Modeling a Multi-Choice Game Based on the Spirit of Equal Job opportunities

Hsiao, Chih-Ru and Chiou, Wen-Lin

6 May 2009

Modeling a Multi-Choice Game Based on the Spirit of Equal Job opportunities

CHIH-RU HSIAO
Department of Mathematics, Soochow University
Taipei 11102, Taiwan
and

WEN-LIN CHIOU
Department of Mathematics, Fu-Jen University
Taipei 24205, Taiwan

Abstract. The H&R multi-choice Shapley value defined by Hsiao and Raghavan for multi-choice cooperative game is redundant free. If the H&R value is used as the solution of a game, there won't be any objection to a player's taking redundant actions. Therefore, the spirit of the law on equal job opportunities is automatically fulfilled.

Keywords. *Shapley value, multi-choice cooperative game, redundant free.*

Introduction. Motivated by calculating the power indices of players in different levels of joint military actions, in [3](1992), Hsiao and Raghvan extended the traditional cooperative game to the multi-choice cooperative game and extended the traditional Shapley value to the Shapley value for multi-choice cooperative games. In short, we call the shapley value for multi-choice cooperative games the **multi-choice Shapley value**. Some authors call the multi-choice Shapley value defined by Hsiao and Raghavan the **H&R** Shapley value.

Based on the spirit of the law on equal job opportunities, in [3] Hsiao and Raghavan allowed players to have the same number of levels of actions. Some authors slightly extended [3] to a multi-choice game where the players have different numbers of options. However, in this article, we will prove that the H&R Shapley value is **redundant free**. If the H&R multi-choice Shapley is used as the solution of a game, it makes no difference to the players whether they have the same number of options or not.

There are another three extensions of the Shapley value for multi-choice games proposed by Derks and Peters[2] (1993) (**D&P** value), Nouweland et al. (1995) (**V&D** value) and Peters and Zank (2005)(**P&Z** value), respectively. In his article, we will rewrite the definitions and the formula in [3] by allowing the players to have different numbers of actions, and show that all the above multi-choice Shapley value are **dummy free of players**.

When players are playing a game, first thing first, they have to decide who are allowed to play the game, what kinds of games they are playing, how many actions they are allowed to have. Fortunately, All the multi-choice Shapley values defined in [2], [3], [5] and [6] are dummy free of players. If one of the values is used as the solution for a game, there won't be any objection to a dummy player's participating in the game. Therefore, the spirit of the law on equal job opportunities will be automatically fulfilled. Furthermore, if the H&R Shapley value is used as the solution for a game, there won't be any objection to a player's taking redundant actions.

Definitions and Notations.

We believe that all the readers are familiar with the traditional mathematical symbols. Therefore, from cognitive point of view, in this article, we will use the traditional mathematical symbols and notations to modify the multi-choice game in order to acquire better meta-cognitive.

Let U be the universal set of players. Without loss of generality, given a finite set of n players $N \subset U$ where $N = \{1, 2, ..., n\}$, we have the following definitions and notations. Any subset $S \subset N$ is called a coalition. Other than what we did in [3], we now allow players to have different numbers of actions. We allow player j to have $(m_j + 1)$ actions, say σ_0, σ_1, σ_2, ..., σ_{m_j}, where σ_0 is the action to do nothing, while σ_k is the option to work at level k, which has higher level than σ_{k-1}. In this article, we assume that there are finitely many players with finitely many choices.

For convenience, we will use non-negative integers to denote the players' actions. Let I_+ denote the set of all finite non-negative integers. Let $\boldsymbol{\beta}_j = \{0, 1, \ldots, m_j\}$, with $m_j > 0$, be the action space of player j. Given $\mathbf{m} = (m_1, m_2, ..., m_n) \in I_+^n$, with $m_j > 0$ for all j, the action space of N is defined by $\Gamma(\mathbf{m}) = \prod_{j \in N} \boldsymbol{\beta}_j = \{(x_1, ..., x_n) \mid x_i \le m_i$ and $x_i \in I_+,$ for all $i \in N\}$. Thus $\mathbf{x} = (x_1, ...x_n)$ is called an action vector of N, and $x_i = k$ if and only if player i takes action σ_k.

Definition 1. A multi-choice cooperative game in characteristic function form is the pair $(\mathbf{m}, v)$ defined by, $v : \Gamma(\mathbf{m}) \to R$, such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, 0..., 0)$.

We may consider $v(\mathbf{x})$ as the payoff or the cost for the players whenever the players take action vector $\mathbf{x}$. Sometimes, we will denote $v(\mathbf{x})$ by $(\mathbf{m}, v)(\mathbf{x})$ in order to emphasis that the domain of v is $\Gamma(\mathbf{m})$.

We can identify the set of all multi-choice cooperative games defined on $\Gamma(\mathbf{m})$ by $G \simeq R^{\prod_{j=1}^{n}(m_j+1)-1}$.

Since we do not assume that action σ_2 is say, twice as powerful as action σ_1, and since we do not assume that the difference between σ_{k-1} and σ_k is the same as the difference between σ_k and σ_{k+1}, etc., giving weights (discrimination) to actions is necessary.

Let $m = \max_{j \in N}\{m_j\}$, and let $w : \{0, 1, ..., m\} \to R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \le w(2) \le \ldots \le w(m)$, then w is called a **weight function** and $w(i)$ is said to be a **weight** of σ_i.

In general a multi-choice cooperative game need not be non-decreasing. When too many players overwork there can be a total system breakdown. Now, we are ready to consider the power indices of the players. Instead of regarding the power index of a game as a vector, we regard the power index or value of a game as a matrix-type table, of course essentially a vector.

$\phi^w : G \to R^{\sum_{j=1}^{n} m_j}$ be the function such that

$$\phi^w(v) = \begin{pmatrix} \phi^w_{1,1}(v) & \phi^w_{1,2}(v) & \ldots & \phi^w_{1,n}(v) \\ \phi^w_{2,1}(v) & \phi^w_{2,2}(v) & \ldots & \phi^w_{2,n}(v) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \phi^w_{m_2,2}(v) & \vdots & \vdots \\ \phi^w_{m_1,1}(V) & & \ddots & \vdots \\ & & & \phi^w_{m_n,n}(v) \end{pmatrix}$$

$$= (\vec{\phi^w}_1(v), \ldots, \vec{\phi^w}_n(v)) \tag{1}$$

and

$$\vec{\phi^w}_i(v) = \begin{pmatrix} \phi^w_{1,i}(v) \\ \phi^w_{2,i}(v) \\ \vdots \\ \phi^w_{m_i,i}(v) \end{pmatrix}$$

Essentially,

$$\phi^w(v) = (\phi^w_{1,1}(v), .., \phi^w_{m_1,1}(v), \phi^w_{1,2}(v), .., \phi^w_{m_2,2}(v), .., \phi^w_{1,n}(v), .., \phi^w_{m_n,n}(v)) \tag{2}$$

The (2) looks much more concise than (1). However, (1) gives us the motivation of redundant free property.

Here $\phi^w_{j,i}(v)$ is the power index or the value of player i when he takes action σ_j in game v.

In fact, we neglect $\phi^w_{0,i}(v)$ and assign $\phi^w_{0,i}(v) = 0$, for all $i \in N$ as does the traditional Shapley value (1953b).

Rewrite [3], we can show that when w is given, there exists a unique ϕ^w satisfying the following four axioms.

Axiom 1. Suppose $w(0), w(1), \ldots, w(m)$ are given. If v is of the form

$$v(\mathbf{y}) = \begin{cases} c > 0 & \text{if } \mathbf{y} \geq \mathbf{x} \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^w_{x_i,i}(v)$ is proportional to $w(x_i)$.

Axiom 1 states that for binary valued (0 or c) games that stipulate a minimal exertion from players, the reward, for players using the minimal exertion level is proportional to the weight of his minimal level action.

We denote $(\mathbf{x} \mid x_i = k)$ as an action vector with $x_i = k$. Given $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbf{m})$, we define $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$ where $x_i \vee y_i = \max\{x_i, y_i\}$ for each i. Similarly, we define $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$ where $x_i \wedge y_i = \min\{x_i, y_i\}$ for each i.

Definition 2. A vector $\mathbf{x}^* \in \Gamma(\mathbf{m})$ is called a **carrier** of v, if $v(\mathbf{x}^* \wedge \mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$. We call $\mathbf{x}^0$ a **minimal carrier** of v if $\sum x_i^0 = \min\{\sum x_i \mid \mathbf{x} \text{ is a carrier of } v\}$.

Definition 3. Player i is said to be a **dummy player** if $v((\mathbf{x} \mid x_i = k)) = v((\mathbf{x} \mid x_i = 0))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and for all $k = 0, 1, 2, ..., m_i$.

The following is a version of the usual efficiency axiom that combines the carrier and the notions of dummy player.

Axiom 2. If $\mathbf{x}^*$ is a carrier of v then, for $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ we have

$$\sum_{\substack{x_i^* \neq 0 \\ x_i^* \in \mathbf{x}^*}} \phi^w_{x_i^*,i}(v) = v(\mathbf{m}).$$

By $x_i^* \in \mathbf{x}^*$ we mean x_i^* is the i-th component of $\mathbf{x}^*$.

Axiom 3. $\phi^w(v^1 + v^2) = \phi^w(v^1) + \phi^w(v^2)$, where $(v^1 + v^2)(\mathbf{x}) = v^1(\mathbf{x}) + v^2(\mathbf{x})$.

Axiom 4. Given $\mathbf{x}^0 \in \Gamma(\mathbf{m})$ if $v(\mathbf{x}) = 0$, whenever $\mathbf{x} \not\geq \mathbf{x}^0$, then for each $i \in N$ $\phi^w_{k,i}(v) = 0$, for all $k < x^0_i$.

Axiom 4 states that in games that stipulate a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded.

Definition 4. Given $\mathbf{x} \in \Gamma(\mathbf{m})$, let $S(\mathbf{x}) = \{i \mid x_i \neq 0, x_i \text{ is a component of } \mathbf{x}\}$. Given $S \subseteq N$, let $\mathbf{e}(S)$ be the binary vector with components $e_i(S)$ satisfying

$$e_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we let the standard unit vectors $\mathbf{e}(\{i\}) = \mathbf{e}_i$, for all $i \in N$, and let $|S|$ be the number of elements of S.

Definition 5. Given $\Gamma(\mathbf{m})$ and $w(0) = 0$, $w(1), \dots, w(m)$, for any $\mathbf{x} \in \Gamma(\mathbf{m})$, we define $\|\mathbf{x}\|_w = \sum\limits_{r=1}^{n} w(x_r)$.

Definition 6. Given $\mathbf{x} \in \Gamma(\mathbf{m})$ and $j \in N = \{1, 2, \dots, n\}$, we define $M_j(\mathbf{x}; \mathbf{m}) = \{i \mid x_i \neq m_i,\ i \neq j\}$.

Following [3], we have

$$\phi^w_{ij}(v) = \sum_{k=1}^{i} \sum_{\substack{x_j = k \\ \mathbf{x} \neq \mathbf{0} \\ \mathbf{x} \in \Gamma(\mathbf{m})}} \left[\sum_{T \subseteq M_j(\mathbf{x};\mathbf{m})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum\limits_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \times [v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_j)]. \qquad (\star)$$

Remark 1. It is well-known that the traditional Shapley value has applications in many fields such as economics, political sciences, accounting and even military sciences. Of course, our multi-choice Shapley value also has the same applications as the traditional Shapley value does.

However, the weight function w has different meanings in different fields. In military sciences, we may treat $w(j)$s' as parameters to modify the differences due to different levels of military actions.

Main Results. The matrix-type table (1) of the multi-choice value and t the law on equal job opportunities give us the motivation that we should avoid discrimination among the players and allow the players to try the same number of actions. We have Definitions as follow.

Definition 7. Given a game $(\mathbf{m}, v)$, the action σ_{m_i} is said to be a **redundant action** for player i if $v((\mathbf{x} \mid x_i = m_i)) = v((\mathbf{x} \mid x_i = m_i - 1))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$.

Given a solution ψ for $(\mathbf{m}, v)$, suppose we allow player i to have one more action which is redundant for player i, say σ_{m_i+1},

Let $\mathbf{m}^* = (m_1, m_2, ..., m_{i-1}, (m_i + 1), m_{i+1}, ..., m_n)$, then we have a new action vector space $\Gamma(\mathbf{m}^*) = \{(x_1, \cdots, x_i, \cdots, x_n) \mid x_j \leq m_j, x_j \in I_+ \text{ for all } j \neq i, \text{ and } x_i = 0, 1, 2, \cdots, m_i + 1\}$. We may extend $(\mathbf{m}, v)$ to $(\mathbf{m}^*, v^R)$ such that $v^R(\mathbf{x}) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and $v^R((\mathbf{x} \mid x_i = m_i + 1)) = v((\mathbf{x} \mid x_i = m_i))$, for all for all $\mathbf{x} \in \Gamma(\mathbf{m}^*)$. The solution ψ is said to be **redundant free** if and only if $\psi_{k,\ell}(v^R) = \psi_{k,\ell}(v)$ for all $\ell \in N$, and $k = 1, 2, ..., m_\ell$, and $\psi_{(m_i+1),i}(v^R) = \psi_{m_i,i}(v)$. Otherwise, the solution is say to be dependent on redundant action.

Note 1. Since in [3] we assumed that players have same number of actions, please note that the definition of **redundant free** in this article is quite different from the definition of **dummy free of action** in [3].

Theorem 1. The**H& R** multi-choice Shapley value is redundant free.

Proof. Omitted

We now consider **dummy free of player** properties Following [3], we have the definition as follows. Given $N = \{1, 2, \ldots, n\}$, $\mathbf{m} = (m_1, \ldots, m_n)$, and a multi-choice cooperative game $(\mathbf{m}, v)$, suppose $\phi_{i,j}(v) = a_{i,j}$ for feasible $i \in \boldsymbol{\beta}_j$. Now, allow a dummy player, say $(n+1)$ with $\boldsymbol{\beta}_{n+1} = \{0, 1, \ldots, m_{n+1}\}$to join the game. Let $N^D = \{1, \ldots, n, n+1\}$ and $\mathbf{m}^D = (m_1, \ldots, m_n, m_{n+1})$, then we have a new game $(\mathbf{m}^D, v^D)$ such that $v^D((\mathbf{x} \mid x_{n+1} = i)) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and all $i \in \boldsymbol{\beta}_{n+1}$. $(\mathbf{m}^D, v^D)$ is called a dummy player extension of $(\mathbf{m}, v)$.

Suppose $\phi(v^D) = b_{i,j}$, for feasible $i \in \boldsymbol{\beta}_j$ and $j \in N^D$, we could ask whether $a_{i,j} = b_{i,j}$ for all $i \in \boldsymbol{\beta}_j$ and all $j \in N$. A solution of a multi-choice cooperative game is said to be **dummy free of players** if (i) $b_{i,n+1} = 0$ for all $i \in \boldsymbol{\beta}_{n+1}$ and (ii) $a_{i,j} = b_{i,j}$ for all $i \in \boldsymbol{\beta}_j$ and all $j \in N$; otherwise the solution is said to be dummy dependent of players. In [3] Hsiao and Raghavan showed that the H&R multi-choice Shapley value is dummy free of players.

In [5] (1995)Van den Nouweland et al. introduced a solution on G which associates with each $(\mathbf{m}, v) \in G$ and each player $i \in N$ a value $\phi(\mathbf{m}, v)$, they called it the N&P

Shapley value. Following an elegant explicit formula given by Calvo and Santos[1](2000), we rewrite the V&D value as follows.

$$\phi_i(\mathbf{m}, v) = \sum_{\substack{\mathbf{x}\leq\mathbf{m}\\ x_i\neq 0}} m_i\cdot\left(\frac{(|\mathbf{x}|-1)!(|\mathbf{m}|-|\mathbf{x}|)!}{|\mathbf{m}|!}\right)\cdot\left[\prod_{\substack{k\in S(\mathbf{m})\\ k\neq i}} \binom{m_k}{x_k}\right]\cdot\binom{m_i-1}{x_i-1}\cdot[v(\mathbf{x})-v(\mathbf{x}-\mathbf{e}_i)], \tag{3}$$

where $|\mathbf{x}| = \sum_{i\in N} x_i$, $S(\mathbf{m}) = \{i|m_1 \neq 0\}$ and the traditional mathematical notation $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Now, suppose $N = \{1, 2\}$, $\mathbf{m}^* = (2, 2)$, and $(\mathbf{m}*, v^R)$ be such that $v^R((0, 0)) = v^R((1, 0)) = v^R(2, 0) = v^R((0, 1)) = v^R((0, 2)) = 0$, $v^R((1, 1)) = v^R((1, 2)) = 2$ and $v^R((2, 1)) = v^R((2, 2)) = 3$. Then by formula (3), the N&P Shapley value $\phi(\mathbf{m}^*, v^R) = (\phi_1(\mathbf{m}^*, v^R)), \phi_2(\mathbf{m}^*, v^R)) = (\frac{11}{6}, \frac{7}{6})$.

Hence, the V&D Shalpy value is not redundant free. However, the V&D Shapley value is, in some sense, dummy free of player.

Theorem 2. Given $N = \{1, \dots, n\}$, $\mathbf{m} = (m_1, \dots, m_n)$ a game $(\mathbf{m}, v)$ and its N&P Shapley value $\phi(\mathbf{m}, v) = (\phi_1(\mathbf{m}, v)), \dots, \phi_n(\mathbf{m}, v))$, let$(\mathbf{m}^D, v^D)$ be a dummy player extension of $(\mathbf{m}, v)$ with $N^D = \{1, \dots, n, n+1\}$, $\mathbf{m}^D = (m_1, \dots, m_n, m_{n+1})$, then $\phi_i(\mathbf{m}, v) = \phi_i(\mathbf{m}^D, v^D)$, for $i = 1, \dots n$ and $\phi_{n+1}(\mathbf{m}^D, v^D) = 0$.

Proof. Omitted.

Conclusion. The real world is full of discrimination, therefore, we need the law to amend the discrimination . Based on the spirit of the law on equal job opportunities, when modeling a multi-choice game and its solution, we have to focus on dummy free properties and redundant free properties.

If a consultant in the real world propose a solution to his/or her clients (players) and the solution is not dummy free of players or redundant free, then the solution will be very controversial or even against the law on equal job opportunities.

References

[1] Calvo E, Santos JC (2000) A value for multichoice games. Mathematical Social Sciences 40:341-354

[2] Derks J, Peters H (1993) A Shapley value for games with restricted coalitions. International Journal of Game Theory 21,351-360

[3] Hsiao, Chih-Ru and T.E.S. Raghavan (1992), Monotonicity and Dummy Free Property for Multi-Choice Cooperative Games. **21**, International Journal of Game Theory, pp. 301-312.

[4] Hwang Y-A, Liao Y-H (2008) The solutions for multi-choice games: TU games approach "Economics Bulletin" Vol. 3, No. 43: 1-7.

[5] Nouweland A van den, Potters J, Tijs S, Zarzuelo JM (1995) Core and related solution concepts for multi-choice games. ZOR-Mathematical Methods of Operations Research 41,289-311.

[6] Peters H, Zank H (2005) The egalitarian solution for multichoice games. Annals of Operations Research 137,399-409

[7] Shapley, L. S. (1953), A value for n-person Games, In: Kuhn, H. W., Tucker, A.W. (eds.). Contributions to the Theory of Games II, Princeton, pp. 307-317.