Dynamic effects of government expenditure in a finance constrained economy: A Note

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Abstract

Gokan [Dynamic effects of government expenditure in a finance constrained economy, J. Econ. Theory 127 (2006) 323-333] introduces constant government expenditure (financed by labor income taxes) in Woodford's model with capital-labor substitution and investigates how local dynamics near two steady states depend upon the elasticity of substitution between capital and labor. In this paper, we show that the local dynamics will change dramatically if the government transfers its revenue to the households (workers) in a lump sum way. In particular, we question the result that the rate of money growth has no impact on the model dynamics. In a numerical example, we illustrate that the result previously obtained is not robust to the alternative assumption.

Keywords: a lump sum transfer; indeterminacy.

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1. Introduction

In a recent article, Gokan (2006) investigates how exogenous government expenditure influences local dynamics near two steady states, depending upon the elasticity of substitution between capital and labor. And he finds that if the elasticity of capital-labor substitution is high, the high steady state might be always determinate and the low one always indeterminate for some sub-interval of government expenditure. Moreover, the low steady state displays indeterminacy through flip or Hopf bifurcations when the government expenditure takes some specific values.

In this note, we complete his analysis by asking whether his results extend to the case where the government transfers its revenue to the households (workers) in a lump sum way. Our answer is negative, as we prove that the local dynamics change dramatically if the households (workers) receive the constant government revenue. Particularly, Gokan (2006) points out that if the government expenditure is financed with a mixture of money and labor income taxes, local dynamics in his model will not be affected. Unfortunately this fact may not hold in our modified finance-constrained model, as we can *only* explicitly solve the model under the assumption of constant money supply as in Woodford (1986).

The fact that multiple steady states arise in our model is due to the presence of endogenous labor income tax rates and constant government revenue. In view of our results in the following sections, for both of the steady states, the (in)determinacy results are different from those in Gokan's model.[1] And when the lump sum transfer goes up from 0 to some critical value, for the high (resp. the low) steady state, the bifurcation parameter in our model moves in the opposite direction, in constrast with the case of government expenditure studied in Gokan's model.

The paper is organized as follows. In section 2, we describe the framework of our model. In section 3, we study the model dynamics with a geometrical method of bifurcation analysis and provide some interpretations of our results. Section 4 concludes.

2. Framework

The model we use is a monetary one-sector model featuring two classes of households called workers and capitalists, a financial constraint that prevents workers from borrowing against wage earning and constant government revenue (transfer) financed by labor income taxes. The key assumption is that workers discount the future more than capitalists.

2.1. Workers

The workers' problem is to maximize their intertemporal utility function

$$\max \sum_{t=0}^{\infty} (\gamma_w)^t \left[\frac{(c_t^w)^{1-\phi}}{(1-\phi)} - \gamma_w \frac{N_t^{1+\zeta}}{(1+\zeta)} \right] \tag{1}$$

where c_t^w and N_t denote the consumption and labor supply, and $\gamma_w \in (0,1)$ the discount factor. Moreover, the assumptions $\phi \in (0,1)$ and $\zeta > 0$ are used to ensure that the elasticity of labor supply with respect to the real wage is positive. In addition, they are subject to the budget constraint and borrowing constraint.

$$c_t^w + \left[k_{t+1}^w - (1-\delta)\, k_t^w\right] + M_{t+1}^w / P_t = r_t k_t^w + (1 - \tau_{wt})\, w_t N_t + M_t^w / P_t + T, \tag{2}$$

and

$$c_t^w + \left[k_{t+1}^w - (1-\delta)\, k_t^w\right] \leq r_t k_t^w + M_t^w / P_t, \tag{3}$$

where T (constant in real terms) is the lump-sum transfer from government financed by labor income taxes, M_t^w and k_t^w represent the nominal money balances and the physical capital (δ being the depreciation rate) held by workers in period t. P_t, w_t, and r_t are the nominal price of the numeraire good, the real wage and the real rental rate of capital. In equilibrium, the borrowing constraint is binding ($c_t^w = M_t/P_t$ and they hold no capital) and the workers' offer curve is,

$$\left(c_{t+1}^w\right)^{(1-\phi)}[1-\frac{T}{(1-\tau_{wt})\,w_t N_t+T}]=\left(N_t\right)^{(1+\zeta)}. \tag{4}$$

2.2. Capitalists

Similar to Gokan (2006), capitalists do not work and their problem is to maximize the logarithmic intertemporal utility function

$$\sum_{t=0}^{\infty}\beta^t \ln c_t^c, \tag{5}$$

where c_t^c denotes their consumption and $\beta \in (0,1]$ the discount factor. Their budget constraint can be stated as follows

$$c_t^c+\left[k_{t+1}^c-(1-\delta)\,k_t^c\right]+M_{t+1}^c/P_t \leq r_t k_t^c+M_t^c/P_t, \tag{6}$$

where the superscript c stands for capitalist. As in Gokan (2006), we assume that $\beta = 1$. Therefore, their optimal choices (in equilibrium, capitalists hold no money) are

$$c_t^c=0,\; k_{t+1}=\left[r_t+(1-\delta)\right]k_t. \tag{7}$$

2.3. Firms and Government

On the production side, a unique good is produced by combining labor N_t and the capital stock k_t resulting from the last period. The technology exhibits constant returns to scale and the output is given by $y_t = N_t f(a_t)$, where $a_t \equiv k_t/N_t$. The production function $f(a_t)$ is continuous for $a \geq 0$, C^m for $a > 0$ and m large enough, with $f'(a_t) > 0$ and $f''(a_t) < 0$. In equilibrium, we obtain $r_t = r(a_t)$ and $w_t = w(a_t)$. The government needs to balance the budget in each period, $\tau_{wt} w_t N_t + \frac{\mu_t M_t}{P_t} = T > 0$, thus the labor income tax rate and the rate of monetary growth ($\mu_t = \frac{M_{t+1}-M_t}{M_t}$) are endogenously adjusted in order to make the budget balanced.

3. Local Dynamics

Following Woodford (1986), we let $M > 0$ be the constant quantity of outside money ($\tau_{wt} = T/w_t N_t$).[2]In this case, one gets the dynamical system of (k_t, a_t) as: ($R(a_t)$ is the real gross rate of return on capital)

$$k_{t+1} = [r(a_t) + (1-\delta)]\, k_t = R(a_t)\, k_t \tag{D-1}$$

$$\left(c_{t+1}^{w}\right)^{(1-\phi)} [1 - \frac{T}{w(a_t)\, k_t / a_t}] = (N_t)^{(1+\zeta)}\,. \tag{D-2}$$

Different from Gokan (2006), the workers' budget constraint and (D-2) imply that $w(a_t)\, N_t = c_t = M_t/P_t$, which is also the equilibrium condition of money market. The good market clearing condition is $N_t f(a_t) = c_t + k_{t+1} - (1-\delta)\, k_t$.

From (D-1), the steady state capital-labor ratio can be obtained by solving $R(a^*) = 1$. That is, $a^* = k^*/N^*$ depends only on the technology, not on the utility curve nor on government lump sum transfer(s) T. In the steady state, (D-2) implies that

$$(w(a^*)\, N^*)^{1-\phi} \left\{1 - \frac{T}{w(a^*)\, N^*}\right\} = (N^*)^{1+\zeta}\,. \tag{8}$$

Proposition 1. *Under the assumption $\phi \in (0,1)$ and $\zeta > 0$, there exist two non-trivial steady states $0 < N_1^* < N_2^*$ for $0 < T < T_{2s}$. They coalesce together when T goes up to T_{2s} and disappear for $T > T_{2s}$.*

Proof. From equation (8), N^* is the point of intersection for two curves: $f_1 = (N^*)^{1+\zeta}$ and $f_2 = (w(a^*)\, N^*)^{1-\phi} \left\{1 - \frac{T}{w(a^*) N^*}\right\}$. $\zeta > 0$ implies that f_1 is convex and passing the point $(0,0)$.

When $\phi \in (0,1)$, the first and second derivatives of f_2 with respect to N (in the steady states) are

$$\begin{aligned}
\frac{df_2}{dN^*} &= A(1-\phi)(N^*)^{-\phi} - B\phi(N^*)^{-\phi-1} > 0, \\
\frac{d^2 f_2}{dN^{*2}} &= (N^*)^{-\phi-2}\left[-A\phi(1-\phi)N^* + B\phi(\phi+1)\right] < 0,
\end{aligned}$$

where $A = [w(a^*)]^{1-\phi} > 0$ and $B = -T\left[w(a^*)\right]^{-\phi} < 0$. Figure 1 validates this proposition. Then the corresponding (two) steady states of k, c, and y are computed using $k^* = a^* N^*$, $w(a^*)N^* = c^*$, and $y^* = f(a^*)N^*$. ■

Insert Figure 1 here

We now linearize the dynamic system around the two steady states and analyze the stability properties of the Jacobian matrix of (D). Let $\varepsilon_w = aw'(a)/w(a)$ be the elasticity of the marginal product of labor and $\varepsilon_R = a\left|R'(a)\right|/R(a)$ be the elasticity of the real gross rate of return on capital.

Proposition 2. *The linearized dynamics for the deviations $dk \equiv k - k^*$ and $da \equiv a - a^*$, are determined by*

$$\begin{cases}
dk_{t+1} &= dk_t - N_i^* \left|\varepsilon_R\right| da_t, \\
da_{t+1} &= \frac{1}{N_i^*}\frac{\zeta+\phi-\frac{1-G\left(k_i^*,a^*\right)}{G\left(k_i^*,a^*\right)}}{(1-\phi)(\varepsilon_w-1)} dk_t + \frac{\left[\left|\varepsilon_R\right|-1-(\zeta+\phi\left|\varepsilon_R\right|)-\frac{\left(1-G\left(k_i^*,a^*\right)\right)(1-\varepsilon_w)}{G\left(k_i^*,a^*\right)}\right]}{(1-\phi)(\varepsilon_w-1)} da_t,
\end{cases}$$

where $G(k_t,a_t) \equiv [1 - \frac{T}{w(a_t)k_t/a_t}]$. We let G_i be $G(k_i^*,a^*)$ for $i = 1,2$.

It is easy for us to have trace (T_i) and determinant (D_i) of the Jacobian matrix,

$$T_i = \frac{\varepsilon_w + \left|\varepsilon_R\right| - 1}{\varepsilon_w - 1} - \gamma\frac{1}{\varepsilon_w - 1} + \frac{1}{(1-\phi)}\widetilde{T}_i^*, \tag{9.1}$$

$$D_i = \gamma \frac{|\varepsilon_R| - 1}{\varepsilon_w - 1} + \frac{\varepsilon_w - |\varepsilon_R| - 1}{(1-\phi)(\varepsilon_w - 1)} \widetilde{T}_i^*, \tag{9.2}$$

where $\gamma = (1+\zeta)/(1-\phi) > 1$ and $\widetilde{T}_i^* = \frac{1-G_i}{G_i} = \frac{T}{c_i^* - T}$ is the steady state share of lump-sum transfers to the workers' after-tax income. As in Gokan (2006), we set γ to be constant and the bifurcation parameter in this model is $\widetilde{T}_i^*$ that is made to vary by moving T up from 0 to T_{2s}. That is because government revenue influences the local stability through its effect on the steady state share of the transfer to the after-tax income. From (9.1) and (9.2), we have the following:

Lemma 1. *For any values of N^*, the point $(T_i,\ D_i)$ is located on the line Δ,*

$$D_i = \Theta(a^*) [T_i + \Pi(a^*)],$$

where $\Theta(a^*) = \frac{\varepsilon_w - |\varepsilon_R| - 1}{\varepsilon_w - 1}$, and $\Pi(a^*) = \gamma \frac{|\varepsilon_R| - 1}{\varepsilon_w - |\varepsilon_R| - 1} - \frac{\varepsilon_w + |\varepsilon_R| - 1 - \gamma}{\varepsilon_w - 1}$. Suppose $T \to 0$. Then we have $D_2 \to \gamma \frac{|\varepsilon_R| - 1}{\varepsilon_w - 1}$ and $T_2 \to \frac{\varepsilon_w + |\varepsilon_R| - 1}{\varepsilon_w - 1} - \gamma \frac{1}{\varepsilon_w - 1}$ in the high steady state, while in the low steady state $T_1 \to +\infty$ and $D_1 \to +(-)\infty$, if $\frac{\varepsilon_w - |\varepsilon_R| - 1}{(1-\phi)(\varepsilon_w - 1)} > (<) 0$.

Proof. $\widetilde{T}_i^* = (1-\phi)(T_i - \frac{\varepsilon_w + |\varepsilon_R| - 1 - \gamma}{\varepsilon_w - 1})$. Using this equation, we have

$$D_i = \frac{\varepsilon_w - |\varepsilon_R| - 1}{\varepsilon_w - 1}(T_i + \gamma \frac{|\varepsilon_R| - 1}{\varepsilon_w - |\varepsilon_R| - 1} - \frac{\varepsilon_w + |\varepsilon_R| - 1 - \gamma}{\varepsilon_w - 1}).$$

Suppose $T \to 0$. In the high steady state, when $T = 0$, we have $\widetilde{T}_2^* = 0$ since $N_2^* \neq 0$ and $c_2^* \neq 0$. So $D_2 \to \gamma \frac{|\varepsilon_R| - 1}{\varepsilon_w - 1}$ and $T_2 \to \frac{\varepsilon_w + |\varepsilon_R| - 1}{\varepsilon_w - 1} - \gamma \frac{1}{\varepsilon_w - 1}$. From (D2), we know that $(N_1^*)^{\varsigma+\phi} = (w(a^*))^{1-\phi} \left(\widetilde{T}_1^* + 1\right)^{-1}$. In the low steady state, we have $N_1^* \to 0_{(+)}$ from above, which implies that $\left(\widetilde{T}_1^* + 1\right)^{-1} = (N_1^*)^{\varsigma+\phi} / (w(a^*))^{1-\phi} \to 0_{(+)}$ from above. Then $\widetilde{T}_1^* \to +\infty$ (as $T \to 0$). So $T_1 \to +\infty$ and $D_1 \to +(-)\infty$ if $\frac{\varepsilon_w - |\varepsilon_R| - 1}{(1-\phi)(\varepsilon_w - 1)} > (<) 0$. ■

When $T = T_{2s}$, these exists a unique steady state. In this case, $\widetilde{T}_2^*(T_{2s}) = \frac{T_{2s}}{c_2^* - T_{2s}}$ is the critical value of the saddle-node bifurcation.

Lemma 2. *The value of T_{2s} can be determined using $\widetilde{T_2^*}(T_{2s}) = \zeta + \phi$.*

Proof. When $T = T_{2s}$, there exists a unique steady state, $\overline{N}$, which satisfies $\left(w\left(a^*\right)\overline{N}\right)^{1-\phi}\left\{1 - \frac{T_{2s}}{w(a^*)\overline{N}}\right\} = \left(\overline{N}\right)^{1+\zeta}$, or $\left(\overline{N}\right)^{\phi+\zeta} = \left(w\left(a^*\right)\right)^{1-\phi}\left\{1 - \frac{T_{2s}}{w(a^*)}\frac{1}{\overline{N}}\right\}$. We assume that $\widehat{f}_1 = \left(\overline{N}\right)^{\phi+\zeta}$ and $\widehat{f}_2 = \left(w\left(a^*\right)\right)^{1-\phi}\left\{1 - \frac{T_{2s}}{w(a^*)}\frac{1}{\overline{N}}\right\}$. In the unique steady state, the slopes of $\widehat{f}_1$ and $\widehat{f}_2$ are equal, i.e., $(\phi+\zeta)\left(\overline{N}\right)^{\phi+\zeta-1} = \left(w\left(a^*\right)\right)^{-\phi} T_{2s}\left(\overline{N}\right)^{-2}$. Since $\widetilde{T_2^*}(T_{2s}) = T_{2s}/\left(w\left(a^*\right)\overline{N} - T_{2s}\right)$, we have $w\left(a^*\right)\overline{N}/T_{2s} = 1 + 1/\widetilde{T_2^*}(T_{2s})$. Using the above relationships, we can have the following:

$$\begin{aligned}
\frac{1}{\widetilde{T_2^*}(T_{2s})} + 1 &= \frac{\left(w\left(a^*\right)\right)^{1-\phi}}{(\phi+\zeta)\left(\overline{N}\right)^{\phi+\zeta}} = \frac{\left(w\left(a^*\right)\overline{N}\right)^{1-\phi}}{(\phi+\zeta)\left(\overline{N}\right)^{1+\zeta}} \\
&= \frac{1}{(\phi+\zeta)\left[1 - \frac{T_{2s}}{w(a^*)\overline{N}}\right]} = \frac{1}{\phi+\zeta}[\widetilde{T_2^*}(T_{2s}) + 1]
\end{aligned}$$

where $[1 - \frac{T_{2s}}{w(a^*)\overline{N}}] = [\widetilde{T_2^*}(T_{2s}) + 1]^{-1}$. Then it is easy for us to have $\widetilde{T_2^*}(T_{2s}) = \zeta + \phi$. ■

Suppose that T increases from 0 to T_{2s}. For the high steady state, the ratio $\widetilde{T_2^*}$ goes up from 0 to $\widetilde{T_2^*}(T_{2s}) = T_{2s}/\left(c_2^* - T_{2s}\right) = \zeta + \phi$ and thereby the corresponding point $(Trace, Det)$ moves along the line Δ. While for the low steady state, the ratio $\widetilde{T_1^*}$ goes down from $+\infty$ to $\widetilde{T_2^*}(T_{2s}) = \zeta + \phi$ and thus $(Trace, Det)$ moves along the line Δ. In the next subsection, we will define the part of Δ on which $(T_i,\ D_i)$ moves as Δ_i, and dicuss the stability properties when T varies.[3]

3.1. A Geometrical Method of Bifurcation Analysis and Local Stability

Following Grandmont et al. (1998), we can obtain the following relationships: $\varepsilon_w = s/\sigma$ and $|\varepsilon_R| = \delta\left(1-s\right)/\sigma$, where s is the share of capital in total output $s \equiv a^* r(a^*)/f\left(a^*\right)$, and σ is the elasticity of capital-labor substitution evaluated at a^*. For the ease of interpretation, we use the following transformation.

$$T_i = \widetilde{Tr} + \frac{\widetilde{T}_i^* - \widetilde{T}}{1-\phi}, \text{ where } \widetilde{Tr} = \frac{\varepsilon_w + |\varepsilon_R| - 1 - \gamma}{\varepsilon_w - 1} + \frac{\widetilde{T}}{1-\phi} \text{ and } \widetilde{T} = \zeta + \phi, \tag{10.1}$$

$$D_i = \widetilde{Det} + \frac{\varepsilon_w - |\varepsilon_R| - 1}{(1-\phi)(\varepsilon_w - 1)}\left(\widetilde{T}_i^* - \widetilde{T}\right), \text{ where } \widetilde{Det} = \gamma\frac{|\varepsilon_R| - 1}{\varepsilon_w - 1} + \frac{\varepsilon_w - |\varepsilon_R| - 1}{(1-\phi)(\varepsilon_w - 1)}\widetilde{T}. \tag{10.2}$$

To describe Δ, as in Gokan (2006), we should analyze the slope $\Theta(a^*)$ and the end point $\left(\widetilde{Tr}, \widetilde{Det}\right)$.[4] We can easily verify that (1) $\left(\widetilde{Tr}, \widetilde{Det}\right)$ lies on the line AC ($\widetilde{Det} = \widetilde{Tr} - 1$) and; (2) the slope of Δ ($\Theta(a^*)$) is $1 - \frac{\delta(1-s)}{s-\sigma}$ and $\widetilde{Det}$ is $2\gamma - 1 - \frac{\gamma s - \delta(1-s)}{s-\sigma}$. Here as in Grandmont et al. (1998), we focus on the case where $\frac{\delta(1-s)}{s} < 1$. As σ varies in the interval $(0, +\infty)$, we summarize the variations of $\Theta(a^*)$, $D_2^{\widetilde{T}_2^*=0}$, $T_2^{\widetilde{T}_2^*=0}$ and $\widetilde{Det}$ in the following table.[5]

σ	0	$s-\delta(1-s)$	$s-\delta(1-s)/2$	s	$+\infty$
Δ_1	below AC			above AC	
Δ_2	above AC			below AC	
slope	$1-\frac{\delta(1-s)}{s} \searrow$	$0 \searrow$	$-1 \searrow$	$-\infty, +\infty \searrow$	1
$\widetilde{Det}$	$\gamma - [1-\frac{\delta(1-s)}{s}] \searrow$	$\gamma[2-\frac{s}{\delta(1-s)}] \searrow$	$1-2\gamma[\frac{s-\delta(1-s)}{\delta(1-s)}] \searrow$	$-\infty, +\infty \searrow$	$2\gamma - 1$
$D_2^{\widetilde{T}_2^*=0}$	$\gamma\frac{\delta(1-s)}{s} \in (0,\gamma) \searrow$	$\gamma\left[2-\frac{s}{\delta(1-s)}\right] \searrow$	$\gamma[3-2\frac{s}{\delta(1-s)}] \searrow$	$-\infty, +\infty \searrow$	γ
$T_2^{\widetilde{T}_2^*=0}$	$1+\frac{\delta(1-s)}{s} \in (1,2) \searrow$	$2+\gamma[1-\frac{s}{\delta(1-s)}] \searrow$	$3+\gamma-\frac{2\gamma s}{\delta(1-s)} \searrow$	$-\infty, +\infty \searrow$	$1+\gamma$

where *slope* $= 1 - \frac{\delta(1-s)}{s-\sigma}$, $\widetilde{Det} = 2\gamma - 1 - \frac{\gamma s - \delta(1-s)}{s-\sigma}$, $D_2^{\widetilde{T}_2^*=0} = \gamma[1 + \frac{s-\delta(1-s)}{\sigma-s}]$, and $T_2^{\widetilde{T}_2^*=0} = 1 + \gamma + \frac{\gamma s - \delta(1-s)}{\sigma - s}$. All this generates essentially several subcases.

Case 1. In order to compare our model with that of Gokan (2006), we follow Grandmont et al. (1998) and use the crudely calibrated values of $\delta = 0.1$ and $s = \frac{1}{3}$.[6] And we find that when $\sigma > s$, namely, $\sigma \in \left(\frac{1}{3}, +\infty\right)$, the high (resp. the low) steady state is always a saddle (resp. a source) for every $0 < \widetilde{T}_2^* < \zeta + \phi$ (resp. $\zeta + \phi < \widetilde{T}_1^* < +\infty$) as $\gamma > 1$. (see figure 2).

Insert Figure 2 here

Case 2. If $\sigma \in (0, s - \delta(1-s))$, namely, $\sigma \in (0, 4/15)$, the possible locations of the line Δ are shown below. (see figure 3.0).[7] $\mathbf{B}(-2,1)$, $\mathbf{C}(2,1)$, and $\mathbf{A}(0,-1)$ describe the stability triangle as in Grandmont et al. We should mention several important things before we derive our (in)determinacy results. For any $\widetilde{T}_2^* \in (0, \zeta+\phi)$, (1) if (T_2, D_2) lies above (resp. below) the point B, $\Pi(a^*) > \frac{1}{\Theta(a^*)}+2$ (resp. $\Pi(a^*) < \frac{1}{\Theta(a^*)}+2$) holds, (2) if (T_2, D_2) lies above (resp. below) the point C, $\Pi(a^*) > \frac{1}{\Theta(a^*)}-2$ (resp. $\Pi(a^*) < \frac{1}{\Theta(a^*)} - 2$) holds, and (3) if (T_2, D_2) lies above (resp. below) the point A, $\Pi(a^*) > -\frac{1}{\Theta(a^*)}$(resp. $\Pi(a^*) < -\frac{1}{\Theta(a^*)}$) holds. Moreover, we need consider whether the starting point ($T_2^{\widetilde{T}_2^*=0}$, $D_2^{\widetilde{T}_2^*=0}$) of the half line Δ_2 lies in the left hand side of the line AB or not. And we find that if it does so, $T_2^{\widetilde{T}_2^*=0} < -D_2^{\widetilde{T}_2^*=0} - 1$ holds, which is equivalent to $(1-30\sigma)\gamma < (30\sigma - 11)$.[8]Therefore, we have the following: as $\sigma > 1/30$ and $\gamma > \gamma^L \equiv \max\{1, \frac{30\sigma-11}{1-30\sigma}\} = \begin{cases} \frac{30\sigma-11}{1-30\sigma}, & \frac{1}{30} < \sigma < \frac{1}{5} \\ 1, & \frac{4}{15} > \sigma > \frac{1}{5} \end{cases}$, the starting point ($T_2^{\widetilde{T}_2^*=0}$, $D_2^{\widetilde{T}_2^*=0}$) lies in the left hand side of the line AB. There are four subcases when $\sigma > 1/30$ and $\gamma > \gamma^L$ hold. (a) If (T_2, D_2) lies above the point B, only a flip bifurcation can be expected to occur along the half line Δ_2 as $\widetilde{T}_2^*$ passes through its flip bifurcation value $\widetilde{T}_{2F}^*$. This requires that $(5-30\sigma)\gamma > (9-30\sigma)+4(4-15\sigma)(5-15\sigma)$. (see figure 3.1).[9](b) If (T_2, D_2) goes through lines AB and BC, i.e., $\frac{1}{\Theta(a^*)}-2 < \Pi(a^*) < \frac{1}{\Theta(a^*)}+2$, flip and Hopf bifurcations may be expected to occur along the half line Δ_2 as $\widetilde{T}_2^*$ passes through the corresponding flip bifurcation value and Hopf bifurcation value ($\widetilde{T}_{2H}^*$) respectively. This requires that $(9-30\sigma) < (5-30\sigma)\gamma < (9-30\sigma)+4(4-15\sigma)(5-15\sigma)$. (see figure 3.2). (c) If (T_2, D_2) goes through lines AB and AC, i.e., $-\frac{1}{\Theta(a^*)} < \Pi(a^*) < \frac{1}{\Theta(a^*)} - 2$, only a flip bifurcation would occur along the half line Δ_2 as $\widetilde{T}_2^*$ passes through the flip bifurcation value. This requires that $-1 < (5-30\sigma)\gamma < (9-30\sigma)$. (see figure 3.3). And (d) (T_2, D_2) lies below the point A, i.e., $-\frac{1}{\Theta(a^*)} > \Pi(a^*)$. This requires that $-1 > (5-30\sigma)\gamma$. In this subcase, only a flip bifurcation can arise along the half line Δ_1 as $\widetilde{T}_1^*$ passes through its flip bifurcation value $\widetilde{T}_{1F}^*$ (see figure 3.4). Before we summarize the numerical results, we define some notations as follows: $\gamma_B^A \equiv [(9-30\sigma)+4(4-15\sigma)(5-15\sigma)]/(5-30\sigma)$, $\gamma_C^A \equiv \frac{9-30\sigma}{5-30\sigma}$, and $\gamma_A^A \equiv \frac{1}{30\sigma-5}$. The subcase in Figure 3.1 appears if

$$\gamma > \max\{\gamma^L, \gamma_B^A\} = \begin{cases} \gamma^L = \frac{30\sigma-11}{1-30\sigma}, & \text{as } \frac{1}{30} < \sigma < 0.051,\ (\gamma^L > \gamma_B^A) \\ \gamma_B^A, & \text{as } 0.051 < \sigma < \frac{1}{6},\ (\gamma_B^A > \gamma^L) \end{cases}.$$

The subcase in Figure 3.2 appears if $\frac{30\sigma-11}{1-30\sigma} = \gamma^L < \gamma < \gamma_B^A$ (when $0.051 < \sigma < 0.11$, $(\gamma_B^A > \gamma^L > \gamma_C^A)$) or $\gamma_C^A < \gamma < \gamma_B^A$ (when $0.11 < \sigma < \frac{1}{6}$, $(\gamma_C^A > \gamma^L)$). The subcase in Figure 3.3 appears if $\frac{30\sigma-11}{1-30\sigma} = \gamma^L < \gamma < \gamma_C^A$ (when $0.11 < \sigma < \frac{1}{6}$) or $\gamma_A^A > \gamma > \gamma^L = \frac{30\sigma-11}{1-30\sigma}$ (when $1/6 < \sigma < 1/5$). The subcase in Figure 3.4 appears if $\gamma > 1$ (when $4/15 > \sigma > 1/5$) or $\gamma > \gamma_A^A$ (when $1/6 < \sigma < 1/5$).

Case 3. We consider the case in which $\sigma \in (s - \delta(1-s), s)$, namely, $\sigma \in (4/15, 1/3)$ holds. $\widetilde{Det} = D_2^{\widetilde{T}_2^*=0} = -3\gamma < -3$ holds when $\sigma = 4/15$. Since both $\widetilde{Det}(\sigma)$ and $D_2^{\widetilde{T}_2^*=0}(\sigma)$ are decreasing functions of σ when $\sigma \in (4/15, 0.3)$, we can infer from this fact that for any $\sigma \in (4/15, 0.3)$, $\widetilde{Det}$ and $D_2^{\widetilde{T}_2^*=0}$ are less than -3.[10]This means that only a flip bifurcation can arise along the half line Δ_1 when it crosses the line AB and $\widetilde{T}_1^*$ passes through its flip bifurcation value. See figure 4.1 about this case. When $\sigma \in (0.3, 1/3)$, the slope is less than -1, flip and Hopf bifurcations can not arise along the lines Δ_1 and Δ_2. See figure 4.2 about this case.

Case 4. We need discuss the case where $\sigma \in (0, 4/15)$, $(T_2^{\widetilde{T}_2^*=0}, D_2^{\widetilde{T}_2^*=0})$ lies in the right hand side of the line AB and only a Hopf bifurcation can occur when (T_2, D_2) crosses the line BC (see figure 4.3). If it happens, $T_2^{\widetilde{T}_2^*=0} > -D_2^{\widetilde{T}_2^*=0} - 1$ holds. That is to say, $\gamma(1-30\sigma) > 30\sigma - 11$ holds. (a) When $1 - 30\sigma > 0$, for any $\gamma > 1$, $\gamma(1-30\sigma) > 30\sigma - 11$ always holds. The inequality

$\frac{1}{\Theta(a^*)} - 2 < \Pi(a^*) < \frac{1}{\Theta(a^*)} + 2$ implies that $\sigma < 1/6$, and $1 < \gamma_C^A < \gamma < \gamma_B^A$. Then figure 4.3 appears when $0 < \sigma < 1/30$ and $\gamma_C^A < \gamma < \gamma_B^A$. (b) When $1 - 30\sigma < 0$, it is easy to verify that when $1/30 < \sigma < 1/5$, $\frac{30\sigma-11}{1-30\sigma} > 1$ holds. This means that when $1/30 < \sigma < 1/5$, we need $\frac{30\sigma-11}{1-30\sigma} > \gamma > 1$ to guarantee that $\gamma(1-30\sigma) > 30\sigma - 11$ holds. Also, $\sigma < 1/6$ and $1 < \gamma_C^A < \gamma < \gamma_B^A$ should hold in order for (T_2, D_2) to cross the segment BC. Then figure 4.3 appears when $1/30 < \sigma < 0.051$ and $\gamma_C^A < \gamma < \gamma_B^A$ hold, or when $0.051 < \sigma < 0.11$ and $\gamma_C^A < \gamma < \gamma_L = \frac{30\sigma-11}{1-30\sigma}$ hold.

Insert Figures 3.0 through 4.3 here

Our results can be summarized in the following proposition.

Proposition 3. *Under the assumption of $\delta = 0.1$ and $s = \frac{1}{3}$ as in Grandmont et al. (1998).*

(1) Figure 2 appears if the pair (σ, γ) falls in the following interval: $1/3 < \sigma$ and $\gamma > 1$. The high steady state is always a saddle when $0 < \widetilde{T}_2^* < (\zeta + \phi)$. But the low steady state is always a source when $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.

(2) Figure 3.1 appears if the pair (σ, γ) falls in the following two intervals: (a) $1/30 < \sigma < 0.051$ *and $\gamma > \gamma_L = \frac{30\sigma-11}{1-30\sigma}$, and* (b) $0.051 < \sigma < 1/6$ *and $\gamma_B^A < \gamma$. The high steady state is a saddle when $0 < \widetilde{T}_2^* < \widetilde{T}_{2F}^*$, and a source when $\widetilde{T}_2^* > \widetilde{T}_{2F}^*$. A flip bifurcation occurs at $\widetilde{T}_2^* = \widetilde{T}_{2F}^*$. But the low steady state is always a saddle when $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.*

(3) Figure 3.2 appears if the pair (σ, γ) falls in the following two intervals: (a) $0.051 < \sigma < 0.11$ *and $\gamma_B^A > \gamma > \gamma_L = \frac{30\sigma-11}{1-30\sigma}$, and* (b) $0.11 < \sigma < 1/6$ *and $\gamma_C^A < \gamma < \gamma_B^A$. The high steady state is a saddle when $0 < \widetilde{T}_2^* < \widetilde{T}_{2F}^*$, and a sink when $\widetilde{T}_{2H}^* > \widetilde{T}_2^* > \widetilde{T}_{2F}^*$. A flip bifurcation occurs at $\widetilde{T}_2^* = \widetilde{T}_{2F}^*$. A Hopf bifurcation occurs at $\widetilde{T}_2^* = \widetilde{T}_{2H}^*$. The high steady state is a source when $\widetilde{T}_2^* > \widetilde{T}_{2H}^*$. But the low steady state is always a saddle when $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.*

(4) Figure 3.3 apprears if the pair (σ, γ) falls in the following two intervals: (a) $0.11 < \sigma < 1/6$

and $\gamma_C^A > \gamma > \gamma_L = \frac{30\sigma-11}{1-30\sigma}$, *and (b)* $1/6 < \sigma < 1/5$ *and* $\gamma_L = \frac{30\sigma-11}{1-30\sigma} < \gamma < \gamma_A^A$. *The high steady state is a saddle when* $0 < \widetilde{T}_2^* < \widetilde{T}_{2F}^*$, *and a sink when* $\widetilde{T}_2^* > \widetilde{T}_{2F}^*$. *A flip bifurcation occurs at* $\widetilde{T}_2^* = \widetilde{T}_{2F}^*$. *But the low steady state is always a saddle when* $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.

(5) Figure 3.4 apprears if the pair (σ, γ) *falls in the following two intervals: (a)* $1/6 < \sigma < 1/5$ *and* $\gamma > \gamma_A^A$, *and (b)* $1/5 < \sigma < 4/15$ *and* $1 < \gamma$. *The low steady state is a saddle when* $\widetilde{T}_{1F}^* < \widetilde{T}_1^* < +\infty$, *and a source when* $\widetilde{T}_1^* < \widetilde{T}_{1F}^*$. *A flip bifurcation occurs at* $\widetilde{T}_1^* = \widetilde{T}_{1F}^*$. *But the high steady state is always a saddle when* $0 < \widetilde{T}_2^* < (\zeta + \phi)$.

(6) Figure 4.1 appears if the pair (σ, γ) *falls in the following interval:* $4/15 < \sigma < 0.3$ *and* $\gamma > 1$. *The high steady state is always a saddle when* $0 < \widetilde{T}_2^* < (\zeta + \phi)$. *The low steady state is a saddle when* $\widetilde{T}_{1F}^* < \widetilde{T}_1^* < +\infty$, *and a source when* $\widetilde{T}_1^* < \widetilde{T}_{1F}^*$. *A flip bifurcation occurs at* $\widetilde{T}_1^* = \widetilde{T}_{1F}^*$.

(7) Figure 4.2 appears if the pair (σ, γ) *falls in the following interval:* $0.3 < \sigma < 1/3$ *and* $\gamma > 1$. *The high steady state is always a saddle when* $0 < \widetilde{T}_2^* < (\zeta + \phi)$. *But the low steady state is always a source when* $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.

(8) Figure 4.3 appears if the pair (σ, γ) *falls in the following three intervals: (a)* $0 < \sigma < 1/30$ *and* $\gamma_B^A > \gamma > \gamma_C^A$, *(b)* $1/30 < \sigma < 0.051$ *and* $\gamma_B^A > \gamma > \gamma_C^A$, *and (c)* $0.051 < \sigma < 0.11$ *and* $\gamma_C^A < \gamma < \gamma_L = \frac{30\sigma-11}{1-30\sigma}$. *The high steady state is a sink when* $0 < \widetilde{T}_2^* < \widetilde{T}_{2H}^*$, *and a source when* $\widetilde{T}_2^* > \widetilde{T}_{2H}^*$. *A Hopf bifurcation occurs at* $\widetilde{T}_2^* = \widetilde{T}_{2H}^*$. *But the low steady state is always a saddle when* $(\zeta + \phi) < \widetilde{T}_1^* < +\infty$.

Now we need explain the contrast between Gokan's results and ours. The first point to be emphasized is that considering the case in which the government transfers its revenue to the workers will make the local dynamics become much more complicated. The result that the rate of money growth doesn't affect the model dynamics *may* not hold. The reason is that the Euler equation of the workers' problem will be affected by the government revenue. If the latter is financed by a mixture of money and labor income taxes, this modified Woodford's model can not be explicitly solved. As Pintus (2004) points out, if we consider the extreme case in which the predetermined government revenue is financed only by labor income taxes (that is, $M_t = M$ at all dates), the tax rate is endogenous and satisfies $\tau_{wt} = T/w_t N_t$ as in Schmitt-Grohe and Uribe (1997). Under the assumption of constant money supply, we numerically solve the model and find that the local dynamics are quite different from those in Gokan's model. Roughly speaking, the existence of constant government transfer reduces the range of the elasticity of capital-labor substitution inducing endogenous fluctuations.[11]

4. Conclusion

In this paper, we show that the local dynamics in Gokan's model change dramatically if the government transfers its revenue to the households in a lump sum way. Our analysis indicates that (1) the existence of constant government transfer reduces the range of the elasticity of capital-labor substitution inducing endogenous fluctuations, and (2) the result that the rate of money growth has no impact on local dynamics in Gokan's model *may* not hold under the alternative assumption.

Notes:

1. It includes that (1) in contrast with the case in Gokan's model, the bifurcation parameter– the steady state share of government revenue to after-tax income– takes different values when the system undergoes Fip or Hopf bifurcations and; (2) for the high (resp. the low) steady state, the sub-intervals of σ (the elasticity of capital-labor substitution) and γ (the value of the wage-elasticity of labor supply $\frac{1}{\gamma-1}$), in which we discuss the (in)determinacy results, will be quite different from those in Gokan's model.

2. If we consider the case where government revenue is financed with a mixture of money and labor income taxes, our model will not be solvable.

3. The points (T_i, D_i) $(i = 1, 2)$ move on Δ_1, and Δ_2 towards the line (AC) and disappear when T goes up through T_2. See Grandmont et al. (1998) for a description of ABC triangle.

4. We should investigate if the half line Δ_1 for the low steady state and the half line Δ_2 for the high steady state cross the triangle ABC in the diagram. If they cross the line [BC], a Hopf bifurcation arises. If they cross the line [AB], a Flip bifurcation arises.

5. $D_2^{\widetilde{T}_2^*=0}$, $T_2^{\widetilde{T}_2^*=0}$ are the values of D_2 and T_2 as $\widetilde{T}_2^* = 0$. Since the position of the half line depends on σ and γ, but not on T, we should investigate how the half line locates when σ varies in the interval $(0, +\infty)$ and γ in the interval $(1, +\infty)$. That is, we need impose some restrictions on γ, when we observe the variation of the bifurcation parameter.

6. Under this assumption, $\Theta = 1 - \frac{\delta(1-s)}{s-\sigma} = \frac{4-15\sigma}{5-15\sigma}$, $\widetilde{Det} = 2\gamma - 1 - \frac{\gamma s-\delta(1-s)}{s-\sigma} = 2\gamma - 1 - \frac{5\gamma-1}{5-15\sigma}$, $D_2^{\widetilde{T}_2^*=0} = \gamma\left(1 + \frac{s-\delta(1-s)}{\sigma-s}\right) = \gamma\left(1 + \frac{4}{15\sigma-5}\right)$, $T_2^{\widetilde{T}_2^*=0} = 1 + \gamma + \frac{\gamma s-\delta(1-s)}{\sigma-s} = 1 + \gamma + \frac{5\gamma-1}{15\sigma-5}$, and $\Pi = \gamma\frac{2\delta(1-s)-s}{s-\delta(1-s)-\sigma} + \frac{\gamma s-\delta(1-s)}{s-\sigma} - 1 = \gamma\frac{5-30\sigma}{(4-15\sigma)(5-15\sigma)} - \frac{6-15\sigma}{5-15\sigma}$. We use the numerical case to make the model analytically solvable since the model dynamics depend on the magnitude of $\delta(1-s)/s$.

7. Note that the lines with the starting point (♦) lie in the right hand side of the line AB, which implies that flip bifurcations can not occur, while the lines with the starting point ($\circ$) lie in the left

hand side of the line AB, and flip bifurcations may occur to Δ_2. Moreover, as $\sigma < s$, the starting point should lie in the the left hand side of the line AC. As $\widetilde{Det} < -1$ (under the restrictions $\sigma < s$), the high steady state is always a saddle, while the low steady state can pass through a flip bifurcation (from a saddle to a source).

8. $T_2^{\widetilde{T}_2^*=0} = D_2^{\widetilde{T}_2^*=0} + 1 + \frac{\delta(1-s)}{\sigma-s}(\gamma-1) < -D_2^{\widetilde{T}_2^*=0} - 1$ can imply this.

9. $\Pi(a^*) > \frac{1}{\Theta(a^*)} + 2$ can imply this.

10. The slope decreases from 0 to -1.

11. As Gokan suggested to us, the low steady state is always a source for the range of elasticity of the capital–labor substitution higher than the capital share in production and thus the range of the parameter leading to indeterminacy and local bifurcations significantly shrinks compared with the one in the case of constant government expenditure financed by endogenous labor income taxes as considered in Gokan (2006).

References

[1] Gokan Y. (2006) Dynamic effects of government expenditure in a finance constrained economy. *Journal of Economic Theory* 127, 323–333.

[2] Grandmont J.M., P. Pintus, and R. de Vilder (1998) Capital-labor substitution and competitive nonlinear endogenous cycles. *Journal of Economic Theory* 80, 14–59.

[3] Pintus P. (2004) Aggregate instability in the fixed-cost approach to public spending. unpublished manuscript.

[4] Schmitt-Grohe S. and M. Uribe (1997) Balanced-budget rule, distortionary taxes and aggregate instability. *Journal of Political Economy* 105, 976-1000.

[5] Woodford M. (1986) Stationary sunspot equilibria in a finance constrained economy. *Journal of Economic Theory* 40, 128–137.

5. Figures

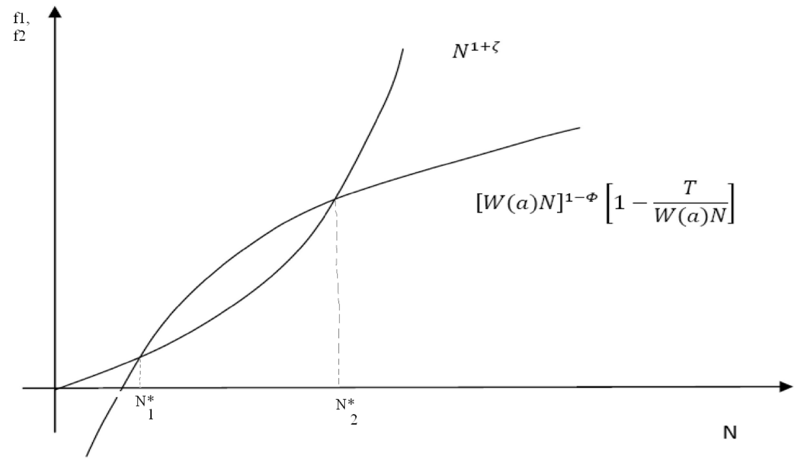

Figure 1.

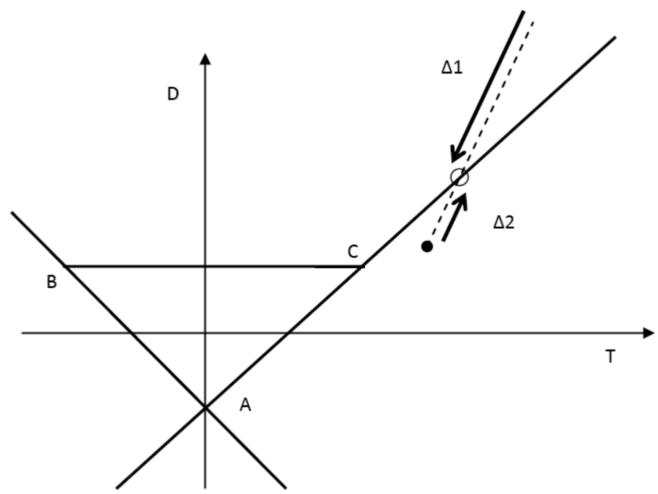

Figure 2. $\sigma > s$.

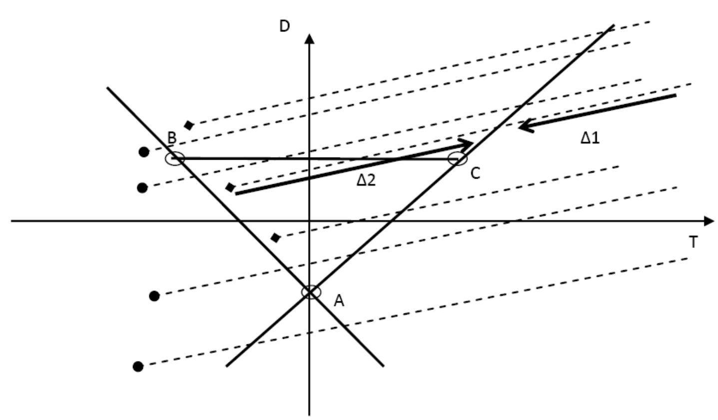

Figure 3.0. Possible Dynamics

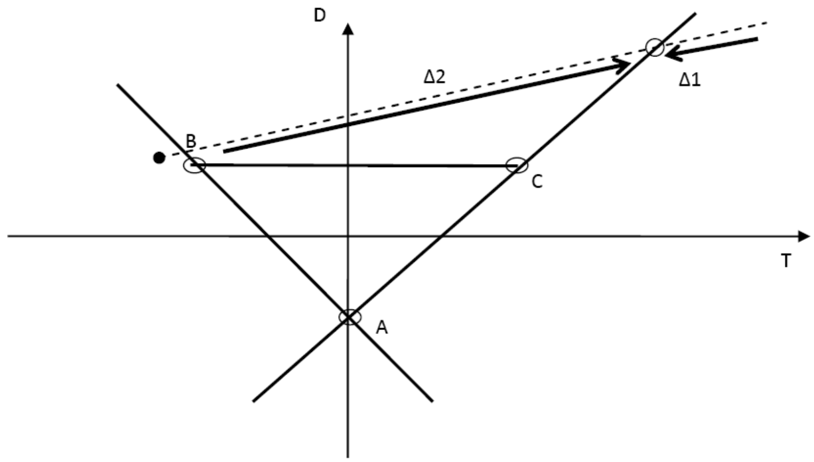

Figure 3.1. Only a Flip bifurcation occurs to Δ_2

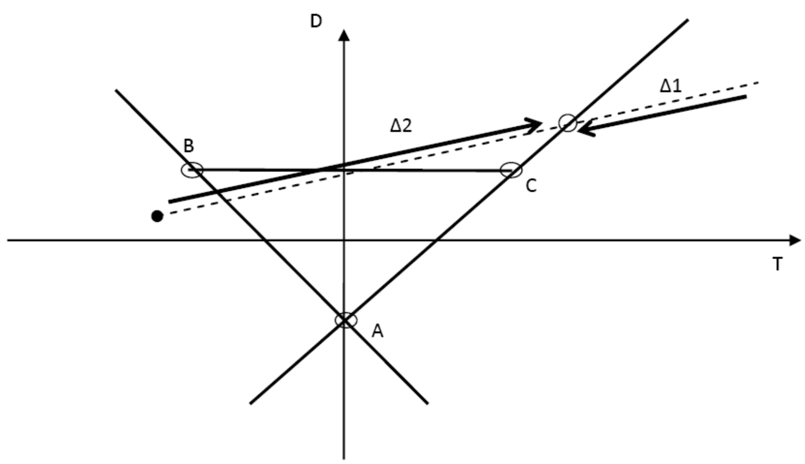

Figure 3.2. Flip and Hopf bifurcations may occur to $\Delta 2$

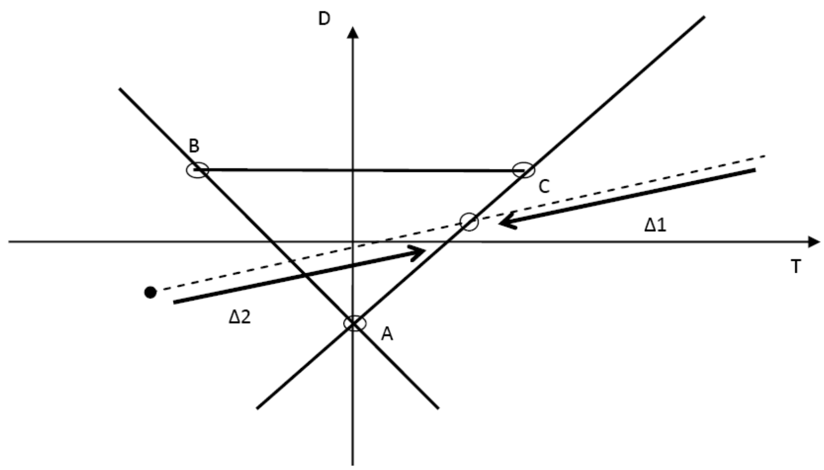

Figure 3.3. Only a flip bifurcation occurs to Δ2.

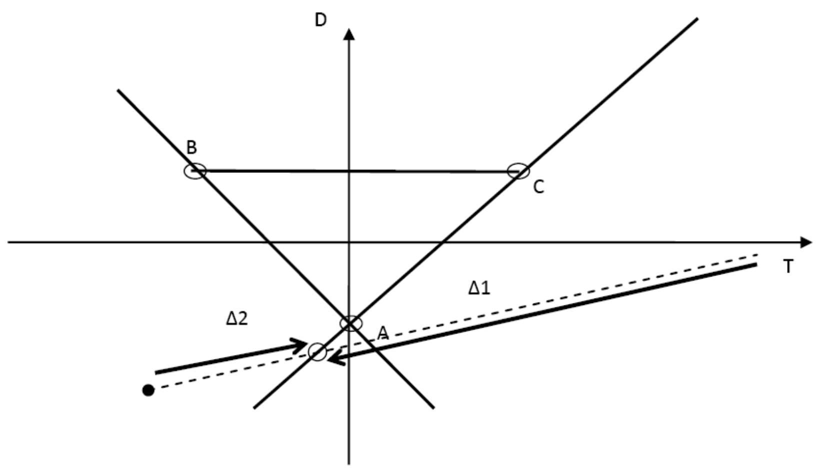

Figure 3.4. A Flip bifurcation can occur to Δ1

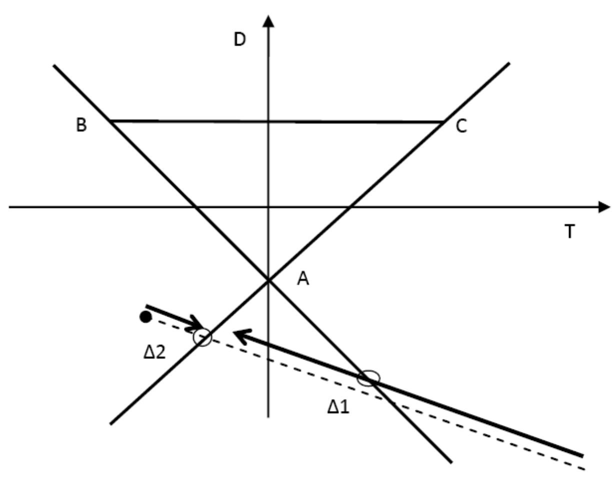

Figure 4.1. when $-1 < slope < 0$, only flip bifurcations can occur to Δ_1.

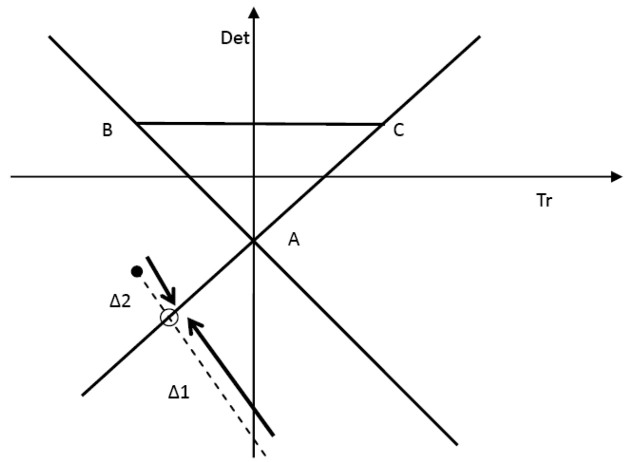

Figure 4.2. the slope is less than -1, bifurcations can not arise along the half lines.

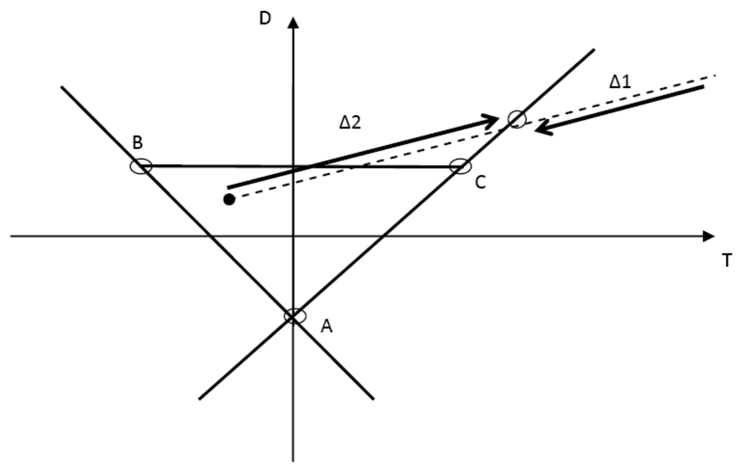

Figure 4.3. Only a Hopf bifurcation can arise when $(T_2^{\widetilde{T}_2^*=0}, D_2^{\widetilde{T}_2^*=0})$ lies in the right hand side of the line AB.