# Elegance with substance

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# Elegance with substance

Mathematics and its education designed for Ladies and Gentlemen

What is wrong with mathematics education and how it can be righted

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Colignatus is the preferred name of Thomas Cool in science.



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# Prologue

Our pupils and students are best treated as ladies and gentlemen with elegance and substance. Providing them with equal mathematics is our much valued objective.

Ideally mathematics would be perfect and unchanging and just be there to be discovered. Mathematics however is as much a discovery as an art. It is made. It is a creation, in the way that cavemen carved their scores in bones and that we create virtual reality with supercomputers. In the interaction between what we do and what we understand almost all of the weight is on what we do, which then imprints on our mind. It appears tedious and hard work to go a bit in the reverse direction, to even get where we are now, let alone develop a notion of perfection.

Given this fragile and historic nature of mathematics it should not come as a surprise that what we currently call mathematics actually appears, on close inspection, to be often cumbersome or even outright irrational. Clarity and understanding are frequently blocked by contradictions and nonsense that are internal to current mathematics itself. Who has a problem mastering mathematics should not be surprised.

Over the years, while teaching mathematics and writing my notes that now result in these pages, there were many moments that I felt frustrated and at times even quite annoyed about the straightjacket of current mathematical conventions. One is supposed to teach mathematics but it is precisely the textbook that blocks this prospect. For many pupils and students the goal is impossible from the outset not because of their lack of capability but because of awkward conventions that only came about for historical reasons.

The flip side is that this is a Garden of Eden for didactic development. What is awkward can be hammered into something elegant. What is irrational can be turned rational and consistent. What is dark and nonsensical can be thrown out and replaced by clarity. There is beauty and satisfaction in redesign.

This didactic reconsideration also changes what we call ‘mathematics’. The interaction between what we do and what we understand shifts to a new equilibrium, a higher optimum at a more agreeable level for both teacher and students. It will still be mathematics since it can be recognized as mathematics. It will be stronger and more efficient mathematics too but it will no longer be the old one.

The criterion for change lies in elegance with substance. Elegance without substance creates a dandy. Elegance ought to signal substance. Mathematics concentrates on the elegance and specific fields of study like economics concentrate on the substance. But mathematics needs to have some substance of itself too. The criterion is tricky since some people see it in the present mathematical conventions too, where awkwardness $A$ plus awkwardness $B$ gives inconsistency $C$. However, we will compare the old ways with the suggestions of the new ways and let the criterion speak for itself. This should open some eyes. Otherwise we just stay in the Garden of Eden.

Which leaves me to thank my own teachers and colleagues who trained and helped me in the old ways. A redesign starts from something and when the old is replaced then this implies that it was valuable to start with. I thank in particular my pupils and students for what they taught me.

# Contents in brief

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# I. Introduction

## 1. Natural limitations to a noble art

A distinction that comes natural to us is between empirical reality and abstract thought. The first is the subject of the empirical sciences, the latter the realm for mathematics and ideal philosophy. This distinction comes with the observation that mathematicians are little trained in empirical issues.

Our subject is the education in mathematics.

Didactics, and in particular the didactics of mathematics, deals with real pupils and students. Didactics requires a mindset that is sensitive to empirical observation – which is not what mathematicians are trained for.

## 2. As far as the mind can reach

Mathematics is a great liberating force. No dictator forces you to accept the truth of PythagoraenTheorem. You are free to check it for yourself. You may even object to its assumptions and invent non-Euclidean geometry. Mathematical reasoning is all about ideas and deductions and about how far your free mind will get you – which is amazingly far.

But you have to be aware of reality if you say something about reality.

The education in mathematics is not without some empirical study. It is proper to recall the Van Streun (2006) *In Memoriam* of A.D. de Groot. It is a painful point however that such exceptions prove the rule.

### ***For the record***

The stock market crash in Autumn 2008 caused criticism on mathematicians and the 'rocket scientists' by Mandelbrot & Taleb (2009), Taleb (2009) and Salmon (2009). The mathematicians involved overlooked the difference between their models and reality. Accents differ a bit, Mandelbrot more on other solutions on the assumptions on the law of large numbers, Taleb more on risk, Salmon more on correlation. It remains amazing that the mathematicians at the banks and hedge funds did not issue a warning somewhere in the processs and it would be obvious that those cannot evade part of the responsibility. Of course, there is a lot of blame to go around. Like the rest of the world, Taleb (2009) and Salmon (2009) are also critical on economists and lawyers in bank management and financial regulation. Fortunately, I am one those economists who issued a warning.

With respect to ecological collapse, Tinbergen & Hueting (1991) presented an approach to monitor how the economy affects the environment and to keep account of ecological survival. Their economic approach pays attention to statistics and real risks as indicated by ecologists. Alternatives came notably from modellers with a mathematical mindset who put emphasis on elegant form and easy notions of risk. Those models suggest that there are no relevant risks on the ecology, which is an agreeable suggestion for most policy makers. Final responsibility falls on those policy makers and society who allow this to happen but it remains strange that those modellers think that they contribute more than only their own assumptions. See Colignatus (2009).

With respect to logic and democracy, Colignatus (2007ab, 2008b), updated from 1981 / 1990, considers statements by mathematicians that have been accepted throughout academia and subsequently society on the basis of mathematical authority. It appears however that those statements mix up true mathematical results with interpretations about reality. When these interpretations are modelled mathematically, the statements reduce to falsehoods. Society has been awfully off-track on basic notions of logic, civic discourse and democracy. Even in 2007, mathematicians working on voting theory wrote a *Letter to the governments of the EU member states* advising the use of the Penrose Square Root Weights (PSRW) for the EU Council of Ministers. See Colignatus (2007c) on their statistical inadequacy and their misrepresentation of both morality and reality.

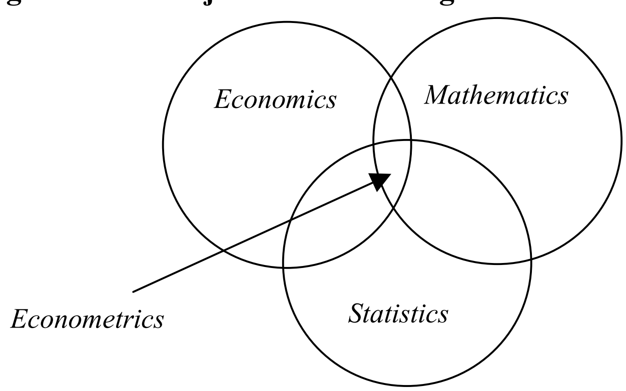
Over the millenia a tradition and culture of mathematics has grown that conditions mathematicians to, well, what mathematicians do. Which is not empirical analysis. Psychology will play a role too in the filtering out of those students who will later become mathematicians. After graduation, mathematicians either have a tenure track in (pure) mathematics or they are absorbed into other fields such as physics, economics of psychology. They tend to take along their basic training and then try to become empirical scientists.

The result is comparable to what happens when mathematicians become educators in mathematics. They succeed easily in replicating the conditioning and in the filtering out of new recruits who adapt to the treatment. For other pupils it is hard pounding.

## *Definition of econometrics*

My own training in mathematics, as a student of econometrics starting Autumn 1973, was with the students of mathematics, physics and astronomy. Thus I do not feel any shortcomings here. My mathematical results e.g. in Colignatus (2007ab) are quite nice even though not developed axiomatically. I have limited affection for pure mathematics but am aware of the hesitations on their part. At least I have the training not to claim more than can be proven, to distinguish fact and hypothesis. For me, however, this holds both in mathematics and about reality. For readers not familiar with the notion of econometrics, I can usefully replicate the diagram by Rijken van Olst, see **Figure 1**.

**Figure 1: The Rijken van Olst diagram for econometrics**



Some see econometrics as a specialisation but actually it is a generalisation that allows one to work on all angles. Specialists in one of the angles can get deeper results and generalisation comes with modesty, but this generalisation is the only way if we want to tackle reality in scientific fashion.

### *Beams of light (through a glass darkly)*

One of the beauties of a sound education in mathematics is that you learn to see that a good argumentation exposes the dependency on assumptions. Officially, mathematicians are aware of this. They are the first to admit "well, if you change one of the assumptions, of course you may get another result". They will say the same, in reconstruction, for assumptions on reality, whether it is the stock market crash, ecological collapse, destruction of democracy, corruption of the subject of mathematical logic, or even mathematics education itself. If only that they would be aware of it sooner and that society would not be swayed so easily by their seeming competence.

On occasion there is a mathematician who is not only officially aware of mathematical shortcomings but who also goes a long way in developing answers. Writing this book got me to reading Krantz (2008) *Through a Glass Darkly* at arXiv again, and it was gratifying indeed. It is advised reading for proper digestion of this present book.

From his conclusions:

> "So it may be time to re-assess our goals, and our milieu, and indeed our very *lingua franca*, and think about how to fit in more naturally with the flow of life. Every medical student takes a course on medical ethics. Perhaps every mathematics graduate student should take a course on communication. This would include not only good language skills, but how to use electronic media, how to talk to people with varying (non-mathematical) backgrounds, how to seek the right level for a presentation, how to select a topic, and many of the other details that make for effective verbal and visual skills. Doing so would strengthen us as individuals, and it would strengthen our profession. We would be able to get along more effectively as members of the university, and also as members of society at large. Surely the benefits would outweigh the inconvenience and aggravation, and we would likely learn something from the process. But we must train ourselves (in some instances re-train ourselves) to be welcoming to new points of view, to new perspectives, to new value systems. These different value systems need not be perceived as inimical to our own. Rather they are complementary, and we can grow by internalizing them."

In such a future, didactics in mathematical education may come about more naturally. In the mean time however we are confronted with the current situation and the current stock of mathematicians. This is what this book is about.

### *Understanding the main advice*

Please do not understand me wrong. This is a book about the education in mathematics, not an evaluation of mathematics by itself and what they have done since Sumer 5000 years ago. We will not look into what mathematicians have done positively in all kinds of areas and neither will we look into what horrors the empirical sciences have wrought by applying inadequate math. These other issues are not relevant here. We will merely consider the current state of the education in mathematics. This book is about solutions to the problems that we find there.

Please do neither misjudge me. My nature is quite cheerful and I tend to rise each morning in good humour and expectation of the interesting events that the day will bring. I have had my share of things but while these add to experience they don't affect my nature and *savoir vivre*. When I employ expressions like "the dismal state of math education" and "let parliament take action" then you might imagine a dishevelled character waving a protest banner. While in truth I am enjoying music and a cup of coffee, carefully compose this text with shades and dashes, and find satisfaction in completing a rational argument to its proper conclusion. Do not read more in the text.

This said, let us get down to business and consider education in mathematics. The subject is fascinating and enlightening. There is a Garden of Eden for all kinds of improvements and advancement indeed. It is liberating to see what causes the viscosity and to see what can be done about it.

## 3. An art and an industry

Mathematics is as much an ideal art as an industry. The art targets the intellectual jolt when an insight and its truth strike the mind. PythagoreanTheorem causes a sense of wonder. Alongside there is the industry of creation, application and teaching. Struik's (1977) *Concise history of mathematics* clarifies that mathematics developed within society as all other arts and sciences. When this math gets taught there are similar influences. Ernest (2000) *Why teach mathematics?* recognizes at least five interest groups in the teaching of mathematics and uses these labels: Industrial Trainers, Technological Pragmatists, Old Humanists, Progressive Educators and Public Educators. His opening statement reads:

> "First of all I want to argue that school mathematics is neither uniquely defined nor value-free and culture-free. School mathematics is not the same as academic or research mathematics, but a recontextualised selection from the parent discipline, which itself is a multiplicity (Davis and Hersh 1980). Some of the content of school mathematics has no place in the discipline proper but is drawn from the history and popular practices of mathematics, such as the study of percentages (Ernest 1986). Which parts are selected and what values and purposes underpin that selection and the way it is structured must materially determine the nature of school mathematics. Further changes are brought about by choices about how school mathematics should be sequenced, taught and assessed. Thus the nature of school mathematics is to a greater or lesser extent open, and consequently the justification problem must accommodate this diversity. So the justification problem should address not only the rationale for the teaching and learning of mathematics, but also for the selection of what mathematics should be taught and how, as these questions are inseparable from the problem." Ernest (2000)

### *The approach in this book*

This book will consider the two faces of the ideal art versus the industry. Our subject is the *education* in mathematics but the ideal art will be guiding and it may be that we first have to change *mathematics itself* before we can adapt its education. Apparently this does not fit in easily with the Ernest categories.

To understand what we will do, consider the case of the decimal separator that can be either the comma (France) or the point (England). The long standing choice by the

International Organization for Standardization (ISO) has been the comma but since 2006 it compromised by allowing the point as well. A Technological Pragmatic approach is that anything works as long as it is standard, even when the standard is double. Here however we will ask which of the two is better as seen from mathematical elegance. Practical considerations have to weigh in too but a change of an ISO standard should be no restriction, and neither the change of textbooks in other subjects that use mathematics. For us the ideal art will be guiding. In this example the decimal marker is not much of mathematics but the idea in this book is that we are willing to change anything if it gives better math. How does the industry deal with the decimal marker ? Of the industry, we primarily meet with teachers of mathematics and the authors of textbooks. They follow their country. Highschool math and didactics in principle are a different world from universities per se (see below on developing brains). Professors of mathematics may already tend to use decimal points even though they live in decimal comma countries. Highschool mathematics in comma countries implicitly assume that their students are more versatile than professors and can deal with both comma's in textbooks and points in internet resources.

## *The industry*

The organizational structure in the education in mathematics is somewhat Byzantine. There is a forest of governments, committees, mathematical associations, exam boards, textbook authors, institutes of education of teachers, journals, a self-created almost world government style *International Commission on Mathematical Instruction* (ICMI), and what have you. Attitudes range from 'teaching to the test' to the Ernest five groups. Each tree holds on to its roots in order to survive. Suggestions for redesign have to convince that forest. Most suggestions in this book may seem rather bold so that adoption will not be very likely. There is no alternative but to convince that forest. The following arguments and structure of argumentation will be used:

(1) To show that mathematics fails we do not require statistics but can look at the math itself. Officially we require a statistic that competing textbooks use the same math but for the sake of simplicity we trust that ICMI has had some success, and my small sample has not disproven this.

(2) A corner stone is that mathematics is man made. It is a building made over time such that all kinds of conventions have crept in. If we were to redesign the building anew, many of those conventions would be deleted. People living in that building – the mathematicians – will mostly not discover by themselves how strange those conventions can be. Others looking on from the outside – for example physicists and economists with mathematical training – can recognize them sooner.

(3) This book shows that redesign of math will result into better mathematics.

(4) At the meta-level and by implication, this shows that there is something amiss with the current industry. Improvements are not easy to bring about and the price of current conventions and procedures is very high. Mathematics can be beautiful and contribute to confidence, competence and joy in life. If the mathematical industry does not serve its customers well, it fails its own stated objectives and may meet with public criticism.

Mathematicians will conventionally regard argument 3 as the only convincing one. They might be the first to recognize the improvements in mathematics and didactics presented here. Mathematical tradition clearly is an improvement from alchemy and astrology.

Most people will also tend to let the professors and teachers decide on whether these items are improvements indeed. It is tempting to conclude that the system then works: an improvement is proposed, it is recognized, and eventually will be implemented. This approach however takes a risk with respect to potential future changes. With the present failure and analysis on the cause we should rather be wary of that risk. We'd better regulate the industry of mathematics education in robust manner. And, actually, the mathematical examples presented here can be understood in principle by anyone with a highschool level of mathematics.

If this were a competitive market, where nobody can change the going price, then it would only *seem* chaotic and there would be the invisible hand working for the good. Instead, markets for ideas and education are regarded in economic theory as *monopolistic competition* and in some cases *natural monopolies* and such markets require more regulation. Many see regulation as a restriction of freedom but it actually liberates and enhances quality. Thus, we have cause to consider regulations and changing them.

## *Implementing change*

It would be unwise to leave the restructuring of the industry to the mathematicians themselves. They are not in the position to look at themselves from the outside. They cannot 'think out of the box'. Teachers and professors of mathematics can do their work with love and acuteness but they have not succeeded, internationally and jointly, to cleanse mathematics and the teaching of mathematics from cumbersomeness and irrationality. Instead, the math teachers, having been trained in their conventions, implement those conventions again and condition their students in the same. When students encountered problems and complained about them, they were not listened to and subjected to further conditioning. Mental anguish and even tears by damsels in distress carried no weight, mathematical convention was sacred and all blame was put onto the students and their supposed lack of mathematical capability.

Realism suggests that we have a system that actually works. Annually millions of students get their highschool diploma including some math, so apparently the system works to a high degree. Our advanced society could not exist otherwise. But, sobering, do graduates leave school with mathematics or is it only seeming 'mathematics' ? That it 'works' and that teachers of mathematics tend to be decent people is by itself no argument to neglect criticism. The evidence in this book carries some weight. Awkwardness and irrationality in 'mathematics' also have consequences for other subjects that use mathematics. We spill a lot of time and energy in education because of the state of mathematics. Many kids suffer. Those who pass their math exams actually are much stunted in their mathematical development. The economy suffers with such low development of mathematical knowledge, skill and attitude. It is rather impossible to quantify the loss and counterproductiveness.

Supposedly, as it is a problem that affects each nation, it would be a task for each national parliament to start the wheels of change. Parents are advised to write their representative. Parliament is not asked to determine the next digit of $\pi$ but to rearrange the institutional setting so that our kids get math without pain.

The suggestion causes people to raise eyebrows. People elect parliament but seem to dislike it and not regard it as a useful place to resolve bottlenecks. The present situation is a chance for parliament to enhance its standing. Decisive action on the failure of mathematics education will set an example.

# 4. Limitations to this study

While mathematics has its limitations, this book suffers some too.

### *Setting, experience, anecdote, bias*

Mathematics itself is international. I have taught mathematics for four years at the international college level with students from all over the world. Nevertheless the location was in Holland. From my own foreign exchange student year in California I know that American highschool is very different from the Dutch system. My observations will still be biased by necessity. Though the present discussion tries to be as general as possible my main experience is bound to create some idiosyncracies. Specific references to Holland will be reduced to a minimum. Holland might be used sometimes as an example however when this can be enlightening.

For example, there is now a discussion in Holland about the choice in elementary school between the algorithmic long-division and the "realistic math method" (pejorative "guesstimate"), where pupils are supposed to find the answer by trial and error relying on their understanding of the problem and self-creation of method. Clearly, teachers in secondary education suffer the consequences of what is done at the elementary level. But there is no joint responsibility and management of the whole column. Teachers at elementary schools appear to have problems with mathematics themselves. The Minister of Education allows a "Math C" profile level for their certification, which is not very much of math. Hence, those elementary school teachers may tell their pupils that mathematics is very difficult and not worth your effort. While the situation in Holland is not our focus, the example clarifies that it is advisable to consider the whole industry and to keep an open mind for the subtle influences between the ideal art and that industry.

### *Limitations exist internationally*

Braams (2001), on the evaluation of research into K-12 mathematics education:

> "A practicing scientist might think that reform efforts could, should, and probably would be guided by a respected body of research into what works and what does not, although within such a body of research there might still be significant differences in research focus, methodology and results. With that in mind I started looking for appropriate research, and this letter is a little report on my search. I'll say right away that the outcome has been entirely negative. (…) To be sure, there are plenty of efforts in mathematics education research. Many of them provide results that are of anecdotal and perhaps of inspirational value. Many appear to be tightly linked to a particular implementation of some reform, limiting their scientific standing. It really looks as if all the recent United States efforts in education research have not produced a single respected comprehensive study of the kind outlined above, let alone a body of authoritative research that provides firm empirical guidance for mathematics pedagogy.
>
> Fortunately we still have our common sense to guide mathematics education. Unfortunately (but it would take us too far afield to discuss it further here) present trends towards discovery-based learning and constructivist pedagogy seem as little rooted in mathematicians' common sense as they are rooted in education research."

With this in mind, I can usefully express that the method chosen here is logic. I draw information from my own experience and reading but, since this would be anecdotal indeed, all conclusions and advices are only based upon logic. And, OK, upon common sense.

I am aware of the Watkins (1995) paper on the US *Follow Through evaluation*, the Hattie (1999) meta-analyses in particular on the influences on student learning, the Anderson, Reder and Simon (2000) evaluation of applications and misapplications of cognitive psychology to mathematics education. Writing these lines I realize their dates. The point however is context awareness. While this book concentrates on what and how we are teaching when we are teaching mathematics, education is a rich context that will always have to be taken account of.

## 5. The order of discussion

While teaching I kept notes. When I grew aware of some regulaties in those notes the idea arose to collect them more systematically. From a list of potentially more cases some could be selected that were particularly useful for the purpose at hand: proving the need for change. I still feared that I had only issues and no unity. It appeared possible to categorize the notes into more unifying chapters. The regularities materialized but it still was a surprise to suddenly see how the perspectives themselves were linked. At some point the unity simply shone out and it became obvious that the whole should be presented to an international audience. This book retains that effect, you will have the same surprise. Though you miss out on the surprise to have to rewrite this Introduction.

This book has a didactic set-up. We already presented the main message. Now we get down to the evidence. We work from the small upwards to the more complex. The small issues should be fun and eye-opening. They prime the mind to become sensitive to the more complex. By allowing readers to digest the examples and arguments the overall reasoning has more chance to be understood.

The chapter on redesigning mathematics itself only gives summaries and then refers to the relevant sources elsewhere. However, the paper on derivatives has been rewritten and is now included as a new chapter of itself.

The first eight chapters number their paragraphs for easier reference to specific points.

This Introduction summarizes the book. A much shorter summary and condensed abstract are in the appendix, for backup.

*Now, however, forget about this Introduction.* Let us consider the education in mathematics afresh. Suppose you are a teacher or student facing the blackboard with some texts, formulas, tables and graphs. What to make of them ? Are they clear, how do we communicate effectively ?

# II. Issues of notation

## 6. The decimal point

The decimal notation was invented by Simon Stevin (1548-1620) who aspired at clarity. He would be upset about what is done with his invention. For decimal marker, the British use the point and the French use the comma. The ISO standard followed the French but since 2006 points are accepted notably for texts in English, see Baum (2006). To allow either a comma or a point is a standard, of course, but actually somewhat loose.

This book uses English. Conventionally we would use the dot and be done with it. At issue now is however to consider the matter from the angle of elegance with substance. Let us avoid getting lost by French – British disputes and diplomacies and let us try to determine what we want.

In Europe we see that textbooks use comma's, computers have to be set to comma's, but internet resources in English will use the point. Pupils and students apparently learn how to deal with it (or fade from view). But it is an awkward situation and weak students suffer a needless burden. Perhaps legal documents require a single format and we have to teach students to use that format. But it is not clear why a course in mathematics should suffer the inability of the legal world to adopt a single notation. The best solution remains that the world adopts one notation and be done with it.

There is already a mathematical standard application for the comma. For a two-dimensional point we use the notation $\{x, y\}$. This is clear for the point $\{2, 0\}$ but it gets less clear for $\{2,5, 4,32\}$ so that some start writing $\{2,5 ; 4,32\}$. English readers will not be familiar with these Byzantine consequences and it may open their eyes to the larger problem.

Hence it is best to use the decimal point.

It might be a compromise to use the dot raised to the middle of the text line, like in 2·5, as I saw this in the medical literature, but this is not advisable since there is no need to change a good practice in the English speaking world of science in general.

# 7. Brackets

Brackets belong to the most important symbols. Consider $(a + b)(c + d)$ if we did not have brackets. There is also the notation for a function $f(x)$. Thus $a(x)$ could be both a function and $a\ x$ … which is inconsistent (unless the function would be very specific and $a$ would remain non-numeric).

Most students can learn to deal with context-dependency and most would guess that $a(x)$ is a function. What about $a(c + d)$ ? Is this a multiplication or a functional expression ?

Some people might object "normally we don't write sums in argument, so your example of $a(c + d)$ is a crafted and irrelevant exception". But what about the differential $f(x + h) - f(x)$ ? Mathematicians should admit that they themselves are confused because of the ambiguity of $a(c + d)$ and they are irrational when they don't admit it.

Issues like these arose in the design of computer algebra languages. Computers are strict and require unambiguous input. The language *Mathematica* (my standard reference) chose to use straight brackets for functions, thus $f[x]$.

A standard reaction by mathematicians is (a) straight brackets are ugly, (b) it is only for computers. Hence, indeed, *Mathematica* later developed the "traditional form" such that the hidden input uses straight brackets for clarity while the display has round brackets.

The proper answer is: (ad a) what is ugly is to a large extent a matter of convention, (ad b) people are much like computers. There is no difference in getting stucked. For consistent thought, people require unambiguity too which is something else than saying that they are computers.

When confronted with $f(x)$ people can do more than computers and work around corners. There is the hidden rule that letters like $f, g, h, \ldots$ are conventionally used for functions so that the expression would be unambiguous. Or at one point $a$ is defined to be a function so that $a(x)$ can be recognized. It are rules like these that are not explicitly mentioned in textbooks but that students have to figure out if they want to pass. "Read carefully," the standard mathematician might say. The key point remains that this is exactly that: *working around corners*. It puts a burden on the weaker students to acquire that additional competence. They are told that mathematics would be clear and consistent, they are confronted with a clearly inconsistent notation, and when it gives them a hard time then they are told that they are themselves to blame.

The supposed esthetics of the round brackets in the notation of a function is merely conditioning – and that conditioning is so strong that a software firm went a long way to satisfy it. A solution might be to design esthetic brackets that still look different.

Some mathematicians might admit to all of this but continue to torture their weak students, using the argument that they ought to be able to read conventional math papers. Now, clearly, weak students will not tend to read such papers anyway while the smart students who will read the historical papers of Euler etcetera would well be able to adapt on the spot.

PM. The meaning of $f(x)$ thus depends upon context frames. Perhaps it is a key mathematical skill to be able to switch frames quickly – for example since notation may frequently be ambiguous anyway. That skill is no explicit target in math education. We will return to this.

# 8. Brackets (2)

Conventionally, the notation ($x, y$) can be used for the two-dimensional point and for the open interval from $x$ to $y$. In Holland this ambiguity is solved by using <$x, y$> for the open interval. As [$x, y$] is the closed interval, in France ]$x, y$[ gives the open interval.

If ($x, y$) is a point then something can be said in favour of using $f(x, y)$ for the function on that point. Unfortunately this breaks down for the single dimension $f(x)$ and the other use of round brackets.

The straight brackets in $f[x, y]$ in *Mathematica* might cause a confusion with the closed interval. Hence *Mathematica* has the notation / object Interval[$x, y$] which is a bit inelegant. With function call $f[x, y]$ we might expect [$x, y$] to be used for the two-dimensional point but instead *Mathematica* uses $\{x, y\}$. For this notation, *Mathematica* has an option to distinguish ordered (default) and unordered sets. Potentially there is a difference between $f[x, y]$ and $f[\{x, y\}]$.

We do not need to resolve these issues here. We merely indicate the problem of the consistent use of brackets and let us hope that an international committee finds a solution.

In this book we adopt *Mathematica's* notation of the two-dimensional point $\{x, y\}$.

# 9. Fractions

There is the expression *two-and-a-half* or 2½  alongside the expression 2√2. The first is the addition 2 + ½ and the latter is the multiplication 2 * √2. A blanc is multiplication and thus there arises the following issue.

Try to spot when 2 times ½ = 1 turns into 2 plus ½ = 2½ . How large can the space be ?

| | | |
|---|---|---|
| $2 \qquad \text{½}$ | $2 \quad \frac{1}{2}$ | $x \qquad y$ |
| $2 \quad \text{½}$ | | $x \quad y$ |
| $2\ \text{½}$ | $2\frac{1}{2}$ | $x\ y$ |
| $2\text{½}$ | | $xy$ |

Somewhat teasingly too this book will write 2 ½ because I actually prefer a bit more space inbetween but clearly this would be confusing since it could read as 2 * ½.

The improvement would be to consistently write 2 + ½ and to stop using 2½. In the same way we already write 2 + √2. In intermediate steps we would often use 5 / 2 rather than 2 + ½ but the latter is the best presentation of the end result.

This would fit not only with notation in general but also with the actual calculation, e.g. of 2½ * 1¼  = (2 + ½) (1 + ¼).

In computer languages *xy* stands for a single variable in the same way as 34 stands for a single number. Mathematical textbooks however can write *a (b + c) = ab + ac* where the latter are multiplications. It is better that they drop this habit and insert a blanc. For example, with multiplication *ab* and *a* = 3 and *b* = 4 children play by calculating 34 = 3 * 4 = 12 = 1 * 2 = 2.

Conversely, one can argue that the use of smaller fonts unambiguously indicates fractions, and that writing a number directly close up to the fractional line (which would anyway be normal for larger numbers) can unambiguously mean that this is addition. On this ground there is no need for change. The latter might be valid – as apparently people with mathematical ability learn to switch frames when we compare 2½ and 2√2.

However, a lot of math education time is wasted by the current notation of fractions. (a) The switching of frames requires mental space and energy without a contribution of substance. (b) While textbooks have neat typesetting with larger and smaller fonts, and can parse neatly with and without intermediate (half-) blancs, the handwriting by students is less accurate and frequently causes confusion. (c) The handwriting by teachers may not be as neat as well but then a hidden algorithm is used: "this calculation should give 2.5 and thus we write 2 1/2 and then we stop thinking since we have reached the end of the calculation" – while proper reading should give outcome 1.

Of course, at the grocery students may see the notation 2½ and thus will have to know what it might mean. But it suffices to explain in class that this is an old notation. Draw a red square around it, explain that it means 2 + ½. But don't let them use it themselves.

Fractions are important. Only the current notation is no good.

# 10. The cult of the radical sign

When it has been clarified that 2^2 = 4 then it is straightforward to explain that 2 = 4^(1/2), and subsequently develop exponents in general. This direct development of exponents is a clear and straightforward route. Students can use good practice on this. It takes time and energy to learn to write the exponent at the right height, fractions already were a bit difficult and the notion of this type of inverse has to sink in.

Instead, school mathematics has developed the cult of the radical sign.

The latin word for root was *radix*, it was abbreviated to *r,* this was written as √, and subsequently it got generalized into the *p*-radical sign $\sqrt[p]{x}$ . Teachers of mathematics apparently seem unable to imagine a life without this sign. Students are submerged in its use and tricks.

Apparently the apotheosis of this cult is that students are told that $\sqrt[p]{x^q} = x^{q/p}$. But we could have gotten there also via the direct route in the first paragraph, with the aside that $x^{q/p}$ is sometimes written by some people as $\sqrt[p]{x^q}$ .

Admittedly, the square root sign is useful in two-dimensional geometry, notably with fast and clear labelling of the lengths of sides using Pythagorean theorem. And the notion of a “root” is fine too. But apart from that it is clutter.

The radical sign has created a life of itself, outside its realm of usefulness, and with counterproductive results. For example, it is considered ugly or unconventional to write $\sqrt{}(a + b)$ with brackets and subsequently a lot of time is spent in having the pupils extend the upper root line to the end of the expression under the root, with hopefully a small drop to indicate that the end has been reached. For example, the equivalence of *p*-roots and exponents does not sink in fast and students lose time in translating the one to the other, and trying to figure out whether this also means that the properties are transferred. Eventually, good students understand that the radical sign is merely a different way of writing fractions for exponents – but really, what is the mathematical insight ? What sense of wonder is this supposed to generate and how is this supposed to contribute to the motivation to learn more ?

It is a valid argument that the notion of “root” best sinks in with the use of a symbol that explicitly is called “root”. Indeed, use √2. Without bound though this is like believing that the notion of an accident best sinks in with the use of a symbol that says “accident” and is printed on all pictures of an accident. No, this confuses convention and efficiency. A photographer may use stamps “accident” and “art” to categorize his collection but this is not how the pictures and understanding came about.

In judging the cult of the radical sign, we compare the gain in knowledge, skill and attitude with the investment of time and energy. Since the exponential notation has to be mastered as the principal notation anyway, the use of the radical sign adds little. It has its cause mainly in convention. Thus, the radical sign can be kept for (a) historical reasons, (b) the name “root” and (c) fast and clear notation in geometry. But there it should stop.

Let us eradicate this cult of the radical sign.

# 11. Pi or theta

The mathematical symbol $\pi$ (Greek "pi") is defined on a circle as the ratio of the circumference to the diameter. Angles are measured in 360 degrees or $2\ \pi$ radians.

It is useful to define $\Theta$ (Greek capital "theta") as the ratio of the circumference to the radius. Thus $\Theta = 2\ \pi$.

The advantage of using $\Theta$ is twofold:

(1) It is easier to think in terms of whole circles than half circles. As $\pi$ radians are an angle of 180 degrees, or a straight line, it carries with it a notion of non-completeness. Using $\Theta / 2$ carries the notion of only a half turn. Indeed, the name $\pi$ is taken from "perimeter" and it has succeeded only half.

(2) There is more outward clarity on the linkage with calculus. The integral of $x$ is $\frac{1}{2}\, x^{2}$. With radius $r$ the circumference of a circle is $\Theta\ r$ and its surface is $\frac{1}{2}\ \Theta\ r^2$. Admittedly, when you look for it then the calculus relationship can also be seen when using $\pi$ but the advantage of $\Theta$ is that you don't really have to look for it since it tends to stand out more by itself.

Independently, Palais (2001ab) came to the same view (see also his animated website [1]). Palais introduced the three-legged  but this is bound to cause writing and reading errors, let alone confusion, and I remain with $\Theta$.

Here it suffices to point out the mere benefits of using $\Theta$. We will return to trigonometry later on when discussing the measurement of angles, see page 58.

PM 1. Some students confronted with $2\ \pi$ have the tendency to complete this by applying the calculator and returning 2 * 3.14… = 6.28… With $\Theta$ it would be easier for them to stop, and wonder whether the exact $\Theta$ is required or the numerical approximation. However, they will meet much of the same problem when they are confronted with $\Theta / 2$. Hence this issue must be dealt with separately.

PM 2. Rather write $x\ \Theta$ than $\Theta\ x$. Current convention is to write $2\ \pi\ r$ but there is advantage in recognizing $\Theta$ as an indication of the full turn as a unit of measurement.

[1] http://www.math.utah.edu/~palais/cossin.html

# 12. Text, function, table and graph

When exploiting the linkages between text, function, table and graph, the current convention requires an unnecessary switch in orientation.

For the graph, we use $x$ for the horizontal axis and $y$ for the vertical axis.

For the table, convention puts $x$ on top and $y$ below (with no further explanation).

This layout of the standard table causes a switch in orientation with respect to the graph. Students have to glance from the numbers in the table to the graph, check values and, now, in addition, have to translate *up* to *down* and in reverse.

Why ? Merely because of the convention that text lines in a book run from the top of the page to the bottom of the page, and that for functions the $x$ values cause the $y$ values and hence come first. There isn't more to it. But this thoughtless convention comes at a price. Young brains that have few memory places and that need to learn to compact their concepts and actions would be served with the same orientation. Also, the distinction between cause and effect does not fully correlate with the order of the lines on a page. It is more instructive to create a table with an explicit legend, see **Table 1**. Suggestion: try this on **Figure 2**.

**Table 1: Improved layout of the table used to draw a function**

| Effect $y = f[x]$ | $f[0]$ | … | … |
|---|---|---|---|
| Cause $x$ | 0 | … | … |

In addition, when a slope is determined with $\Delta y$ / $\Delta x$ then the current convention with $x$ on top and $y$ below causes another reorientation. The format in **Table 1** retains the orientation of numerator and denominator.

PM 1. The latter might be objected to with the argument that it allows thoughtless execution of a (simpler) algorithm. Of course we want students to know what they do. Eventually. But they have to learn to do too. The algorithm is best learned if it isn't cumbersome and actually supports learning. There is great value in learning to perform the algorithm and then look back and wonder: "OK, they told me what it was. What was it again that they said ?"

PM 2. See page 46 for the important issue of *text*.

PM 3. A good standard format is to put text, formula, table and graph on one single page, in four blocks, in that order. Textbooks tend to put the items at random, going for the flashy.

# 13. Verbs versus nouns

*To ride* is a verb and *ride* is a noun. Riding is an activity and a ride is something completed, an abstraction of the activity. Clearly, a ride also implies an activity but there remains that subtle difference.

Computer programmers have noted this distinction early on. In the language Algol a statement "X := 5" means that variable X is set to the value of 5 while a statement "X = 5" would be a logical expression that evaluates to True iff X is 5. Earlier in history there was the distinction between the potential and actual infinite. Above, however, we saw that 2 + ½ was seen (also by teachers) not as the result itself but as an instruction to further simplify to 2½. We saw that students had to learn to recognize 2 π as a result on itself instead of an instruction for continued calculation.

The distinction between verbs and noun can be stated mathematically as the distinction between a procedure / algorithm and its outcome / result.

Mathematics is full of switches between verbs and nouns. It is a pity and also rather a shame that this is not pointed out didactically as frequently. As a teacher I noted that pupils and students have a hard time to deal with these switches. There are two conclusions. (1) The first is the general insight that educators in math must pay more attention in general to this distinction and how it affects learning by students. (2) The second is that it will help to introduce innovations at particular spots to support this.

PM 1. Pierre van Hiele and Dina van Hiele-Geldof developed a theory of learning mathematics, by concrete, ordering and abstract levels:

> "the process of learning proceeds through three levels: (1) a pupil reaches the first level of thinking as soon as he can manipulate the known characteristics of a pattern that is familiar to him/her; (2) as soon as he/she learns to manipulate the interrelatedness of the characteristics he/she will have reached the second level; (3) he/she will reach the third level of thinking when he/she starts manipulating the intrinsic characteristics of relations." (FU wiki (2008))

Textbooks should better recognize the points where level jumps tend to occur or are required to occur. The verb / noun distinction is such a point. Sometimes the noun will be the abstract of the verb, sometimes in reverse. The individual learning process may differ from the reconstruction of a general process in more standardized terms.

PM 2. Independently, Gray & Tall developed this distinction into the idea of a "procept". Tall (2002) seems to embed the "procept" into the 2nd Van Hiele level:

> "The Symbolic-Proceptual World of symbols in arithmetic, algebra and calculus that act both as PROcesses to do (eg 4+3 as a process of addition) and conCEPTs to think about (eg 4+3 as the concept of sum.)" (Tall (2002))

I have a small problem with this use of vocabulary, in that a "concept" is not necessarily static and may well be a process too. It is not necessary to limit the distinction between verbs and nouns to symbols only. It is not entirely clear whether it is really useful to use a new word "procept" to indicate that verbs and nouns are connected, and that processes hopefully give a result and that results tend to be created by processes. That said, the Gray & Tall papers remain an important source.

# 14. Verbs versus nouns – square root

A key example is the square root, for example √2.

The equation $x^2 = 4$ has two solutions, $x = 2$ and $x = -2$. At this stage in the curriculum, students are not aware of the distinction between a function (for each $x$ there is a single $y$) and a correspondence (for each $x$ there may be more $y$). It would be better if they were introduced to this distinction. The solution of $x^2 = 4$ would be easier with the correspondence Do√ or "take the root" such that $x^2 = 4$ solves into $x$ = Do√4 => {2, -2}, the solution set. This inverse can be shown by mirroring $x^2$ along the line $y = x$.

In the current situation punching in √4 on the calculator looks like a procedure and the students get confused (a) between the noun / number and verb / procedure, and (b) between solving and simplifying. Students are inclined to take the square root of 4 and to write 'solutions' √4 = 2 and √4 = -2, which they check by squaring both sides. In mathematical convention this is false since √4 has to be a nonnegative number. √4 can be simplified to 2, and simplification is not solving. Thus √4 = -2 is a false statement.

We can also write Sim√4 = √2 to distinguish the number √2 from the procedure of simplifying the square root of 4.

For the instruction in the current situation it would help to write the solution of $x^2 = 4$ as $x = \sqrt{4} = 2$ or $x = -\sqrt{4} = -2$. Curiously, this is not really done much. In some books we can see that functions $f[x] = x^2$ and $g[x] = \sqrt{x}$ are discussed but with little discussion of their relation, and in other books there is more discussion but it tends to be confusing.

Currently the radical sign denotes the passive number and equation solving gives the active process. In itself it is a strong distinction. But expressions like "take the root" must then be avoided (which is somewhat difficult since roots are used).

> PM 1. Students find it hard to distinguish between the number notion and the procedure that is available on their calculator. Mathematics teachers think that students are confused between exact and approximate results but here it would rather be the distinction between verb and noun. If you recognize √2 as information and stop seeing it as a command then *there you are*. See page 32 for the issue of approximation itself.

For exponents in general we would have DoExp[$x$, $1/n$] so that there is a distinction between the noun / number $4^{1/2} = 2$ and the verb / process DoExp[4, 1/2] => {2, -2}.

> PM 2. It might actually be a suggestion to define $\sqrt[p]{x^q} \equiv$ DoExp[$x^q$, 1 / $p$]. This means that the radical sign becomes the solution operator instead of the completed number. That implies that $\sqrt[2]{4}$ = Do√4 => {2, -2} so that this sign differs from √4 = 2. It remains to be seen whether the profession is willing to make the change. Likely Do$\sqrt[p]{x^q} \equiv$ DoExp[$x^q$, 1 / $p$] is a good choice then.

# 15. Verbs versus nouns – division

Western mathematics had to wait till 1200 AD before the zero came from India via Arabia together with the Arabic digits – where both “zero” and “cipher” are jointly derived from the Arabic “sifr” = “empty”. Arabic numerals are easier to work with than Roman numerals, e.g. try to divide MCM by VII, yet this advance came with the cost that the zero caused a lot of paradoxes. Western math solved most problems by forbidding division by zero. However, we might also try some algebra.

Dijksterhuis (1990) suggests that the ancient Greeks did not develop algebra – and subsequently analytical geometry – since they used their alphabet to denote numbers. Thus $\alpha + \alpha = \beta$ already had the meaning $1 + 1 = 2$, whence it would be less easy to hit upon the idea to use $\alpha$ as a variable. We too would consider it strange to use e.g. 15 as a variable ranging over $-\infty$ to $+\infty$. This explanation is not entirely convincing since the Greeks did use names like “Plato” or “Aristotle” and thus might have used a name to denote a variable – like “Variabotle” – though this then should not be a number again. Notation clearly was one of the obstacles to overcome. Let us now assume that we are familiar with algebra and that someone announces the new invention of the zero.

Let us distinguish the passive division result from the active division process. In the active mode of dividing $y$ by $x$ we may first simplify algebraically under the assumption that $x \neq 0$ while subsequently the result can also be declared valid for $x = 0$. This means extending the domain, and not setting $x = 0$.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined as the above, for $x = 0$. Mathematicians will tend to regard $y / x$ as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Thus the active notion would be new. Denote it as $y // x$. Others who aren’t professional mathematicians will tend to take $y / x$ as an active process and they might denote $y // x$ for the passive result. All in all, it would not matter much, since we might continue to write $y / x$ and allow both interpretations depending upon context. In that way the paradoxes of division by zero are actually explained, i.e. by confusion of approach or perspective.

### *Definition*

To make this strict, let $y / x$ be as it is used currently by mathematicians and $y // x$ be the following process or program:

> $y // x \equiv \{\ y / x,$ unless $x$ is a variable and then: assume $x \neq 0$, simplify the expression $y / x$, declare the result valid also for the domain extension $x = 0\ \}$.

Thus simplification only holds for variables but not for numbers. Thus $x // x = 1$ but $4 // 0$ generates $4 / 0$ which is undefined. $x / x$ is standard undefined for $x = 0$.

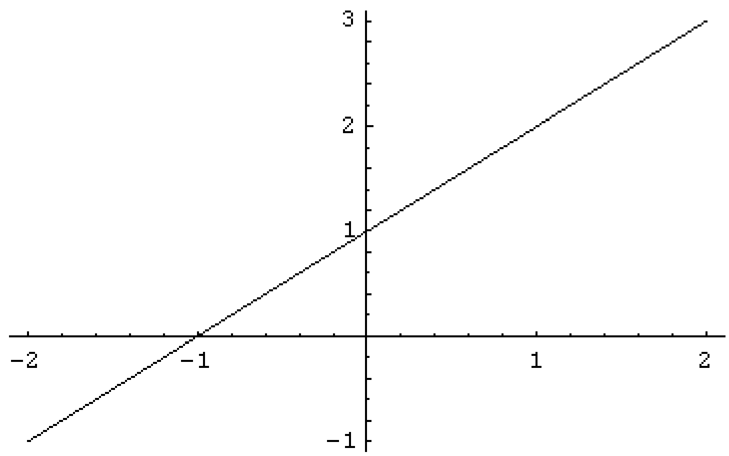
### *General application*

There is no need to be very strict about always writing “//”. Once the idea is clear, we might simply keep on writing “/”. An expression like $(1 - x^2) / (1 - x)$ would be undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 2**. The current teaching and math exam practice is to use the division $y / x$ as a hidden code that must be cracked to find where $x = 0$ but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also skip the undefined point, see **Figure 2**, requiring us to give the special point if we really want a hole.

**Table 2: Symplification and continuity**

| *Traditional definition overload* | *With the dynamic quotient* |
|---|---|
| $f(x) = (1 - x^2) / (1 - x) = 1 + x$ if $x \neq 1$<br>$f(1) = 2$ | $(1 - x^2) // (1 - x) = 1 + x$ |

**Figure 2: Graph of $(1 - x^2) // (1 - x)$**




### *Subtleties*

The classic example of the inappropriateness of division by zero is the equation $(x - x)(x + x) = x^2 - x^2 = (x - x)\, x$, where division by $(x - x)$ causes $x + x = x$ or $2 = 1$. This is also a good example for the clarification that the rule that we should never divide by zero actually means that we must distinguish between:

- creation of a fraction by the choice of the *infix* between $(x - x)(x + x)$ and $(x - x)$
- handling of a fraction such as $(x - x)(x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as $2 = 1$, the second is only the application of the rules of algebra. In this case, $x - x = 0$ is a constant and not a variable, so that simplification generates a value Indeterminate for both / and //.

Also $a (x + x) / a$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $a (x + x) // a$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to write $a = x - x = 0$, since then it is an instant and not a variable.

Another conclusion is that calculus might use algebra and the dynamic quotient for the differential quotient instead of referring to infinitesimals or limits, see page 75.

## *Requirements*

Clearly, mathematics education already takes account of these kind of aspects in some fashion. In early excercises pupils are allowed to divide $2\,a\,/\,a\;=2$ without the definition overload. At a certain stage though the conditions are enforced more strictly. The topic of discussion is not only that this stage can be a bit later but also that the transition can be smoother, also for the rest of the education, by the distinction between / and //. For the mathematically inclined pupils or students graduating at highschool one would require that they are aware that $x\,/\,x$ is undefined for $x = 0$ and that they can find such points.

# III. Opaque or confusing terms

## 16. Logarithm versus Recovered Exponent

Around 1600, Simon Stevin created many terms in the Dutch language that better clarified the Greek and Latin phrases of then-traditional mathematics. For example, for ‘mathematics’ he coined the Dutch word “wiskunde” – meaning the art of certainty, as “mathesis” means “what we have learned”. Nowadays, with probability theory and statistics, the Dutch would also need the word “giskunde” – the art of uncertainty. What anyhow remains is that it helps to use self-explanatory terms.

John Napier’s term “logarithm” is singularly opaque. My suggestion is to use the term “recovered exponent”. [2]

With $10^3 = 1000$ the exponent 3 disappears into the result 1000, while it is recovered by the operation ${}^{10}\log(1000) = 3$.

Instead of Log[1000, 10] = 3 we would write Rex[1000, 10] = 3.

PM 1. Current teaching practice would be to use Log[*x*] with standard base 10 and then introduce Ln[*x*] with standard base *e.* This reflects the phenomenon that it is cumbersome to continuously write the base. Indeed, some graphical calculators curiously don’t have Log[*x, b*] but require the use of Log[*x*] / Log[*b*]. Didactically, though, it would be wise to start with Rex[*x, b*] and continue writing the base. This helps students in realizing that the function is defined with respect to that base. Eventually they see that it is cumbersome to continuously write the base and use the shorter Rex[*x*] with default base *e.*

PM 2. Dutch textbooks are prim on *e.* It is only presented in grade 11. This compares to the equally special number π that is introduced much earlier. I would suggest that there remains a difference between being able to ride a bicycle and explaining how in terms of Newtonian physics. (See page 48+.)

[2] Dutch: “teruggevonden exponent” tge[*x*]

## 17. National idiosyncracies

There are idiosyncracies that differ by nation and that cannot be discussed in general but only by example. Each nation would benefit however by cleaning up their clutter.

For example, in Holland, the expression $2 < 3$ is translated inaccurately as “two is smaller than three” while the English language is accurate with “two is less than three”. The Dutch language at school confuses size with order. Dutch students get into problem when considering $-100 < 3$ where $-100$ clearly is less than 3 but not smaller in absolute size.

The Dutch curiously have a good alternative. Do not say “twee is kleiner dan drie” maar “twee is minder dan drie”.

Historically it can be understood, since Dutch grandchildren are “kleinkinderen” which expresses order rather than size. But it is equally clear that we better avoid this since this usage is quite exceptional in the Dutch language.

Once I attended a class given by an English math teacher who explained how to distinguish the various polygons by counting their sides (triangle, square, etcetera). Apparently this was not a didactic gimmick but he had survived his education himself by not knowing that the Greek *gonos* means *corner*. Quite likely the Greeks had already discovered that it may be didactically easier to count angles anyway since the pointy bits stick out so clearly. Perhaps it is an idiosyncracy of the English language that so many of the opaque Greek and Latin terms have survived. It causes great pride in the breasts of the Greek but it may not really help the English pupils. One may suppose that there have been English variants of Simon Stevin who haven’t had the impact and it likely would be very beneficial to overcome some needless conservatism here. Admittedly, English since William the Conqueror and William Shakespeare contains both the courtly French and the popular Anglo-Saxon which adds to the richness of the language. Mathematics in the English speaking world would benefit from using Shakespeare’s example and use more popular terms alongside the lofty phrases.

# 18. The vertex of a parabola

Sullivan (1999:496) defines the vertex of the parabola from the intersection of the parabola and its axis of symmetry. Angel (2000:594) has: "The *vertex* is the lowest point on a parabola that opens upward or the highest point on a parabola that opens downward". The latter definition avoids the opaque terms *concave* (hollow seen from below, h-shaped) and *convex* (bulging seen from below, b-shaped).

Both definitions still take a risk on *vertex*. Mathematicians often grab a word from the language soup and stick it onto their own well-defined notions. It is dubious whether that is the right procedure. The vertex of a parabola is mathematically well-defined but the general notion might be a bit confusing. The English language itself is a bit ambiguous about what a "vertex" means.

Hornby (1985) has:

> "highest point; top; point of a triangle, cone, etc. opposite to the base".

Hornby uses the adjective "highest" which suggests an orientation. This is not really the mathematical intention. The mathematical instruction and the normal English instruction are inconsistent. Angel's students who consult Hornby might want to look for the highest point also in a parabola that opens upward. Sullivan's students may see it as a property.

For the plural *vertices* Hornby refers to *vertex*, correctly implying e.g. that a triangle might have multiple highest points (notably when one angle is the base), but digressing from the mathematical usage that a vertex in general may be just a corner (or even be defined at liberty depending upon the subject).

Partridge (1979) has etymologically:

> "L. uertex, ML v-, a whirl, e.g. a whirlpool, hence, app from a supposed whirling centre, the pole of the heavens, hence a summit (e.g. the crown of the head), the top or crest (…)"

All ambiguity can be solved by using the word "turning point".

Dutch textbooks use the label "top" to indicate the vertex of a parabola. We can imagine indeed that a hat or cone has a top, whatever the orientation of the object. However, someone may hold the hat upside down, ask you to indicate the top, and thus trick you with the ambiguity. Dutch textbooks do not use the English distinction of opening upwards or downwards but put more emphasis on the orientation by using the distinction between "mountain" and "valley" parabolas. In Dutch highschool mathematical lingo there are *valley parabolas* that have a *top* – while everyone knows that only mountains have tops and valleys have bottoms. The Dutch thus do much worse than the English. Dutch math teachers and exam requirements succeed in mixing up two analogies without noticing that it creates lunacy and increased problems for their pupils. They must be applauded for their wish to avoid the terms *concave* and *convex* but they have not been sufficiently critical on how they did this.

It is advisable to follow Shakespeare and mix lofty language with the popular, so that we indeed can pick pieces from the language soup. But we have to remain critical in picking the right pieces to avoid confusing associations and conventions.

# 19. Exactness and approximation

One would hope that *exactness* would be an exact notion. Textbooks still can create some vagueness.

The popular story, true or false, is that Pythagoras thought that everything in the cosmos could be expressed in numbers and all numbers again in ratios (fractions), but was shown, by use of his very own theorem, that this does not hold for $\sqrt{2}$. It is a wonderful story since it shows the power of proof. That someone apparently got murdered for leaking the secret also shows human nature. The story clarifies that $\sqrt{2}$ is the exact number and that it can only be approximated, by fractions or decimals.

The standard story is also that fractions would be more exact than decimals. For example, 1/3 is more exact than 0.333…

Issues however get mixed up. The proper distinction is between = and ≈. Math textbooks persistently use the equality sign where they better use approximation. Let us observe:

(1) The number 0.25 is just as exact as 25/100 and only written differently. The number 0.25 is not simplified to ¼ but is simplified to a decimal form so that it is clearer in relation to other decimal forms. For example 0.25 and 0.20 compare a bit better than ¼ and 1/5.

(2) If $a = 1/3 + 10^{-5}$ then 1/3 is only an approximation of $a$. Thus we cannot hold that 1/3 is always the exact number.

(3) Obviously, $b = 1/3$ is an exact number seen just by itself. Thus the approximation of $a$ is an exact number. But only $a \approx b$.

(4) $a \approx 0.333$ differs from $a \approx 0.333\ldots$ with the necessity of approximation in digits.

Students have a tendency to regard 0.333… as more exact or accurate than 1/3, and 3.14… as more exact than π, likely since the digits better relay where the number is located on the real axis. This tendency is pervasive. This is not a simple issue but reflects the difference between engineers and pure mathematicians. An engineer will use $\sqrt{2}$ in intermediate steps, and rejoice when it can be eliminated to simplify a result, but when $\sqrt{2}$ turns up in the final answer then the engineer wants to know where it is at. Students with insecure mathematical skills will resort to piecemeal-engineering and use the calculator on $\sqrt{2}$ in intermediate steps as well.

This is an issue of sensitivity to what language means. The "exact sciences" are not just mathematics. The percentage of engineers is much larger than the percentage of pure mathematicians. Civilization produced economic growth when the engineers liberated themselves from the reign of the pure mathematicians. My inclination is to let them have the word "exactness" and then use the word "perfect number" for 1/3, π and $\sqrt{2}$ that in decimals can only be approximated. Perfection better expresses what is intended. Ask your students. Apologies to the small branch in Number Theory that already employs the "perfect number" label and that will have to switch to "ancient Greek perfect number".

PM. Dutch textbooks use the phrase "solve algebraically" as equivalent to "solve exactly". The phrase derives from the choice between the use of pen and paper versus the use of the graphical calculator. This interpretation of "algebra" differs from widely held notions about algebra. The calculator may also use algebraic routines and even computer algebra. The phrase is inaccurate, arbitrary, pedantic and superfluous, and can be ditched.

# 20. Slope of a line: more words for the same

The inclination of a line can be measured by its angle or functions thereof, as we already know from trigonometry. If we denote the line with a formula within a system of co-ordinates then the tangent is the useful measure. Take two points on the line and then deduce that the coefficient of $x$ is $\Delta y / \Delta x$ which is that tangent. It can be mentioned that this particular form of the tangent is also called the difference quotient. Tangent and difference quotient are the same, this is called *slope* to distinguish it from *angle*, and to distinguish this oriented tangent from the general notion. The slope is also the average increase over an interval, which average must be the same everywhere along the line.

The last paragraph uses different terms and aspects. However, while the above seems like a clear and straightforward approach, the textbooks create a wildernis with entirely different compositions and accentuations. Notably:

(a) slope, defined by Angel (2000:426): "The *slope of a line* is a ratio of the vertical change of the horizontal change between any two selected points on the line"
(b) difference quotient, defined on itself as $\Delta y / \Delta x$.
(c) average increase over an interval, as a general notion
(d) tangent, defined as the ratio of the length of the opposite side to the length of the adjacent side
(e) for Dutch textbooks: coefficient of direction, defined as the coefficient of $x$ in the formula $y = a\, x + b$. Which is also the definition of "slope" by Sullivan (1999).

We see the same terms arise as in the first paragraph above but with clear distinctions: (1) The idea that degrees could be used is not mentioned, (2) There is no link to the known concept of the tangent either, (3) The difference quotient is created out of the blue as a supposedly independent concept, (4) The latter may happen with the average increase as well, (5) There can be idiosyncracies like "coefficient of direction", (6) These terms and properties can be used in all kinds of combinations.

It was a discovery when the Morning Star appeared to be the Evening Star – i.e. both the planet Venus. This was a question on nature. It must be doubted whether multiplicity must be increased in the realm of the mind to provoke similar sensations of discovery. A richness in concepts can help understanding but there is also a danger. Overabundance has some curious effects:

(A) A student may think that something is new, but not see that it is old. The student does not understand it, is not saved from misunderstanding, and has more material to create new blockages to understanding.

(B) A student may think that something has to be new, but only see the old. The student concludes not to understand it (and indeed loses understanding).

(C) A student may think that something has to be new, but only see the old. While understanding, he feels cheated for his time and energy, and loses motivation.

A teacher may entertain her students a whole year with concepts that are essentially the same and most of the students won't notice anything. Is this education ? Would the notion of "education" not require that you explain that they are being entertained with concepts that are essentially the same ? And for those few who otherwise notice the lack of advance: will they not feel cheated ?

My colleague educators will hesitate. Sameness will be 'obvious' for us but this may only be because of our training. The sameness of (a) to (e) may be explained in perhaps less than five minutes to a novice to these terms but then it will be passive knowledge only and for a limited period only. It requires the immersion into the various aspects to acquire active knowledge, skill and attitude. Multiplicity serves a purpose.

The current approaches have some logic as well. With (a) the definition then (b) is the implementation, and a useful stepping stone to the differential quotient. Then (c) is an interpretation that helps to understand what the slope means. The use of the tangent (d) might be seen as confusing. Better not discuss the tangent since some students will start to calculate the angle and say that this is the slope. Finally (e) uses a formula and thus uses an entirely different formalization than (a).

Maybe. Let me refer to the first paragraph on the former page. Check the logic and how it hangs together.

There are two empirical hypotheses that can be tested in practice:

(i) The current axiom: Students have to be exposed continuously with the various perspectives, even when those are essentially the same, even while students are not told that they are essentially the same, in order to challenge their brains to grow and to bring about the required integration of concepts.

(ii) My conjecture: Those brains are growing and adapting anyway and under smart exposure to the material it is only a matter of time before they will bring about the required integration of concepts.

Most likely, there are different groups (i) and (ii) so that it is rather a matter of determination what student falls into what group. Most likely there are different degrees of "smart exposure" as well. Admittedly, the latter is a vague concept but in the context of this book it would be clear what I intend. For example, use the first paragraph. For example, when lines and slopes are used formally in other subjects than mathematics then the math class can save time on practice. However, the force of the argument is that current practice is too far into the (i) direction while it could move towards (ii). Instead of beating students about the bush we better streamline the information and offer them the opportunity to work on the steps that are not sufficiently clear yet.

Current practice has grown over time. It may be thought that (ii) has been tried in the past but has failed, no "smart exposure" has been found, so that experience has shown that students have to take the long circuitous route. I doubt that this is true. There may of course be particular effects. When the chapter on trigonometry contains all kinds of complexities that many students turn averse about, then it might be a psychological gimmick to start lines and slopes with newly defined difference quotients that seem entirely different. The alternative course would then be to rather save the complexities of trigonometry to a later chapter. Likely there are all kinds of options here that have not been tried yet.

Hence: (1) The current approach on the *slope of a line* is a mess, (2) There is a lack of evidence and documentation that the current approach would be the best, (3) There is a clear alternative, stated in the first paragraph on the former page, that purely on logical ground should rather be the null hypothesis and the basis to start collecting the evidence. (4) This example on *the slope of a line* is an instance of a more general phenomenon. (5) Personally, though, I would actually prefer to use "inclination" and "slope" to be equivalent, and allow these to be measured by either angle or gradient (tangent).

# IV. Breaking the chain of understanding

## 21. Inconsistent names for parameters

Textbooks often use $y = a\, x + b$ for the line and $y = a\, x^2 + b\, x + c$ for the parabola. Do you spot a possible source of confusion ?

Might it not be an idea to use the line $y = b\, x + c$ instead ?

It is only a small difference, and mathematically irrelevant, but it would didactically help students who associate $a$ with the slope. Are we to make life difficult for them and test their real understanding just now and use that as a criterion for advance, or are we going to help them and allow understanding and skill grow over the years ?

It would be an advantage to be able to teach that a parabola with $a = 0$ reduces to a line, without losing time on showing by various substitutions that it does indeed. The Quadratic Formula cannot be used when $a = 0$ and it can be pointed out to students that there exist tricky test questions where they have to test on this condition.

Why do those textbooks use the notation $y = a\, x + b$ ? Most likely because of the order of the letters of the alphabet and the fact that the line is presented before the parabola.

(There is no direct relation in terms of derivative or integral, as for example holds for velocity $v$ and acceleration $a$.)

Another textbook uses line $y = m\, x + c$ instead. This still does not link up to the parabola in a straightforward manner.

Perhaps my colleague math teachers will pose that students have to learn that parameters can be indicated with different letters. In that case, my response is that we should not confuse two learning objectives. The relation between a line and a parabola is one thing. Dealing with arbitrary letters is another thing. Indeed, for the latter it would be useful to see more Greek letters.

PM. It is an option to use standard order $a + b\, x + c\, x^2 + d\, x^3 + \ldots$ Of course the $c$ stands nicely for “constant”. Decisions, decisions.

# 22. The line subservient to the function

When the line is presented as $y = b\ x + c$ then we need $x = m$ as a "special line", namely the vertical line with an undefined or infinite slope.

The proper general formula rather is $k\ y = b\ x + c$ so that $k = 0$ and $x = - c / b = m$ just fall under this general framework.

Why is it that textbooks opt for the broken approach instead of the general formula ? The cause must be that students are not presented with the notion of a *correspondence,* see page 24. Students are only told about functions. With $k\ y = b\ x + c$ we find that $y$ cannot be written as a function when $k = 0$ and $x = m$.

It must be doubted that pupils and student would be incapable of understanding the difference between a function and a correspondence. Instead, it need not be doubted that we do wrong in withholding that insight. Since we withhold it, students suffer the difficulty of entertaining a "distinction" between $y = b\ x + c$ and $x = m$.

The broken approach to lines actually breaks down in the chapter on linear programming. Here we need the general formula of the line anyway.

The treatment of the line is strange and cumbersome.

Gladwell (2008:239) has a discussion about how a student learns that a vertical line has an infinite slope. The setting does not display any particular deep mathematical insight but is entirely caused by the framing of the question. Presenting lines in this manner combines both their mathematical formulation with difficult notions in the infinite. It would be more enlightening for the student to know that the angle is $\Theta$ / 4. Gladwell's basic story is that students learn more when they are persistent, which is OK. Let us encourage persistence but also allow for a lower slope in clutter and a higher gradient in learning.

PM. One might ask whether also $k\ y = a\ x^2 + b\ x + c$ is more general. In that case $k = 0$ reduces to the 0, 1, 2 solution points of the parabola, and the vertical lines through them. A discussion of this might be part in explaining the difference between a function and a correspondence.

## 23. Chaos with co-ordinates, complex numbers and vectors

At already an early stage in his mathematical education, the student is introduced to the system of co-ordinates, the $x$-axis and the $y$-axis, on which he draws his lines and parabolas.

Likely even earlier, PythagoreanTheorem has been discussed, i.e. with the $a^2 + b^2 = c^2$ of the sides of a triangle with a right angle.

A logical development would be to consider the addition of co-ordinates, as in {1, 2} + {3, 4} = {4, 6}. Arrows can be drawn from {0, 0}. Subsequently, the lengths of the arrows can be calculated with Pythagoras. Finally, students can be told that they can sound wise and competent by using the phrase "adding vector $A$ = {1, 2} to vector $B$ = {2, 4} gives vector $C$ = {4, 6}". Let John come up front, say this, and let the class give him a great applause. Let Mary come up front, say this, and let the class give her a great applause. Perhaps a volunteer ? In advance of the class, inform the adjacent teachers that you will be teaching vectors today.

The difficulty doesn't lie in the mathematics but in understanding why this type of calculation and modelling would be so useful for practical applications. The marble that rolls over the deck of a ship however remains a helpful example.

Nothing would thus be simpler than to show that "calculation with vectors" is exactly the same as the "calculation with co-ordinates". The mathematical difficulty starts with multiplication – that leads to matrix algebra.

My sample may be small but I have not seen a textbook that proceeds in this manner.

Rather, the textbooks introduce the "entirely new concepts" of complex numbers and vectors. This is another example of "More words for the same" – see page 33. It is destructive. Now with the added zing that the natural growth of the understanding of space and the development from co-ordinates to matrix algebra is broken.

For example, Sullivan (1999) develops matrix algebra from systems of equations. But a linear equation actually is an improduct so it is better to start with vector multiplication in the system of co-ordinates.

PM 1. It would also be simple to show complex numbers as a historically interesting reformulation for the two-dimensional plane, with

$$z = \{x, y\} = x + i\, y = |\, z\,|\, (\cos \varphi + i \sin \varphi) = |\, z\,|\ \ \mathrm{Exp}[i\, \varphi] = \mathrm{Polar}[|\, z\,|\, , \varphi]$$

The implementation of the imaginary number as $i = \sqrt{-1}$ remains problematic with $-1 = i^2 = \sqrt{-1}\ \sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$. The implementation $i$ = {0, 1} does not suffer this problem.

PM 2. Students are taught that the Quadratic Formula has no solution for a negative discriminant. Later they are told that there is a complex solution. It should be feasible to mention the complex solution directly. To know that it exists is different than doing exercises with it. Perhaps we need a course Geography of Mathematics with all the countries we never go to but still know about. You learn to wash your hands and only later may have a chance to look under the microscope to see germs.

# 24. Needlessly slow on derivatives

The discussion about Superficial Calculus (rules only), Serious Calculus (Cauchy limits) or Deep Calculus (Weierstraß) has a long history. Let us consider the current state and a suggestion for improvement.

Students currently find the turning point of a parabola with a formula that either is merely supplied or derived by moving the parabola so that the turning point is on the horizontal axis. (Thus, a single point of intersection, choosing *c* in the Quadratic Formula such that Discriminant is zero without actually calculating any *c*.) The same formula can also be found by differentiation but this is taught only later in the course. The reasoning on this learning plan must be that students first require some mathematical skill, to be developed on the parabola, before they can grasp the notion of the derivative, which will help them to reflect on their earlier learning on the parabola. There is indeed a small effect of amazement when students discover that the derivative gives the already known result.

I beg to differ. In an alternative learning plan the rules of differentiation are presented at a much earlier phase. When they are applied to the parabola to find the turning point then also the ancient way to find it can be presented alongside, both for corroboration and historical perspective, and clearly both approaches will sink in much better at the same time.

Slopes are important. That is why they are in the programme. The rules of differentiation are an important discovery not only because they are fairly simple but also because they generate important results and generate them fast. It pays to command those rules as soon as possible. For example, in economics, to differentiate the parabola of profits, set the derivative to zero, and find maximal profits. Why it works ? Well, there are levels of understanding. Clearly the slope is zero at a maximum, minimum or inflection point. Why these rules give the slope ? Well, we will get to this later on in the course.

Recall that we allow people to drive a car without knowing how it works. People are vaguely aware of the different kinds of electrical current but only sufficiently to prevent appliances to blow up. We play soccer without much knowledge of Newtonian physics and aerodynamics. It is not evident that all of this would be different for mathematics. It is a nice ethic that you want to prove everything but (a) clearly this is *not done now* in highschool, and (b) the selection of *what is proven now* is arbitrary, superficial, traditionalistic, unconvincing. It is valuable that pupils feel that some argument is given, and an explanation helps memory. But an explanation "derivatives help to find the slope" may be as adequate as the explanation in biology "people breathe because they need oxygen". Eventually a lot can be explained and proven but soon it becomes a specialty and it runs against economic laws that everyone can be a specialist in everything. Thus, in the same way, we can teach how to find the derivative without detailing why it works. It is already quite a mathematical competence to know the rules and how to apply them.

The true story about the current situation is that students first memorize the rule for the turning point before they discover that they had better memorized the rules of differentiation for finding such points in general. Thus there is more memorizing than needed and less time spent in competence.

Admittedly, for calculus in the English speaking world, I have available only Hughes-Hallett et al. (2000) for universities and colleges. This course in calculus would be separate from a course in algebra (e.g. Sullivan (1999) and Angel (2000)). For highschools I have to rely on my experience in Holland. Dutch highschools have four tracks of math: D for the advanced level (somewhat linking up to above English sources), B for normal math (always taken by D too, including Serious Calculus), A for economists (Superficial Calculus) and C for students of art (no derivatives).

In Holland the main distinction between A/C and B/D thus has already been made. At issue here is only the order of presentation. My suggestion is to always start with the rules, in track A and B/D alike, already when discussing the parabola, and only later provide a more formal justification for the B/D group. We want these students to get serious mathematics – which however means that we also want to enhance elegance with substance, and avoid a cumulation of cumbersome convolutions.

I can understand the mathematical urge to introduce some more formal math at the highschool level, albeit not Weierstraß then at least Cauchy. We see the same with Hughes-Hallett et al. (2000), where first the formal definition is presented (though Cauchy only) while only the subsequent chapter provides the short-cut rules. The driving force in this reasoning is the urge by (pure) mathematicians that the derivative needs a good definition *before* it can be applied. It seems to be part of the mathematical ethic and decency not to use something that hasn't been clearly defined first. They are not fundamentalist on this, they are willing to compromise, they don't insist on Weierstraß and accept Cauchy, and they let the A/C tracks go their way. Nevertheless, the urge is there. In my view this urge is didactically unwise, not only for the B/D track but also for the A/C track, since all tracks get the rules on differentiation too late. Much time in the early phases of the current programme is lost on fractions and radical signs. It is much better to spend that time on learning the important rules of differentiation.

Perhaps course designers also feel that when students know the rules they would not be interested any more in the formal definition. Indeed, once the formal definition is presented it is hardly used any more and all attention goes to the rules and their application. Nevertheless, students in the B/D track would most certainly have the attitude to be interested – as it also is an interesting subject. But a bit later.

With Van Hiele, first concreteness, then ordering, then analysis.

Below, we will look a bit deeper into the formal definition of the derivative.

PM. Dutch students in the B/D track have only derivatives and integrals of a single variable and miss out on the distinction between partial and total derivatives. The latter should rather be in the program.

# V. Like the stepmother in the fairy tale

## 25. Probability and statistics

This may be a Dutch idiosyncracy. In the Dutch highschool programme, see page 38 and following, probability and statistics are put in Track A and are not included in Track B. Perhaps the physics professors want to be able to develop quantum mechanics by themselves. Physicists may have limited understanding of probability and statistics:

> "There appears to exist a strange miscommunication between physics and mathematics. Gill quotes Suppes: "For those familiar with the applications of probability and mathematical statistics in mathematical psychology or mathematical economics, it is surprising indeed to read the treatements of probability even in the most respected texts of quantum mechanics. ... What is surprising is that the level of treatment in both terms of mathematical clarity and mathematical depth is surprisingly low. Probability concepts have a strange and awkward appearance in quantum mechanics, as if they had been brought within the framework of the theory only as an afterthought and with apology for their inclusion." (P. Suppes, 1963). Gill suggests that this is still the case in 1998." (Colignatus (2005:81) footnote 64.)

Students from both the A and the B tracks are not introduced to the "abstract" notation of elementary probability. A textbook need not mention the formal definition and notation for the conditional probability $P[X \mid Y] = P[X, Y] \,/\, P[Y]$ while this would be important for proper understanding. Even worse, students are submitted to complex language constructions that supposedly code for conditional probabilities. Thus they have to learn both the concept of conditional probability and dubious linguistic codes.

The following example translates well. A textbook has a crosstable on injuries at a sports club. $A$ is the probability that an arbitrary member of the club "was younger than 20 years and had more than one injury". $B$ is the probability that an arbitrary member of the club "that had more than one injury, was younger than 20 years". Thus $A = P[X, Y]$ and $B = P[X \mid Y]$. One awkward point is that the language construct uses a comma for the conditional while the mathematical convention uses the comma for the joint probability. Students are encouraged to write "*P*[that had more than one injury, was younger than 20 years]" which will require some unlearning again later on. Another awkward point is that the clear statement "the probability that a member is younger than 20 years given that he or she has more than one injury" is not used. The textbook uses a construct that admittedly might be used. We should hope that people who use that construct indeed intend the conditional probability. However, the construct will be rather unfamiliar for students in a first course on the subject. To avoid the ambiguity and parallel learning of both mathematics and language, it is much better to concentrate on the mathematics and use language for clear communication. The expression "given that" provides that clarity and indeed links up to the formal expression of conditionality.

# 26. Ambiguous dice

There is a distinction between a perfect die with probabilities 1/6 and empirical dice of which the probabilities per die have to be determined empirically and that could be approximated by observed frequencies while assuming similar conditions.

Many discussions and test questions don't mention the label "perfect" and expect students to be able to determine from the context whether it applies or not.

We note a subtle shift in learning goal. The math course is no longer targetted at math itself but apparently on "reading well" – with always the gamble on what the author really intended.

Supposedly when the exam question is about a die factory doing quality tests then we might presume students to be so smart as to understand that factories cannot produce perfect dice. A question like "John throws two dice. What is the chance that he throws less than 4 ?" is already tricky on language students who will hold that two dice are less than four dice so that the probability must be 1. Assuming that the digit codes for the outcome, the question might presume perfect dice so that John is only an imaginary figure created for literary purposes. Otherwise we would not know what that chance could be since we have not been able to test those dice. Perhaps there is a hidden convention that factories are real and person names imaginary.

It is advisable to distinguish the learning of math from the learning of context. For the math section there could be a statement like "all dice are perfect unless it is explicitly stated that they are real" or "all dice are real unless it is explicitly stated that they are perfect". For the context section there could be a statement like "determine from the context whether dice are perfect or real".

# 27. Mathematics and economics

Textbooks on mathematics must develop a position with respect to textbooks on economics. Economics is often seen as a useful application of mathematics – though historically many impluses went the other way – and thus textbooks on mathematics contain such topics. While, an example from economics might occasionally be used to highlight a mathematical point, hopefully, though, mathematics is supposed to support economics and not the other way around.

The Cambridge economist Alfred Marshall (1842-1924) created the diagram of demand and supply, put the cause *price* on the vertical axis and the effects *demand* and *supply* on the horizontal axis. Textbooks in economics faithfully copy him to this day.

The international scientific and mathematical convention is to put the cause on the horizontal axis and the effect on the vertical axis.

It would be obvious that textbooks in economics better adapt to the international scientific standard. It would reduce the confusion for their students between the classrooms in economics and mathematics.

It would be rather simple for economics to adapt. They could start in textbooks for highschool, and trust that these students will not read the historical books and the international journals. When the train gets going then it will be as simple to adapt the textbooks for university and college. At that level, students will be sufficiently versed in the subject to understand the older literature.

Teachers of mathematics apparently are confused themselves too and don't seem to realize the inverted use in economics. They are a bit like a hair-dresser who offers his services but appears to know only one cut. Of course there is the Cournot model where companies set quantities rather than prices. However, the common discussion is about the competitive model where the price is given. Diagrams in economics then have a horizontal line. Discussion of this case in math textbooks creates the curious situation that they want to draw a vertical line and still reproduce the economics diagram. They manage to somehow talk around it, but obviously at great confusion for the student.

Mathematicians should urge the economists to adapt. In the mean time textbooks in mathematics better (a) keep following the international scientific standard, (b) refrain from messing up economic models, (c) explain to students about cause and effect and (d) explain the differences in conventions in mathematics and (old) textbooks in economics.

# 28. A shopping list on content

Textbooks in mathematics clutter with the dust of ages and the efforts by mathematicians to understand something about mathematics and to formulate it clearly. While the sand flows in the hour-glass, and hour-glasses themselves slide through our fingers, time in class is lost to tradition, and hardly any time is left to discuss new things that would actually be useful to discuss. There is a balance between tradition and adaptability. Let us see what could be included in the mathematics programme, preferably in highschool.

### *(a) Logic and set theory*

Sullivan (1999) fortunately contains some set theory but curiously logic and set theory have disappeared from Dutch highschools – only to resurface a bit in the new programme for track C. The formal representations of logic and sets are crucial results for the history of mankind but curiously they are not mentioned. To me it feels like a criminal act – a "white board crime" – to withhold these results from students. Apparently, set theory already belonged to the exam programme for a while in the Dutch past but then was reduced to needlessly complex issues of notation. It is advisable to try again. I must refer to Colignatus (2007a) *A logic of exceptions* (ALOE) since this redesigns logic. Thus it is advisable not to start with traditional logic. ALOE has been written for first year students at a university or college. Also, fuzzy logic deserves attention too.

### *(b) The axiomatic method*

There is a distinction between the axiomatic method as a topic of content versus as a possibly didactic way of teaching. As content, an axiomatic system is a rational reconstruction of a body of experience that also contains a lot of irrationality. Teaching this content will increase competence in reasoning. As a didactic method, it is of dubious quality. The next section will say more on the method. Traditional mathematics has a tendency to fuse the two. The Van Hiele theory however, reduced to rough simplicity, has the levels of concreteness, ordering and analysis. What is it, what can you do with it, how does it work and why does it work ? Analysis only comes at a later stage. This amends the traditional way of the education in mathematics. Possibly pupils with mathematical ability have a fast route to the analytical phase so that the earlier phases might be neglected more but that is a different kind of discussion.

In my own highschool days (I am from 1954) there was much more reliance on the axiomatic method or at least the Form with definition, theorem, proof. Checking those books again this method does not strike me as so didactically useful indeed. It is hard to tell, of course, since my analytical capabilities must have been influenced by that background, for better or worse. I think anyway that we have strayed too far from abandoning the topic itself. Hughes-Hallett et al. (1999) for example present the rule of L'Hopital and then proceed with "To justify this result (…)". It is a nice literary trick. One might turn it more formal.

Using the Form in normal discourse is pedantic and should be avoided. But in mathematics the objective is to develop and support reasoning. There the Form is on target. If a proof is given then it would support this notion by providing the Form. When to apply it ? The criterion would be that students have already had the first two stages of Van Hiele and are ready for the analytical phase.

If students would get worried and ask whether they are required to provide such proofs themselves then the answer can be (i) only sometimes, (ii) if so, do not worry, for we can follow the old Greek advice: assume that what needs to be proven is already proven, write down all the properties that you known (also given that assumption), reorder a bit, and the proof will click in place, (iii) remember, the idea behind the mathematical method is the liberating force that no authority can impose a rule but that only you yourself can check its validity – and with this liberty also comes the duty to prove to others what you would like them to believe, which means that you better work on some competence to provide proofs.

#### Intermezzo: Is anyone to blame ?

Some mathematicians are inclined to explain the disappearance of the Form to the pressure of social and economic developments. We can refer to Ernest (2000) again. On Dutch history, Goffree et al. (2000) is obviously relevant too. It are just as well the weak backs of the mathematicians themselves who have not defended their field.

At his retirement after 40 years of teaching Groen (2003) states, in my translation:

> "In the last decennia the call has become louder and louder that education must offer knowledge that is directly applicable and useful. Economic use as the measure of all things is incontestable. Talk about products in cases where this never happened before (university graduates, train connections, medical treatment, overnight stay in a hotel) is only considered strange by unwordly and unadjusted poor souls. This had its influence on programmes for mathematics. In the past we were satisfied with the proposition that education in mathematics greatly contributed to Bildung without being able to show concretely what the effects of that forming value were. Nowadays we want to see that forming value expressed into recognizable, profitable applications. This has also led to the almost complete disappearance of the emphasis on theorems, definitions and proofs that existed in the past almost directly from the 7$^{th}$ grade in Lyceum. It has been replaced by quasi socially relevant calculation about heating bills and angles of sight. The return of planar Euclidean geometry as a context for exercises on proofs is an effort to do something about that again, but that only begins in senior highschool. To me this seems dangerously late for the development of the required reasoning capabilities."

Mathematics teachers may also conclude that they as guardians of that Bildung have failed collectively in defending it. Society is liable to be gullible if we would hand them that responsibility afresh. What guarantee is there that they now will steer the right course ? My suggestion is to put the responsibility with a council of not only mathematicians but also the other sciences and humanities, teachers, parents and students.

### *(c) Algorithms and computer programming*

Algorithms are key in mathematics. A proof for example is an algorithm to check the theorem. An algorithm is a longer chain of logic, possibly extended with text, formulas, tables and graphs, to identify problems and solve them. Landa (1998) is an important source here. Landa's core idea is (a) observe experts, (b) disect their actions in small steps, (c) analyse these steps, optimize them, formulate everything as an algorithm, (d) allow students to execute this algorithm so that they can perform immediately like experts. Subsequently, students greatly extend their scope when they learn to create algorithms themselves and recognize this as a tool for enhancing their understanding. For explorative cases (deterministic) algorithms are replaced by (probabilistic) heuristics. Note that the terms algorithm and heuristic are not part of the student vocabulary. Using these words may at first put them off. Learning them is part of the understanding.

Textbooks in mathematics generally provide students with algorithms to solve the test questions in the book, but they are modest in discussing algorithmic design. Students will learn a lot from computer programming as part of the course since the choices for problems and solutions are more varied. Working with the computer is interactive with direct feedback. A programmer has to think about the overall target and the small steps at the same time. You tinker with it till it works. If programming is to be educational it must be done in a serious language and not with drop-down menu clicking or the use of strange codes. A modern course will take a computer algebra language such as *Mathematica* or Maple which allows flexibility to explore the different kinds of programming (functional, object-oriented, rule based).

Current education tends to use the graphical calculator. This is penny wise and pound foolish. It seems a good bargain but it has limited capacity, does not allow good programming, reduces effort to a lot of senseless punching, kills motivation.

### *(d) Use of language*

Textbooks can contribute to a sharper use of language. With text, formulas, tables and graphs, the first element does not get sufficient attention. We already have seen the examples of the vertex of a parabola and the perfect die. Textbooks better use sharper language themselves but it would also be an improvement when they provide educational material to increase student awareness.

Mathematicians hold the idea that language is vague and formulas will be exact. This idea however runs counter to good didactics since it would imply that we may give up trying. Instead it is better to sharpen language as well.

Reality is not neat. Data have to be collected and pruned to become evidence. Formulas and graphs don't fall from the sky but have to be hunted and crafted. Texts can be very messy. There is no reason to single out language as the element to neglect.

Accuracy also applies to math test questions. It is no rare occurrence that a question is opaque except under a particular interpretation that suddenly gives all that is required. The student then is tested on finding that particular interpretation and not really on mathematical insight. "Reading well" is a soft criterion. Math test questions should provide all information, and actually also some redundancy to allow a double check. Admittedly, it may be difficult to provide all information without giving away the answer but it will generally be clear what kind of questions actually should not be asked. Opaque questions might be asked to query mathematical creativity – which is anyway a difficult property to test.

### *(e) Theory of democracy and voting*

Democracy is an important concept. The mathematics of voting is somewhat complex. It would be beneficial for society when its citizens understand more about the mathematics behind election results. Students in the USA have a Government class where such aspects can be indicated. Political Science as a subject has not reached highschool in general. Much can be said in favour of including the subject in economics, since the aggregation of preferences into a social welfare function is a topic of Political Economy. See page 57 and Colignatus (2007b) *Voting theory for democracy* (VTFD) for details and other references. Most economists will be unfamiliar with the topic and its mathematics though and thus it may well be practical to include it in the mathematics programme.

### *(f) New subjects of the last two decades*

There seem to be no other new subjects of the last two decades that students should do in depth. This shows the elementary nature of the current program.

However, there are subjects of the category "useful to have seen the major relevance and results". Such subjects tend to date back longer but apparently take a while to diffuse into textbooks. Those are fractals, chaos as opposite to randomness, cryptography. Topology with the fixed point, useful for the definition of *e.* Graphical models with conditional independence are a useful addition, and a combination of graphs and probability theory.

Economics may want to spend more time on finance theory and stock market crashes, possibly desiring mathematical support for the Black-Scholes model for option pricing and the critique by Mandelbrot & Taleb on the too simplistic interpretation of the law of large numbers.

# 29. A shopping list on method

Next to content there is the way how mathematics is taught. Some aspects hold for education in general but some will be specific to mathematics.

### *(a) Dealing with developing brains*

Jolles et al. (2006) "Brain lessons", its website http://www.brainandlearning.eu/ and other initiatives around the world provide a fresh angle, alongside other developments on evidence based education.

I was also struck by Gladwell (2000), while reporting the known fact that small kids enjoy watching the same tv program over and over again (e.g. Sesame Street), also mentioning that they see different things each time, which is an angle I had not considered before. The phenomenon can actually also be observed at highschool, where much of the same material is presented in the different grades, over and over again (and continues to give problems). Apparently brains value a decent amount of repetition and in particular when they develop.

We already discussed the aspect of "More words for the same" – see page 33 – and suggested "smart exposure" as an alternative. Thus, brains must be stimulated to grow but they must not be forced on topics for which they are not ripe and that will come about rather by themselves over time. This is a nice general statement and possibly everyone agrees as long we are vague on specifics. Randomized controlled trials would be a way to work out the details, provided that parents will offer their kids to such experiments. A key point of this book is that, when designing such trials, we better don't do it with mathematics that is inherently cumbersome and irrational, but with the elegance with substance that we expect from good mathematics.

One aspect is cognitive dissonance, see Aronson (1992). It is a pervasive human property and must affect education too. The brain is an information processing machine with conditions of energy efficiency, and one of the cheapest ways to deal with new information is to neglect it. One example might be textbooks used in 9th grade and 10th grade. Dutch textbooks are not by subject but collect the material used in a grade for the different subjects. In 10th grade it might be instructive to run through the textbooks of the 9th grade again, and refresh what already should be known. The kids might consider this childish though and below their standards. Some might argue that a whole new book provides the chance to create a new environment afresh, a new start, a new dawn, and when much of the same material is treated again then this gives pupils a chance who missed out the last time. Perhaps. An alternative is to arrange textbooks by subject, such that a discussion at the level of the 9th grade is followed by a discussion of more advanced aspects at the level of the 10th grade. This avoids the cognitive dissonance that it would be childish to look into the book of last year, repetition comes about naturally, and we can save a lot of time on actual repetition because of these two effects. Of course kids would have more books. Are we *penny wise, pound foolish* ?

### *(c) Overall didactic awareness*

Overall didactic awareness: it seems obvious but may amount to a paradigm shift in the teaching of mathematics. Textbooks of mathematics still suffer from the tradition that

Euclid's axiomatic (re-) construction of geometry defines the Nature of Mathematics and is The Way, not only for Presenting Results but also for Teaching and Learning.

We already mentioned the Van Hiele approach to allow room for levels of understanding. We also mentioned Landa's algorithmic and heuristic approach as subjects to learn, but they also are a methods of teaching and learning. Including with other writers on didactics, research on the brain and cognitive psychology, there is a strong alternative to The Way.

Old ways die hard. An example may be taken from a Dutch textbook where the derivative of $a^x$ is introduced. It is not stated first that $(a^x)' = a^x$ Rex[$a$] – see page 29 – but it is derived formally. The differential quotient gives an expression where the natural logarithm cannot be used yet since $e$ has not yet been defined. The purpose of the exercise precisely is the definition of $e$. The book solves the problem by defining $f(x) = a^x$ and then presents the solution that $f'(x) = f'(0)\ a^x$. The original problem of finding the derivative of $a^x$ is further unsolved and dropped from consideration. The section proceeds with determining $e$ and only the next section completes with determining that $f'(0)$ = Rex[$a$]. Hence it is proven in general that $(a^x)' = a^x$ Rex[$a$]. For a reminder, note:

$$f'(x) = \lim_{h \to 0} \frac{a^h - 1}{h} . a^x = f'(0) . a^x$$

The reasoning is sound and will appeal to the mathematically trained who wants to check. Possibly the mathematical ethic also requires that we should not discuss things that have not been defined properly. Possibly it saves time and energy doodling about with concepts only to find out later that they are ill defined / not defined / not definable at all. However, it does not save time when the abstract soup prevents understanding. Here I would rather follow Van Hiele and allow the students to first play around with what it all means both concretely and in terms of interrelationships, before concluding with the proof why things actually are so. Thus:

(1) There is a fixed point in differentiation with $f'(x) = f(x)$

(2) In particular there is a number $e$ such that $(e^x)' = e^x$ — on the computer Exp[$x$]

(3) All numbers can be expressed as a power of $e$. Thus there is only one such fixed point in differentiation. The number $e$ = 2.718… is as special to mathematics as Θ.

(4) We define Rex[$x$] = Rex[$x$, $e$]

(5) For all exponential functions we find $(a^x)' = a^x$ Rex[$a$]

(6) Check that $(e^x)' = e^x$ Rex[$e$] = $e^x$ indeed, since Rex[$e$, $e$] = 1.

(7) Graphics and exercises, to explore what it means

(8) Provide the proof using above proper differential, to show why. Calculate $e$

(9) Graphics and exercises to let it sink in, so that we do it in full understanding.

To me it would be obvious to proceed in this manner. But I referred to a serious textbook and they mess it up. They also clutter the argument by first discussing translations of logarithmic functions, suggesting that it seems like the major point of the chapter while this is a minor topic that may come in an appendix. You don't have to be able to translate a logarithmic function to master $e$.

### *(c) Class and quality*

At university thirty years ago I attended my math lectures in an oratorium with possibly 150 students but then of course we were no teenagers no more, and after the lecture we had our practica in smaller groups.

Class size depends upon national regulations and possibilities of scheduling. In Holland highschool teachers of mathematics accept classes with a maximal size of 30 pupils. Apparently it works somewhat, witness the state of the Dutch economy (with natural gas resources). I would still hold that math is not the same as French or geography (of mathematics). Learning to think and to reason and catching the subtleties in the personal route towards understanding are served by a class of maximal 15 pupils or students.

Allowing only 15 pupils or students requires more math teachers. There can be savings in (a) a quality program requires less contact hours, (b) rely on more independent work with the computer, (c) shift non-core-business such as repetition of exercises back to the subject fields such as economics and physics where those actually belong, (d) recruit (good) older grade students to help younger grade students, (e) relieve the task of checking exams, by more computerized tests. If a class of 30 students has an hour of geography and subsequently an hour of mathematics, then it can be split and we need two teachers of mathematics where we now schedule one. In practice a class might have for example 25 students, 15 would go the contact hour, some might consult their student-assistant if she is scheduled to be available, others work on the computer.

Overall, though, some increase in the number of math teachers seems advisable. Mathematics is important, and good mathematics saves on the demands on other subjects. It is said that there is a shortage of teachers of mathematics but this is a use of language that is low on analysis. The better statement is that salaries are too low and that more must be done to let it become education in mathematics indeed.

### *(d1) Support and testing by computer – the problem*

The world abounds with computer programs and materials for mathematics. This should not be surprising since computers were developed by mathematicians and computer science engineers. Nevertheless, the relation of mathematics and the education in mathematics to the computer is actually rather a problem.

We have e.g. Excel, Java, typesetting LaTeX, html or xml with MathML, *Mathematica*, Maple and MapleTA, Matlab, Maxima, Wiris, Derive, Scientific Workplace, open source Sage,[3] and the graphical calculators as well. All these have their various applications that users often put on the internet. MathBook or OpenMath/MathDox, see RIACA, TU Eindhoven, accept various computer algebra systems and build a layer on top, which seems useful but requires additional attention for the uninitiated and seems unnecessary for who already has a system. Geometry programs are Cabri and free Geogebra.[4] Class management systems are Blackboard/WebCT and open source Moodle. In Holland examination on the computer is already partially allowed for graduation and there are steps to further develop that. Systems that combine class

[3] http://www.sagemath.org/
[4] http://www.geogebra.org/

management with instruction and testing are WIMS,[5] MapleTA and (likely) Wiris in combination with Moodle.[6] Textbook publishers are starting to provide their own systems. Schools tend to have their own system to administer students and their grades.

The key notion is insularity. Each participant is defined by own objectives, own resources and own restrictions and it appears very difficult to arrive at a common goal, pool resources and overcome the restrictions. To name a few points:

(i) Countries have their own languages and national regulations.
(ii) Nations have their own school districts. Education is a sensitive issue for parents.
(iii) Publishers have their own authors and websites.
(iv) Teachers have their own students and particular issues.
(v) Programmers have their own computer languages.
(vi) Associations of mathematics must be diplomatic about sensitivities.
(vii) As this book shows, issues need not be simple, with different grades, levels of understanding and competence, aspects of didactics.
(viii) It is not correct to only consider mathematicians since it are governments and national parliaments who determine how important they judge this issue and how many resources they make available.
(ix) Since mathematics rather is an international venture it actually is the international community that is responsible.

Educators are peddlers and drugdealers. First you are encouraged to "graduate" from elementary school if your life is to be any good, but once you have done so then you are told that you have to graduate from highschool. That done, you are told that you better graduate from college or university if you want to have some perspective. With that document secured, you are told that the minimum is a Ph D. Eventually you may discover that you may have learned a lot but still know very little. Plenty dealers around to peddle a course that you really should take. The moral is that we may as well be relaxed about all this, even concerning mathematics.

Perhaps the situation compares with soccer clubs that do not co-operate to form one super club. Soccer clubs are focussed on competition and thus mathematicians would be a more agreeable lot, perhaps only a bit more critical than soccer clubs on the aspects where they disagree. But let us see what can be done for mathematics.

### *(d2) Support and testing by computer – a direction for solution*

Given the importance, there is a separate chapter on this, see page 69. However, at this point it is more relevant to develop the underlying notions:

(1) In education, feedback is important and differs in kind and intensity depending upon the individual. Nowadays the teacher gives feedback, students look in the booklet with answers and they ask around. The idea is that the computer will be a great tool to take away the tedium and to provide new levels of interactivity. Mathematics will continue to require much testing with pen and paper and teachers will want to see what their students are doing in that manner to better judge their knowledge, skill and attitude. But at various points even multiple choice questions can be used if only for preparation and to set entrance levels. (i) Teachers will have to take the

[5] http://wims.math.leidenuniv.nl/wims/
[6] http://www.wirisonline.net/

psychological barrier and recognize that the feedback from written tests is relevant but limited. Computer tests can be relatively smart by responding to the level of the student and by monitoring how often the same kind of multiple choice test is done in an effort to pass purely by randomness. (ii) The second barrier to take is organization. Schools and universities have to create computer test rooms, with special supervisors who check on identity card, login procedure, mobiles and usb-sticks if it is a formal test. This applies for all subjects but also mathematics. Since the creation / modification of test questions is fairly simple for mathematics (e.g. plug in numbers selected in random manner) students may be allowed to do tests at liberty. It seems rather strange, but a major bottleneck towards advance in quality in teaching of mathematics are the costs of such test supervisors and other concierges for school opening hours. This relates to my economic analysis in Colignatus (2005). In economics, everything hangs together. (iii) Rather general experience with WIMS, and also my own, shows that students don't use its availability on the internet and its possibility of feedback if they cannot earn points. Hence, procedures are designed such that students learn that it is wise to do such testing especially if they lack in competence. One option is to require an entrance test in advance of a written test, which entrance test is done on the computer under supervision (otherwise a friend might use the internet). Another option is that a failed exam is counted as a worse failure if there is no record of sufficient advance self-testing. Another option is to give the new system a chance, let students get used to it, create an attitude and culture that they use computer feedback, and subsequently talk with the students who don't and their parents.

(2) Schools can best stop using graphical calculators since what those can do can hardly be called mathematics. Proper is the switch to mini laptops with open source linux, open-office, open source Sage / Python and free Geogebra. This will support instruction and feedback from interactivity. Feedback from actual tests will not be automated yet. It is a start and we can work from there. See page 69.

(3) The use of those mini laptops during official examination will be problematic since students would be free to put anything on the hard disk or perhaps even create a wireless connection. Reformatting and reinstalling is tedious and actually somewhat unfriendly towards the hard working student who includes all kinds of material. Alternatives are (a) the use of the common test room, (b) have a sample of mini laptops in minimal configuration purely for such tests.

(4) There are three additional advantages of using mini laptops: (a) programming – see page 45, (b) integration with other subjects such as economics and physics, since computer algebra is much more versatile than the graphical calculator, (c) overall mathematical accuracy. Above we saw the distinction between $f(x)$ as multiplication $f\ x$ (dropping the brackets) and $f(x)$ as the function call $f[x]$. Who works with a computer algebra system will see many more cases where accuracy can be improved.

(5) Computer programmers are insufficiently aware of the golden rule in programming: do not program to others what you would not want to be programmed to yourself. The rule should be basic to the education of programmers. Perhaps the basic education for programmers is to engage them in social activities (since programming tends to come to them naturally anyway – see Krantz (2008) again).

(6) The integration of computer algebra in mathematics education is not a small issue. A small example is notation. A capable mathematician and teacher of mathematics can switch relatively easy between the various notations, e.g. between the various books, the textbook, graphical calculator, the computer algebra system, and, indeed the writings of the students. Students however are learning mathematics and rely on consistent notation. Students are very sensitive to differences between the textbook and computer programs. The choice of a program is a crucial one and not easily changed. See page 69. Textbooks will have to adapt to the computer as well.

(7) Textbooks are rather expensive but that is also because they nowadays provide their own websites and software. If publishers had done that much earlier then software producers would not have stepped in – and now they are competing for market share, driving up costs and reducing quality. There is an increasing tendency to refer to free sources on the internet. The internet seems to provide an abundance of applications indeed. This is rather an illusion. Many applications are in Java and thus very specific, not easy to adapt, and not suited as building bricks for a more complete system. The only sound step is to switch to using a computer algebra program, see page 69. This conclusion does not disqualify or diminish the efforts by teachers and other producers of those other programs and their discussions of manuals and didactic qualities. Indeed, when we consider the various resources created e.g. in Holland by e.g. the Freudenthal Institute, [7] Mathadore [8] or Kennisnet [9] even apart from the main three commercial publishers and other sources, the fragmentation seems to prove the need for a single working environment. In fact, this is already obvious for the last 15 years if not earlier.

(8) For computer algebra we can distinguish between the mathematical language – that would be uniform over the world – and the computer program that interpretes this language and evaluates this. Current programs tend to proprietize mathematics by using slightly different codings. That menus differ and that different programs have different capacities and layouts would be acceptable and subject to competition in the market place. However, a criterion should be that there is a uniform, text based, simple language for mathematics, that can be used as input and output. See Colignatus (1999, 2000). Personally, I am in favour of using *Mathematica* as the base of that mathematical language, and hope that there can be put a shell on top of Sage / Python, or whatever. I imagine that others think otherwise. The Sage language does not strike me as sufficiently elegant for doing mathematics on the computer. But it is an improvement upon graphical calculators and we may work from there. See page 69.

(9) For senior highschool and up, mathematics would likely be done in English for most countries in the world. With this complexity of mathematics it might not pay to translate all of it. This would affect the other subjects like economics and physics that use mathematics. Likely those subjects face the same kind of problems with respect to computer assisted support and testing. Countries face tough decisions about the costs of maintaining their national languages in education. My advice is to be relaxed about it since national identity is very strong and will not be rocked by this influx of English.

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[7] http://www.fi.uu.nl/nl/

[8] http://www.mathadore.nl/

[9] http://digischool.kennisnet.nl/community_wi

(10) Best is to design a mechanism to transport applications to the public domain. Applications written in the uniform mathematical language would be put on the internet as an open source contribution. This however creates an unbalance between the investments and costs for the producer and the use by the free-riding world. For quality we require higher investments but those costs will not be covered – as already is the case. Computer assisted education has been in the doldrums for decades because of the inability of society to create the proper market structure. The solution is that countries contribute funds to either a national authority or an international authority that (i) awards contributions and (ii) tenders projects with the objective to put results into the public domain. The use of applications can be monitored and good use can be properly awarded again. Countries can do so on a national basis but then have to accept that other nations ride free on them.

(11) The latter is actually derivative of a more general proposal. The economy will benefit much if individual creativity is released in more areas than just programming for mathematics. We may for example consider the situation of scientific publishing, where governments subsidize universities but the output disappears behind the gates of publishers in the private sector. Similarly, the publication of textbooks for mathematics can be managed differently. Texts would be in the public domain, awarded for that, publishers could compile courses, and be awarded for that again.

(12) Current computer keyboards have a layout that is little better than QWERTY with a special pad for data punchers. Nowadays they could add some rows with the most relevant mathematical symbols for easy access. And a key to toggle between the Latin and Greek alphabets. Apparently the standing of mathematics is low even amongst the engineers who make the computers – it is time to enhance it.

# VI. Redesigning mathematics itself

## 30. Introduction

The chapters above rearrange standard material but leave known mathematics intact. The current chapter creatively innovates mathematics, in a way that is relevant for education.

I am actually not interested in doing research in mathematics. My focus for research is on economics, in scientific manner with econometrics. There have been four impulses that set me on a course that eventually caused these new results anyway.

The first case when this happened was when I was still a student of econometrics and followed lectures on philosophy, logic and the methodology of science. The logical paradoxes caused me to write a book on logic. The typescript was shelved in 1981 but turned up again in 2006 when moving house. I found time to type it over and program the logical routines in *Mathematica*. It is now Colignatus (2007a) *A logic of exceptions* (ALOE). See the discussion by Gill (2008). The news is a development of three-valued logic that remains free from Liar paradoxes itself.

The second case was in 1990, at the Central Planning Bureau, when I had cause to consider Kenneth Arrow's Impossibility Theorem with respect to the voting paradoxes. The subject started as the economic question about the social welfare function to use in economic models but ended up in a rejection of Arrow's analysis. Arrow's Theorem is mathematically valid but Arrow's verbal interpretation does not cover it, and when that interpretation is formalized then it fails. See Colignatus (2007b) *Voting theory for democracy* (VTFD). Part of the news is also a suggestion for a compromise voting procedure that many are likely to be able to live with – the Borda Fixed Point method.

The third case arose in 2008 seeing students struggle with trigonometry. I hadn't used the subject for a long while and apparently could approach it afresh. The news is the measure Unit Meter Around (UMA) alongside degrees and radians. The functions Xur[α] = Cos[α Θ] and Yur[α] = Sin[α Θ] eliminate a lot of clutter and tedious calculation.

The fourth case can be mentioned last though it arose in 2007 as well. While teaching mathematics, various questions had come up naturally. Most of those issues belong to the earlier chapters. While retyping ALOE and thinking about paradoxes again, the idea came up to reconsider also the paradoxes of division by zero, in particular in relation to the differential quotient and the problems encountered by students. In economics there is the distinction between statics and dynamics. In 1981 in ALOE I had already applied that distinction to (static) propositions and (dynamic) inference. This also fitted the experience in programming between identity (=) and assignment (:=), see page 24. Thus the idea arose to algebraically distinguish the act of dividing (//) from the result after division (/), see page 26. The news is that calculus can be formulated algebraically without use of limits or infinitisemals.

# 31. A logic of exceptions (ALOE)

*A logic of exceptions,* Colignatus (2007a), is intended for use in the first year of college or university. The last two chapters require a more advanced level that is worked up to. For highschools, ALOE is advisable for teachers and textbook authors but for implementation for pupils the notions in the book need to be translated.

ALOE provides the concepts and tools for sound inference. Discussed are: (1) the basic elements: propositional operators, predicates and sets; (2) the basic notions: inference, syllogism, axiomatics, proof theory; (3) the basic extra's: history, relation to the scientific method, the paradoxes. The new elements in the book are: (4) a logic of exceptions, solutions for those paradoxes, analysis of common errors in the literature, routines in *Mathematica*.

Logic is used not only in science and mathematics but also in business and sometimes in politics and government. Logic and inference however can suffer from paradoxes such as the Liar paradox "This sentence is false" or the proof-theoretic variant by Gödel "This statement is not provable" or the Russell set paradox of "The catalogue of all catalogues that don't mention themselves". This book explains and solves those paradoxes, and thereby gives a clarity that was lacking up to now. The author proposes the new approach that a concept, such as the definition of truth or the notion of proof or the definition of a set, also reckons with the exceptions that may pertain to its very definition. The approach to keep exceptions in the back of one's mind is a general sign of intelligence.

A quote from this book:

> "Since the Egyptians, mankind has been trying to solve the problem of bureaucracy. One frequent approach is the rule of law, say, that a supreme law-giver defines a rule that a bureaucracy must enforce. It is difficult for a law however to account for all kinds of exceptions that might be considered in its implementation. Ruthless enforcement might well destroy the very intentions of that law. Some bureaucrats might still opt for such enforcement merely to play it safe that nobody can say that they don't do their job. Decades may pass before such detrimental application is noticed and revised. There is the story of Catherine the Great regularly visiting a small park for a rest in the open air, so that they put a guard there; and some hundred years after her death somebody noticed that guarding that small park had become kind of silly. When both law-givers and bureaucrats grow more aware of some logic of exceptions then they might better deal with the contingencies of public management. It is a long shot to think so, of course, but in general it would help when people are not only aware of the rigour of a logical argument or rule but also of the possibility of some exception."

The computer environment has these advantages:

(a) Three-valued logic, that normally is rather opaque, can be handled now with clarity.
(b) The student can create more complex algorithms using the routines.
(c) ALOE has not Questions & Answers. But interactive variation is possible.

# 32. Voting theory for democracy (VTFD)

*Voting theory for democracy,* Colignatus (2007b), can be used in college or university. The last chapters require a more advanced level that is worked up to. For highschools, VTFD is advisable for teachers and textbook authors but for implementation for pupils the notions in the book need to be translated.

VTFD provides the concepts and tools for democratic decision making. Voting is used not only in politics and government, but also in business - and not only in the shareholders' meetings but also in teams. Voting however can suffer from paradoxes. In some systems, it is possible that candidate *A* wins from *B, B* from *C,* and *C* from *A* again. This book explains and solves those paradoxes, and thereby it gives a clarity that was lacking up to now. The author proposes the new scheme of 'Pareto Majority' which combines the good properties of the older schemes proposed by Pareto, Borda and Condorcet, while it adds the notion of a (Brouwer) 'fixed point'. Many people will likely prefer this new scheme over Plurality voting which is currently the common practice.

The literature on voting theory has suffered from some serious miscommunications in the last 50 years. Nobel Prize winning economists Kenneth Arrow and Amartya Sen created correct mathematical theorems, but gave incorrect verbal explanations. The author emphasises that there is a distinction between 'voting' and deciding. A voting field only becomes a decision by explicitly dealing with the paradoxes. Arrow and Sen did not solve the paradoxes and used them instead to conclude that it was 'impossible' to find a 'good' system. This however is a wrong approach. Once we understand the paradoxes, we can find the system that we want to use.

This book develops the theory of games (with Rasch - Elo rating) to show that decisions can change, even dramatically, when candidates or items are added to the list or deleted from it. The use of the fixed point criterion however limits the impact of such changes, and if these occur, they are quite reasonable. Groups are advised, therefor, to spend time on establishing what budget they will vote on.

See also Colignatus (2008b) *Review of Howard DeLong (1991), "A refutation of Arrow's theorem".*

The computer environment has these advantages:

(a) Voting routines are computationally cumbersome but can be handled now with clarity.
(b) The student can create more complex algorithms using the routines.
(c) VTFD has not Questions & Answers. But interactive variation is possible.

# 33. Trig rerigged

I am not much of a fan of trigonometry. Apparently I am neither too rational, for the smart way would be to neglect it and proceed with the fun stuff. On the other hand, it was a bad itch that felt like scratching. We already discussed the choice of $\Theta = 2\ \pi$, see page 22. But we can do more.

For students it is a bit confusing that angles are measured counterclockwise. It would be too complex to change this, e.g. also with derivatives. Perhaps there is a moment later on to try it but now we let this rest.

In thinking about angles, people naturally think in turns, half turns, quarter turns. Mathematicians have considered the case, and don't listen. An angle is *defined* as a plane section between two intersecting lines. But it is *measured* (in a dubious distinction with *definition*) with either (a) sine, cosine and tangent, or (b) the arc of the unit circle. A unit circle has radius 1. The circumference can be subdivided in 360 degrees, deriving from the Babylonian measurement of the year and maintained over the ages since 360 is easy to calculate with. Subsequently, it is seen as an "innovation" – the advancement of grade 11 over grade 10 – that the said perimeter can also be subdivided in Θ radians.

Most mathematicians would hold that radians and π are dimensionless numbers. For example π would be defined as the ratio of a circumference $2\pi\ r$ to the diameter $2\ r$ of any circle. Since numerator and denominator are measured in say meters, the unit of measurement drops out. I would oppose this, first by holding that a 'meter around' is something else that a 'meter in one direction'. Secondly, when we consider a unit circle, then that unit has to be something. Everyone can imagine a circle and also imagine a measuring rod, and each image will be quite arbitrary. But it is curious to argue that this would be without a unit of measurement – precisely since such a measuring rod is imagined too. For communication it helps to use the already existing unit of measurement, the meter. We can also use a circle with a circumference of 1 meter and thus a radius $r = 1 / \Theta \approx 16.16$ cm. Thus the unit would be "unit meter around" (UMA) and not degrees or radians. Here we have our turns, half turns, quarter turns. (Potentially the UMA has the meter dimension and the turn has none.) When drawing a sine function the student can plot out one meter instead of measuring out $\Theta = 6.28...$ meters.

We cannot wholly eliminate the unit circle because of sine and cosine and their neat derivatives. Sine and cosine are OK for triangles in arbitrary orientiation. With co-ordinates, they indicate $y$ and $x$ on the unit circle. Thus let us call them so too.

**Figure 3** and **Figure 4** give the situation. By choosing β on the Unit Circumference Circle (UCC) and co-ordinates on the Unit (Radius) Circle (UR), $x_{ur}$ = xur[β] = cos[β Θ] and $y_{ur}$ = yur[β] = sin[β Θ]. These functions thus translate the β turn to the $\{x, y\}$ co-ordinates on the unit circle.

It remains to document this a bit more and to show that exercises become more tractable. I have considered including the paper Colignatus (2008a) in this book but, as said, trigonometry is not my favourite subject and it suffices to refer.

NB1. π not only clutters traditional expressions but those expressions also implicitly use π to indicate the measurement in radians, letting students guess. NB2. Textbooks manage to write sin($x$) and cos($x$) where $x$ then both signifies the angle and the co-ordinate.

A typical question is: Solve $\cos(\alpha)^2 - \cos(\alpha) = 0$. Solved by: $\cos(\alpha)\ (\cos(\alpha) - 1) = 0$. Thus $\cos(\alpha) = 0$ or $\cos(\alpha) = 1$. Thus $\alpha = \pi/2 + k\,\pi$ or $\alpha = 2\,\pi\,k$ rad.

This now becomes: Solve $xur[\beta]^2 - xur[\beta] = 0$. Solved by: $xur[\beta]\ (xur[\beta] - 1) = 0$. Thus $xur[\beta] = 0$ or $xur[\beta] = 1$. Thus $\beta = ¼ + ½\,k$ or $\beta = k$ UMA. Less cryptic: $\beta = 0$, ¼ or ¾, and each subsequent full turn from there.

**Figure 3: The unit circle ($r$ = 1) and Xur and Yur**

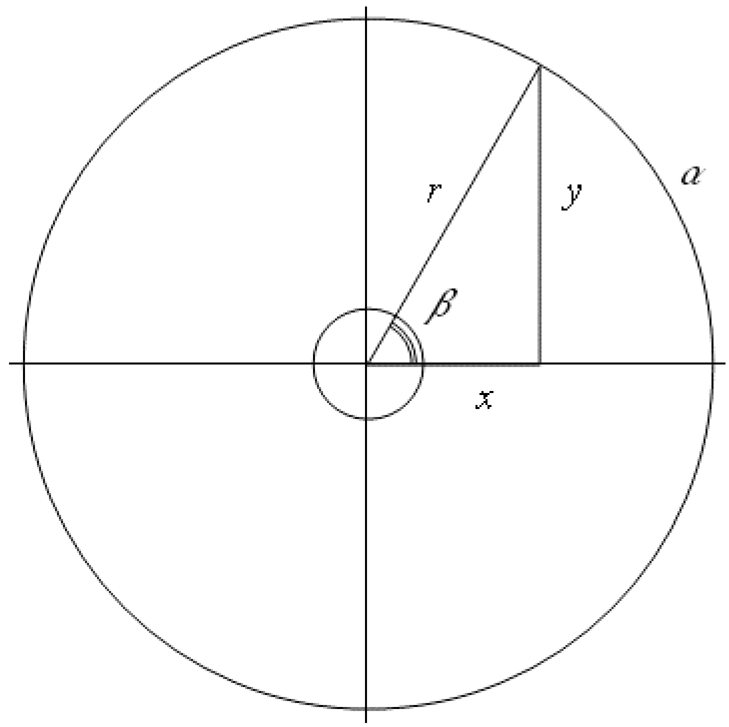
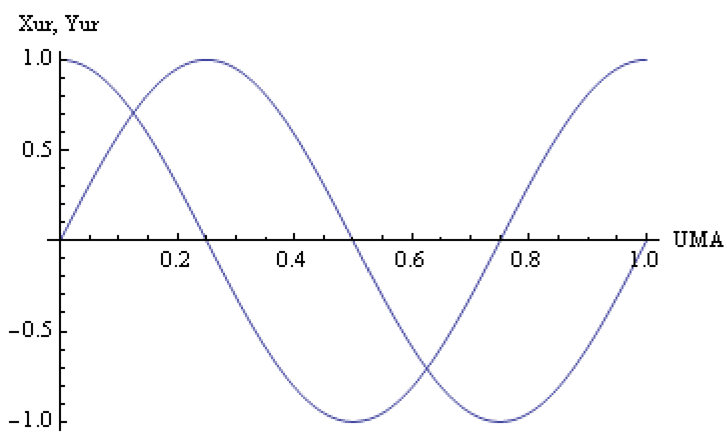



**Figure 4: The functional graphs of Xur and Yur**



# 34. The derivative

Calculus can be developed with algebra and without the use of limits and infinitesimals.

Define $y / x$ as the "outcome" of division and $y // x$ as the "procedure" of division (see page 26). Using $y // x$ with $x$ possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplication. The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ and then setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $| x |$ traditionally is undefined at $x = 0$ but would algebraically be sgn[$x$], and so on.

This longer discussion can best be put in a separate chapter, see page 75. That discussing improves upon a version of July 2007 on my website.

# VII. Questions for evidence based education

## 35. What to test ?

A new trend is evidence based education (EBE), by analogy of evidence based medicine, while the stock market crash and economic-financial crisis has caused the call for evidence based finance. In academic hospitals care, study and training are combined, and by analogy we best get academic schools where education of pupils is combined with study on their education and training of their teachers. A good friend of mine has warned though that pupils and students tend to be much too diverse, not only across time and culture, but also in personal histories, to allow for much accuracy even with huge sample sizes. Thus let us be cautious. And let us be aware of the issues of equity involved – which kids will get the increased attention ?

The institutes of education themselves can be subject to closer study too. A study on institutional set-up may be easier and more productive than studying specifics (e.g. textbook page A versus variant page B).

## 36. Test questions

The following issues crossed the mind as suggestions for such research in set-up:

(1) School organization depends crucially upon the concierge and other facilitators. Generally their wage costs are out of line, causing a reduction of services such as opening hours. The economic analysis in Colignatus (2005) helps to free resources.

(2) Schools follow a model developed in medieval times for the elite, with full time learning. Why not allow an integration of work and study at already younger ages ?

(3) Dronkers [10] observed: In a greying society, the stock of teachers is confronted with fewer students, which might cause schools to allow more students into the higher tracks of education, causing a drop in general quality.

(4) The greying of society and the rapid development of ICT affects the gap between teachers and students, between what is done and could be done.

(5) Teenagers apparently have a different biological clock.

(6) European textbooks still do not deal properly with backgrounds of migrants.

(7) There can be more democracy at schools, see Colignatus (2007b).

(8) Empowerment of teachers will affect quality. Will teachers have influence on what questions are researched in EBE ?

(9) I found Gladwell (2000, 2008) illuminating on, as already mentioned, repetition, but also on (a) organization size of 150 people, (b) enrollment per half-year instead of per year, (c) too long summer vacations (at least in the USA), (d) Asian counting, (e) rice paddies and the impact of persistence on math competence.

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[10] http://www.eui.eu/Personal/Dronkers/ - lost the actual reference. Dutch / Flemish: http://www.o-zon.be/teksten/proefdrukmanifest/profjaapdronkers/index.html

# 37. Number sense

### Language and brain memory

Most of this book can be subjected to EBE of course but there is one point that deserves explicit mentioning. Gladwell (2008:228):

> “(…) we store digits in a memory loop that runs for about two seconds.”

English numbers are cumbersome to store. Gladwell quotes Stanislas Dehaene:

> “(…) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits.”

Apparently fractions in Chinese are clearer too. Instead of *two-fifths* it would use *two-out of-five*. First creating *fifths* indeed is an additional operation. Perhaps the West is too prim on the distinction between the ratio 2:5 and the number 2/5. Perhaps it does really not make a difference except in terms of pure theory – the verb of considering the ratio and the noun of the result (called “number” when primly formalized in a number theory).

On addition:

> “Ask an English-speaking seven-year-old to add thirty-seven plus twenty-two in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add three-tens-seven and two-tens-two, and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It’s five-tens-nine.”

I am not quite convinced by the latter. Thirty-seven can be quickly translated into three-tens-seven and twenty two in two-tens-two. The “thir” and “ty” are liguistic reductions of “three” and “ten”. There is no need to create the digital image of the numbers. I can imagine two tracks: pupils who learn to mentally code thirty (sound, and mental code too) as three-tens (brain meaning) and pupils who follow the longer route via the digits. That said, the Western way is a bit more complicated.

The problem has a quick fix: Use the Cantonese system and sounds for numbers. It would be good EBE to determine whether this would be feasible for an English speaking environment (for starters, located in Hong Kong).

### Sounds and pictures

There is a bit more to it, though, and also relevant for this EBE.

In Gladwell’s case the pupils apparently are given a sum via verbal communication. This differs from a written question. There are two ways to consider a number. 37 can be seen as a series of digits only and pronounced as three-seven or it can be weighed as thirty-seven or three-tens-seven. We have to distinguish math from the human mind.

(a) For the mathematical algorithm of addition only the first suffices since the order already carries the weights. The mathematically neat way starts with the singles, as indeed Gladwell first mentions 2 plus 7 is 9.

(b) But a human mind tends to have different priorities and is interested in size. The human mind tends to use the weights and to focus on the most important digit. Witness “nine thousand four hundred twenty six”. In a written question this tendency is easier to suppress. In a verbal question the tendency is stimulated. Depending upon the circumstances there can be more focus on the size. The actual algorithm / heuristic that a pupil uses can be special, like first adding up the thousands, then the hundreds, tens, singles, and then resolve the overflow. The Asian child might indeed start with three plus two is five.

The distinction also shows from our uses of ten, hundred, thousand, ten thousand etcetera. Counting in traditional / verbal manner uses these infixes to indicate the place and the unit of counting. The weight infixes are more intended for communicating size and would be redundant for merely transmitting the number – though redundancy can help for checking. In a digit system it suffices to say one-zero, one-zero-zero, etcetera. Expressions with weights still can be ambiguous. With 100 million = 100 times 10^6 it follows that 123 pronounced as *hundred twenty three* can be understood as 100 times 23 = 2300. Clearly 23 is not a normal base but the potential ambiguity is there. Some people carefully say *one hundred and twenty three*.

A deeper issue is that the West writes and reads text from the left to the right while Arabic numbers are from the right to the left. Thus *fourteen* is 14.

English already adapted a bit, with *twenty one* and 21. Dutch still has “een en twintig” up to “negen en negentig”. From hundreds onwards Dutch follows the Arabic too, for example “vijf honderd een en twintig”. French of course still has the special “quatre-vingt” for 80 and “quatre-vingt-treize” for 93.

There are two key properties of the Arabic order:

- The mental advantage is that the most important digit is mentioned first.
- The disadvantage is that addition and multiplication work in the opposite direction from reading. It goes against the flow. And it also affects overflow. For example 17 + 36 = 53 has overflow 7 + 6 = 13 and this has to be processed from the right to the left.

The requirement on eye, ear & hand co-ordination again shows the importance of Kindergarten – see the work by economist Heckman, e.g. his Tinbergen Lecture, who confirms what Kindergarten teachers have been telling since ages.

### Flow and overflow

The flow and overflow problem is a bit awkward. It would be interesting – when we are considering changing to Cantonese – to see whether it can be solved at the same time. Thus, can we write numbers in the opposite way ? Let us use the word “Novel” when we write “123” for the Arabic number 321 (and try not to get confused).

Something strange happens.

On close inspection, say for Arabic 5,310,000, the eye traverses first from the left to the right to determine how many digits there are, the pupil deduces that 7 digits are millions, then either calls out the number from memory or the eye goes back, from the right to the left to the beginning, and then the pupil reads it off. Possibly there are parallel processes, as the eye picks out words rather than letters. What remains though is that to say “five million, 3 hundred ten thousand” is not exactly following the reading order since there is a jump somewhere. The Jump is unavoidable since the number of digits has to be counted. As the mind focusses on the most important digit, the speaking order will reflect the order in the mind – which is independent of the reading order.

Thus, where had the distinction between the mathematical algorithm and the human mind we now see a parallel distinction between reading order and order of pronunciation.

Let us first work silently on paper, or only pronounce the digits in stated order without pronouncing the whole number. To distinguish the Novel from the Arabic it will be most useful to write them in mirror image (perhaps as they are intended to be read if you change the reading order). Thus 19 becomes 91 . It does not take much time to get used to and **Table 3** contains the first practice.

### Table 3: Novel versus Arabic notation and addition

| | |
|---|---|
| 4321 | 1234 |
| 765 | 567 |
| 98 | 89 |
| 0981 | 1890 |

Overflow thus is processed neatly in the reading direction. This is straightforward. Thus, to repeat: the mathematical algorithms for addition and multiplication basically work on the digits and not on how the whole numbers are pronounced. Addition in Arabic 17 + 36 = 53 works with the digits as "one-seven plus three-six gives five-three". Addition in Novel works with digits as *seven-one plus six-three is three-five*. The difference in the latter case is only that the overflow is processed in the reading order.

#### The problem is pronunciation

The tricky question appears to depend upon pronunciation. There is no pronunciaton for a written test question. Digit-wise pronunciation, provided that the Arabic / Novel convention is in place, is even feasible. Pronunciation causes problems when a number is communicated (verbally) with weights. Even a written question may carry this problem if the number is not merely processed in an algorithm but subvocalized. Subvocalization tends to happen as part of the process of understanding when the mind wonders what the number means.

The true questions are how we would pronounce these Novel numbers and how pronunciation with size interferes with the neat algorithms. If we follow the Novel reading and writing order, our mind still wants to pronounce it starting with the most important digit. In that case the speaking order is opposite to the reading order. This seems like a burden. But it is a subtle matter again, because of the Jump.

There are four options: writing Arabic/Novel and pronoucing leftward/rightward. The current situation is that the number is written Arabic and spoken rightward (from the left to the right). The option to write Arabic and pronouce leftward (from the right to the left, as Arabic is written in Arabia) is not relevant since we lose the advantage of pronoucing the most important digit first, without any benefit. Let us consider the two other options.

#### Writing Novel and pronoucing from the left to the right

In this case 91 is pronounced nine-one-ten. We stick to the text direction and the linguistic translation of numbers essentially mentions the digits as they appear and adding the weight. This approach has the drawback that the largest value appears at the end.

There are some epi-phenomena here. People may have a tendency to drop infixes and this may cause ambiguity. *One-two-three-hundred* that drops the *ten* could also be understood as *one-two-three hundred*, which then would be 32100. It seems that this kind of ambiguity could be prevented by first mentioning the base, as in “million 5.31”.

**Writing Novel and pronoucing from the right to the left**

The other possibility is to write 000,01Ɛ,ꙅ and still say “five million, 3 hundred ten thousand”, i.e. temporarily reading from right to left. This would combine the Novel way (so that addition and multiplication follow the reading order) with starting the pronunciation with the biggest digit. There would be a small added advantage in that you first count the digits and then have the option to say “about 5  million” if that is adequate, without resorting to reading it wholly in reverse direction. Writing from dictation would be more involved, requiring the dictator to either start with the lowest digit or stating the number of places in advance. It seems like a do-able system.

## Conclusion

We will not quickly drop the Arabic numbers and writing order. But EBE on these aspects will help.

# 38. Memory

Yates (1974) relates that society used to be built upon the training of memory. Orators like Cicero are inspiring examples but law makers, lawyers and bureaucracy alike in the ancient and medieval world would require it in mundane fashion. The art of memory tended to rely on the trick to foster memorabilia and associate new matters with those. One could for example visit a temple or church, memorize the statues and their locations, and associate the steps of a mathematical proof with the separate points along the physical walk. The art of memory was embedded in a wider culture of learning, philosophy and ethics, in which, indeed oratory played an important part. However, when the printing press was invented and the abundance of bibles facilitated the rise of Luther and Calvin, with reliance on the bible instead of authority, the protestant iconoclasts did not only destroy the statues in the churches but also their images in memory, since also the classical education was reformed and pruned from the old ways. Society became dependent upon the printing press, a world faded and the art of memory with it.

For evidence based education it would be interesting to determine whether a rekindling of perhaps some modified form of the Art of Memory would not be beneficial.

PM. Symbols and notation in mathematics are also anchors for memory, which explains part of their importance. Writing perhaps started from accounting and subsequently was hijacked by the literary people who now regard anything that isn't text as an abomination. See Barrow (1993).

# VIII. Re-engineering the industry

## 39. Introduction

Countries differ in histories, regulations, organizations, conventions. I am only vaguely aware how they differ. It is relatively easy to download material on mathematical content from the internet but it is rather more complex to understand the situation elsewhere. My base is Holland and I only tentatively write for an international audience, precisely to get more abstraction. Readers from other countries will go for the abstraction but may nevertheless find some aspects interesting that pertain to Holland.

## 40. Goal

Economists distinguish between competitive markets where participants have no influence on price and quality and non-competitive markets such as oligopoly and monopoly where participants have influence. A hybrid combination is monopolistic competition where products are so special that each seller is a monopolist in the niche while buyers are budget constrained and still have to choose amongst sellers.

Our subject is the education in mathematics in a country. Education is quite specialised and thus non-competitive with many features of monopolistic competition. A market like this cannot be left to itself and requires a market manager and clearing house. Markets for food and medicine are already quite regulated and the same would hold for education. Economics emphasizes the advantages of free enterprise and competition. People should be free to set up a school, appoint teachers, collect materials and enroll students, and hope that employers accept the graduation certificates. But there are standards and the market only works well if properly regulated. Aspects are didactics, quality, norms, levels, standards versus implementations, evidence based education. Projects must be contracted out, managed, evaluated. There are economies of scale and scope while freedom can be enhanced by smart social engineering. For example, products can be acquired centrally and put in the public domain.

It is useful to have a market manager and clearing house for the education in mathematics. There is a letter soup of existing organizations for niches in the education in mathematics, and their role needs monitoring and evaluation.

## 41. Governance

The Ministry of Education would supervise education in general only. For the branch of the education in mathematics there would be a national institute named *Mathematics Education Name of the Country* (MENC) – like the national statistical offices have managed to call themselves *Statistics Name of the Country*. The MENC runs ME. The MENC will also have the authority to set the standards, specifications and details of the computer algebra language in the open domain that is also used in education, obliterating any claim by commercial parties, also potential claims based upon the past.

The **MENC Council** is open to society. It has seats for (1) representatives of (a) parents, (b) pupils and students, (c) business and labour, (d) the arts and the media, (2) presidents of recognized associations of (e) mathematicians in general, (f) teachers of mathematics, (g) institutes of education (employers), (h) professions who use mathematics, (i) producers of educational materials such as textbook authors and programmers, and their publishers.

The **MENC User Parliament** consists of (a) teachers of mathematics and (b) producers of educational materials such as textbook authors and programmers. Each year a quarter in replaced by elections in the constituencies.

The **MENC Executive** has at least a quarter of its employees in parttime teaching.

## 42. Finance

Finance comes from the national treasury. Reasons are: (i) Bildung, (ii) key role for other subjects, (iii) economies of scale and scope, (iv) contribution to the national economy, (v) necessity. The necessity follows from the economic observation above. Improvements don't come about when there are no funds. Teachers are no entrepreneurs. They write and teach and can program software but this remains fragmentated in niches when there is no organization and when there are no funds.

## 43. A Dutch experience

Bear with me. Last Autumn I proposed to create a *Simon Stevin Institute* (SSI) for this basic infrastructure, see Colignatus (2008c) – when the idea to call it *Mathematics Education Netherlands* (MEN) had not occurred yet. Independently and at almost the same time, Poelman et al. eds. (2008) came with a *Masterplan Wiskunde* (MPW) with main support by (President of the Royal Academy of Sciences) Dijkgraaf, (Social Economic Council chairman) Rinnooy Kan, and (internationally known mathematician) (J.K.) Lenstra. My budget is EUR 10 million per annum and the masterplan requires EUR 18.5 million but does more on female participation (WoMEN ?). One recent development following that masterplan is the creation of a *Platform Wiskunde Nederland* (PWN) where two mathematical associations KWG and NVvW start working closer together to reduce fragmentation. The main difference is that MEN / SSI opens up the world of mathematics to society at large while MPW considers itself fantastic and wants to do more *public relations* to the multitudes out there who do not understand yet that mathematics is so important. Interestingly, mathematicians have a captive audience of the whole population during their six to twelve formative years, but they still manage to foul it up and then conclude that the cause must be not *us* but *them*.

The Dutch Minister of Education, Culture and Science, Plasterk was so kind to react to this suggestion of a MEN / SSI and even kinder to qualify it as "interesting and thoroughly developed" (letter 2009-11-26, BOA/EBV/82918). His reaction is that it would create a new layer of superfluous bureaucracy with respect to the various existing institutes. Clearly I didn't explain sufficiently clear that the MEN / SSI has been targetted to actually reduce bureaucracy. Perhaps this book gives a second chance. Hopefully we have our integrated textbook / computer algebra environment by 2015.

I agree with one idea of *public relations*. Other subjects like physics, economics and psychology depend upon mathematics. Their (women ?) professors will be respected by mathematicians. I move that some of the *masterplan* funds are used to distribute copies of this book to them. In the kind and warm light of reason flowers will grow.

# IX. Beating the software jungle

### *The need for a common computer algebra language*

As said:

> "The MENC will also have the authority to set the standards, specifications and details of the computer algebra language in the open domain that is also used in education, obliterating any claim by commercial parties, also potential claims based upon the past" (page 67).

This authority is useful (a) to set a common standard, (b) to prevent any confusion or commercial hold-up. Above on page 51+ we already saw the importance of the computer algebra language. Its notation must fit the textbook. It must be uniform across schools for economies of scale (more students) and scope (more applications) but primarily for didactic reasons – in that pupils and students do not switch easily between formats. See how hard it is to switch between 2 $\pi$ and $\Theta$. For example, society regulates that cars have (at least) four wheels, mirrors, brakes, drive on the one side of the road, and such. In the same way there is a national committee on spelling the language – not a popular committee though – since it matters both for education but rather also for legal documents. We need similar rules for doing mathematics on the computer.

### *The problem*

There is the distinction between the single common language and various commercial engines that can interpretate the language and evaluate it to produce results. The engines are the place for commercial competition. The problem that occurs is that commercial companies start mixing the two.

The major topic of this chapter is the commercial appropriation of the language of mathematics. The computer algebra languages are mainly created in the USA where there is a strong litigation culture. Such companies have a tendency to evade conflicts of copyright by creating new issues of copyright. By consequence it becomes rather impossible to do mathematics on the computer without paying for copyrights.

### *There is a good language available*

In 1993 I selected the commercial computer program *Mathematica* because it seemed better, closer to the language of mathematics. I have been using this program consistently since then. Looking at alternatives again in 1999 and 2009 still gives the same conclusion. The language used in the *Mathematica system for doing mathematics on the computer* is a straightforward implementation of the age-old mathematical conventions. There are some particulars but that is because people differ from computers, or that computers differ from other environments.

When mathematics adapts to the environment – speech, wax or clay tablets, papyrus, blackboard, printing press, typewriter, computer – then this does not imply copyrights for any particular firm. Mathematics is free for common use and without copyrights.

Above, the open source Sage / Python language has been noted. The suggestion was to start using this and work from there. The key words are "work from there". In other words, Sage / Python is not perfect. In particular, as a language Sage / Python appears rather ugly. The main question then is: why not use *Mathematica* ? Because it would be copyright protected ? Would you really be able to copyright mathematics ?

Other people have (developed) a preference for other (cheaper) computer algebra languages such as Maple or Maxima or Wiris. For the present discussion this is immaterial. In the following I shall write "*Mathematica*" and "WRI" (Wolfram Research Inc., the makers of *Mathematica*), and the reader can substitute the personal preference. What is important is that society arrives at a standard computer algebra language for education.

## *The major policy questions*

The major policy questions remain:

(i) Will society accept appropriation of the language of mathematics by WRI ? Will it accept a possible commercial claim by WRI on the mathematical language used in the *Mathematica system for doing mathematics on the computer* ?

(ii) If society accepts such a claim, will it accept the associated costs of using *Mathematica,* or incur the costs of alternatives (including the costs of an alternative language for mathematics) ?

(iii) If society does not accept such a claim, will it stimulate other producers to create engines that use the language of mathematics on the computer ?

These policy questions are answered either explicitly or implicitly. Current decisions are left to the unregulated oligopolistic market. By implication choice (ii) surfaces, with the associated high costs.

In 1999 and 2000 I wrote two papers on these policy questions. I will restate the summaries and provide the proper links to where the papers can be found. They are dated with respect to particulars but still relevant on the analysis and choice criteria.

## *A suggestion done in 1999*

The summary of my paper *Beating the software jungle. Selecting the economics software of the future* Colignatus (1999) reads: [11]

> "Currently there is a jungle of software for economics, for both professional and educational software, and including the supportive mathematics and statistics. A comparison of 1993 showed and now in 1999 shows again - at least to this author - that *Mathematica* is the most useful and promising software, both for its elegant language and its breadth of application. A problem with *Mathematica* is its current price of about $1500 for a professional licence. Part of the solution would be to separate the language and interface and the engine. Once the *Mathematica* language is adopted as the lingua franca of science software, for which there are no legal barriers, there can be competition in front ends, interpreters and compilers. Another part of the solution in the short term would be coherent and determined discussion of the economics community

[11] http://econpapers.repec.org/paper/wpawuwpgt/9904001.htm

> (software users and purchasing departments) with Wolfram Research Inc. (WRI), the makers of *Mathematica*. Also, as there might still be a (natural or lock in) monopoly, there could be regulatory action that creates a public service utility. WRI could name its price for becoming a public utility company, and we might see whether *Mathematica* users are willing to pay that."

The current price of *Mathematica for students* is EUR 160. As said Sage / Python is the available open source program and we may use it to get going. As a language it is pretty ugly but beggars can't be choosers.

## *An analysis done in 2000*

The other relevant article is *The Disappointment and Embarrassment of MathML* Colignatus (2000), with summary: [12]

> "W3C is about to release MathML 2.0. This should have been a joyous occasion, but it appears to be a horror. They created a horrible way to do mathematics on the internet. It is Byzantinely complex, unintuitive, unesthetic, highly undocumented, it requires complex software support, etcetera. A quite perfect alternative already exists in *Mathematica*: simple, elegant, intuitive, highly documented etcetera - and users of Maple may think similarly about Maple. W3C is reinventing the wheel, making it square, and putting the horse behind the cart. Their talk about providing a 'service to the scientific and educational community' is pure nonsense, as they precisely do the opposite. The real reason why W3C developed MathML is (a) that they didn't do their homework, (b) that they didn't *really* deal with the makers of *Mathematica* (or Maple). We can only solve this situation by have a serious discussion of the copyright status of mathematics. A short run pragmatic solution is to use a <mathematics use=Mathematica> and </mathematics> bracket in HTML (with possible other values, like Maple). This may be 'expensive' in the short run, but much cheaper and beneficial in the longer term. Update: This discussion now includes answers to reactions of others. Readers should keep in focus that this paper concludes to the proposal to the scientific community that we have a discussion on the question: Are we going to accept this gift from W3C, or is it something like the Trojan horse, that will actually destroy the intellectual freedom of mathematics ?"

The paper is dated on some aspects but still valid on the situation and the criteria. For example: [13]

> "The expression $(a+b)^2$ in MathML is to read as (see op.cit. for the explanation):

```
<msup>
   <mfenced>
          <mrow>
                 <mi>a</mi>
                 <mo>+</mo>
                 <mi>b</mi>
```

[12] http://www.dataweb.nl/~cool/Papers/MathML/OnMathML.html
[13] See also http://www.w3.org/Math/

```
            </mrow>
    </mfenced>
    <mn>2</mn>
</msup>
```

Conversely, the *Mathematica* Inputform is: (a+b)^2."

The MathML argument is that the latter is ambiguous between an exponent, an index or a footnote. They neglect (1) that that 'ambiguity' does not arise in *Mathematica* (there it is already defined to be InputForm), (2) that *Mathematica* already works on the computer, and (3) that MathML then doesn't deal with input by people.

The true story of MathML is that the math community is afraid of copyright claims. The reason to recall this is that it may happen again now with Sage / Python. 'Open source' sounds like a bargain but society may fall in the trap of *penny wise, pound foolish*.

The larger picture is a lack of regulation, at the cost of the freedom of mathematics and the education in mathematics. From this 2000 paper:

> "The general idea of this paper is that dealing with the language of mathematics is an issue of market structure. The current W3C solution is to "program around" market structure. The W3C solution would be 'open', and all other languages might be turned into property rights. This is an approach that is in direct violation with the tradition of mathematics itself, and that might indeed cause a market structure that we would not want."

## *Advance in 2009 from 1999-2000*

In 2009 there is no change on the fundamental data since 1999-2000:

(1) In terms of language, mathematics is free for common use and cannot be put under copyrights. The commercial product *Mathematica* uses a language that is a straightforward implementation of the age-old mathematical conventions.

(2) There remains the distinction between the single common language and various commercial engines. The engines are the place for commercial competition.

(3) The news in 2009 is (a) that MapleTA has advanced in the field for testing of pupils and students, where WRI, the maker of the *Mathematica* engine, apparently is absent, (b) that Sage / Python now are available as open source environment and engine.

(4) As language, Sage / Python is no improvement. Well, if something has already been done, it is hard to beat it, especially when you are afraid of copyright issues.

(5) Sage apparently could be produced quite quickly by use of the various bits and pieces of software that various mathematical programmers had already put on the internet. It still remains quite an enterprise to further develop and support it for a great variety of potential users. It may be doubted whether the open source community can provide the support on the applications that are required for education. The current community of users of Sage seems to be more of the variety of computer-wise math university students and graduates who differ, it may be noted, from junior high pupils.

Apart from the sad conclusion that the news indicates progressed fragmentation, it also reflects the tough choices facing the math community and educators in mathematics.

### *Comparing some costs*

As an example of costs: It might be cheaper when each pupil or student buys a copy of *Mathematica* at EUR 160 than:

(a) all the work to create Sage / Python (well, OK, it has already been created, but then the subsequent versions)

(b) suffer the difficulties and limitations of the Sage / Python language and engine (see for example the recent discussion that $x$ / $y$ should stand rather for normal division instead of giving the floor integer)

(c) suffer the (temporary) differences for pupils and students between Sage / Python and the MapleTA testing environment. (If this point has much weight, the overall choice might be Maple instead of Sage / Python. I have not checked what it's current price is.)

(d) create all kinds of applications (such as an own testing environment but e.g. also for economics and physics) but eventually change those again to the language as used in *Mathematica* anyway because of its more agreeable character. (In this scenario, a language interpreter is put on top of Sage, thus still with a non-integrated engine.)

Relevant are also the costs when we don't do anything. The above assumes the optimistic scenario that Sage / Python is selected so that at least something will happen. It is more likely though that stagnation and fragmentation continue if parliament doesn't re-engineer the industry.

### *Managing the industry*

I did not perform a survey in the mathematical industry how they think about these issues. This is beyond my means and a bit beyond the immediate relevance. It is rather useless to ask views when people are not aware of the issues. For Holland, a good point of reference is the *Masterplan Wiskunde* (MPW) by the Dutch academic mathematical community (see on page 68). The plan does not mention computer algebra. It mentions an initiative without additional budget for more co-operation in the exact sciences on computional science, which is something else. As said the plan also mentions more attention from the academia for highschools but one of the major instruments is *public relations*.

Let us state some common sense hypotheses on views in the different layers:

- Kids in elementary school would actually already be able to use computer algebra, as they learn arithmetic and, according to Van Hiele, can master vectors. But teachers at elementary schools will hardly be aware of computer algebra.
- Teachers at highschools are aware of its existence but will still have little use for it. In Holland, highschools got stuck by selecting the graphical calculator. It is hard to get out of this because of the software jungle and the divergence in lock in interests.
- Professors at university will focus on 'real math' and will see computers as interesting topics for computer science only. For highschool math they rather want to see the same. They are not bothered much by students outside of mathematics, except that if non-mathematics students get math then they must still be taught by real mathematicians.

By consequence the important contribution of computer algebra for highschool pupils and non-mathematics students at university or college is lost. The didactic importance of algorithms, interactivity and feedback, the quality difference in math instruction between graphical calculators and computer algebra, the advantages of computerized testing (at liberty), the integration of subjects: the industry will not be interested.

Which has indeed been the case for the last 15 years.

## *A caveat*

*Mathematica*'s quality got me to use it. Another relevant quote from (1999):

> "While the discussion is open minded, it turns out that it still centers around *Mathematica*. The reader should be aware that a lot of my work thus is with *Mathematica*, and I even sell application software for it, see (…), so that I may have a personal lock in bias. Please check whether I am still level-headed. Please be aware too, that I do not want cross relations with Wolfram Research Inc. (WRI), the only providers of *Mathematica*, the product that my work relies on. So when I suggest to differentiate and to lower the price of the product, to separate the *Mathematica* language from front end and engine, and perhaps cutting up the company, I may still be biased in trying to be friends."

# X. The derivative is algebra

Improving the logical base of calculus on the issue of "division by zero"


### *Abstract*

Calculus can be developed with algebra and without the use of limits and infinitesimals.

Define $y / x$ as the "outcome" of division and $y // x$ as the "procedure" of division (see page 26). Using $y // x$ with $x$ possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplication. The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ and then setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $| x |$ traditionally is undefined at $x = 0$ but would algebraically be $\text{sgn}[x]$, and so on.


PM. The present discussing improves upon a version of July 2007 on my website.

### *Introduction*

Since its invention, the zero has been giving trouble. Mathematicians solved the paradoxes by forbidding the division by zero. But the problem persisted in calculus, where the differential quotient relies on infinitesimals that magically are both non-zero before division but zero after it. Karl Weierstraß (1815-1897) is credited with formulating the strict concept of the limit to deal with the differential quotient.

Consider the following expressions, three well-known and the fourth a new design.

(1) The difference quotient $\Delta f / \Delta x = (f[x + \Delta x] - f[x]) / \Delta x$ for $\Delta x \neq 0$. Note that one would see this as a result and not as a procedure.

(2) The differential quotient or derivative $f'[x] = \boldsymbol{df} / \boldsymbol{dx} = \lim(\Delta x \to 0)\ \Delta f / \Delta x$.

(3) The current theoretical true meaning of the derivative with outcome value $L$: $\forall \varepsilon > 0\ \ \exists \delta > 0$ so that for $0 < | \Delta x | < \delta$ we have $| \Delta f / \Delta x - L | < \varepsilon$.

(4) The new suggestion: $f'[x] = \boldsymbol{df} / \boldsymbol{dx} = \{\Delta f // \Delta x$, then set $\Delta x = 0\}$. This means first simplifying the difference quotient and then setting $\Delta x$ to zero.

Let us consider the various properties.

### *The old approaches*

The theory of limits is problematic. The limit of e.g. $x / x$ for $x \to 0$ is said to be defined for the value $x = 0$ on the horizontal axis yet not defined for actually setting $x = 0$ but only for $x$ getting close to it, which is paradoxical since $x = 0$ would be the value we are interested in. Mathematicians get around this by defining a special function $f[x] = x / x$ with split domain but this requires a separate "*f*" and it is faster to write $x // x$.

Also, the interpretation given by Weierstraß can be rejected since that definition of the limit still excludes the value (at) $\Delta x = 0$ which actually is precisely the value of interest at the point where the limit is taken.

While the Weierstraß approach uses predicate logic to identify the limit values, the new alternative approach uses algebra, the logic of formula manipulation.

Leibniz, Newton, Cauchy and Weierstraß were trained to regard $y / x$ as sacrosanct such that it indeed doesn't have a value for $x = 0$. They worked around that, so that algebraically $y / x$ could be simplified before $x$ got its value. While doing so, they created a new math that appeared useful for other realms. These new results gave them confidence that they were on the right track. Yet, they also created something overly complex and essentially inconsistent. Infinitesimals are curious constructs with no coherent meaning. Bishop Berkeley criticized the use of infinitesimals, that were both quantities and zero: who could accept all that, need, according to him, "not be squeamish about any point in divinity". The standard story is that Weierstraß set the record straight. However, Weierstraß's limit is undefined at precisely the relevant point of interest. "Arbitrary close" is a curious notion for results that seem perfectly exact. When we look at the issue from this new algebraic angle, the problem in calculus has not been caused by the "infinitesimals" but by the confusion between "/" and "//".

The present discussion can be seen as riviving the Cauchy approach but providing another algebraic interpretation that avoids the use of "infinitisemals". The impetus comes from the notion of the dynamic quotient in algebra. We cannot change properties of functions but we can change some interpretations. Undoubtedly, the notion of the limit and Weierstraß's implementation remain useful for specific purposes. That said, the discussion can be simplified and pruned from paradoxes.

Struik (1977) incidently states that Lagrange already saw the derivative as algebraic. See there for details and why contemporaries thought his method unconvincing.

## *The algebraic approach*

In a way, the new algebraic definition is nothing new since it merely codifies what people have been doing since Leibniz and Newton. In another respect, the approach is a bit different since the discussion of "infinitesimals", i.e. the "quantities vanishing to zero", is avoided.

The derivative deals with formulas too, and not just numbers. It uses both that $\Delta f // \Delta x$ extends the domain to $\Delta x = 0$ and that the instruction "set $\Delta x = 0$" subsequently restricts the result to that point.

Since we have been taught not to divide without writing down that the denominator ought to be nonzero, the following explanation will help for the proper interpretation of the derivative: first the expression is simplified for $\Delta \mathrm{x} \neq 0$, then the result is declared valid also for the domain $\Delta \mathrm{x} = 0$, and then $\Delta \mathrm{x}$ is set to the value 0. The reason for this declaration of validity resides in the algebraic nature of the elimination of a symbol, as in $\mathrm{x} // \mathrm{x} = 1$, and the algebraic considerations on "form".

The true problem is to show why this new definition of ***df*** / ***dx*** makes sense.

### *Stepwise explanation of the algebraic approach*

Let us create calculus without depending upon infinitesimals or limits or division by zero.

(1) We distinguish cases $\Delta x \neq 0$ and $\Delta x = 0$, and the (*) implicit or (**) explicit definition of relative error $r[\Delta x]$.

(2) Let $F[x]$ be the surface under $y = f[x]$ till $x$, for known $F$ and unknown $f$ that is to be determined (note this order). For example $F[x] = x^2$ gives a surface under some $f$ and we want to know that $f$.

(3) Then the change in surface is $\Delta F = F[x + \Delta x] - F[x]$. When $\Delta x = 0$ then $\Delta F = 0$.

(4) The surface change can be approximated in various ways. For example $\Delta F \approx \Delta x\ \ y = \Delta x\, f[x]$, or $\Delta F \approx \Delta x\, f[x + \Delta x]$, or inbetween with $\Delta y = f[x + \Delta x] - f[x]$, $\Delta F \approx \Delta x\ \ (y + \Delta y/2)$.

(5) The error will be a function of $\Delta x$ again. We can write $\Delta F$ in terms of $y = f[x]$ (to be found) and a general error term $\varepsilon[\Delta x]$, where the latter can also be written as $\varepsilon[\Delta x] = \Delta x\ r[\Delta x]$ where $r[\Delta x]$ is the relative error. When $\Delta x = 0$ and thus $\varepsilon[\Delta x] = 0$ then the relative error can be seen as undefined so it will be set to zero by definition.

(6) We have these relations where we multiply by zero and nowhere divide by zero or infinitesimals.

| | (*) *Implicit definition of r* | (**) *Explicit definition of r* |
|---|---|---|
| $\Delta x \neq 0$ | $\Delta F = y\ \Delta x + \varepsilon[\Delta x]$ | $r[\Delta x] \equiv \Delta F\ /\ \Delta x - \ y$ |
| $\Delta x = 0$ | $\Delta F = 0 = c\ \Delta x + \varepsilon[\Delta x]$<br>for any $c$; select $c = y$ | $r[\Delta x] \equiv 0 = c - y$<br>for $c = y$ |

(7) Simplify $\Delta F\ /\ \Delta x$ algebraically for $\Delta x \neq 0$ and determine whether setting $\Delta x = 0$ gives a defined outcome. When the latter is the case, take $c$ as that outcome.

(8) Thus $c = \{\Delta F\ //\ \Delta x$, then set $\Delta x = 0\}$.

(9) We then find $c = y = f[x]$ which can be denoted as $F'[x]$ as well.

For example, the derivative for $F[x] = x^2$ gives $\boldsymbol{d}F\ /\ \boldsymbol{d}x = \{(x + \Delta x)^2 - x^2)\ //\ \Delta x$, then $\Delta x := 0\} = \{2x + \Delta x,$ then $\Delta x := 0\} = 2x$. This contains a seeming "division by zero" while actually there is no such division.

The selection of $c = y$ is based upon "formal identity". This is a sense of consistency or "continuity", not in the sense of limits but in the sense of "same formula", in that (*) and (**) have the same form (each seen per column) irrespective of the value of $\Delta x$.

The deeper reason (or "trick") why this construction works is that (*) evades the question what the outcome of $\varepsilon[\Delta x]\ //\ \Delta x$ would be but (**) provides a definition when the error is seen as a formula. Thus, (*) and (**) give exactly what we need for both a good expression of the error and subsequently the "derivative" at $\Delta x = 0$. The deepest reason (or "magic") why this works is that we have defined $F[x]$ as the surface (or integral), with both (a) an approximation and (b) an error for *any* approximation that still is accurate for $\Delta x = 0$. When the error is zero then we know that $F[x]$ gives the surface under the $c = y = f[x] = F'[x]$ which is the function that we found.

In summary: The program is $F'[x]$ = $\boldsymbol{d}F$ / $\boldsymbol{d}x$ $\Leftrightarrow$ {$\Delta F$ // $\Delta x$, then set $\Delta x = 0$}. The definitions (*) and (**) give the rationale for extending the domain with $\Delta x = 0$, namely form.

## *Implications*

Perhaps other approaches can be found in the same manner. In the mean time it seems that the proper introduction to calculus is to start with a function that describes a surface and then find the derivative. Since we only use equivalences, this also establishes that the reverse operation on the derivative gives a function for the surface.

By implication, derivatives have no immediate association with slopes. Traditionally the derivative is created from the question to find the slope at some point of a function. This also suggests a separate development for the integral, e.g. with Riemann sums. Instead, here we find that the slope comes as a fast corollary – seeing that $\Delta F$ // $\Delta x$ would be the tangent if it is defined.

Let us look closer into the difference between starting from slopes or from surfaces.

The derivative of $| x |$ is traditionally undefined at $x = 0$ but would algebraically become sgn[$x$]. For $x \neq 0$, we can consider the various combinations and find the normal result, sgn[$x$]. For $x = 0$ the dynamic quotient gives ($|x + \Delta x| - |x|$) // $\Delta x$ = $|\Delta x|$ // $\Delta x$ = sgn[$\Delta x$]. Setting $\Delta x = 0$ gives 0. Hence in general $| x |'$ = sgn[$x$].

The traditional approach to $| x |$ is a bit complicated. Cauchy naturally gives 0 at 0 too. However, there is a multitude of "tangent" lines at 0, that is, when tangency is not defined as having the same slope as the function (which slope is undefined at 0) but as having a point in common that is no intersection. Traditionally the derivative is used for finding slopes and then the amendment on Cauchy was to hold that the right derivative differs from the left derivative, hence traditionally there is no general derivative.

In our approach, when we are interested in slopes, then it remains proper to consider these left and right derivatives. However, better terms are derivatives "to the left" and "to the right". We do not need to speak about limits but merely can point to the different values of the derivative sgn[$x$] in the intervals $(-\infty, 0)$, $[0]$, $(0, +\infty)$. Depending upon the definition of "tangent": (a) "Tangent" lines that have the point {0, 0} in common without intersection then can have slopes from $-1$ to 1. (b) "Tangent" lines that have the same slope as the function however have only the three slopes $-1$, 0, 1.

The dynamic quotient is the leading impetus here and the issue starts with algebra so that slopes come in only second. $| x |$ is the surface under some function $f$. Any approximation of changes in the surface, when the surface value is $| 0 | = 0$, finds a perfect answer with zero relative error by requiring $f[0] = 0$. The general function appears to be sgn[$x$]. The choice to extend the domain of $\Delta x$ with value 0 at $x = 0$ derives from a notion of consistency of the *form* of the relative error in the approximation. This is sufficient though not necessary. One could argue that the relative error is not defined when $\Delta x = 0$ but this runs counter to our choice to define it as 0. This choice again relates to the form of the relations in step (6).

## *Students*

Generations of students have been suffering. Teachers of math seem to have overcome their own difficulties and thereafter don't seem to notice the inherent vagueness.

Students not only suffer from the vagueness but also from the notation. Many forget to write "lim($\Delta x \to 0$)" as the first part of each differential quotient, each separate line again and again for each step of the deduction, assuming that stating it once should be sufficient to express that they are taking the limit. Some 'take the limit' so that for them $\Delta x$ has become 0, and then, just to be sure, they still mention "… + $\Delta x$" arguing that it should not matter when you add 0. Those 'official mathematical errors' will be past.

Conversely, if the new notation of dynamic division is adopted also for general purposes, see page 26, then the algebraic origin of the derivative will be sooner recognized, strengthening the insights in logic and algebra. Time can be won for more relevant issues.

Teachers may be less tempted to distinguish between 'those who know the truth' (Deep Calculus, the ε and δ) (who thus actually are wrongfooted) and 'those who only learn the tricks' (Superficial Calculus).

Didactics remain an issue. Above nine steps are somewhat elaborate while the short program {$\Delta F$ // $\Delta x$, then set $\Delta x = 0$} sums it up and suffices. Possibly some randomized controlled trials in education would bring more light in the question what explanation works where.

## *The derivative of an exponential function*

For exponential functions the dynamic quotient ($a^h - 1$) // $h$ or ($e^{h \text{ Rex}[a]} - 1$) // $h$ (see page 48) does not easily simplify. My current intuition would be to look into Lagrange's original Taylor development and use mathematical induction. Admittedly, this is still a vague suggestion only. The notion of a limit by itself still has its values of course. For example for the limit to infinity, and by implication for 1 // 0 again. It would not be right not to mention limits in education. And perhaps they still are the best approach of the exponential function.

## *Conclusion*

History is a big subject and we should be careful about drawing big historical lines. But the following seems an acceptable summary of the situation where we currently find us after the introduction of the zero.

Historically, the introduction of the zero in Europe around AD 1200 gave so many problems that once those were getting solved, those solutions, such as that one cannot divide by zero, were codified in stone, and pupils in the schools of Europe would meet with bad grades, severe punishment and infamy if they would sin against those sacrosanct rules. Tragically, a bit later on the historical timeline, division by zero seemed to be important for the differential quotient. Rather than reconsidering what "division" actually meant, and slightly modifying our concept of division, Leibniz, Newton, Cauchy and Weierstraß decided to work around this, creating the concepts of infinitesimals or the limit. In this way they actually complicated the issue and created paradoxes of their own.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and the derivative's shorthand lim($\Delta x \to 0$) $\Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken.

Logical clarity and soundness can be restored by distinguishing between the (formal) act of division and the (numerical) result of division. Using $\Delta f$ // $\Delta x$ and then enlarging the domain and setting $\Delta x = 0$ is not paradoxical at all.

The distinction between static and dynamic division suggests that the Weierstraß purity may be overly pedantic for the main body of calculus. The exact definition of the limit is of great value but not necessarily for all of calculus. Indeed, "most" derivatives can be found without the Weierstraß technical purity and "many" courses already teach calculus without developing that purity. Thus there is ample cause to bring theory and practice more in line.

# XI. Residual comments

This chapter collects comments that do not find a natural place in the other parts of the book but that still seem useful to include.

## *Scope*

### Kind of mathematics

This book does not cover all angles in mathematics. Its math tends to be a bit literal, with logic, notation and procedure. We should also consider geometry, shape, patterns, symmetry, regularity, order vs chaos, topics in probability and statistics. Pierre van Hiele has been arguing that kids at elementary school can already work with vectors, if only they are allowed to. The scope for improvement is large indeed. Perhaps even the abstractions of category theory. This book stays rather close to the traditional curriculum, it is actually quite conservative and it might well be that a more fundamental change is better.

### Notation

This book puts some weight on issues of notation. Notation in itself seems a trivial issue. Mathematics is done in the mind (or subconsciously, with the conscious part mostly in the spectator role). The mind codes addition and other operators in a different manner than we on paper. Notation however is important for communication. It becomes especially important in learning, especially for the weak student. Confusion quickly sets in, and wrong habits are hard to undo. We also have seen the link from notation to the more complex issues. Thus, the notational examples might seem trivial but their triviality also reminds us that those issues should have been solved long ago.

### Notation and psychology

Part of the issue can be seen in the Dutch distinction in math tracks, math A and B. These tracks cater to different psychological capacities (that are stimulated by tracking them). Track A relies for understanding on context, tends to a (vague) helicopter view and is less analytical. Track B is less influenced by or sensitive to context, or too sensitive so that it is better reduced, cannot do without an overview but digs analytically deeper. The good mathematician and in particular the econometrician does both, takes the context, makes the model, derives results, has an eye for detail, maintains that helicopter view, and also sees the purely mathematical properties behind all of it. Not everyone can play two instruments. Concessions must be made for practical education, and then issues of notation start playing an important role.

## *General phenomena and properties*

The examples are not just issues by themselves but can be caused by deeper processes, sometimes making them instances of those processes.

In some notational issues that we have considered the underlying property was that you have to develop a local schizophrenia. The key example is writing 2 + ½ as 2 ½. Some pupils and students can do so but then are conditioned so that they do no longer see what they do. It is a prerequisite of becoming a mathematician. An alternative example is the switching between a verb and a noun. In this case the switch is potentially productive instead of burdersome. The general property may well be cognitive dissonance, or reflect fundamentally how a brain works.

Mathematicians can be observant of confusions but once they have defined confusion away then they can be less observant in seeing the value in what people continue to tend to do in opposition to those definitions. Pupils and students over many generations have been right about the cumbersomeness and irrationality in mathematics. Supposedly they could not put a finger on precisely what the problem was but that was not their responsibility. What is crucial is that they got not listened to.

The best mathematicians would be happy with what they had learned themselves and would focus on new problems. Teachers face another trap. It is kind of natural to think that when you are the teacher then you must tell others what they must learn. But putting up a wall is something else.

## *Conjectures and refutations*

### Pretentions

A reader of an earlier Dutch version thought it pretentious to say "what is called mathematics but actually isn't so". But my examples were not refuted. The examples are not just examples, they are cases. They build up to *sufficient evidence* that an enquiry by parliament is desirable. They are called *examples* since other cases can and likely will show up. Each example likely could be handled without parliament sitting in but the total adds up. Math textbooks really should look different from what they are now.

### More research

It has been suggested that the issue requires more research, and that I would do this myself. However, my role is limited, and definitely my resources. I already referred to some sources on the history of mathematical education, the policy changes over the decennia, developments in didactics. This is sufficient. My role is precisely to clarify that need for enquiry. The first step is an enquiry by parliament.

### Prodding with questions

Another suggestion was that I would pose questions rather than solutions and opinions. This book indeed has a more open style than the original in Dutch that was argued more directly. Indeed, questions are more friendly than direct critique. They are necessary for teaching, also outside of the classroom. Prodding with questions may have a larger effect since readers discover themselves that something may have to change. My reaction is that I am a bit beyond the posing of questions. Questions have been posed for decennia. Kids have been brought to tears for failing math and because this affected their self-image and life. Mathematics allows clarity. Let us use that clarity. Mathematics fails. The education in mathematics does not give what we would expect from it. This is not a criticism per se but an expression of standards. Pointing to successes in the number of students that currently pass their highschool graduation is a bit awkward since they have been taught 'mathematics' that isn't really mathematics. *Lies, damn lies, and statistics.*

### Respect for the efforts made

Readers might feel that this book is disrespectful of the efforts by mathematicians and teachers of mathematics. If that is the impression indeed then let me correct it. There is great respect and gratitude. I stated that mathematicians are not quite blind to the issues raised here. They notice the hardship with their students. My problem is merely that they don't dig deeper. They keep the illogical material and then work on better didactics so that students can learn the illogical more easily. When mathematicians operate like this they are not tuned to the logic of empirical observation but to blind obedience to traditional authority. My idea is to set them free. It is the mathematical thing to do. Mathematicians will only convinced if they see that the mathematics can be improved.

### A note, hopefully subtle enough

Admittedly, some mathematicians in Sumer, 5000 years ago, and other mathematicians working in the middle ages, 700 years ago, were doing real math, even though their tools were, in the terms of this book, cumbersome and illogical. It would not be correct to say that these were only astrologers and alchemists since there was real effort (at times) at abstract thought and deduction. Without their cumulative effort we would be nowhere. But this is not the topic of discussion. It would become the topic of discussion if you would suggest that highschool math is replaced by Sumerian math, since, as you would hold, the only goal of education would be to teach pupils to think, and Sumerian math would be mathematics too (we agreed on that, in some respects). This kind of discussion tends to become awkward. People must have the right to vote. Children are people. Hence children must have the right to vote. In reponse, I would rather point to the concrete arguments and amendments given in this book. Current mathematics would be mathematics in the sense that you can think about it abstractly and do deductions on it, but at the same time it would not be real mathematics because of the errors exposed and the kind of attitude that shows from those errors.

### Barbarians at the gate

Some circles appear to regard parliament as barbarians. Totally unfit to judge about mathematics education in any way or other. Well, in that case: the barbarians are at the gate! As mathematicians haven't succeeded in bringing their house in order, these barbarians have every reason to think that they belong there too.

## *Industrial aspects*

### Bringing about change

We have mentioned various points that require further development and testing. We argue for change but have not given a blueprint for a new textbook. Eevidence based education sets new standards. When the mathematics industry starts processing our comments, then critique will turn into a self-critical-attitude. It is too simple though to assume that this will happen just by itself. Society will have to do some prodding.

### Education versus didactics

The current infrastructure around the education in mathematics creates its own problems. The capacity for self-organisation is limited, the product ill-defined and the moment of transaction rather vague. In the past there were more competing books, a teacher wrote her own book, found a publisher, and that was it. The size of the market has grown and specialisation is determined by the size of the market. Nowadays there seems to arise a distinction between education and didactics, with real education done in class ("what" to do – e.g. use 2 ½) and didactics done as a university science executed by Ph D's far away from class ("how" to do – e.g. test the various methods to torture students on learning fractions). The journals on didactics may hardly be read by teachers, and it is a science again to translate results from meta-analyses to possible application in class. Society feels helpless. It has provided the funds to improve on the education in mathematics but it may have created a new bureaucracy. Allowing each his or her ways keeps the peace. The bureaucracy has its uses, for there are "experts" and people like that idea. In a way it functions. The professional didactics earn their wages, articles are published, websites maintained, teachers are invited for their annual refreshing day. Little seems changed since the tale *Gulliver's Travels* by Jonathan Swift.

Even the size of the world may not be sufficiently large to create a decent free market for the ecucation in mathematics. Education for highschool may seem rather standard and one textbook on highschool algebra might well suffice for some hundred million highschool students. But the markets are fragmented, likely by their very nature.

### *Economics of education*

There is a subfield in economics, the economics of education. Given my economic comments on the education in mathematics it might be expected of me that I would delve deeper into that. However, I haven't felt the urge. For me it is a matter of logic. (1) The suggestions here would be an improvement. (2) Making the change would come at some cost. (3) The people who would bear the costs (teachers) would not be the ones who would benefit (students, future society). (4) Current society might compensate teachers, in an intertemporal cost-benefit analysis. (5) It is no use doing such calculations if there is no awareness of the issue and no perspective that it will be understood. (6) Such calculations would be difficult anyway since how would we score the effects of writing 2 + ½ instead of 2 ½ ? (7) Nevertheless, once the awareness of the problem has grown and once objections to a possible changes are based not on the miscomprehension of the content but on true costs, then it would be sensible to see if we can agree on the economic benefits.

### *Consequences*

We mentioned the consequences of the mathematical attitude for ALOE 1981, VTFD 1990, the stock market crash in 2008, and the environment.

Pure mathematicians will hold that they have no involvement with real world data and that it are the other sciences that deal with those. It is a valuable notion. There are also deeper philosophical aspects on how mathematics relates to the world. This 'refutation' however seems to misunderstand this book. This is not what we have argued. This book argues that mathematics suffers from itself. Even these pure mathematicians got lost on logic, 2 + ½ and the derivative. Subsequently, what to do with monks who claim to distance themselves from the world but who still want to eat and drink ? The Dutch *Masterplan Wiskunde* (MPW) referred to above, page 68, puts a lot of emphasis on the relevance for society – and they don't mind mentioning stochastic diffusion for option pricing just at the moment while the stock market crashed. So we are talking grey areas here and mathematics cannot evade part in the key responsibilities here.

### *Public relations and the power of math*

There is a TV series ("*Numb3rs*") where the hero mathematician helps to catch thugs by the use of mathematical techniques. It is good public relations. The logic is somewhat convoluted though. Undoubtedly mathematical theories can enlighten situations but that does not make mathematics an empirical science. Fortunately, the hero shines out as a person with outstanding ability. The common professor in mathematics would not be able to translate the empirical thug situation to the right mathematical format. Hopefully Hollywood script writers find inspiration to create an interesting series on math education in class. Gladwell (2008:239) contains an example.

### *A note on Barrow 1993*

Barrow (1993:1) on the power of math:

> "A mystery lurks beneath the magic carpet of science, something that scientists have not been telling, something too shocking to mention except in rather esoterically refined circle: that at the root of the success of twentieth-century science there lies a deeply 'religious' belief – a belief in an unseen and perfect transcendental world that controls us in an unexplained way, yet upon which we seem to exert no influence whatsoever."

I think that Barrow is a bit mistaken on empirical science. Only reality proves what abstraction is relevant for reality. Implications of abstractions are only relevant if the assumptions fit the bill.

But abstract thought is important. And the philosophical issues are worthy of a good discussion.

### *The stock market crash and the classroom*

The crash clearly has not been caused by mathematics taught in highschool. Teaching math in highschool is interesting because the math is of a fundamental nature and because of the didactics and the interaction with the pupils. Universities carry the burden of complexity and professional integrity. Some mathematicians might hold that highschools should return to hard-core axiomatics to ingrain the proper attitude. Alternatively, this book argues and shows that highschool math already suffers the non-communicative tendencies that are not corrected by universities. Returning more to the axiomatic method would by itself not be the cure.

### *Statistics*

Perhaps empirically and statistically it does not matter so much. Suppose we adopt all the suggested improvements. This only means that there is more scope for better teaching and learning but now the burder falls on the implementation in the classroom. Then the statistics become of prime importance. Perhaps all improvement disappears in the error soup of individual diversity. This book takes a logical position while using the available information but, indeed, it might not be enough. We can likely only tell after changes have been tried.

## *The farmer and his animals*

At a young age children assume that the farmer exists to look after the animals. Old presumptions die hard. At some age reality gets through and it is seen that the animals are there for the farmer. A waste processing plant created aerial dioxine that dropped on meadows around it. Quality controllers on milk wanted to forbid further grazing by cows there. The farmers protested that it was not their fault and that the milk should be allowed. Thus, also consumers are there for the farmer.

It is a good trick of the educational community to have some central exam requirement. For now it are the parents who want their kids to qualify instead of the teachers looking for a job. The professional sets the standards and the demand side has to qualify.

There can be all kinds of arguments of a self-serving nature that can be used to defend the bastion. The simplest is to argue against change since that would be difficult for the pupils and students while in reality it would only be a hassle for the teacher.

The golden rule in education is not to kill the natural interest in learning new things. Somehow this rule is broken regularly. It indicates an imbalance in the distribution of power between demand and supply. A teacher depending upon results would have every incentive to keep kids interested in learning.

Of course, as the literature on incentives shows, they may have unintended effects. This holds as well for current incentives. The master / apprentice relationship seems the most sturdy model and this book is not on that aspect. But it can be usefully mentioned.

# XII. Conclusions

## Intermediate conclusions

Mathematics is man made. Education is man made. Pupils and students are people too. We can only say something about education in mathematics when we have the proper empirical attitude and attention for reality.

Education has progressed a lot but in this day and age we don't get good results and there is still much to improve.

We found:

- Spatial sense and understanding is hindered and obstructed by subservience of the line to the function, inconsistent names of parameters, switches in orientation of tables and graphs, opaque or inconsistent terms, cumbersome treatment of derivatives, maltreatment of co-ordinates, vectors, complex numbers and trigonometry.
- Algebraic sense and competence are hindered and obstructed by inconsistent brackets, switches in plus / times with fractions, language idiosyncracies, the cult of the radical sign, intractable terms, untenable conventions of exactness and approximation.
- Logical sense and the competence in reasoning are hindered and obstructed by above confusions and cumbersomeness, the withholding of explicit discussion of logic and set theory, the withholding of the basic calculus of probability, by not surporting the development of mathematical ability in general by means of such formalizations.

What is seen as mathematics appears to be illogical and/or undidactic. Hence it has to be redesigned. It is no use to improve on the didactics of bad material, it better is replaced. We also considered only a number of topics, a selection of ideas that this author found interesting to develop a bit. More can be found. We should allow for the possibility that teachers have more comments and suggestions themselves (though our critique is that either they don't have them or don't follow up on them). The situation is wanting.

This book looks at the result rather than at how this situation could have come about. Still, if the result is inadequate, the conclusion is warranted that something is wrong.

One of the most important human characteristics is the preference for what is known and familiar – and mathematicians are only human. They adapt to new developments and are are critical and self-critical, not only with respect to what is discussed but also on how things will change. Nevertheless, key issues got stuck, and the industry as a whole is incapable of freeing itself from grown patterns. New entrants in the industry are conditioned to the blind spots, and pupils and students suffer them.

The situation is not such that there are no mathematicians to improve on content and that we lack researchers in didactics to improve on that angle. This book will hopefully be read by some in both groups and contribute to improvements. But it would be wrong for governments to think that it would suffice to leave the matter to the industry, and possibly give more subsidies for more of the same. More funds may well mean more outgrowth of awkwardness, cumbersomeness, irrationality. A call for more teaching hours may well mean more hours to mentally torture the students even more. Given this whole industry and the inadequate result the conclusion is rather that the whole industry is to be tackled.

Indeed, it sounds so well. Mathematicians will hold that only they are capable of deciding what is ‘mathematics’. Researchers in the education of ‘mathematics’ will hold that they do the research and nobody else. Will they regard this book as ‘research in the education in mathematics’ ? *Quis custodet custodes ?* It will be a mis-judgement to provide the industry with more funds without serious reorganization.

In sum, we have considered the work of men and found them to be men. It is a joy to see all these issues to improve upon. Let us hope that mathematicians proceed in this direction indeed. Let economists and the other professions support them.

## Final conclusion

My final conclusion definitely applies to Holland. I tend not to judge about other countries. But the same cumbersome and illogical issues can also be seen internationally. There is a structure to it. It is part of the economics of regulation. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture condition mathematicians to see what they are conditioned to see. The industy cannot handle its responsibility. This must hold internationally, country by country. A parliamentary enquiry is advisable, country by country.

Parents are advised to write their representative – and not only those who pay for extra private lessons. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

# Epilogue

It is useful to be aware of the following. With respect to the Rijken van Olst **Figure 1** it can be observed that an econometrician has more scope to be misunderstood on more angles.

My books ALOE and VTFD referred to above have not received much attention. This holds in general, also for my fellow economists but also for mathematicians, who would be potential readers with respect to the theorems discussed there. Some readers might think that this explains my criticism on mathematics and mathematicians. So let me recall what I wrote, in the Introduction about my nature. It is not affected. It is not logical to interprete lack of attention and / or appreciation into something that is targetted at my person and that would have to affect the way how I feel. OK, I miss out on some satisfaction of the meeting of minds but the potential readers who neglect ALOE and VTFD miss out on some good books and fundamental theory, and their attitude and misunderstanding rather reflects on them than on me.

With this established, it is useful to be specific on these points.

- For VTFD I refer to a text on my website. [14] The mathematicians who clearly did not understand voting theory later participated in the already mentioned *Letter to the governments of the EU member states* advising the use of the Penrose Square Root Weights (PSRW) for the EU Council of Ministers. This letter was misleading in argument and professionally deficient on VTFD, see Colignatus (2007c). In a parallel track, there was a sorry episode with wikipedia – quite sad for its users. The main perpetrator was a math student from MIT. [15]
- For ALOE, I can refer to ALOE itself as it explains what happened in 1981. In short, see page 55 above. In ALOE I already applied the distinction in economics on static and dynamic analysis to propositions (static) and inference (dynamic) in logic. We see this return in the distinction between verbs and nouns. The professor who did not appreciate ALOE in 1981, in the discussion back then did however appreciate that distinction. He later got the Spinoza Award for a project "Logic in Action" from 1997 to 2001. It is not clear to me whether there is proper reference.

What happens with all of this is not so material by itself. Though it is relevant to observe that science apparently lacks adequate avenues to channel problems with professional conduct. [16]

[14] http://www.dataweb.nl/~cool/Thomas/English/Science/Letters/SCT-working-group.html
[15] http://www.dataweb.nl/~ (…) /Letters/2006-03-20-Comments-RfC.pdf
[16] With theft and peddling drugs you can call the police but when a professor repeats falsehoods ? You can write a paper clarifying this politely, put it on the internet and send the professor the link. Thereafter the professor tells not only falsehoods but also lies. Freedom of speech differs from graft. He says he hasn't read the paper and doesn't have to. But it belongs to the scientific mores that the other party looks into it. You are a competent econometrician and have taken the time to explain the issue. He may say that you are not a mathematician, and then doesn't know his Venn-diagrams. He may say that it is not peer-reviewed, but you approach him to do that. What next ?

I must confess to one important personal point of interest however, relating to writing this current book in 2009, that might affect it by way of conflicting interests that might contribute to bias. I avow that this is not the case, but, maybe I am not in a position to judge. This concerns my economics book on unemployment, DRGTPE, Colignatus (2005). This book is not getting sufficient attention by my fellow economists. I have a vested interest in getting this afloat.

One element in the current situation is that mathematicians pay insufficient attention to my work in ALOE and VTFD. The books show that standard texts are incorrect but they don't read the books. It would help that they did and subsequently could tell the economists that it is sound indeed. To the effect of "say, this is good work, why are you ignoring work by one of your own who is capable of something ?" It would help not being ignored on all sides. It would help to have some support on the minor confusions in current mathematics and then be able to face the major misunderstandings in current economics on the main problem in society.

In this case mathematicians started ignoring in 1981, perhaps they can be the first to restore this, following Gill (2008). A subtle point is of course my location so that the current situation may have come about by the idiosyncracies in Holland. It is no use to attribute to mathematics in general what happened in this small country (that is, on ALOE and VTFD, not education in general).

In the Introduction, I listed some major real world problems in which mathematicians have been busy, the stock market crash, ecological collapse, destruction of democracy, perversion of logic. This book adds education in mathematics. This epilogue adds the indirect contribution to mass unemployment (without stock market crashes).

Mathematicians thus are depicted here like lifeguards, who you'd expect to jump into the water to save a drowning person (mathematical theorem, which is their job), but who don't do so – while it also happens that this person holds on to some papers and yells "save these papers!" (economic theory). Perhaps some information overload ? Or merely more interested in their pet theories, the ladies on the beach (other theorems) ?

Please get the drift. This book is about both the education in mathematics and what is considered to be mathematics. A key aspect in the analysis is the diagnosis on the non-empirical training and attitude by mathematicians. Another key aspect is that ALOE and VTFD change conceptions about what mathematics is. ALOE implies (amongst others) that you must keep account of exceptions even in formal systems. VTFD implies (amongst others) that you should not confuse a theorem with your interpretation of it. A change on these aspects will, as a corollary, have effects on other issues as well. Such as on perceptions of my fellow economists on my analysis on unemployment. I think that it is important to be aware of that corollary. When you go to London for a holiday then it is a corollary that you are in Europe. When waking up in the hotel you might decide that a trip to Paris is actually a nice surprise. It would be not correct to infer that your trip was targetted for Paris – as it would be inaccurate to say that this book and its 'creative destruction of mathematics' was written with the idea to get my fellow economists to study DRGTPE. But there could be a wonderful bonus for the unemployed.

# Appendices

## What is new in this analysis ?

'New' is taken in comparison to others, and thus includes points also made in my earlier publications on this analysis. New are:

1) A list of examples / cases in mathematics that are cumbersome or illogical. Clarification and resolution.
2) Associated suggestions for better notation, such as decimal dot in comma-using countries, better brackets, 2 + ½ instead of 2 ½, eradication of the cult of the radical sign, better tables for drawing graphs, Rex[*x*] instead of Log[*x*], DoExp[*y*, 1/*n*] for solution by taking roots, *y* // *x* for dynamic division, say *turning point* instead of *vertex* of a parabola.
3) *New math:* ALOE and VTFD. Highly relevant and easy for education.
4) *New math:* A redesign of trigonometry with Θ, unit meter around (UMA), Xur and Yur. Much greater ease.
5) *New math:* Clarification that the derivative is algebra, as opposed to using limits and infinitisemals. Much greater ease for education.
6) Explanation of the fundamental causes. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture condition mathematicians to see what they are conditioned to see.
7) Suggestions for structural redesign of highschool mathematics.
8) Identification and direction of solution of main problems in ICT. Suggestion for a world standard in computer algebra. Creation of computer test rooms. Resolution of the problem of supervision and the costs of concierges for supervision and school opening hours.
9) Suggestions of research questions for evidence based education (in mathematics).
10) Suggestions for re-engineering the industry of mathematics education.

## Abstract

Education in mathematics fails. What is called ‘mathematics’ often is illogical. Pupils and students are tortured and withheld from proper mathematical insight and competence. Professors and teachers of mathematics apparently cannot diagnose this themselves. The economic consequences are huge. Let each national parliament take action starting with an enquiry.



## Summary

**Subject:** The education in mathematics, its failure and how to redesign it. Mathematics seen as an art and an industry that requires better regulation. Political economy of the education in mathematics.

**Method:** We do not require statistics to show that mathematics education fails but can look at the math itself. Criticism on mathematics itself can only succeed if it results into better mathematics. Similarly for the didactics of mathematics. Proof is provided that the mathematics that is taught often is cumbersome and illogical. It is rather impossible to provide good didactics on what is inherently illogical.

**Basic observations:** We would presume that school mathematics would be clear and didactically effective. A closer look shows that it is cumbersome and illogical. (1) This is illustrated here with 22 examples from a larger stock of potential topics. (2) It appears possible to formulate additional shopping lists for improvement on both content and didactic method. (3) Improvements appear possible with respect to mathematics itself, on logic, voting theory, trigonometry and calculus. The latter two improvements directly originate from a didactic approach and it is amazing that they have not been noted earlier by conventional mathematics. (4) What is called mathematics thus is not really mathematics. Pupils and students are psychologically tortured and withheld from proper mathematical insight and competence. Spatial sense and understanding, algebraic sense and competence, logical sense and the competence in reasoning, they all are hindered and obstructed. Mathematics forms a core element in education and destroys much of school life of pupils and students in their formative years.

**Basic analysis:** This situation arises not because it is only school math, where mathematics must be simpler of necessity, but it arises because of the failure of mathematicians to deliver. The failure can be traced to a deep rooted tradition and culture in mathematics. Didactics requires a mindset that is sensitive to empirical observation which is not what mathematicians are trained for. Psychology will play a role in the filtering out of those students who will later become mathematicians. Their tradition and culture conditions mathematicians to see what they are conditioned to see.

**Higher order observations:** When mathematicians deal with empirical issues then problems arise in general. The failure in education is only one example in a whole range. The stock market crash in 2008 was caused by many factors, including mismanagement by bank managers and failing regulation, but also by mathematicians and “rocket

scientists" mistaking abstract models for reality (Mandelbrot & Taleb 2009). Another failure arises in the modelling of the economics of the environment where an influx of mathematical approaches causes too much emphasis on elegant form and easy notions of risk and insufficient attention to reality, statistics and real risk (Tinbergen & Hueting 1991). Improvements in mathematics itself appear possible in logic and voting theory, with consequences for civic discourse and democracy, where the inspiration for the improvement comes from realism (Colignatus 2007). Economics as a science suffers from bad math and the maltreatment of its students – and most likely this is also true for the other sciences. Professors and teachers of mathematics – or at least 99.9% of them – apparently cannot diagnose their collective failure themselves and apparently 'blame the victims' for not understanding mathematics. The other scientific professions are advised to verify these points.

**Higher order analysis:** Application of economic theory helps to understand that the markets for education and ideas tend to be characterized by monopolistic competition and natural monopolies. Regulations are important. Apparently the industry of mathematics education currently is not adequately regulated. The regulation of financial markets is a hot topic nowadays but the persistent failure of mathematics education would rather be high on the list as well. It will be important to let the industry become more open to society. Without adjustment of regulations at the macro-level it is rather useless to try to improve mathematics education and didactics at the micro level. Mathematical tradition and culture creates a mindset, and mathematicians are like lemmings that are set to go into one direction. Trying to micro-manage change with some particular lemmings will not help in any way. An example layout is provided how the industry could be regulated.

**Conundrum:** Mathematicians might be the first to recognize the improvements in mathematics and didactics presented here. Mathematical tradition clearly is an improvement from alchemy and astrology. Most people will also tend to let the professors and teachers decide on whether these items are improvements indeed. It is tempting to conclude that the system then works: an improvement is proposed, it is recognized, and eventually will be implemented. This approach however takes a risk with respect to potential future changes. With the present failure and analysis on the cause we should rather be wary of that risk. We better regulate the industry of mathematics education in robust manner. The mathematical examples presented here can be understood in principle by anyone with a highschool level of mathematics. They are targetted to explain didactically to a large audience how big the failure in the education in mathematics actually is.

**Advice:** The economic consequences are huge. National parliaments are advised to do something about this, starting with an enquiry. Parents are advised to write their representative. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

# Literature

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