_N(0,1) = 4(1-p) + (2-c)p$, the first condition is met when $c > 2$. The second condition is always met with $0 \leq p \leq 1$. Thus, a separating equilibrium exists for $c > 2$.

Semi-separating Equilibrium

If $c < 2$ and $p > 2/3$, we can use the decision trees to determine the strategies that form a Bayesian-Nash semi-separating equilibrium. Nova's equilibrium strategy is determined by rolling back the decision tree one level as shown in Figure 5. Solving the following equation for m_W) ensures that Oldstar will not have a preference for any particular value of a_C:

$$\frac{m_W \cdot p + 2p - 2}{m_W \cdot p - p + 1} = 0 \ .$$

Similarly, using the decision tree for Nova in Figure 5, Oldstar's equilibrium strategy is determined by solving for a_L in

$$4a_L - 2 - c = 0 \ .$$

The results are $m_W = (2-2p)/p$ and $a_C = (2+c)/4$. The decision tree representation provides a convenient tool to establish the equations required to solve for the equilibrium strategies. By using the decision trees in Figure 6 or the expressions in (1) and (2) and substituting the optimal strategies, we can determine the expected payoffs in the game as $EV_N = (0.5 - 0.5p)(c+6)$ for Nova and $EV_O = 4 + c - p - cp$ for Oldstar.

The example in this section has described how the modified decision-theoretic approach can be used to obtain the same Bayesian-Nash equilibrium strategies as a game-theoretic approach. The VBM decision-theoretic approach will not be used to solve the signaling game because of the difficulty of representing the multi-stage nature of the solution approach in this framework. In the VBM approach, the chance nodes representing the weak Nova's message in Figure 4 would be replaced with a decision node with three branches. The three branches would represent the pure strategies $m_W = 1$, $m_W = 0$, and a mixed strategy where $m_W \in (0,1)$. The remainder of the game involves Oldstar following a strategy of $m_W = 1$ or $m_W = 0$ with an action a_C or a_N. However, this action cannot be represented with one chance node after the $m_W \in (0,1)$ branch because this branch represents a combination of the strategies $m_W = 1$ and $m_W = 0$.

The next section presents the Bayesian-Nash equilibria in a more general version of the signaling game.

Table 4: The payoffs to each player in the signaling game.

	Type 1 Sender (t_1) with probability p			Type 2 Sender (t_2) with probability $1-p$	
Receiver	Left (L)	Right (R)	Receiver	Left (L)	Right (R)
Up (u)	v_{11}, w_{11}	v_{12}, w_{12}	Up (u)	v_{13}, w_{13}	v_{14}, w_{14}
Down (d)	v_{21}, w_{21}	v_{22}, w_{22}	Down (d)	v_{23}, w_{23}	v_{24}, w_{24}

Table 5: Descriptions for the parameters in the two-type signaling game.

Parameter	Description
p	Probability the sender is type 1 (t_1)
m_1	Type 1 Sender's strategy (% of time t_1 plays Message 1 (L))
m_2	Type 2 Sender's strategy (% of time t_2 plays Message 1 (L))
a_L	Receiver's strategy after observing Message 1 (L) (% of time receiver plays u)
a_R	Receiver's strategy after observing Message 2 (R) (% of time receiver plays u)

3.2 General Signaling Game

The payoffs and a description of the parameters for a more general version of the two-player signaling game with two types of Senders are provided in Tables 4 and 5, respectively. Notation used here is similar to that employed by Gibbons (1992).

Nature selects the type of Sender that the Receiver ultimately faces in the game, with p representing the probability that the Sender is Type 1 (t_1). The Sender's strategies are the signals Left (L) or Right (R). The Receiver's strategies are the Up (u) and Down (d) responses to these signals. In Table 4, the first set of payoffs are those realized when the Sender is Type 1 (t_1), whereas the second set of payoffs ensues when the Sender is Type 2 (t_2). In each pair of payoffs in Table 4, the first value is the payoff to the Receiver, while the second value is the payoff to the Sender. For example, the payoff to the Receiver playing Up (u) when the Type 1 Sender (t_1) plays Left (L) is v_{11}, while the payoff to the Sender in this scenario is w_{11}. An extensive form representation of this game is shown in Figure 7. The dashed lines in this diagram represent the information sets for the Receiver, which will observe the signal of the Sender, but not the Sender's type.

The Receiver's strategy a_L is interpreted as the conditional probability that action u is initiated given L is played, or $P(u \mid L)$. The receiver's strategy a_R is interpreted as the conditional probability that action u is initiated given R is played, or $P(u \mid R)$. Likelihood probabilities for message type given player type are assigned according to the Sender's strategy, as shown in Table 5. The marginal probabilities of observing Message 1 (L) and Message 2 (R) are determined as

$$q_1(m_1, m_2) = m_1 \cdot p + m_2 \cdot (1-p) \quad \text{and} \quad 1 - q_1(m_1, m_2) = (1-m_1) \cdot p + (1-m_2) \cdot (1-p) \, .$$

The Receiver's probabilities for player type given the observed signal are calculated us-

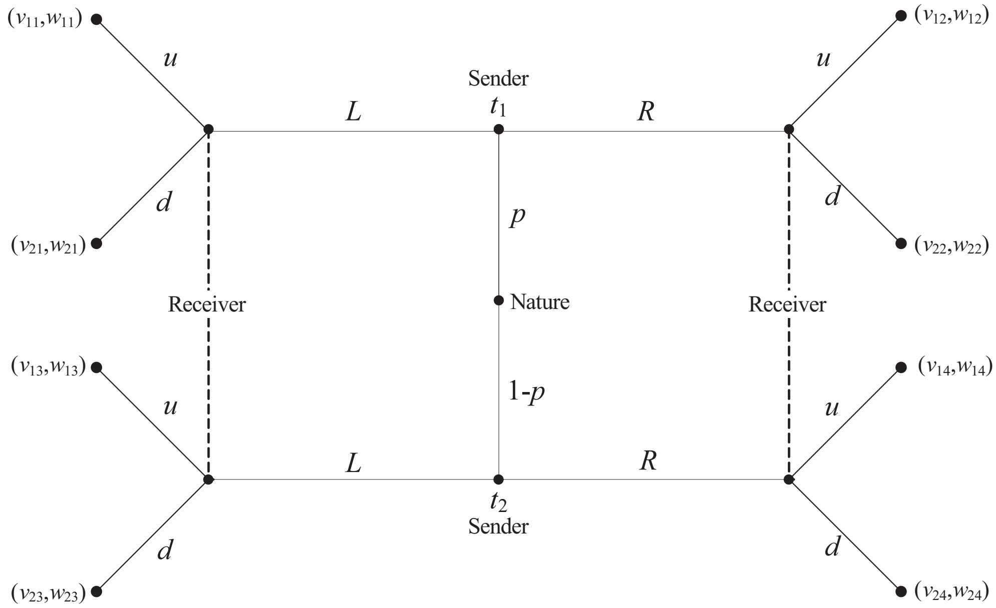

Figure 7: A game tree representation of the signaling game.

ing Bayes' rule. The conditional probabilities for Sender type given the message received, denoted by r_{ij} for the probability of type i given message j, are determined as

$$\begin{aligned} r_{11}(m_1, m_2) &= \frac{m_1 \cdot p}{m_1 \cdot p + m_2 \cdot (1-p)} \\ r_{21}(m_1, m_2) &= \frac{m_2 \cdot (1-p)}{m_1 \cdot p + m_2 \cdot (1-p)} \\ r_{12}(m_1, m_2) &= \frac{(1-m_1) \cdot p}{(1-m_1) \cdot p + (1-m_2) \cdot (1-p)} \\ r_{22}(m_1, m_2) &= \frac{(1-m_2) \cdot (1-p)}{(1-m_1) \cdot p + (1-m_2) \cdot (1-p)} \ . \end{aligned}$$

3.2.1 Game-Theoretic Approach

In this section we will describe a game-theoretic approach to solving a signaling game. Since general signaling game solutions are sensitive to assumptions about player payoffs and beliefs, we provide a conditional partial solution. Our purpose here is not to provide a complete solution to our chosen game, but to show how a solution process works. In Section 3.2.2 we use a modified decision-theoretic approach to solve the same game. Of course, we get the same solutions for the same set of initial conditions in either section.

Typically, signaling games can be solved with a Bayesian–Nash approach, though this is not the only way of solving games of this class. We use a Bayesian–Nash approach to solve this game. The fundamental principle of a Nash equilibrium—no player has an incentive to deviate from an equilibrium strategy—remains. The main innovation in this class of games is that players form beliefs about a sender's type by using a Bayesian process.

In the following three sections, we provide representative solutions for three different classes of a Bayesian–Nash equilibrium.

Pooling Equilibrium

In this section we focus on the conditions under which a pooling equilibrium is plausible. For our game pooling may occur with both types of Senders playing L or R. We choose to illustrate the Bayesian–Nash approach as it applies to a pooling equilibrium by focusing on the possibility of pooling on L. Any solution must satisfy the fundamental Nash principle—there is no incentive for any player to deviate.

Suppose a pooling occurs on L. In other words $m_1 = m_2 = 1$. Then the Receiver's belief of the Sender's type, driven by Bayes' rule, is $r_{11} = p$ and $r_{21} = 1 - p$. Given this belief, the Receiver's strategy would be a_L if her expected payoff from playing $a_L = 1$ (Up) exceeds her expected payoff from playing $a_L = 0$ (Down). The expected values to the Receiver from playing $a_L = 1$ and $a_L = 0$ are $pv_{11} + (1-p)v_{13}$ and $pv_{21} + (1-p)v_{23}$, respectively. Thus, the Receiver will play $a_L = 1$ only if $p \geq ((v_{23} - v_{13}))/((v_{11} - v_{13}) - (v_{21} - v_{23}))$. Suppose first that this condition holds. Given the Receiver's strategy, each type of Sender should have no incentive to deviate from pooling, i.e. given the Receiver will play Up, either type of Sender should not deviate from L. This happens only when $w_{11} > w_{12}$ and $w_{13} > w_{14}$. If $p < ((v_{23} - v_{13}))/((v_{11} - v_{13}) - (v_{21} - v_{23}))$ so that the Receiver plays $a_L = 0$, the Type 1 and Type 2 Senders will not deviate from L if $w_{21} > w_{22}$ and $w_{23} > w_{24}$.

Separating Equilibrium

In this section we look at the conditions under which a separating equilibrium is plausible. In our game there are two possible separating equilibria. Type 1 Senders can play L while Type 2 Senders play R, and vice versa. To illustrate a Bayesian–Nash approach to finding separating equilibria we choose to focus on Type 1 playing L while Type 2 plays R, i.e. $m_1 = 1$ and $m_2 = 0$. The underlying belief system of the Receiver is represented by $r_{11} = r_{22} = 1$. Let us posit an equilibrium where $a_L = a_R = 1$. This equilibrium will be stable, i.e. there will never be an incentive to deviate, if and only if $w_{14} > w_{13}$, $w_{11} > w_{12}$ and $v_{11} > v_{21}$, $v_{14} > v_{24}$.

Semi–Separating Equilibrium

In this section, we focus on the conditions under which a semi-separating equilibrium is plausible. This type of equilibrium arises when one type of Sender has a clear pure strategy in equilibrium due to the structure of its payoffs, while the other type plays a mixed strategy. The problem in this scenario arises because the Receiver may observe a particular signal emanating from both types of Sender. The Receiver therefore has to make a clear judgment about the type of the Sender underlying the signal. For the remainder of this analysis, assume that $m_2 = 1$ because $w_{13} \geq w_{14}$ and $w_{23} \geq w_{24}$, which implies that $r_{12} = 1$. This dictates that $a_R = 1$ if $v_{12} \geq v_{22}$, and $a_R = 0$ otherwise. To illustrate the resolution process for a semi-separating equilibrium we will look at the L information set in Figure 7, i.e. we will assume $v_{12} < v_{22}$ so that $a_R = 0$. Note that this class of equilibrium arises only if there is no chance of a pooling equilibrium. In our game, this means $p < ((v_{23} - v_{13}))/((v_{11} - v_{13}) - (v_{21} - v_{23}))$.

The Receiver's belief system is determined by Bayes' rule according to the probabilities r_{11} and r_{12}. The Receiver mixes between Up and Down only if she is indifferent between the expected payoffs from either move. Thus, assuming $m_2 = 1$,

$$\frac{v_{11}(pm_1)}{pm_1+(1-p)} + v_{13}\left(1-\frac{pm_1}{pm_1+(1-p)}\right) = \frac{v_{21}(pm_1)}{pm_1+(1-p)} + v_{23}\left(1-\frac{pm_1}{pm_1+(1-p)}\right) .$$

Solving for m_1 gives

$$m_1^* = \frac{(1-p)(v_{23}-v_{13})}{p(v_{11}-v_{21})} .$$

The Type 1 Sender, given our assumption that $m_2 = 1$, on the other hand mixes only if her expected payoff from playing Left is equal to her expected payoff from playing Right. This means, given $a_R = 0$, that $w_{11}a_L + w_{21}(1-a_L) = w_{22}$. Solving for a_L gives us

$$a_L^* = \frac{w_{22}-w_{21}}{w_{11}-w_{21}} .$$

Note that given our assumptions, neither a Type 1 Sender nor the Receiver has any incentive to deviate from m_1^* and a_L^*. In other words, we have established the existence of a Bayesian-Nash equilibrium.

3.2.2 Modified Decision-Theoretic Approach

This section illustrates a modified decision-theoretic approach to solving the two-type signaling game in cases where only one type of Sender has a pure strategy, i.e. a semi-separating equilibrium exists. We will demonstrate that this technique can provide a convenient framework that represents graphically the reasoning required to determine the equilibrium strategies. Additionally, in the two-type Sender signaling game, these strategies can be calculated from the computations produced by rolling back the decision trees. The decision trees for the Sender and Receiver in the signaling game are shown in Figures 8 and 9.

Rolling Back the Receiver's Tree

This section presents the calculation of the expected value in the game to the Receiver. The expressions presented will later be used to determine any pure strategy equilibria in the game (if any exist), as well as determine Nash equilibrium mixed strategies that are part of semi-separating equilibria.

Based on the decision tree shown in Figure 9, we can calculate the expected values for the Receiver observing L as

$$EV_{LU}(m_1,m_2) = r_{11}(m_1,m_2)\cdot v_{11} + r_{21}(m_1,m_2)\cdot v_{13} = \frac{m_1pv_{11}+m_2(1-p)v_{13}}{m_2(1-p)+m_1p}$$

$$EV_{LD}(m_1,m_2) = r_{11}(m_1,m_2)\cdot v_{21} + r_{21}(m_1,m_2)\cdot v_{23} = \frac{m_1pv_{21}+m_2(1-p)v_{23}}{m_2(1-p)+m_1p} .$$

The overall expected value for the Receiver observing L is

$$EV_L(a_L,m_1,m_2) = a_L\cdot EV_{LU}(m_1,m_2) + (1-a_L)\cdot EV_{LD}(m_1,m_2) .$$

The expected values for the Receiver observing R are

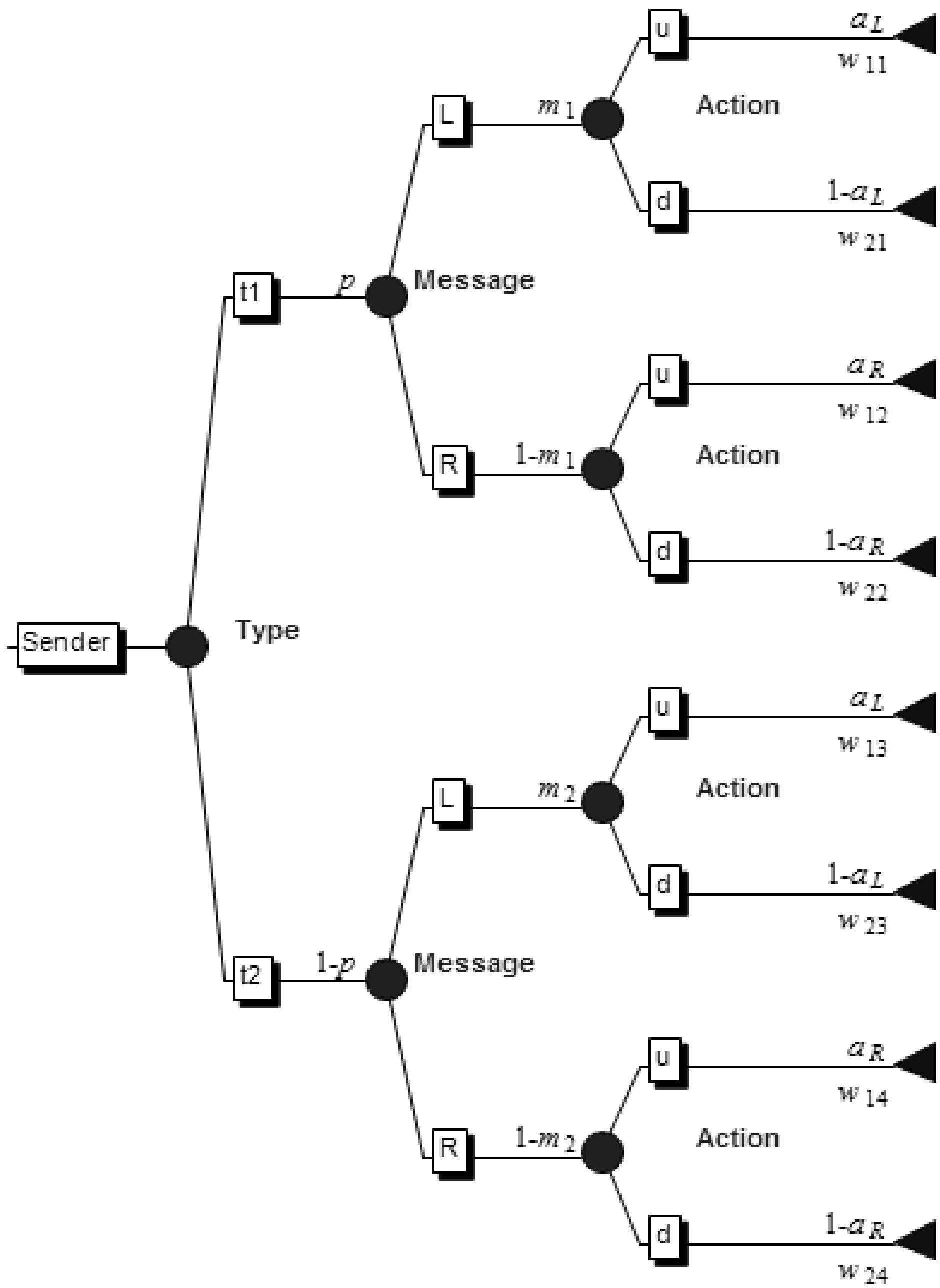

Figure 8: Decision tree model for the Sender in the modified decision analysis approach to solving the signaling game.

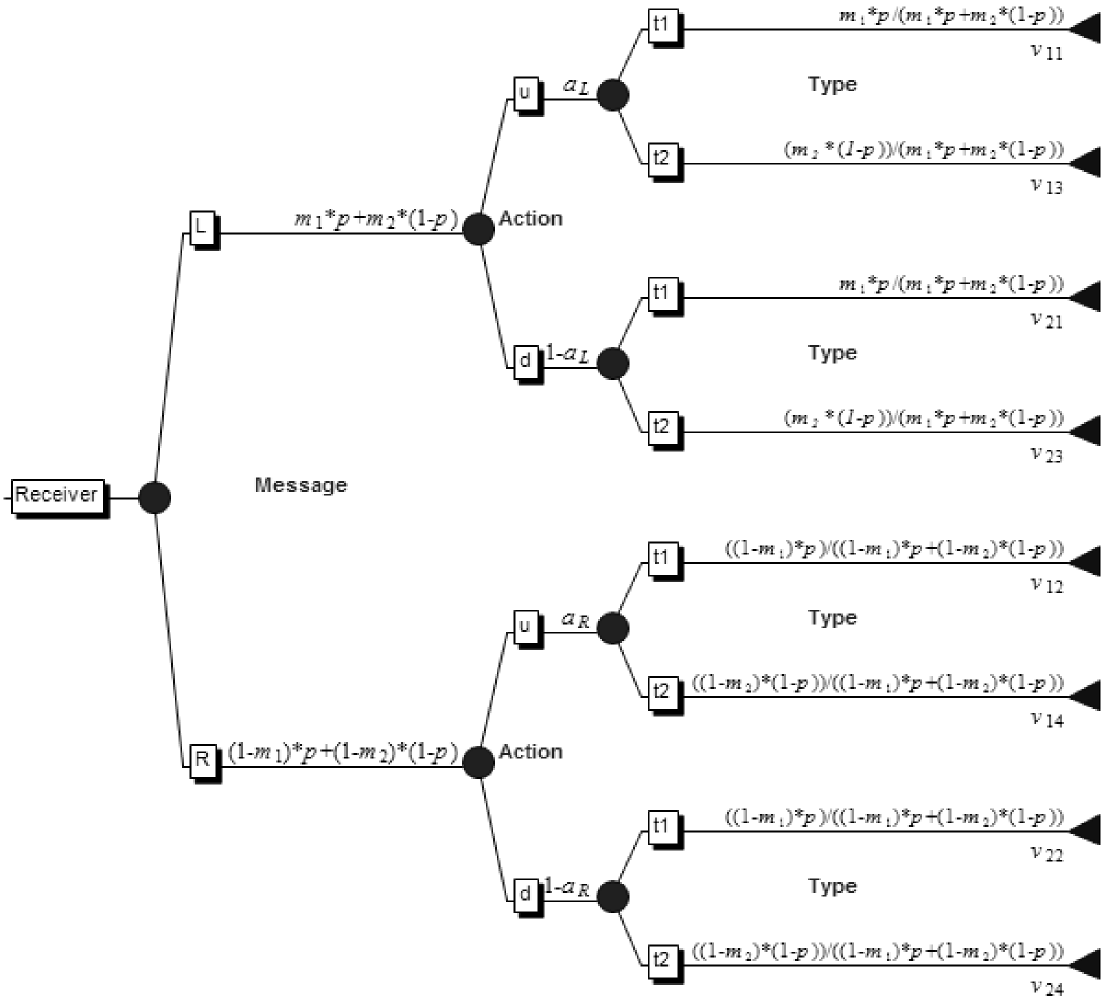

Figure 9: Decision tree model for the Receiver in the modified decision analysis approach to solving the signaling game.

$$EV_{RU}(m_1, m_2) = r_{12}(m_1, m_2) \cdot v_{12} + r_{22}(m_1, m_2) \cdot v_{14} = \frac{(1 - m_1)pv_{12} + (1 - m_2)(1 - p)v_{14}}{(1 - m_2)(1 - p) + (1 - m_1)p}$$
$$EV_{RD}(m_1, m_2) = r_{12}(m_1, m_2) \cdot v_{22} + r_{22}(m_1, m_2) \cdot v_{24} = \frac{(1 - m_1)pv_{22} + (1 - m_2)(1 - p)v_{24}}{(1 - m_2)(1 - p) + (1 - m_1)p} \ .$$

The overall expected value for the Receiver observing R is

$$EV_R(a_R, m_1, m_2) = a_R \cdot EV_{RU}(m_1, m_2) + (1 - a_R) \cdot EV_{RD}(m_1, m_2) \ .$$

The expected value for the Receiver in the game is

$$EV_R(a_L, a_R, m_1, m_2) = q_1(m_1, m_2) \cdot EV_L(a_L, m_1, m_2) + (1 - q_1(m_1, m_2)) \cdot EV_R(a_R, m_1, m_2) \ .$$

Rolling Back the Sender's Tree

Based on the decision tree, we can calculate the expected value for the Sender, given a probability distribution for its type, and a strategy pair for each player. As in the case of the Receiver's decision tree, the expressions presented here will later be used to solve for pure and mixed strategy Nash equilibria.

The expected values for the Type 1 Sender playing L and R are

$$EV_{L1}(a_L) = a_L \cdot w_{11} + (1 - a_L) \cdot w_{21}$$
$$EV_{R1}(a_R) = a_R \cdot w_{12} + (1 - a_R) \cdot w_{22} \ .$$

The expected values for the Type 2 Sender playing L and R are

$$EV_{L2}(a_L) = a_L \cdot w_{13} + (1 - a_L) \cdot w_{23}$$
$$EV_{R2}(a_R) = a_R \cdot w_{14} + (1 - a_R) \cdot w_{24} \ .$$

Applying the strategies of the Sender gives the expected values in the game for the Type 1 and Type 2 players of

$$EV_{S1}(m_1, a_L, a_R) = m_1 \cdot EV_{L1}(a_L) + (1 - m_1) \cdot EV_{R1}(a_R)$$
$$EV_{S2}(m_2, a_L, a_R) = m_2 \cdot EV_{L2}(a_L) + (1 - m_2) \cdot EV_{R2}(a_R) \ .$$

The expected value of the game for the Sender is

$$EV_S(a_L, a_R, m_1, m_2) = p \cdot EV_{S1}(m_1, a_L, a_R) + (1 - p) \cdot EV_{S2}(m_2, a_L, a_R) \ .$$

Pure Strategy Equilibria

A pure strategy Nash equilibrium in the signaling game is defined as a strategy pair

$$\{(a_L = i, a_R = j), (m_1 = k, m_2 = \ell)\} \tag{3}$$

where i, j, k and ℓ are each equal to zero or one. The strategy pair in (3) can be evaluated to determine whether or not it is a Nash equilibrium by using the following function:

$$\Phi(i,j,k,\ell) = \begin{cases} 1 \quad \text{if} & \begin{array}{rl} & (EV_S(i,j,k,\ell) \geq EV_S(i,j,k,1-\ell)) \\ \cap & (EV_S(i,j,k,\ell) \geq EV_S(i,j,1-k,\ell)) \\ \cap & (EV_S(i,j,k,\ell) \geq EV_S(i,j,1-k,1-\ell)) \\ \cap & (EV_R(i,j,k,\ell) \geq EV_R(i,1-j,k,\ell)) \\ \cap & (EV_R(i,j,k,\ell) \geq EV_R(1-i,j,k,\ell)) \\ \cap & (EV_R(i,j,k,\ell) \geq EV_R(1-i,1-j,k,\ell)) \end{array} \\ \\ 0 \quad \text{otherwise} \end{cases} \tag{4}$$

If the function evaluates to true (or $\Phi(i,j,k,\ell) = 1$), then the strategy pair in (3) is a Nash equilibrium.

Example. Consider a separating equilibrium where the Type 1 Sender plays L ($m_1 = 1$) and the Type 2 Sender plays R ($m_2 = 0$). A separating equilibrium with $a_L = a_R = 1$ occurs if $\Phi(1,1,1,0) = 1$. The conditions in (4) are met if the payoffs are structured as follows:

$$(w_{14} \geq w_{13}) \cap (w_{11} \geq w_{12}) \cap (v_{14} \geq v_{24}) \cap (v_{11} \geq v_{21})$$

These are the same conclusions made in Section 3.2.1.

Semi-separating Equilibrium with $m_2 = 1$

If there are no pure strategy Nash equilibria in the game, we must search for a semi-separating equilibrium. First, suppose the Type 2 Sender plays pure strategy $m_2 = 1$ because $w_{13} \geq w_{14}$ and $w_{23} \geq w_{24}$. This induces the Receiver to play one of two pure strategies as $r_{12}(m_1, 1) = 1$ and $r_{22}(m_1, 1) = 0$ for all $m_1 \in [0, 1]$:

1. If $v_{12} \geq v_{22}$, the Receiver plays $a_R = 1$.
2. If $v_{12} < v_{22}$, the Receiver plays $a_R = 0$.

Thus, the Receiver's payoff in the game if it observes R is $\text{Max}\{v_{12}, v_{22}\}$. The remaining strategies to be selected in the game at equilibrium are m_1 for the Sender and a_L for the Receiver. The solution in the sequential game begins with the Sender rolling back the decision tree of the Receiver and selecting its strategy.

Since the Receiver's strategy a_R is completely determined by the payoffs v_{12} and v_{22} when $m_2 = 1$, the Sender determines its optimal strategy by selecting m_1 in such a way that the Receiver's payoff will be unaffected by its choice of the strategy a_L. This can be ensured by solving the following for m_1 using expected values calculated from the decision tree:

$$\begin{aligned} EV_{LU}(m_1, 1) &= EV_{LD}(m_1, 1) \\ \frac{m_1 p v_{11} + (1-p) v_{13}}{1 - p + m_1 p} &= \frac{m_1 p v_{21} + (1-p) v_{23}}{1 - p + m_1 p} \\ m_1 p v_{11} + (1-p) v_{13} &= m_1 p v_{21} + (1-p) v_{23} \; . \end{aligned} \tag{5}$$

The Type 1 Sender's equilibrium strategy is characterized as follows.

Proposition 1 *If $m_2 = 1$, the equilibrium strategy for the Sender is*

$$m_1 = \frac{(1-p)(v_{23} - v_{13})}{p(v_{11} - v_{21})} \ . \tag{6}$$

Proof. If the Sender's strategy m_1 meets the condition in (5), the Receiver will be indifferent between all possible values for its strategy a_L. Solving (5) for m_1 yields the result.

Since the Type 2 Sender's strategy is $m_2 = 1$ because of the restrictions assumed for w_{13}, w_{23}, w_{14}, and w_{24}, the Sender determines its optimal strategy by solving the following for a_L:

$$\begin{aligned} EV_{L1}(a_L) &= EV_{L2}(a_R) \\ a_L w_{11} + (1 - a_L) w_{21} &= a_R w_{12} + (1 - a_R) w_{22} \ . \end{aligned} \tag{7}$$

To determine its equilibrium strategy, the Receiver adjusts the condition in (9) for its optimal strategy a_R played after observing $m_2 = 1$, then searches for a strategy a_L such that the Sender cannot change strategies and improve its payoff. The following propositions ensue.

Proposition 2 *If $m_2 = 1$ and $v_{12} \geq v_{22}$ so that the Receiver plays $a_R = 1$, the equilibrium strategy for the Receiver is*

$$a_L = \frac{w_{12} - w_{21}}{w_{11} - w_{21}} \ . \tag{8}$$

Proof. If the Receiver's strategy a_L meets the condition in (9), the Sender will be indifferent between all possible values for its strategy m_1. Substituting $a_R = 1$ in (9) and solving for a_L yields the result.

Proposition 3 *If $m_2 = 1$ and $v_{12} < v_{22}$ so that the Receiver plays $a_R = 0$, the equilibrium strategy for the Receiver is*

$$a_L = \frac{w_{22} - w_{21}}{w_{11} - w_{21}} \ .$$

Proof. If the Receiver's strategy a_L meets the condition in (9), the Sender will be indifferent between all possible values for its strategy m_1. Substituting $a_R = 0$ in (9) and solving for a_L yields the result.

In summary, a semi-separating equilibrium in the signaling game for the case where the Type 2 Sender has a pure strategy $m_2 = 1$ and $v_{12} \geq v_{22}$ is the following strategy pair $\{(a_L, a_R), (m_1, m_2)\}$:

$$\left\{ \left(\frac{w_{12} - w_{21}}{w_{11} - w_{21}}, 1 \right), \left(\frac{(1-p)(v_{23} - v_{13})}{p(v_{11} - v_{21})}, 1 \right) \right\} \ .$$

Similarly, a semi-separating equilibrium in the signaling game for the case where the Type 2 Sender has a pure strategy $m_2 = 1$ and $v_{12} < v_{22}$ is the following strategy pair $\{(a_L, a_R), (m_1, m_2)\}$:

$$\left\{ \left(\frac{w_{22} - w_{21}}{w_{11} - w_{21}}, 0 \right), \left(\frac{(1-p)(v_{23} - v_{13})}{p(v_{11} - v_{21})}, 1 \right) \right\} \ .$$

Semi-separating Equilibrium with $m_2 = 0$

Next, suppose the Type 2 Sender plays pure strategy $m_2 = 0$ because $w_{13} < w_{14}$ and $w_{23} < w_{24}$. The remaining strategies to be selected in the game at equilibrium are m_1 for the Sender and a_L and a_R for the Receiver. Because the likelihood $P(L \mid t_2) = 0$, the revised probability $P(t_1 \mid L) = 1$, so the Receiver will play one of the following two strategies:

1. If $v_{11} \geq v_{21}$, the Receiver plays $a_L = 1$.

2. If $v_{11} < v_{21}$, the Receiver plays $a_L = 0$.

Thus, the Receiver's payoff in the game if it observes L is $\text{Max}\{v_{11}, v_{21}\}$. The remaining strategies to be selected in the game at equilibrium are m_1 for the Sender and a_R for the Receiver. The solution in the sequential game begins with the Sender rolling back the decision tree of the Receiver and selecting its strategy.

Since the Receiver's strategy a_R is completely determined by the payoffs v_{12} and v_{22} when $m_2 = 1$, the Sender determines its optimal strategy by selecting m_1 in such a way that the Receiver's payoff will be unaffected by its choice of the strategy a_L. This can be ensured by solving the following for m_1 using expected values calculated from the decision tree:

$$\begin{aligned} EV_{RU}(m_1, 0) &= EV_{RD}(m_1, 0) \\ \frac{(1-m_1)pv_{12} + (1-p)v_{14}}{(1-p) + (1-m_1)p} &= \frac{(1-m_1)pv_{22} + (1-p)v_{24}}{(1-p) + (1-m_1)p} \\ (1-m_1)pv_{12} + (1-p)v_{14} &= (1-m_1)pv_{22} + (1-p)v_{24} \ . \end{aligned} \tag{9}$$

The Type 1 Sender's equilibrium strategy is characterized as follows.

Proposition 4 *If $m_2 = 0$ the equilibrium strategy for the Type 1 Sender is*

$$m_1 = \frac{(v_{14} - v_{24}) + p(v_{12} - v_{22} - v_{14} + v_{24})}{p(v_{12} - v_{22})} \ .$$

Proof. If the Sender's strategy m_1 meets the condition in (9), the Receiver will be indifferent between all possible values for its strategy a_R. Solving (9) for m_1 yields the result.

Since the Type 2 Sender's strategy is $m_2 = 0$ because of the restrictions assumed for w_{13}, w_{23}, w_{14}, and w_{24}, the Sender determines its optimal strategy by solving the following for a_R:

$$\begin{aligned} EV_{R1}(a_L) &= EV_{R2}(a_R) \\ a_L w_{11} + (1-a_L)w_{21} &= a_R w_{12} + (1-a_R)w_{22} \ . \end{aligned} \tag{10}$$

To determine its equilibrium strategy, the Receiver adjusts the condition in (10) for its optimal strategy a_L played after observing $m_2 = 0$, then searches for a strategy a_R such that the Sender cannot change strategies and improve its payoff. The following propositions ensue.

Proposition 5 *If $m_2 = 0$ and $v_{11} \geq v_{21}$ so that $a_L = 1$, the equilibrium strategy for the Receiver is*

$$a_R = \frac{w_{11} - w_{22}}{w_{12} - w_{22}} \ .$$

Proof. If the Receiver's strategy a_R meets the condition in (10), the Sender will be indifferent between all possible values for its strategy m_1. Substituting $a_L = 1$ in (10) and solving for a_R yields the result.

Proposition 6 *If $m_2 = 0$ and $v_{11} < v_{21}$ so that $a_L = 0$, the equilibrium strategy for the Receiver is*

$$a_R = \frac{w_{21} - w_{22}}{w_{12} - w_{22}} \ .$$

Proof. If the Receiver's strategy a_R meets the condition in (10), the Sender will be indifferent between all possible values for its strategy m_1. Substituting $a_L = 0$ in (10) and solving for a_R yields the result.

In summary, a semi-separating equilibrium in the signaling game for the case where the Type 2 Sender has a dominant strategy $m_2 = 0$ and $v_{11} \geq v_{21}$ is the following strategy pair $\{(a_L, a_R), (m_1, m_2)\}$:

$$\left\{ \left(1, \frac{w_{11} - w_{22}}{w_{12} - w_{22}}\right), \left(\frac{(v_{14} - v_{24}) + p(v_{12} - v_{22} - v_{14} + v_{24})}{p(v_{12} - v_{22})}, 0\right) \right\} \ .$$

Similarly, a semi-separating equilibrium in the signaling game for the case where the Type 2 Sender has a dominant strategy $m_2 = 0$ and $v_{12} < v_{22}$ is the following strategy pair $\{(a_L, a_R), (m_1, m_2)\}$:

$$\left\{ \left(0, \frac{w_{21} - w_{22}}{w_{12} - w_{22}}\right), \left(\frac{(v_{14} - v_{24}) + p(v_{12} - v_{22} - v_{14} + v_{24})}{p(v_{12} - v_{22})}, 0\right) \right\} \ .$$

There may exist a way to extend the decision theoretic representation beyond this branch in the decision tree to make the tree consistent with the tenets of the VBM approach, but the modification suggested in this paper seems to be a more natural approach, so we will not utilize the VBM approach for the case of the signaling game.

4 Discussion and Conclusions

This paper has presented a modified decision-theoretic approach to solving decision problems where the payoffs are affected by strategic interaction. This approach builds upon the method presented by van Binsbergen and Marx (2007). The difference between our approach and the VBM technique is that the strategies of players are represented by the probabilities assigned to random strategy nodes, whereas VBM use decision nodes to represent strategies. This modification is advantageous in two ways. First, the payoffs at all endpoints in the decision trees are simply payoffs to the players in the game, as opposed to expected values. Second, in certain games, such as the two-type signaling game where at least one type of sender plays a pure strategy, we have shown that the equilibrium strategies can be determined by rolling back the tree without iteration. A more general solution for the two-type signaling game where both sender types play mixed strategies in equilibrium is a topic of ongoing research.

The potential usefulness of the approach presented in this paper is similar to that suggested by VBM for their model. Decision trees and sensitivity analysis are commonly used tools. Using the modified decision-theoretic approach, such tools can be used to model decisions, even when strategic interaction between players is present. In the context of a thorough decision analysis, use of this approach may expand the possibilities for decision analysts to consider how the actions of others influence decision making. This is in contrast to traditional decision analysis where all events not under the control of the decision maker are modeled as random variables with static probability distributions.

The decision analysis process involves defining the problem, establishing appropriate decision criteria, then identifying the four aspects of the problem, as mentioned in Section 1. As random or chance events that affect the outcomes are determined, the decision maker may realize that the outcomes also depend on strategic decisions made by other firm(s). Rather than abandon the decision analysis approach in favor of a game-theoretic approach, the techniques presented in this paper (and by VBM) allow the process to continue using decision analysis models and solution techniques.

The techniques outlined in this paper can be used to model any strategic game among multiple players. The limitations to modeling such games are only those imposed by the decision tree structure. Clearly, as the number of nodes in the tree increases, the size of the tree grows exponentially. Similarly, as the number of outcomes for chance nodes and/or alternatives for decision nodes increase, the size of the tree can become prohibitive. These limitations are well-documented in the decision analysis literature.

In this paper, we have presented solutions to a two-by-two normal form game and the signaling game with two types of Senders. The Nash equilibrium strategies were obtained by rolling back the decision trees. In more complicated games or in absence of certain assumptions, an iterative process may be required to determine equilibrium strategies using the approach presented in this paper. For instance, if the payoffs in the signaling game are such that both types of senders play mixed strategies at equilibrium, the results in this paper do not provide the Nash equilibrium strategies. The models presented in this paper can be an input to such a procedure, and the use of models to solve such games is a topic for future research. The models in this paper retain the advantage of representing the payoffs at all endpoints directly, without having to establish formulas for expected values at the endpoints representing mixed strategies. Use of chance nodes to represent all strategies in the game appears to extend the class of games to which decision analysis can be applied.

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