A Simple Supermodular Mechanism that Implements Lindahl Allocations

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15 January 2008

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May 12, 2009

Abstract

This paper introduces a new incentive compatible mechanism which for general preference environments implements Lindahl allocations as Nash equilibria. The mechanism does not increase in structural complexity as consumers are added to the economy, the minimum dimension of data needed to compute payoffs is smaller than other mechanisms with comparable properties; finally, for quasi-linear environments and appropriate choices of the mechanism parameters, the mechanism induces a supermodular game whose best reply mapping is a contraction. Thus, this new Lindahl mechanism provides a connection between the desirable welfare properties of Lindahl allocations and the desirable theoretical/ convergence properties of supermodular games.

1 Introduction

The reliance on unregulated markets for the provision of public goods presents well known challenges to efficiency. For economists, the existence of this problem continues to motivate the search for alternative institutions which may yield Pareto optimal outcomes. One problem with this approach is that some Pareto outcomes may not be desirable for everyone involved. Some people

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could end up being worse off than they were with their original endowment, a common critique, for example, of the Groves-Ledyard (G-L) mechanism. G-L is an institution that overcomes the free riding problem – the incentive to enjoy the public good's benefits while not sharing in its cost – but the mechanism may leave some participants worse off than before they participated. In contrast, Lindahl allocations, while also Pareto optimal, make no one worse off than he was to begin with (i.e., are individually rational). Lindahl allocations are therefore attractive outcomes when there are public goods.

In addition to being Pareto optimal and individually rational, Lindahl allocations share another important property of Walrasian ("competitive") allocations of private goods: every individual's payment is for each unit of the public good is equal to its marginal value to him. In the Walrasian setting consumers all face the same price and they demand potentially different quantities of a good; in the Lindahl setting they face distinct "personalized" prices and, in equilibrium, each consumer demands the same quantity of the public good. Actually implementing a Lindahl scheme, however is somewhat problematic, since it is not exactly clear how the personalized prices are to be determined. Perhaps one could use surveys, but there may then be incentives for participants in the surveys to misrepresent their preferences in order to pay a lower price. This has led to the development of *incentive compatible* public goods mechanisms.

The purpose of this paper is to introduce a new incentive compatible mechanism which attains Lindahl allocations as Nash equilibria. This is true for economies with an arbitrary number of consumers and general preference environments. In addition to this Nash "implementation" result, the mechanism has several other attractive properties that have been motivated by experimental research: it retains its structural simplicity as the number of consumers increases; the minimum dimension of data needed to compute payoffs is smaller than other mechanisms with comparable properties; the components of the mechanism have a clear economic interpretation; and for quasi-linear preference environments the unique equilibrium is stable under a wide variety of learning algorithms.

The mechanism introduced here is not the first to implement Lindahl allocations. Hurwicz (1979) and Walker (1981) were the first to present such mechanisms, but Kim (1987) has shown that both mechanisms are quite unstable. While some sort of dynamic stability is desired, there is no agreement in the literature about how people's behavior adjusts when out of equilibrium. In mechanism-design experiments, however, a common empirical finding is that in mechanisms with theoretically robust dynamic stability properties, subjects' behavior tends to converge. Supermodular mechanisms have been particularly successful.[1] This empirical regularity was presaged by the theoretical stability results established by Milgrom and Roberts (1990a) for supermodular games.

Chen's (2002) theoretical contribution is of particular interest. She presented the first Lindahl mechanism that is supermodular in quasi-linear environments for some values of the mechanism parameters. Thus she ties the observation that supermodular mechanisms tend to perform better in the laboratory to the welfare properties of Lindahl equilibria. The Chen mechanism has also had some initial success in a laboratory environment. Van Essen, Lazzati, and Walker (2007) experimentally tested three Lindahl mechanisms, including the Chen mechanism, finding that it converged quite close to its equilibrium messages. However, there are several reasons to be dissatisfied with this initial success. The Chen mechanism does not maintain its structure as the number of consumers are added to the economy; the amount of information consumers need to compute their payoffs increases in the number of participants. The experimental evidence suggests that the mechanism generates large amounts of tax waste when not in equilibrium, and participants frequently did worse than their initial endowment when out of equilibrium. Finally, it should be noted that there remain some practical concerns with how to satisfy the Milgrom and Robert's stability conditions when you apply the Chen mechanism (or any similar mechanism). In particular, there are several issues related to the unboundedness of the strategy space which is critical in the implementation part of the proof. The Lindahl mechanism presented in this paper builds upon Chen's contribution by: first, addressing the above concerns in the design stage; and second, by exploring an alternative direction than Chen in ensuring the dynamic stability of equilibrium.

The remainder of the paper will proceed as follows: Section 2 provides a simple definition of a supermodular game and summarizes some of the important properties these games exhibit; Section 3 outlines the basic public goods problem and mechanism environments; Section 4 contains the bulk of the paper's theoretical results concerning implementation; finally, Section 5

[1] For example see Chen and Tang (1998), Chen and Gazzale (2004), Healy (2004), and Van Essen, Lazzati, and Walker (2007).

compares the new Lindahl mechanism with several existing ones in order to clarify the paper's contribution.

2 Preliminaries

Supermodularity plays a significant role in several of the results to follow. In this section we review some definitions, framed in terms of the strategy spaces and payoff functions used in this paper. The strategy spaces are subsets of Euclidean spaces and the payoff functions are twice continuously differentiable (or C^2). More general definitions of a supermodular game can be found in Topkis (1998) or Milgrom and Roberts (1990a).

A normal form game is defined by a set of players, a strategy set for each player, and a payoff function for each player. Denote the set of players I, where $I = \{1, ..., N\}$. Functions belonging to players are indexed by a superscript while arguments are indexed by a subscript. Let $M_i \subseteq \mathbb{R}^2$ be player i's strategy space with an arbitrary element $\mathbf{m}_i = (m_{i1}, m_{i2})$, where $M = \times_{i=1}^{N} M_i$ is the collection of all players' strategy spaces. Last, for each player i let $u^i : M \to \mathbb{R}$ be a payoff function which maps strategy profiles into a numerical payoff.

A supermodular game is one in which the strategy spaces satisfy the above mentioned criteria and the payoff functions satisfy the following two criteria. The first criterion is that a player's marginal utility for increasing an action is increasing in his own actions. This is known as the **supermodularity property**:

Definition 1 *A C^2 payoff function u^i is supermodular if a player's own actions are strategic compliments–i.e. for each i*

$$\frac{\partial u^i(\mathbf{m})}{\partial m_{i1} \partial m_{i2}} \geq 0.$$

The second criterion requires that the marginal utility of each player for increasing an action is increasing in all of their rivals' actions. This is known as the **increasing difference property**:

Definition 2 *A C^2 payoff function u^i has increasing differences if a player's own actions are strategic compliments with the actions of all other players– i.e. for each i*

$$\frac{\partial u^i(\mathbf{m})}{\partial m_{in} \partial m_{jl}} \geq 0$$

for $n = 1, 2$ and $l = 1, 2$.

A game is supermodular if the payoffs for all players satisfy both properties:

Definition 3 *A game is supermodular if for each player i: M_i is a non-empty subset of $\mathbb{R}^2$, and u^i has the supermodularity and increasing difference properties.*

Supermodular games have nice properties which make them interesting for mechanism design. If the strategy space is compact and the payoff function is C^2, then Milgrom and Roberts (1990a) show that:

1. The set of serially undominated strategy profiles has a maximum and a minimum element, and these elements are Nash equilibria;

2. Under a wide class of dynamic adjustment processes the predicted behavior eventually ends up between the two extreme Nash equilibria. These dynamic processes include best-response dynamics, fictitious play, Bayesian learning, and others.

When the Nash equilibrium is unique, the predictive power of these results is increased: the first property implies that the game is dominance solvable and the second property says that the unique Nash equilibrium is "stable" under a wide range of adaptive behavior.

In the next section, I explain the public good environment, which can be thought of as a simplified general equilibrium problem.

3 The Public Good Economy

For simplicity, I restrict attention to economies which have $N \geq 2$ consumers, one private good, one public good, and a constant returns to scale production technology. The quantity of the public good will be denoted by x, and the private good for consumer i by y_i, where consumers are indexed by subscript i. Each consumer is characterized by a convex consumption set $C_i = \mathbb{R}^2_+$, an initial endowment of the private good $\omega^i > 0$, and no initial endowment of the public good. The public good is produced, using the private good as an input (quantity denoted z), with a constant returns to scale production technology $f(z) = \frac{z}{\beta}$ — i.e., each unit of the public good x requires β units ($\beta > 0$) of the private good. Thus β is the constant (real) marginal cost of production. An allocation in this simple economy is an $N+1$- tuple $(x, y_1, ..., y_N) \in \mathbb{R}^{N+1}_+$.

3.1 The Mechanism

A mechanism takes consumers' strategies (or messages), and maps them into an outcome (or allocation). Here I consider a mechanism in which consumers report messages to a "planner" who uses this information to determine an amount of the public good to produce and a tax for each consumer. The message space of consumer i is $M_i = \mathbb{R}^2$ with generic element $\mathbf{m}_i = (r_i, s_i)$. Let $\mathbf{m} = (\mathbf{m}_1, ..., \mathbf{m}_N)$ denote the profile of all players' messages. Consumer i's action r_i should be interpreted as a *request* from the consumer to the planner for r_i units of the public good. Notice that negative requests are allowed. Consumer i's other action, s_i, is interpreted as his *statement* about the amount of the public good that will be produced. Rather than write $(r_1, s_1, r_2, s_2, ...)$ for a strategy profile, I write $(r_1, r_2, ..., r_N, s_1, ..., s_N) = (\mathbf{r}, \mathbf{s})$. These messages are collected by the planner and used to determine an amount of the public good and a tax for each player i according to outcome functions $\chi(\mathbf{r}, \mathbf{s})$ and $\tau^i(\mathbf{r}, \mathbf{s})$ respectively. For any positive real numbers ξ, γ, and δ, let $\varphi^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s}) = \left(\chi(\mathbf{r}, \mathbf{s}), (\omega^i - \tau^i(\mathbf{r}, \mathbf{s}))_{i=1}^N\right)$ be a mechanism with outcome functions defined as follows:

$$
\begin{aligned}
\chi(\mathbf{r}, \mathbf{s}) &= \frac{1}{N}\sum_{i=1}^{N} r_i \\
\tau^i(\mathbf{r}, \mathbf{s}) &= P^i(\mathbf{r}, \mathbf{s}) \cdot \chi(\mathbf{r}, \mathbf{s}) + \frac{\gamma}{2}\left(s_i - \chi(\mathbf{r}, \mathbf{s})\right)^2 + \frac{\delta}{2}\left(s_{i+1} - \chi(\mathbf{r}, \mathbf{s})\right)^2
\end{aligned}
$$

where

$$P^i(\mathbf{r},\mathbf{s}) = \frac{\beta}{N} - \xi\left(\sum_{j\neq i}\frac{r_j}{N-1} - s_{i+1}\right)$$

where ξ, $\gamma > 0$ and $\delta \geq 0$ are positive parameters. Furthermore, interpret $s_{N+1} = s_1$.[2] $P^i(\mathbf{r},\mathbf{s})$ can be thought of as i's personalized price for the public good and the remaining two terms as *statement penalties* i must pay.

In words, the mechanism works as follows: the planner collects each consumer's request and produces an amount of the public good equal to the average request. In addition, the requests and statements are used to determine each consumer's tax, which is the sum of the two penalty terms and the term involving the personalized price. The first statement penalty for consumer i is increasing in the amount by which his own statement differs from the actual amount of the public good produced, and the other statement penalty is increasing in the amount by which his *neighbor's* (consumer $i+1$) statement, s_{i+1}, differs from the actual public good production. Since $\chi(\mathbf{r},\mathbf{s})$ is independent of $\mathbf{s}$ and since preferences are increasing in y_i, it is clear in a Nash equilibrium every consumer's statement will be correct. Consequently, in equilibrium both penalty terms will be zero for every consumer, and that the consumer will therefore simply pay the price $P^i(\mathbf{r},\mathbf{s})$ for each unit of the public good. Note that $P^i\left(\mathbf{r},\mathbf{s}\right)$is independent of both r_i and s_i.

The personalized price function, $P^i\left(\mathbf{r},\mathbf{s}\right)$, has an intuitive economic interpretation. The price is higher for a consumer who is perceived by their neighbor (consumer $i+1$) to demand more of the good than others and the price is lower if he is perceived to request less than others. The term $\sum_{j\neq i}\frac{r_j}{N-1}$ corresponds to the amount of the public good if consumer i did not participate in the mechanism. The term s_{i+1} represents consumer $i+1$'s statement about the level or quantity of the public good. Thus if $\sum_{j\neq i}\frac{r_j}{N-1} > s_{i+1}$, it means that consumer $i+1$ believes that consumer i's request will lower the level of the public good produced. As a consequence, i's personalized price is less than an equal share of the marginal cost. If $\sum_{j\neq i}\frac{r_j}{N-1} < s_{i+1}$, then the reverse is true and consumer i's personalized prize is greater than an equal share of the marginal cost. If $\sum_{j\neq i}\frac{r_j}{N-1} = s_{i+1}$, the personalized price is the

[2]For some non-equilibrium messages the payoffs are not completely well defined. That is they will take consumers outside of their consumption set C_i. This same weakness is shared by the Groves-Ledyard, Hurwicz, Walker, Kim, and Chen mechanisms. However it should be noted that each interior equilibrium there is a neighborhood on which feasibility is assured.

equal share of the marginal cost. Note that the first term in $P^i(\mathbf{r},\mathbf{s})$ is $\frac{\beta}{N}$, the per-capita cost of the public good.

3.2 Preference and Wealth Assumptions

The coupling of the mechanism $\varphi^{\xi,\gamma,\delta}(\mathbf{r},\mathbf{s})$ and a preference environment defines a game. I am interested in two types of preference environments: first, a "regular" environment where preferences satisfy a set of consistency conditions; second, a sub-case of the regular environment that satisfies some additional properties. The definitions of these environments are given below.

Definition 4 *A regular preference environment is one in which for each player has a complete and transitive preference relation $\succeq_i$ that satisfies the following properties:*

1. (Continuity): For every $(\bar{x},\bar{y}_i)\in C_i$, the sets $\{(x,y_i)|(x,y_i)\succeq_i(\bar{x},\bar{y}_i)\}$ and $\{(x,y_i)|(\bar{x},\bar{y}_i)\succeq_i(x,y_i)\}$ are closed in C_i.
2. (Convexity): If $(x,y_i)\succeq_i(\bar{x},\bar{y}_i)$, then $(\lambda x+(1-\lambda)\bar{x},\lambda y_i+(1-\lambda)\bar{y}_i)\succeq_i(\bar{x},\bar{y}_i)$ for any $\lambda\in[0,1]$.
3. (Strictly Increasing in y_i): If $\bar{y}_i>y_i$, then for any x, $(x,\bar{y}_i)\succ_i(x,y_i)$.

Definition 5 *E^Q denotes the set of standard C^2 quasi-linear environments– i.e. those in which, for each i, there is a real valued function v^i such that $u^i(x,y_i)=y_i+v^i(x)$, v^i is C^2, concave and its second derivative is bounded from below by some real number K_i such that $-\infty<K_i\leq\frac{\partial^2 v^i(x)}{\partial x^2}<0$. Note that $E^Q\subset E$.*

Finally, I assume that in equilibrium no consumer can be at his minimum wealth. I formally define this condition below, but in words, this condition states that in equilibrium, it must be possible to find a cheaper, feasible consumption bundle. This assumption is needed to rule out boundary equilibrium.[3]

[3]See Groves and Ledyard (1977) or Groves and Ledyard (1980) for a discussion of this assumption.

Definition 6 *(The "No Minimum Wealth Condition"): If* $(\mathbf{r}, \mathbf{s})$ *is a Nash equilibrium, there exists* y_i *and* (r_i, s_i) *such that* $(y_i, \chi(r_i, \overline{\mathbf{r}}_{-i}, s_i, \overline{\mathbf{s}}_{-i})) \in C_i$ *and* $y_i + \tau^i(r_i, \overline{\mathbf{r}}_{-i}, s_i, \overline{\mathbf{s}}_{-i}) < \omega^i$*.*

4 Implementation

The first result of the paper shows that the game induced by the mechanism $\varphi^{\xi,\gamma,\delta}(\mathbf{r}, \mathbf{s})$ implements Lindahl allocations as Nash equilibrium outcomes. Implementation is an exact correspondence between Lindahl and Nash outcomes. In other words, any Lindahl allocation can be achieved as the allocation of a Nash equilibrium; and at any Nash equilibrium the equilibrium allocation is Lindahl.

Theorem 1 *The mechanism* $\varphi^{\xi,\gamma,\delta}$ *implements the Lindahl allocations for any* $e \in E$*.*

Proof. See Appendix. ■

The Lindahl equilibria are also associated with the two fundamental welfare theorems of the public good economy. First, like Walrasian equilibria, Lindahl allocations are Pareto optimal. Second, any Pareto optimal allocation can be supported as a Lindahl equilibrium through appropriate redistribution of the initial endowment ω. The exact conditions needed for the existence of Lindahl equilibria can be found in Milleron (1972) or Foley (1970). Notice that Theorem 1 does not impose any restrictions on the positive parameters ξ, γ, and δ. These are free parameters which will be manipulated later in the paper to create a family of supermodular Lindahl mechanisms.

In order to illustrate the dual nature of this theorem the next example may be useful.

Consider a two-consumer economy, where each consumer is endowed with $\omega = 20$ units of the private good. Suppose it takes 4 units of the private good y to produce each unit of the public good x (i.e., $\beta = 4$) and that Consumer 1's and Consumer 2's preferences can be represented by the utility functions $u_1(x, y_1) = y_1 - \frac{1}{2}(6 - x)^2$ and $u_2(x, y_2) = y_2 - \frac{1}{2}(8 - x)^2$ respectively. The mechanism $\varphi^{1,1,1}(\mathbf{m})$ implements the Lindahl allocations of this economy.

Implementation of Lindahl allocations requires first that any Lindahl allocation can be achieved as a Nash equilibrium of the mechanism, and at any Nash equilibrium, the equilibrium allocation is Lindahl. For this example, I start with the first requirement.

From the utility functions we solve for both Player 1's and Player 2's demand for the public good (or their marginal rate of substitution) which are $MRS_1 = 6 - x$ and $MRS_2 = 8 - x$ respectively. Using the Samuelson marginal condition (i.e., that at a Pareto optimal quantity of the public good $MRS_1 + MRS_2 = 4$), the Pareto optimal level of the public good for these two consumers is $x^{PO} = 5$. Inserting this quantity into each consumer's demand for the public good, we find that the corresponding Lindahl prices for Consumer 1 and 2 are $\bar{P}^1 = 1$ and $\bar{P}^2 = 3$ respectively. Therefore this example has a unique Lindahl allocation.

Suppose $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium of the game induced by mechanism $\varphi^{1,1,1}(\mathbf{m})$. If the Lindahl allocation is to be achieved as a Nash equilibrium, then two equations must hold: first, the average request must equal the Pareto optimal amount, i.e.,

$$\chi\left(\mathbf{r}, \mathbf{s}\right) = \frac{\bar{r}_1 + \bar{r}_2}{2} = 5;$$

second, Player 1's personalized price function must equal his Lindahl price $\bar{P}^1 = 1$, i.e.,

$$P^1\left(\mathbf{r}, \mathbf{s}\right) = \frac{\beta}{2} - \bar{r}_2 + \bar{s}_2 = 1.$$

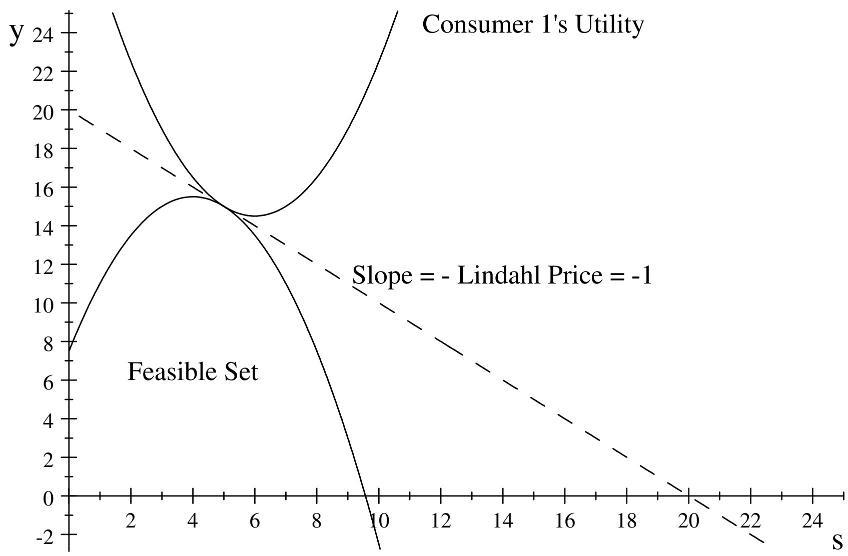
Any equilibrium that achieves this allocation requires Consumer 2's statement to be correct (i.e., $\bar{s}_2 = 5$), it follows from the second equation that $\bar{r}_2 = 6$. Thus, the strategy profile $[(\bar{r}_1, \bar{s}_1), (\bar{r}_2, \bar{s}_2)] = [(4, 5), (6, 5)]$ is the only profile which could achieve the Lindahl outcome as an equilibrium. I now show that this profile is a Nash equilibrium by checking that Consumer 1 is best responding to Consumer 2's strategy and vice versa.

Consumer 1's best response problem is to maximize his utility subject to a feasible set defined by Consumer 2's strategy and the mechanism. Since in a best response Consumer 1's strategy satisfies $s_1 = \frac{1}{2}r_1 + \frac{1}{2}\bar{r}_2$ (i.e., $s_1 = \chi(r_1, \bar{r}_2, s_1, \bar{s}_2)$), Consumer 1's best response problem simplifies to

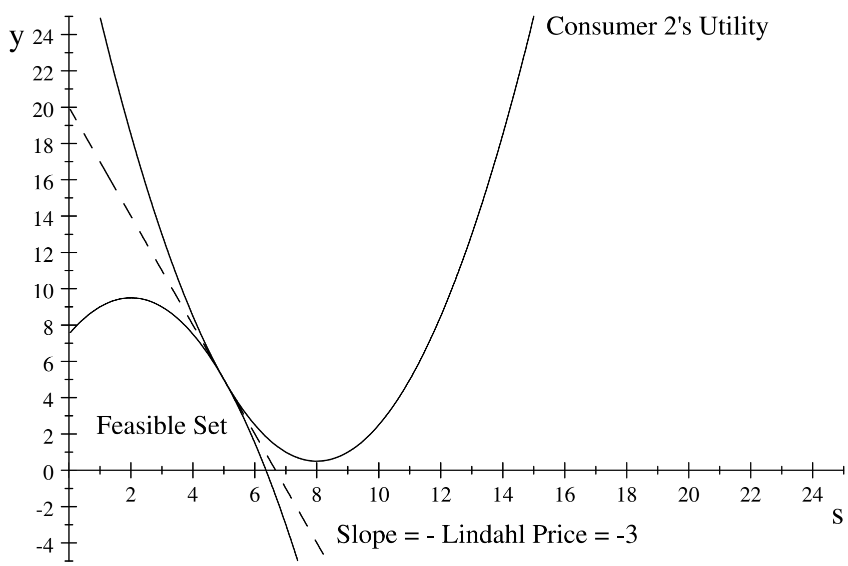
$$\max_{s_1} u_1(s_1, 20 - s_1 - \frac{1}{2}(5 - s_1)^2).$$

The first order condition yields $\bar{s}_1 = 5$, which implies $\bar{r}_1 = 4$ and verifies that Consumer 1 strategy $(4, 5)$ is his best response to Consumer 2's strategy.

A graphical depiction of Consumer 1's best response problem is illustrated below.



A similar argument can be used to show that Consumer 2's best response to $(\bar{r}_1, \bar{s}_1) = (4, 5)$ is $(\bar{r}_2, \bar{s}_2) = (6, 5)$. Notice that Consumer 1's actions define a personalized price equal to the Lindahl price $\bar{P}_2 = 3$ for Consumer 2. The graphical depiction of Consumer 2's best response problem is given below.



Since both players are best responding to each others actions, the unique Lindahl allocation of this example is achieved as a Nash equilibrium.

The second implication of Theorem 1 says that it is also possible to go in the other direction. Namely, if $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium of the mechanism, the equilibrium allocation is Lindahl. To demonstrate this in our example suppose $(\bar{r}_1, \bar{r}_2, \bar{s}_1, \bar{s}_2)$ is a Nash equilibrium. Then the first order condition (with respect to statement s_i) yields $\bar{s}_i = \frac{\bar{r}_1+\bar{r}_2}{2}$ for each i. Inserting this expression into each consumer's first order condition (with respect to their request) we have

$$6 - \frac{\bar{r}_1 + \bar{r}_2}{2} = 2 - \bar{r}_2 + \bar{s}_2$$

and

$$8 - \frac{\bar{r}_1 + \bar{r}_2}{2} = 2 - \bar{r}_1 + \bar{s}_1.$$

for Consumer 1 and Consumer 2 respectively. The unique solution of this pair of equations is $\bar{r}_1 = 4$, $\bar{r}_2 = 6$, which yields the Lindahl equilibrium $\bar{P}^1 = 1$, $\bar{P}^2 = 3$, and $x = 5$. Thus, the Nash allocation is Lindahl, completing the example.

4.1 Implementation in Quasi-Linear Environments

In this section, I show that the new Lindahl mechanism induces a supermodular game in quasi-linear E^Q environments for certain values of the mechanism's parameters. Furthermore, I identify sufficient conditions for uniqueness and the stability of equilibrium in this environment. This aligns the desirable welfare properties of Lindahl equilibrium with a set of desirable behavioral properties one would like in practice. I begin however with the following useful corollary of Theorem 1.

The corollary exploits the strict concavity assumption on the valuation functions to demonstrate a unique Lindahl equilibrium in these environments and as a consequence of Theorem 1 a unique Nash equilibrium.

Corollary 1 *For any $e \in E^Q$, the mechanism $\varphi^{\xi,\gamma,\delta}$ has a unique Nash equilibrium.*

Proof. See Appendix. ■

For N players in the E^Q environment and with an appropriate choice of mechanism parameters, the new mechanism induces a supermodular game. Recall from **Definition 5** that $\frac{\partial^2 v^i}{\partial x^2}$ is bounded from below by K_i for all $x \geq 0$. Theorem 2 therefore gives a *sufficient* condition for the game to be globally supermodular.

Theorem 2 *For any $e \in E^Q$, the mechanism $\varphi^{\xi,\gamma,\delta}$ induces a supermodular game if*

$$\begin{aligned} \gamma &\leq \frac{\delta}{N-1} + \min_{i \in I} K_i \\ \xi &\in \left[\frac{(N-1)}{N} \left(\gamma + \delta - \min_i K_i \right), \delta \right] \end{aligned}$$

Proof. See Appendix. ■

Theorem 2 provides conditions under which the mechanism induces a supermodular game. If the strategy set for each player is a compact rectangle in $\mathbb{R}^2$, then the game induced by the mechanism satisfy the Milgrom and Roberts conditions mentioned previously. However, simply compactifying the strategy set has a number of troubling consequences. Perhaps the most obvious of these is that the uniqueness result in Corollary 1 no longer applies since we exploited an unbounded strategy space in the proof. There may now exist boundary equilibria which are not Lindahl equilibria.[4] In the next section, I discuss a solution to this problem.

4.1.1 Stability

One of the stated goals of this paper is to find preference environments for which the new mechanism induces a game with a unique and stable equilibrium. Thus far it has been shown that, in quasi-linear environments, the mechanism φ has a unique Nash equilibrium and has the increasing difference and supermodular properties. These two properties are typically not enough to guarantee stability of equilibrium. In fact, it is relatively straightforward to cook up mechanisms (with an unbounded strategy space) that

[4]This is an issue since Milgrom and Roberts only show that adaptive behavior converges to the bounds of the outermost Nash equilibria. If there are equilibria on the boundary there may be no predictive power.

are supermodular with a unique, unstable equilibrium.[5] While the existence of such mechanisms would seem to contradict the Milgrom and Robert's stability theorem, it turns out that the unbounded strategy spaces do not meet the criteria of their theorem. Specifically, the strategy space needs to be a complete lattice. If the strategy space is compactified for these problematic mechanisms, there would be boundary equilibria, and the Milgrom and Roberts stability result (which now applies) loses all of its predictive power. We can rule this sort of thing out by creating conditions that ensure compacting the strategy space would not create new equilibria. The most natural method is to look for conditions that make the best reply mapping a contraction.

Definition 7 *Let X be any complete metric space. A best reply map ζ on X is said to be a contraction if there exists a real number $0 < k < 1$ such that*

$$d(\zeta(x), \zeta(y)) \leq k \cdot d(x, y)$$

for all x , $y \in X$.

Specifically, contraction mappings are useful, in this context, due to Banach's fixed point theorem. For completeness, I include the following statement of his theorem.

Theorem 3 (Banach's Fixed Point Theorem) *Let X be a non-empty complete metric space, $\zeta : X \to X$ a function. Suppose ζ is a contraction.*

[5] As an example, consider the following 2 player, 2 dimensional, tweak of the Walker mechanism

$$\begin{aligned}
\chi(r, s) &= r_1 - r_2 \\
\tau_1(r, s) &= (\frac{\beta}{N} - r_2 - s_2) \cdot \chi(r, s) + \frac{1}{2}(s_1 - r_2)^2 + \frac{1}{2}(s_2 - r_1)^2 \\
\tau_2(r, s) &= (\frac{\beta}{N} + r_1 + s_1) \cdot \chi(r, s) + \frac{1}{2}(s_1 - r_2)^2 + \frac{1}{2}(s_2 - r_1)^2
\end{aligned}$$

This mechanism will Nash implement the Lindahl allocations of a general environment and in quasi-linear environments induces a supermodular game with a unique equilibrium, but the unique equilibrium is unstable.

Then there exists a unique point $x^* \in X$ such that $\zeta(x^*) = x^*$. Furthermore if x_0 is any point of X and $x_1 = \zeta(x_0)$, $x_2 = \zeta(x_1)$, $x_3 = \zeta(x_2)$, etc., then

$$\lim_{n\to\infty} x_n = x^*.$$

Two parts of this theorem are of particular interest. First, if the best reply mapping is a contraction the equilibrium will be unique whether the strategy set is $\mathbb{R}^{2N}$ or the compact rectangle in $\mathbb{R}^{2N}$. This observation makes the theorem immediately relevant to the problems observed in the previous section. Second, Banach's theorem provides an algorithm for finding the unique fixed point of the game. I will elaborate on the application of this part of the theorem in Corollary 2.

Clearly, a contraction mapping is a powerful tool. In this section, I provide the somewhat surprising result that if the new mechanism induces a supermodular game, then the best reply map is always a contraction. Then, taking advantage of Banach's fixed point theorem, I show that the sufficient conditions for uniqueness and stability of the Nash equilibrium are satisfied even if the strategy space is compactified.[6] While I will later argue there is no need to compactify the stategy space, the discussion is useful since it highlights several issues in this literature.

The following theorem reports the contraction result.

Theorem 4 *If ξ, γ, and δ satisfy the supermodularity restrictions of Theorem 2, then the best reply mapping is a contraction.*

Proof. See Appendix. ■

Theorem 2 guarantees uniqueness of equilibrium so long as the strategy space is a complete metric space. Throughout the paper the complete metric space $\mathbb{R}^{2N}$ has been used. Consequently, Theorem 2 provides an alternative proof to the uniqueness result in Corollary 1. Since the best reply mapping is a contraction and since the compact rectangle in $\mathbb{R}^{2N}$ is still a complete metric space, we can compact its strategy space and remain confident that our Nash equilibrium is unique (and finite). Therefore, if one were inclined

[6] I am grateful to PJ Healy for pointing out a problem in this proof in an earlier draft as well as for additional comments that have greatly improved this section of the paper.

to compactify the strategy space the mechanism φ will induce a game with a unique equilibrium that also satisfies the Milgrom and Roberts' dynamic stability properties. Thus, at the cost of shrinking the set of applicable preference environments to quasi-linear environments, I gain the property that only rationalizable strategies coincide with the Nash strategies and that the unique equilibrium is stable under "adaptive" learning dynamics such as fictitious play, k-period average best response, and Bayesian learning.

Unfortunately, compactifying the strategy set in this manner is an unacceptable way of guaranteeing stability for this class of mechanisms. Despite the fact that we will always have a unique equilibrium, we cannot be sure that the equilibrium corresponds to the Lindahl outcome unless the strategy sets are compacted in such a way to keep the original equilibrium strategies in the strategy space. A planner would, in general, not have enough information to guarantee that equilibrium messages would be in the interior of the compacted message space. Thus, by arbitrarily compacting the message space, we could actually eliminate the nice equilibrium outcome and prevent rational players from learning to achieve the Lindahl allocation. Fortunately, using the result from Theorem 3, stability of equilibrium can be ensured under some learning dynamics without resorting to compacting the strategy space. We formalize this statement in the following corollary.

Corollary 2 *If ξ, γ, and δ satisfy the supermodularity restrictions of Theorem 2, then the unique equilibrium of the induced game is stable under the myopic best reply learning algorithm.*

Proof. From Theorem 3, we know that if ξ, γ, and δ satisfy the supermodularity restrictions, then the best reply mapping is a contraction. The best reply mapping is a continuous function. Therefore, from Banach's fixed point theorem, we have that the equilibrium of the induced game exists and will be unique. Furthermore, the theorem also states that we can find the equilibrium by starting at any initial point and iterating the best reply mapping. However, this is just the well known Cournot process, or myopic best reply learning algorithm. ■

Thus, in quasi-linear environments, without resorting to compacting procedures, supermodularity of the game induced by the new mechanism actually ensures *existence*, *uniqueness,* and *global stability* of equilibrium.

5 Comparison, Informational Complexity, and Discussion

I now compare the new Lindahl mechanism with the Lindahl mechanisms due to Kim (1993) and Chen (2002).[7] These two mechanisms share a similar game structure with the new mechanism, which makes for a straightforward comparison. I first briefly explain the outcome functions for these mechanisms using the notation we have already developed. Afterwards, I compare them with the new mechanism. I do not go into the proofs of why these mechanisms work. For details I refer the interested reader to the aforementioned articles.

Just as in the new Lindahl mechanism, the other two mechanisms use a message space of $\mathbb{R}^2$, where a generic message for consumer i will take the form of $\mathbf{m}_i = (r_i, s_i)$, where r_i serves as i's request and s_i as his statement about x. The level of the public good is determined by the outcome function $\chi(\mathbf{r}, \mathbf{s})$, and the individual consumer's tax function will again be denoted $\tau^i(\mathbf{r}, \mathbf{s})$.

In the Chen/Kim Mechanism, each consumer i chooses a request and a statement. The request helps determine the level of the public good, and both choices act to determine the level of the tax. The outcome functions of Chen's mechanism $\varphi_C^{\xi,\delta}$ are as follows:

$$\begin{aligned}
\chi(\mathbf{r}, \mathbf{s}) &= \sum_{i=1}^{N} r_i \\
\tau^i(\mathbf{r}, \mathbf{s}) &= P^i(\mathbf{r}, \mathbf{s}) \cdot \chi(\mathbf{r}, \mathbf{s}) + \frac{1}{2}\left(s_i - \chi(\mathbf{r}, \mathbf{s})\right)^2 + \frac{\delta}{2}\sum_{j \neq i}\left(s_j - \chi(\mathbf{r}, \mathbf{s})\right)^2
\end{aligned}$$

where

$$P^i(\mathbf{r}, \mathbf{s}) = \frac{\beta}{N} - \xi \sum_{j \neq i} r_j + \frac{\xi}{N} \sum_{j \neq i} s_j$$

can be thought of as $i's$ personalized price for the public good and $\xi > 0$, $\delta \geq 0$ are constant parameters.

[7] de Trenquale (1989) and Kim (1996) also present stable Lindahl mechanisms, but these mechanisms are not supermodular. Since this paper's focus is on supermodularity I restrict my comparison to the Chen mechanism which includes Kim (1993) as a special case. That said, this paper's debt to all of these previous mechanisms will, hopefully, be obvious to the reader.

The Kim mechanism is simply the Chen mechanism with $\xi = 1$ and $\delta = 0$. Chen recognized that adding additional statement penalty terms to Kim's mechanism created the complimentarities required to make the mechanism supermodular.

There are several key differences between the new mechanism and Chen's mechanism which are worth distinguishing. All the important differences are derived from the choice of the personalized price function.

5.1 Differences in Penalty Structure

For consumer i, the personalized price function of the new mechanism only depends on the statement of consumer $i + 1$. The personalized price of Chen's mechanism depends on the statements off all the other players. This seemingly innocuous choice of personalized price function actually suggests several issues. The first is a potential welfare issue related to the statement penalties.

In order to get the right complementarity between actions in quasi-linear environments, this choice of personalized price requires Chen to include a separate squared difference penalty for *each* consumer in the economy. In other words, a term $\frac{\delta}{2}\left(s_j - \chi\left(\mathbf{r}, \mathbf{s}\right)\right)^2$ is added to the Chen tax function for each consumer $j \neq i$ in the economy. While in equilibrium each of these terms will be equal to zero and drop out of the tax function, when out of equilibrium, even small incorrect statements by each player can quickly increase the taxes each consumer has to pay (the magnitude of the penalties depends on the specific parameterization of the mechanism). This welfare issue was documented by Van Essen, Lazzati, and Walker (2008), where, in an experiment, subjects' incorrect statements often create large losses for all consumers, as well as generated large revenue swings to the government, and overall losses in efficiency.

Since consumers in the new mechanism have only one penalty term connected to the statement of their neighbor, statement penalties for each consumer in a similar (parametric) situation to the situation mentioned above will also be significantly smaller. Additionally, from a welfare perspective, individuals are shielded from large incorrect guesses by everyone other than their partner. In an experiment, it is easy to imagine that one player in a group may be a little slow to correct his statement. In the Chen mechanism, every individual pays for this slowness, while in the new mechanism only one other person is affected. Lastly, since for any economy size N, the new mechanism's personalized price for consumer i depends only on the statement of his neighbor $i+1$, it maintains this bilateral structure as N increases.

This observation also indirectly highlights another important difference, described in the following subsection.

5.2 Differences in Information Requirements

Despite the theoretical assumption of complete information, the majority of public good experiments have been conducted under incomplete information protocols – i.e., subjects were only aware of their own payoff function. This type of experiment seems more consistent with a real world setting where it is unlikely that subjects would have knowledge of one another's payoff functions. The interpretation of Nash equilibrium then becomes one of a steady state of some dynamic learning process (think Cournot best reply) rather than a common knowledge/ introspection argument.

Consider the following thought experiment. Suppose I design an experiment to test the Chen mechanism or the new mechanism in the laboratory under an incomplete information protocol. What is the minimum amount of information a subject would need to compute his own payoff?[8] [9]

Information Requirement for Consumer i (New Mechanism):

1. The total request of all other players

$$\sum_{j \neq i} r_j.$$

2. The statement of their "neighbor" player $i+1$

$$s_{i+1}.$$

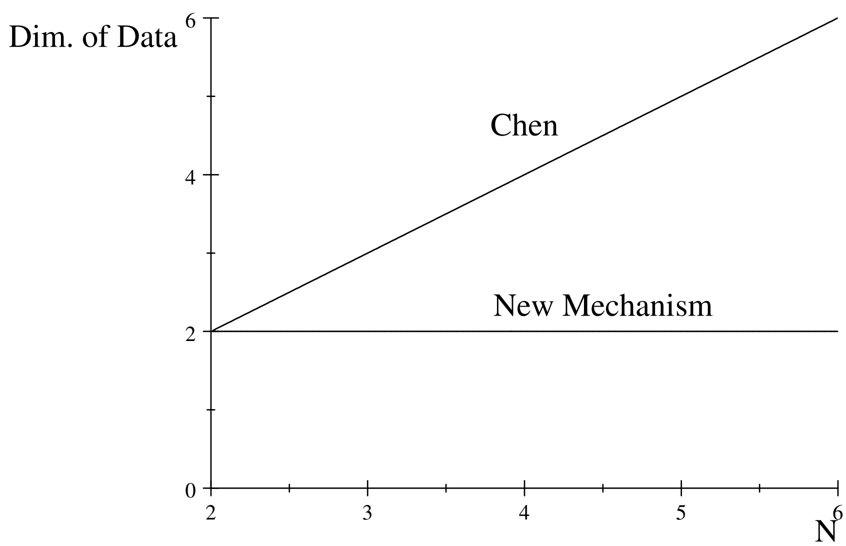
[8] This is not an uncommon question asked by experimenters. For instance, imagine a typical Cournot experiment with N firms. Each firm (played by a subject) does not need to know the quantity decisions of each individual firm to compute his payoff. This would entail providing each firm with $N-1$ pieces of information. The minimum amount of information needed by each firm about rivals' choices in a Cournot experiment is just aggregate quantity produced by all other firms, or one piece of information.

[9] This is not the same as saying: What is the minimum amount of information a subject would need to compute a best response? The other question seems more natural for an experimental setting.

Information Requirement for Consumer i (Chen mechanism):

1. The total request of all other players
$$\sum_{j \neq i} r_j.$$
2. The individual statements of all of the other players. Player 1, for example, would need $s_2, s_3, ..., s_N$.

It is clear that for $N > 2$, these two mechanisms have a different information requirement. Furthermore this difference grows larger as N grows larger. Specifically, as the economy gets larger, participants in the Chen mechanism require N independent pieces of data to be able to compute their payoffs.[10] The information requirement for the new mechanism always stays constant at 2. This difference is illustrated in the graph below.



If I define information complexity as the minimum dimension of data needed to compute one's payoff, the Chen mechanism becomes more complicated as N increases, and the new mechanism does not.

[10] It is possible to reduce this number to three if one does some algebra. Unfortunately this does not simplify the Chen mechanism. For instance, unlike the published version, it is no longer obvious that statements should try and match the level of the public good.

6 Conclusion

This paper introduces a new incentive compatible mechanism capable of implementing Lindahl allocations as Nash equilibria. While a simplified economy with two goods was used for the exposition, it is straightforward to generalize the mechanism to accommodate economies with an arbitrary number of private and public goods. Second, motivated by experimental observations and Chen's 2002 mechanism, I have shown conditions under which this new mechanism satisfies the increasing difference and supermodularity properties. I then use these observations to identify a set of preference environments which will be robustly stable and implement a unique Nash equilibrium. Finally, unlike the Chen mechanism, this new mechanism does not increase in complexity as the number of consumers grows large. The importance of this property is an empirical question and is well posed to be answered by additional experimentation in the laboratory.

Finally, there are several interesting areas for future research. For instance, it is known that stable Lindahl mechanisms can be found in quasi-linear preference environments. And while the stability results in quasi-linear environments are important, it is unknown what is the maximum preference domain for stable environments. A natural extension of the quasi-linear environments could be those defined by generalized Bergstrom-Cornes preferences. It would also be nice to know if it is possible to find a Lindahl mechanism that is stable for some environments and always in budget balance; or a stable, one-choice-variable, Lindahl mechanism. Some answers to this later question have been shown in Healy and Mathevet (2009). They provide an impossibility result for the existence of one dimensional Lindahl and Walrasian contractive mechanisms, but show, in general, it is possible to construct two dimensional contractive mechanisms. Furthermore, they show, in a similar manner to Milgrom and Robert's learning results for supermodular games, that contractive mechanisms induce games for which a wide variety of learning rules converge to the equilibrium bounds in this framework. Finally, there needs to be more experiments on implementation theory. Experiments on mechanisms with various properties will give us a better handle on what mechanism characteristics work or do not work behaviorally when they look to develop new theory.

7 Appendix

The strategy for the proof of Theorem 1 will be as follows: first, I demonstrate that a Nash allocation is Pareto optimal via an argument similar to the one used by Groves and Ledyard (1979); second, using the fact that the Nash allocation is Pareto optimal, I use an "unbiasedness" proof similar to Foley (1970) p. 68-69 and Chen (2002) to establish that the outcome is Lindahl; finally, I show that any Lindahl allocation is achieved as a Nash allocation of the mechanism using a technique I believe was first used by Walker (1981).

Lemma 1 *Suppose the strategy profile* $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ *is a Nash equilibrium of* $\varphi^{\xi,\gamma,\delta}$ *for* $e \in E$ *, where* $(\bar{x}, \bar{y}_i)$ *is consumer* i*'s Nash allocation, then the following statements are true:*

1. *For any bundle* $(x, y_i) \in C_i$*, there is a pair* (r_i, s_i) *such that* $x = \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})$*.*
2. *The private good consumed by consumer* i *in equilibrium is* $\bar{y}_i \equiv \omega^i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$*.*
3. *Consumer* i*'s statement and tax are* $\bar{s}_i = \frac{1}{N}\sum_{i=1}^{N} \bar{r}_i$ *and* $\tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ *for all* i*.*
4. *If a feasible allocation* $(x, y_i) \in C_i$ *is weakly preferred to the Nash allocation* $(\bar{x}, \bar{y}_i)$*, then the preferred bundle is at least as expensive as consumer* i*'s initial wealth (i.e,* $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) \geq \omega^i$*).*
5. *If a feasible allocation* $(x, y_i) \in C_i$ *is strictly preferred to the Nash allocation* $(\bar{x}, \bar{y}_i)$*, then the preferred bundle is more expensive than consumer* i*'s initial wealth (i.e.,* $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) > \omega^i$*).*

Proof.

L1.1 Since χ only depends on the requests of individuals, set $x = \frac{1}{N}(r_i + \sum\limits_{j \neq i} \bar{r}_j)$ or $r_i = Nx - \sum\limits_{j \neq i} \bar{r}_j$.

L1.2 Consider the following two bundles $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \bar{y}_i)$ and $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \hat{y}_i) \in C_i$, where $0 \leq \hat{y}_i < \omega^i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. Since preferences are complete, transitive, and strictly increasing in y_i, I have $(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \bar{y}_i) \succ_i (\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \hat{y}_i)$ for all $\hat{y}_i$.

L1.3 Since (r, s) is a Nash equilibrium, then for each consumer i

$$(\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \omega^i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})) \succeq_i (\chi(\bar{\mathbf{r}}, \mathbf{s}_{-i}, s_i), \omega^i - \tau^i(\bar{\mathbf{r}}, \mathbf{s}_{-i}, s_i)) \text{ for all } s_i.$$

From the functional form of the tax function and since preferences are complete, transitive, and strictly increasing in y_i, for each i, $\bar{s}_i = \frac{1}{N} \sum_{i=1}^{N} \bar{r}_i$. It follows directly that $\tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$.

L1.4 Suppose not. Then $y_i + \tau^i(\bar{\mathbf{r}}_{-i}, \bar{\mathbf{s}}_{-i}, r_i, s_i) < \bar{y}_i + \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \omega^i$. Since τ^i is continuos and using the fact that preferences are continuos, convex, and strictly increasing in y_i, there exists $(\acute{y}_i, \acute{r}_i, \acute{s}_i)$ such that $(\chi(\acute{r}_i, \bar{\mathbf{r}}_{-i}, \acute{s}_i, \bar{\mathbf{s}}_{-i}), \acute{y}_i) \in C_i$, $\acute{y}_i + \tau^i(\acute{r}_i, \bar{\mathbf{r}}_{-i}, \acute{s}_i, \bar{\mathbf{s}}_{-i}) \leq \omega^i$, and $(\chi(\acute{r}_i, \bar{\mathbf{r}}_{-i}, \acute{s}_i, \bar{\mathbf{s}}_{-i}), \acute{y}_i) \succ_i (\bar{x}, \bar{y}_i)$. However, this means that there is an individually feasible bundle which is strictly preferred to the Nash allocation. This contradicts the assumption that $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium.

L1.5 Suppose not. Then $(\bar{r}_i, \bar{s}_i)$ is not a best response which contradicts the assumption that $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium.

■

Lemma 2 *Suppose consumer i could purchase units of the public good at a price of t_i, where t_i is defined as consumer i's equilibrium marginal tax rate (i.e., $t_i \equiv P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$), then the following statements are true:*

1. *If a feasible allocation $(x, y_i) \in C_i$ is weakly preferred to the Nash allocation $(\bar{x}, \bar{y}_i)$ was then the preferred bundle is at least as expensive as the Nash allocation (i.e., $y_i + t_i \cdot x \geq \bar{y}_i + t_i \cdot \bar{x}$).*
2. *If an allocation achieved in the mechanism is less expensive than the Nash allocation (i.e., $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) < \bar{y}_i + \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$) , then the same allocation is less expensive if the public good could be purchased at a price of t_i (i.e., $y_i + t_i \cdot x < \bar{y}_i + t_i \cdot \bar{x}$).*
3. *If a feasible allocation $(x, y_i) \in C_i$ is strictly preferred to the Nash allocation $(\bar{x}, \bar{y}_i)$ was then the preferred bundle is more expensive than the Nash allocation (i.e., $y_i + t_i \cdot x > \bar{y}_i + t_i \cdot \bar{x}$).*

Proof.

L2.1 By definition, $t_i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. By assumption, preference are convex so certainly the set of bundles that are weakly preferred to $(\bar{x}, \bar{y}_i)$ is convex and $(\bar{x}, \bar{y}_i)$ is on the boundary of the set. Let the set of affordable bundles be denoted $B_i = \left\{(x, y_i) \in C_i | y_i + \hat{\tau}^i(x; \bar{\mathbf{r}}_{-i}, \bar{\mathbf{s}}_{-i}) \leq \omega^i\right\}$, where $\hat{\tau}^i(x; \bar{r}_{-i}, \mathbf{s}) = P^i(\mathbf{r}, \mathbf{s}) \cdot x + \frac{\gamma}{2}(s_i - x)^2 + \frac{\delta}{2}(\bar{\mathbf{s}}_{i+1} - x)^2$. B is convex since $\hat{\tau}^i$ is a convex function of x. By part (2) of Lemma 1, $(\bar{x}, \bar{y}_i)$ is on the boundary of set B_i. From part (5) of Lemma 1, we have the intersection of the set of weakly preferred bundles to $(\bar{x}, \bar{y}_i)$ (denote WP_i) and the budget set B_i is empty. From the Separating Hyperplane Theorem, there exists a hyperplane through $(\bar{x}, \bar{y}_i)$ that separates WP_i and B_i. The vector $(t_i, 1)$ defines this hyperplane. Also from the Separating Hyperplane Theorem, I have that $y_i + t_i \cdot x \geq c$ and $\bar{y}_i + t_i \cdot \bar{x} = c$, where $c \neq 0$. It follows that $y_i + t_i \cdot x \geq \bar{y}_i + t_i \cdot \bar{x}$.

L2.2 If $y_i + \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) < \bar{y}_i + \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \omega^i$, I can expand each of these expressions to $y_i + P^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) \cdot \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) + \frac{\gamma}{2}(s_i - x)^2$ $+ \frac{\delta}{2}(\bar{s}_{i+1} - x)^2 < \bar{y}_i + P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) \cdot \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. Let $x = \chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})$ and $\bar{x} = \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$. By construction, the personalized price function $P^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i}) = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = t_i$. I can subtract the two squared terms on the LHS to get $y_i + t_i \cdot x < \bar{y}_i + t_i \cdot \bar{x}$.

L2.3 Suppose not. By part (1) of Lemma 2, I have $y_i + t_i \cdot x = \bar{y}_i + t_i \cdot \bar{x}$. Since preferences are continuous there exists a neighborhood of (x, y_i), denoted $N(x, y_i)$ such that for all $(\hat{x}, \hat{y}_i) \in N(x, y_i) \cap C_i$, $(\hat{x}, \hat{y}_i) \succ_i (\bar{x}, \bar{y})$. The "no minimum wealth assumption," part 1 of Lemma 1 and part 2 of Lemma 2 imply that there exists a bundle $(\acute{x}, \acute{y}_i) \in C_i$ such that $\acute{y}_i + t_i \cdot \acute{x} < \bar{y}_i + t_i \cdot \bar{x} = y_i + t_i \cdot x = \omega^i$. Let

$$G \equiv \{(\hat{x}, \hat{y}_i) \in C_i | \, (\hat{x}, \hat{y}_i) = (\lambda \acute{x} + (1 - \lambda) x, \lambda \acute{y}_i + (1 - \lambda) y_i) \text{ for all } \lambda \in (0, 1)\}.$$

All points in this line between $(\acute{x}, \acute{y}_i)$ and (x, y_i) have a value smaller than $(\bar{x}, \bar{y}_i)$. However since the consumption set is convex it follows that there exists a λ which is small enough such that $N(x, y_i) \cap G$ –i.e. there exists a bundle $(\hat{x}, \hat{y}_i)$ such that $(\hat{x}, \hat{y}_i) \succ_i (\bar{x}, \bar{y})$ and $\hat{y}_i + t_i \cdot \hat{x} < \bar{y}_i + t_i \cdot \bar{x}$ which leads to a contradiction of part (1) of Lemma 2.

■

The next lemma and its proof are almost identical to those in the First Fundamental Welfare Theorem for private good economies (see Debreu 1959).

Lemma 3 *Suppose* $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ *is a Nash equilibrium of* $\varphi^{\xi,\gamma,\delta}$ *for* $e \in E$ *, then the Nash allocation* $\left[\chi\left(\bar{\mathbf{r}}, \bar{\mathbf{s}}\right), (\omega - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}))_{i=1}^N\right]$ *is Pareto optimal.*

Proof. Suppose $\left[\bar{x}, (\bar{y}_i)_{i=1}^N\right]$ is not a Pareto optimal allocation and that $\left[x, (y_i)_{i=1}^N\right]$ is a feasible, Pareto superior allocation. From part 3 of Lemma 2, I have that

$$y_i + t_i \cdot x > \bar{y}_i + t_i \cdot \bar{x} \qquad \text{for all } i.$$

Summing across all consumers, I have

$$\sum_{i=1}^N y_i + \sum_{i=1}^N t_i \cdot x > \sum_{i=1}^N \bar{y}_i + \sum_{i=1}^N t_i \cdot \bar{x}.$$

By construction, $\sum_{i=1}^N t_i = \sum_{i=1}^N P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}) = \beta$. Re-writing the above strict inequality, I have

$$\sum_{i=1}^N y_i + \beta \cdot x > \sum_{i=1}^N \bar{y}_i + \beta \cdot \bar{x} = \sum_{i=1}^N \omega^i.$$

Thus, the Pareto superior bundle is not feasible. ■

Lemma 4 *The affordable feasible set, denoted* F*, is a convex set, where*

$$F = \left\{ \begin{array}{c} (x_1, ..., x_N, y_1, ..., y_N) \,|\, (x_i, y_i) \in C_i \\ \textit{where } x_i = x_j = x \textit{ for all } j \neq i \textit{ and } x \leq \frac{\sum_{i=1}^N \left(\omega^i - y_i\right)}{\beta} \end{array} \right\}.$$

Furthermore, the point $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$*, associated with the Nash equilibrium, is on the boundary of* F*.*

Proof. To show that F is convex choose two arbitrary profiles

$$(x_1, ..., x_N, y_1, ..., y_N), (\acute{x}_1, ..., \acute{x}_N, \acute{y}_1, ..., \acute{y}_N) \in F.$$

For $\lambda \in [0, 1]$, the convex combination of these two vectors is

$$(\lambda x_1 + (1-\lambda)\acute{x}_1, ..., \lambda x_N + (1-\lambda)\acute{x}_N, \lambda y_1 + (1-\lambda)\acute{y}_1, ..., \lambda y_N + (1-\lambda)\acute{y}_N).$$

First, since C_i is convex, $(\lambda x_i + (1-\lambda)\acute{x}_i, \lambda y_i + (1-\lambda)\acute{y}_i) \in C_i$ for all i. Second, because both $x_i = x_j = x$ and $\acute{x}_i = \acute{x}_j = \acute{x}$ for all $j \neq i$, then $\lambda x_i + (1-\lambda)\acute{x}_i = \lambda x_j + (1-\lambda)\acute{x}_j = \lambda x + (1-\lambda)\acute{x}$. Finally, $\beta x \leq \sum_{i=1}^{N}(\omega^i - y_i)$ implies $\lambda\beta x \leq \lambda\sum_{i=1}^{N}(\omega^i - y_i)$. Similarly, $\beta\acute{x} \leq \sum_{i=1}^{N}(\omega^i - \acute{y}_i)$ implies $(1-\lambda)\beta\acute{x} \leq (1-\lambda)\sum_{i=1}^{N}(\omega^i - \acute{y}_i)$. Adding these two conditions together, I have the following inequality,

$$\lambda x + (1-\lambda)\acute{x} \leq \frac{\sum_{i=1}^{N}(\omega^i - (\lambda y_i + (1-\lambda)\acute{y}_i))}{\beta}.$$

verifying that the set F is convex.

To see that $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is in the boundary of the set. Recall from the Lemma 3 that the Nash allocation is Pareto optimal–i.e. $\sum_{i=1}^{N}\bar{y}_i + \beta\cdot\bar{x} = \sum_{i=1}^{N}\omega^i$. Re-arranging this expression, I have $\bar{x} = \frac{\sum_{i=1}^{N}(\omega^i - \bar{y}_i)}{\beta}$, which is clearly on the boundary of F. ■

Proof of Theorem 1. The proof for Theorem 1 is done in two parts. In the first half of the proof I show that if $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium of $\varphi^{\xi,\gamma,\delta}$, the corresponding allocation $\left[\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), (\omega^i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}}))_{i=1}^{N}\right]$ is a Lindahl equilibrium and for each i, $P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is the corresponding Lindahl price. It is first shown that the personalized price associated with the Nash equilibrium per unit tax $P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ defines a separating hyperplane between the feasible allocation set F and the preferred set D; second, I show that the Nash allocation is the allocation that maximizes a consumer's preferences subject to a budget constraint when facing the personalized price $P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$; finally, I show that the tax revenue equals the cost of producing the public good.

In the second half of the proof, I show that if $(\bar{P}^1, ..., \bar{P}^N)$ is the profile of Lindahl prices and $\left(\bar{x}, (\omega^i - \bar{P}^i\cdot\bar{x})_{i=1}^{N}\right)$ is the corresponding Lindahl allocation, then it must correspond to a Nash equilibrium of the mechanism. I do this by first showing that the messages that could achieve this allocation in the mechanism are unique. Subsequently that this profile of strategies is a Nash equilibrium of the game induced by the mechanism.

(Part 1): Consider the point $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ associated with the Nash allocation for each consumer. From Lemma 4, I have that the feasible set F is convex and that the point $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is on its boundary. Similarly from Lemma 5, I have that the set D is convex and point $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is on the boundary. Notice that the intersection of the interiors of F and D have no points in common. To see this suppose that these sets do have points in the interior that are common. Then there is a strictly cheaper feasible point that is weakly preferred by all consumers. However, this contradicts the fact that $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is Pareto optimal (Lemma 3). Therefore by the Separating Hyperplane Theorem, there exists a vector $(p_1^x, \cdots, p_N^x, p_1^y, \cdots, p_N^y) \neq 0$ and $c \in \mathbb{R}$ such that for all points in the weakly preferred set D,

$$(\sum_{i=1}^{N} p_i^x) \cdot x + \sum_{i=1}^{N} p_i^y \cdot y_i \geq c.$$

In addition, since the vector $(\bar{x}_1, ..., \bar{x}_N, \bar{y}_1, ..., \bar{y}_N)$ is in the boundary of both F and G,

$$(\sum_{i=1}^{N} p_i^x) \cdot \bar{x} + \sum_{i=1}^{N} p_i^y \cdot \bar{y}_i = c.$$

Since $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium, the hyperplane that crosses through $(\bar{x}, \bar{y}_i)$ is defined by the vector of $(p_i^x, p_i^y) = (t_i, 1)$ for each i where $p_i^x = t_i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ (Lemmas 1 and 2). This should be thought of as consumer i's personalized price.

Next, I show that the bundle $(\bar{x}, \bar{y}_i)$ maximizes the preferences of consumer i subject to i's budget constraint when facing $P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ as his personalized price.

Suppose $(x_i, y_i) \succ_i (\bar{x}, \bar{y}_i)$ while $x_j = \bar{x}$ and $y_j = \bar{y}_j$ for all $j \neq i$. This point is in set D. From the separating hyperplane defined above I have,

$$\left(\sum_{i=1}^{N} P^i(\bar{r}, \bar{s})\right) \cdot x + \sum_{i=1}^{N} y_i \geq \left(\sum_{i=1}^{N} P^i(\bar{r}, \bar{s})\right) \cdot \bar{x} + \sum_{i=1}^{N} \bar{y}_i.$$

All terms in this expression are the same except those belonging to consumer i. Thus the expression can be simplified to $y_i + P^i(\bar{r}, \bar{s}) \cdot x \geq \bar{y}_i + P^i(\bar{r}, \bar{s}) \cdot \bar{x}$. From part 3 of Lemma 2, since the bundle $(x_i, y_i) \succ_i (\bar{x}, \bar{y}_i)$ equality cannot hold so

$$P^i(\bar{r}, \bar{s}) \cdot x + y_i > P^i(\bar{r}, \bar{s}) \cdot \bar{x} + \bar{y}_i.$$

The personalized price for consumer i is independent i's actions–i.e., $P^i(\bar{r}, \bar{s}) = P^i(r_i, \bar{r}_{-i}, s_i, \bar{s}_{-i})$ for all r_i and s_i. Using this fact I am going to rewrite the above expression to be

$$P^i(r_i, \bar{r}_{-i}, s_i, \bar{s}_{-i}) \cdot x + y_i > P^i(\bar{r}, \bar{s}) \cdot \bar{x} + \bar{y}_i.$$

Now adding two appropriately chosen positive terms on the LHS, I have

$$P^i\left(r_i,\bar{r}_{-i},s_i,\bar{s}_{-i}\right)\cdot x+y_i+\frac{\gamma}{2}\left(s_i-x\right)^2+\frac{\delta}{2}\left(\bar{s}_{i+1}-x\right)^2>P^i\left(\bar{r},\bar{s}\right)\cdot\bar{x}+\bar{y}_i.$$

However, this is equivalent to $y_i+\tau^i\left(r_i,\bar{r}_{-i},s_i,\bar{s}_{-i}\right)>\bar{y}_i+\tau^i(\bar{r},\bar{s})=\omega^i$, where $x=\frac{1}{N}(r_i+\sum_{j\neq i}\bar{r}_j)$ and $\bar{x}=\frac{1}{N}(\bar{r}_i+\sum_{j\neq i}\bar{r}_j)$. Thus, any bundle that is strictly preferred to the Nash bundle is not affordable by the consumer–i.e., the Nash allocation maximizes consumer i's preferences subject to a budget constraint.

The last part of the argument requires tax revenue to equal the total cost of production.

If I add up the tax revenue, I have that

$$\begin{aligned}
\sum_{i=1}^{N}\tau^i(\bar{r},\bar{s}) &= \sum_{i=1}^{N}P^i(\bar{r},\bar{s})\cdot\ \chi(\bar{r},\bar{s})\\
&= \sum_{i=1}^{N}\left(\frac{\beta}{N}-\xi\sum_{j\neq i}\frac{\bar{r}_j}{N-1}+\xi\bar{s}_{i+1}\right)\cdot\chi(\bar{r},\bar{s})\\
&= \left(\beta-\xi\sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{r}_j}{N-1}+\xi\sum_{i=1}^{N}\bar{s}_{i+1}\right)\cdot\chi(\bar{r},\bar{s})\\
&= \left(\beta-\xi N\chi(\bar{r},\bar{s})+\xi N\chi(\bar{r},\bar{s})\right)\cdot\chi(\bar{r},\bar{s})\\
&= \beta\cdot\chi(\bar{r},\bar{s}).
\end{aligned}$$

Thus the allocation is feasible and this is a Lindahl allocation, where $\left(\bar{P}^1,...,\bar{P}^N\right)$ will be the profile of Lindahl prices.

(Part 2): For all i, let $\bar{s}_i=\bar{x}$. Consider the following system of N linear equations and N variables $(r_1,...,r_N)$

$$\begin{aligned}
r_1+r_2+\cdots+r_N &= N\cdot\bar{x}\\
-\sum_{j\neq i}r_j &= \left[\frac{1}{\xi}\left(\bar{P}^i-\frac{\beta}{N}\right)-\bar{s}_{i+1}\right](N-1)\qquad\text{for } i=1,...,N-1
\end{aligned}$$

It is straightforward to verify that the $N\times N$ coefficient matrix of this system of equations is non-singular with a rank of N. Thus, the system has a unique solution which I will call $(\bar{r},\bar{s})$. It remains to show that $(\bar{\mathbf{r}},\bar{\mathbf{s}})$ is a Nash equilibrium.

Since the allocation $(\bar{x}, (\omega^i - \bar{P}^i \cdot \bar{x})_{i=1}^N)$ is Lindahl, $(\bar{x}, \omega^i - \bar{P}^i \cdot \bar{x}) \succeq_i (x, \omega^i - \bar{P}^i \cdot x)$ for all x. Let $x = \frac{1}{N}(r_i + \sum_{j\neq i} \bar{r}_j) = \chi(r_i, \bar{\mathbf{r}}_{-i}, \bar{\mathbf{s}})$, then

$$(\bar{x}, \omega^i - \bar{P}^i \cdot \bar{x}) \succeq_i \left(\frac{1}{N}(r_i + \sum_{j\neq i} \bar{r}_j), \omega^i - \bar{P}^i \cdot \frac{1}{N}(r_i + \sum_{j\neq i} \bar{r}_j) \right)$$

for all r_i.

Similarly, since preferences are strictly increasing in y_i, it is also true that

$$\begin{aligned} & (\bar{x}, \omega^i - \bar{P}^i \cdot \bar{x}) \\ \succeq & {}_i \left(\frac{1}{N}(r_i + \sum_{j\neq i} \bar{r}_j), \omega^i - \bar{P}^i \cdot \frac{1}{N}(r_i + \sum_{j\neq i} \bar{r}_j) - \frac{\gamma}{2}(s_i - \frac{1}{N}\sum_{i=1}^N \bar{r}_i)^2 \right. \\ & \left. -\frac{\gamma}{2}(s_{i+1} - \frac{1}{N} r_i - \frac{1}{N}\sum_{j\neq i} \bar{r}_j)^2 \right) \end{aligned}$$

for all r_i, s_i.

By construction, the public good

$$\bar{x} = \frac{\bar{r}_1 + \cdots + \bar{r}_N}{N} = \chi(\bar{\mathbf{r}}, \bar{\mathbf{s}})$$

consumer i's Lindahl price was

$$\bar{P}^i = P^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})$$

and $\bar{s}_i = \frac{1}{N}\sum_{k=1}^N \bar{r}_k$ for all i.

Plugging in these expressions into the above inequality, we have

$$\begin{aligned} & (\chi(\bar{\mathbf{r}}, \bar{\mathbf{s}}), \omega^i - \tau^i(\bar{\mathbf{r}}, \bar{\mathbf{s}})) \\ \succeq & {}_i(\chi(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{s}_{-i}), \omega^i - \tau^i(r_i, \bar{\mathbf{r}}_{-i}, s_i, \bar{\mathbf{s}}_{-i})) \end{aligned}$$

for all r_i, s_i.Therefore $(\bar{\mathbf{r}}, \bar{\mathbf{s}})$ is a Nash equilibrium of the mechanism. ∎

Proof of Corollary 1. Let Ω be the sum of each individuals initial endowment–i.e., $\Omega = \sum \omega^i$. For any $e \in E^Q$, Pareto optimal levels of the public good will be solutions to the following maximization problem

$$\max_{0 \le x \le \frac{\Omega}{\beta}} \sum_{i=1}^N v^i(x) - \beta x$$

Since each $v^i(x)$ is a strictly concave function, there is a unique solution $\bar{x}$ to the problem. Note that for some production to be optimal we need that $\sum_{i=1}^{N} v^i(0) > \beta$. Since the Pareto optimal amount of the public good is unique and $v^i(x)$ is a strictly concave function, there are unique Lindahl prices $\bar{P}^i = \frac{dv^i(\bar{x})}{dx}$, for each i so long as this price is affordable—i.e., $\omega^i - \bar{P}^i \cdot \bar{x} > 0$. I have a feasible and unique Lindahl equilibrium $\left[\bar{x}, \left(\omega^i - \bar{P}^i\bar{x}\right)_{i=1}^{N}\right]$. From Theorem 1, I know that there is a unique Nash equilibrium that corresponds to this allocation. ■

Proof of Theorem 2. Since $M_i = \mathbb{R}^2$, it is a sublattice of $\mathbb{R}^2$. By definition of being in the E^Q environment u^i is C^2 and therefore trivially satisfies the continuity requirement. To see that u^i has the supermodularity property, I appeal to the fact that the utility function is C^2. I therefore need to check the following cross-partial derivative

$$\frac{\partial^2 u^i}{\partial r_i \partial s_i} \geq 0$$

Checking this, I see that

$$\frac{\partial^2 u^i}{\partial r_i \partial s_i} = \frac{\gamma}{N} \geq 0$$

for each consumer i. The increasing difference property requires checking the following five conditions:

$$\begin{array}{lrcll}
(1) & \dfrac{\partial^2 u^i}{\partial r_i \partial r_j} & \geq & 0 & \text{for all } j \neq i \\
(2) & \dfrac{\partial^2 u^i}{\partial r_i \partial s_j} & \geq & 0 & \text{for all } j \neq i \text{ and } j \neq i+1 \\
(3) & \dfrac{\partial^2 u^i}{\partial r_i \partial s_{i+1}} & \geq & 0 & \\
(4) & \dfrac{\partial^2 u^i}{\partial s_i \partial r_j} & \geq & 0 & \text{for all } j \neq i \\
(5) & \dfrac{\partial^2 u^i}{\partial s_i \partial s_{i+1}} & \geq & 0 &
\end{array}$$

Checking each of these in turn I have

$$\frac{\partial^2 u^i}{\partial r_i \partial r_j} = \xi \frac{1}{N(N-1)} - \frac{\gamma}{N^2} - \frac{\delta}{N^2} + \frac{1}{N^2}\frac{\partial^2 v^i}{\partial x^2}.$$

In order for the above expression to be positive I need

$$\xi \geq \frac{N-1}{N}\left(\gamma + \delta - \frac{\partial^2 v^i}{\partial x^2}\right) \qquad \text{for all } j \neq i.$$

A more compact way of writing this is

$$\xi \geq \frac{N-1}{N}\left(\gamma + \delta - \min_{i \in I} \frac{\partial^2 v^i}{\partial x^2}\right).$$

Condition 2 is trivially satisfied since

$$\frac{\partial^2 u^i}{\partial r_i \partial s_j} = 0 \qquad \text{for all} \qquad j \neq i \text{ and } j \neq i+1.$$

Checking Condition 3 I have

$$\frac{\partial^2 u^i}{\partial r_i \partial s_{i+1}} = -\frac{\xi}{N} + \frac{\delta}{N}.$$

This expression is positive for all i if and only if

$$\xi \leq \delta.$$

Condition 4 and 5 are always satisfied since

$$\frac{\partial^2 u^i}{\partial s_i \partial r_j} = \frac{\gamma}{N} > 0 \qquad \text{and} \qquad \frac{\partial^2 u^i}{\partial s_i \partial s_{i+1}} = 0$$

Therefore, for the mechanism to be supermodular the following is sufficient.

$$\xi \in \left[\frac{N-1}{N}\left(\gamma + \delta - \min_{i \in I} K_i\right), \delta\right]$$

Finally, this interval is non-empty if and only if $\gamma \leq \frac{\delta}{N-1} + \min_{i \in I} K_i$ is true. ■

Proof of Theorem 3. First, we characterize the best replies. Applying the mechanism to each consumer's utility we arrive at the augmented utility function

$$v^i\left(\frac{1}{N}\sum_{k=1}^{N} r_k\right) - \left(\frac{\beta}{N} - \frac{\xi}{N-1}\sum_{j \neq i} r_j + \xi s_{i+1}\right)\frac{1}{N}\sum_{k=1}^{N} r_k$$

$$-\frac{\gamma}{2}\left(s_i - \frac{1}{N}\sum_{k=1}^{N} r_k\right)^2 - \frac{\delta}{2}\left(s_{i+1} - \frac{1}{N}\sum r_k\right)^2.$$

Best responding requires consumers choices to satisfy first order conditions – i.e.,,

$$r_i \quad : \quad \frac{v_1^i}{N} - \left(\frac{\beta}{N} - \frac{\xi}{N-1} \sum_{j \neq i} r_j + \xi s_{i+1} \right) \frac{1}{N} + \frac{\gamma}{N}(s_i - \frac{1}{N} \sum_{k=1}^{N} r_k) + \frac{\delta}{N}(s_{i+1} - \frac{1}{N} \sum r_k) = 0$$

$$s_i \quad : \quad -\gamma(s_i - \frac{1}{N} \sum_{k=1}^{N} r_k) = 0$$

If we let $r_i^*(r_{-i}, s_{-i})$ and $s_i^*(r_{-i}, s_{-i})$ be the solutions to these first order conditions. Clearly, $s_i^*(r_{-i}, s_{-i}) = \frac{1}{N} r_i^*(r_{-i}, s_{-i}) + \frac{1}{N} \sum_{k=1}^{N} r_k$, if we plug s_i^* into the r_i condition. The new r_i condition is

$$\frac{v_1^i(\cdot)}{N} - \left(\frac{\beta}{N} - \frac{\xi}{N-1} \sum_{j \neq i} r_j + \xi s_{i+1} \right) \frac{1}{N} + \frac{\delta}{N}(s_{i+1} - \frac{1}{N} r_i^*(r_{-i}, s_{-i})) - \frac{1}{N} \sum r_k) = 0$$

We can think of each decision being chosen by a seperate agent: 1 agent for $r_i^*(r_{-i}, s_{-i})$ and one agent for the $s_i^*(r_{-i}, s_{-i})$. Since we have already accounted for the interaction between own decisions in the determination of the best replies, we can think of the game as one with $2N$ independent players choosing according to the specified reaction functions. The problem of showing a contraction reduces to the one of Vives.[11] Therefore a sufficient condition for the best reply map to yield a contraction is that, for each i, the absolute total change in $r_i^*(r_{-i}, s_{-i})$ and $s_i^*(r_{-i}, s_{-i})$ (evaluated at any point (r_{-i}, s_{-i})) is bounded by 1. In other words, we require

$$\begin{aligned} \sum_{j \neq i} \left| \frac{\partial r_i^*(r_{-i}, s_{-i})}{\partial r_j} \right| + \sum_{j \neq i} \left| \frac{\partial r_i^*(r_{-i}, s_{-i})}{\partial s_j} \right| &< 1 \\ \sum_{j \neq i} \left| \frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j} \right| + \left| \frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}} \right| &< 1 \end{aligned}$$

[11] See, for example, p. 47.

We compute these slopes directly. Differentiating the new r_i first order condition with respect to r_j.

$$
\begin{aligned}
\frac{v_{11}^i}{N^2}(1+\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j}) + \frac{\xi}{N(N-1)} - \frac{\delta}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j} - \frac{\delta}{N^2} &= 0 \\
v_{11}^i(1+\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j}) + \frac{N\xi}{(N-1)} - \delta\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j} - \delta &= 0 \\
v_{11}^i + \frac{N\xi}{(N-1)} - \delta &= (\delta - v_{11}^i)\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j} \\
\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial r_j} &= \frac{v_{11}^i + \frac{N\xi}{(N-1)} - \delta}{\delta - v_{11}^i}
\end{aligned}
$$

Differentiating with respect to s_{i+1}

$$
\begin{aligned}
\frac{v_{11}^i}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} - \frac{\xi}{N} + \frac{\delta}{N} - \frac{\delta}{N^2}\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} &= 0 \\
-N\xi + N\delta &= \delta\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} - v_{11}^i\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} \\
\frac{\partial r_i^*(r_{-i},s_{-i})}{\partial s_{i+1}} &= \frac{-N\xi + N\delta}{\delta - v_{11}^i}
\end{aligned}
$$

A sufficient condition for $r^*(r_{-i},s_{-i})$ to be a contraction is that

$$
\sum_{j\neq i}\left|\frac{v_{11}^i + \frac{N\xi}{(N-1)} - \delta}{\delta - v_{11}^i}\right| + \left|\frac{-N\xi + N\delta}{\delta - v_{11}^i}\right| < 1.
$$

Suppose γ, δ, and ξ satisfy the supermodularity conditions from Theorem 1, then the slopes are all positive leaving

$$
\begin{aligned}
(N-1)v_{11}^i + N\xi - (N-1)\delta - N\xi + N\delta &< \delta - v_{11}^i \\
0 &< -Nv_{11}^i.
\end{aligned}
$$

This condition is always satisfied since $v_{11}^i < 0$.

Now consider $s_i^*(r_{-i}, s_{-i}) = \frac{1}{N} r_i^*(r_{-i}, s_{-i}) + \frac{1}{N}\sum_{k=1}^{N} r_k$.

$$\begin{aligned}
\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j} &= \frac{1}{N}\frac{\partial r_i^*}{\partial r_j} + \frac{1}{N} \\
&= \frac{1}{N}\left(\frac{v_{11}^i + \frac{N\xi}{(N-1)} - \delta}{\delta - v_{11}^i}\right) + \frac{1}{N} \\
&= \frac{1}{N}(\frac{-(\delta - v_{11}^i) + \frac{N\xi}{(N-1)}}{\delta - v_{11}^i} + 1) \\
&= (\frac{\xi}{(N-1)(\delta - v_{11}^i)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}} &= \frac{1}{N}\frac{\partial r_i^*}{\partial s_{i+1}} \\
&= \frac{1}{N}\left(\frac{-N\xi + N\delta}{\delta - v_{11}^i}\right) \\
&= \frac{-\xi + \delta}{\delta - v_{11}^i}
\end{aligned}$$

Adding up across each player and checking the sufficient condition.

$$\sum_{j \neq i}\left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial r_j}\right| + \left|\frac{\partial s_i^*(r_{-i}, s_{-i})}{\partial s_{i+1}}\right| < 1$$

Supermodularity ensures the slopes are all positive. Therefore,

$$\begin{aligned}
\frac{\xi}{\delta - v_{11}^i} + \frac{-\xi + \delta}{\delta - v_{11}^i} &< 1 \\
\delta &< \delta - v_{11}^i \\
v_{11} &< 0
\end{aligned}$$

and since we have $v_{11}^i < 0$ by assumption, the second condition is satisfied. Since this is true for each i, the best reply map is a contraction. ■

8 Resources

1. Amir, R. (2005). "Supermodularity and Complementarity in Economics: an Elementary Survey." *Southern Economics Journal*, 71.

2. Bailey, M. (1994). “Lindahl Mechanisms and Free Riders.” *Public Choice*, 80, 35-39.

3. Bergstrom, T. and Cornes, R. (1983). “Independence of Allocative Efficiency from Distribution in the Theory of Public Goods.” *Econometrica*, 51, 1753-1765.

4. Chen, Y. and Tang, F. (1998). “Learning and Incentive Compatible Mechanisms for Public Good Provision: An Experimental Study,” *Journal of Political Economy*, 106: 633-662.

5. Chen, Y., (2002). “A family of supermodular Nash mechanisms implementing Lindahl allocations,” *Economic Theory 19,* 773-790.

6. Chen, Y. and Gazzale, R. (2004). “When does Learning in Games Generate Convergence to Nash Equilibria? The Role of Supermodularity in an Experimental Setting.” *American Economic Review,* 94, 1505-1535.

7. Debreau, G. (1959). *Theory of Value.* New York: John Wiley & Sons.

8. de Trenqualye, P. (1989). “Stable Implementation of Lindahl Allocations.” *Economic Letters* 29, 291-294.

9. de Trenqualye, P. (1994). “Nash implementation of Lindahl allocations.” *Social Choice Welfare* 11, 83-94.

10. Foley, D. (1970). “Lindahl's Solution and the Core of an Economy with Public Goods.” *Econometrica*, 38, 66-72.

11. Groves, T. and Ledyard, J. (1977). “A Solution to the "Free Rider" Problem.” *Econometrica*, 45, 783-809.

12. Groves, T. and Ledyard, J. (1980). “The Existence of Efficient and Incentive Compatible Equilibria with Public Goods.” *Econometrica*, 48, 1487-1506.

13. Healy, P., (2004). “Learning Dynamics for Mechanism Design: An Experimental Comparison of Public Goods Mechanisms,” *Journal of Economic Theory.*

14. Healy, P., and Mathevet, L. (2009). “Designing Stable Mechanisms for Economic Environments.” The Ohio State University Working Paper.

15. Hurwicz, L. (1979). “Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points.” *The Review of Economic Studies* 46, 217-224.

16. Kim, T. (1987). “Stability Problems in the Implementation of Lindahl Allocations.” Doctoral Thesis. University of Minnesota.

17. Kim, T. (1993). “A Stable Nash mechanism implementing Lindahl allocations for quasi-linear environments.” *Journal of Mathematical Economics, 22, 359-371.*

18. Kim, T. (1996). "A Stable Nash Mechanism for Quasi-Additive Public-Good Environments." *Japanese Economic Review* 47, 144-156.

19. Lindahl, E. (1919). “Just Taxation–A Positive Solution.” English translation of portion of *Die Gerechtigkeit der Besteuerung*, in Classics in the Theory of Public Finance, edited by R. Musgrave and A. Peacock. New York: MacMillan, 1958, 168-176.

20. Milgrom, P., and Roberts, J. (1990a). “Rationalizability, Learning, and Equilibrium in Games with Strategic Complimentarities.” *Econometrica,* 58, 1255-1277.

21. Milgrom, P., and Roberts, J. (1990b). “Adaptive and Sophisticated Learning in Normal Form Games.” *Games and Economic Behavior,* 3, 82-100.

22. Milleron, Jean-Claude. (1972). "Theory of Value with Public Goods: A Survey Article." *Journal of Economic Theory* 5, 419-477.

23. Muench, T. and Walker, M. (1983). “Are Groves-Ledyard Equilibria Attainable?” *The Review of Economic Studies* 50, 393-396.

24. Namatame, A. and Tse, E. (1981). “Adaptive Expectations and Dynamic Adjustment in Noncooperative Games with Incomplete Information.” *Journal of Optimization Theory and Applications* 34, 243-261.

25. Rosen, J. B. (1965). “Existence and Uniqueness of Equilibrium Points in Concave N-person games.” *Econometrica* 33, 520-34.

26. Rosenlicht, M. (1968). Introduction to Analysis. *Scott, Foresman and Company.*

27. Smith, V. (1979). "An Experimental Comparison of Three Public Good Decision Mechanisms." *Scandinavian Journal of Economics*, 81, 198-215.

28. Smith, V. (1980). "Experiments with a Decentralized Mechanism for Public Good Decisions." *American Economic Review* 70, 584-599.

29. Sundaram, R. (1996). *A First Course in Optimization Theory.* Cambridge University Press.

30. Topkis, D. (1998). *Supermodularity and Complimentarity.* Princeton: Princeton University Press

31. Van Essen, M., Lazzati, N., and Walker, M. (2008). "Learning to Achieve the Lindahl Allocation." University of Arizona Working Paper.

32. Varian, H. (1994). "A Solution to the Problem of Externalities When Agents Are Well-Informed." The American Economic Review, 84, 1278-1293.

33. Vega-Redondo, F. (1989). "Implementation of Lindahl Equilibrium: An Integration of the Static and Dynamic Approaches." *Mathematical Social Sciences*, 18, 211-228.

34. Vives (1990). "Nash Equilibrium with Strategic Complimentarities." *Journal of Mathematical Economics*, 19, 305-321.

35. Vives (2001). *Oligopoly Pricing, Old Ideas and New Tools*, The MIT Press.

36. Walker, M., (1980). "On the Nonexistence of a Dominant Strategy Mechanism for Making Optimal Public Decisions." *Econometrica*, 48, 1521-1540.

37. Walker, M., (1981). "A Simple Incentive Compatible Scheme for Attaining Lindahl Allocations," *Econometrica 49, 65-71.*