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SKEWED LIBOR MARKET MODEL AND GAUSSIAN HJM EXPLICIT APPROACHES TO ROLLED DEPOSIT OPTIONS

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Abstract. A simple exotic option (floor on rolled deposit) is studied in the shifted log-normal Libor Market (LMM) and Gaussian HJM models. The shifted log-normal LMM exhibits a controllable volatility skew. An explicit approach is used for both models. Using approximations the price in the LMM is obtained without Monte Carlo simulation. The more precise approximation uses a twisted version of the perdictor-corrector adapted to explicit solutions. The results of the approximation are surprisingly good.

1. Introduction

The Libor Market Models (LMM) were introduced at the end of the 90's (independently in [2], [17] and [14]). They are nowadays very popular and almost standard interest rate models.

Even if they are *twin brothers* (see [6]), the LMM and Gaussian Heath-Jarrow-Morton (HJM) models usually lead to very different numerical implementations techniques. Monte Carlo simulations is the main tool for the former while explicit formulas or trees are the standard for the latter.

Unfortunately the Gausssian HJM models can not be calibrated to the skew present in the cap market. One model that can be (partly) calibrated to the skew is a LMM with rate's dynamic based on a displaced geometric Brownian motion also called shifted log-normal LMM.

The market standard Black formula to price caps can be viewed as one extreme where a simple product is priced with a simple model through a very simple formula. On the other extreme is the pricing of exotic products with a LMM incorporating the skew through a numerically heavy Monte Carlo simulation. The latter has created a vast literature, in particular on how to approximate efficiently the rate dependent drift present in the Libor equation (see for example [19] for an overview and [13], [4], [15] for recent results). It seems that between those two continents the ocean is empty, no intermediary approach is proposed. This article describes an island in the ocean. A simple exotic product is priced in the skewed model through a simple formula using approximation.

The skew obtained with this new approach is very difficult to distinguish from the Monte Carlo skew. The implied skews are very similar and certainly within market bid-offer but with the immense advantage of the efficiency and stability in the pricing.

The approach is based on a initial model which is a shifted log-normal model with a very specific shift. This specific shift simplifies the drift term present in the LMM to a deterministic one. The model can be viewed as a the LMM version of the continuous time Ho and Lee model (Appendix B.3). The model is a Gaussian HJM for which the bond prices are log-normal. It is also known as the Bond Market Model (BMM, see [1] for this name).

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The explicit price of the option on rolled deposit is presented for that version of the model in Section 3.3. The price in that model will be used as a yard stick for the rest. The other prices will be embedded in that model using an implied volatility view.

The prices in the skewed LMM with other skews are first obtained using a simplistic approximation. The path dependent coefficients are frozen to their initial value. This can be viewed as the equivalent to the one step Euler scheme in Monte Carlo approach. The prices obtained in this way are acceptable for *at-the-money* options but exhibit the same flat skew as the initial model. This approximation simply restate the volatility in the reference model (like a normalized volatility in the Black model) and used it for the price.

As very well known, in the LMM the coefficients are generally *path dependent*. The second approach to simplify the path dependency proposed here find his inspiration in the Runge-Kutta approach introduced in finance in [16] and described in [19] in the context of the LMM. The Monte Carlo approach is not used and the path dependency is not analysed at the individual path level. The aim is to obtain an explicit formula and for this reason the path dependency should be valid only at an *average level*. Consider a cap type product starting *out-of-the-money*. When a path ends out-of-the-money its contribution to the average is known, it is a constant. For that path there is no need to study the path dependency. The paths of interest are the one that reach (and go beyond) the strike. For those paths an information on their end value is known and our approximation takes advantage of that information. The path dependent coefficients are approximated by the average value between the initial value and one *at-the-money* value. This simple change incorporates the information available and is enough to create an approximation surprisingly efficient. Like the predictor-corrector approach it is very simple to implement and improve significantly the results.

As a by-product of the simple way in which the approximation is implemented the price is presented as the price in the BMM with a strike dependent (implied) volatility. This is a situation similar to the SABR model [7] where a model incorporating more market characteristic than a simple model is described through an implied volatility in the simple model (Black in the SABR case, BMM here).

The instrument studied here is an option on rolled deposits. It is a simple exotic instrument which involves several Libor rates fixed at different dates and one payment date.

The rolled deposit pays a floating rate (typically the Libor) compounded on several consecutive periods. The period dates are $0 \leq t_1 < t_2 < \cdots < t_n$. The rates are fixed at the dates $s_i \leq t_i$. The accrual factors for the periods $[t_i, t_{i+1}]$ are δ_i. The rate for the period viewed from s is denoted L^i_s. The *rolled deposit* pay-off in t_n is

$$\prod_{i=1}^{n-1}(1+\delta_i L^i_{s_i}) \tag{1}$$

It can be viewed as a *discretely-rebalanced bank account*. It is also called *composition*. In practice the time-discretisation for this type of product is often three months. In theory the results can also be applied to over-night deposits.

The payment is subject to a floor (or a cap). Without the floor, the value of the instrument would simply be 1 in t_1, like a floating rate note. What is special here is that the floor is on the compounded rate, not on each individual fixing. It is an option on an average, i.e. an Asian type option. It involves the correlations between the different rates. For a floor with an amount K, the payment at maturity is

$$\max\left(\prod_{i=1}^{n-1}(1+\delta_i L^i_{s_i}), K\right). \tag{2}$$

In its floored version, the product is called Floored Instrument on Rolled DEposit (FIReD).

Those products are not traded in the interbank market. They have a real utility for investor in short-term money-market type products requiring a minimal return over the long term. A typical FIReD product will have a total maturity of one or two years and a quarterly fixing. The fixing reference is usually Libor (plus a spread, depending of the issuer). At the moment of writing the one year rate is around 5.30%. A FIReD could have a floor at 5% in rate term or 1.05 in price. The investment is mainly a floating rate investment, but if the rate moves really lower on average over the live of the instrument, the floor kicks in.

The appendices are devoted to more technical existence and non-existence results for the LMM. To our knowledge those short results have never been published even if shifted log-normal LMM have been used in different places (in particular [19, Chapter 11] and [5]).

Appendix B.1 proves that not all shifted LMM can be embedded into a HJM framework. Or in other words, not all LMM exists in theory. If needed the equations are modified far away from any realistic rate to ensure the theoretical foundation of the model but with no impact on any practical computations.

Appendix B.3 is dedicated to the BMM. In its simplest version it is equivalent to the continuous time version of the Ho and Lee [11] model. The equivalence is explained in the appendix.

2. Model and hypothesis

The two instances of the HJM framework used in this article are described in this section. They are a multi-factors LMM type with displaced log-normal Libor as base equations and a Gaussian version (deterministic volatility).

In general, the HJM framework describes the behavior of $P(t,u)$, the price in t of the zero-coupon bond paying 1 in u ($0 \leq t \leq u \leq T$). When the discount curve $P(t,.)$ is differentiable in a weak sense (absolutely continuous in the non-linear analysis jargon), which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t,u)$ such that

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{3}$$

The Heath et al. idea [8] was to exploit this property by modeling f with a stochastic differential equation

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)\cdot dW_t$$

for some suitable (stochastic) μ and σ and deducing the behavior of P from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The volatility and the Brownian motion are m-dimensional while the drift and the rate are 1-dimensional. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of [12].

The probability space is $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a m-dimensional standard Brownian motion $(W_t)_{0\leq t\leq T}$. To simplify the writing in the rest of the paper, the bond volatility is denoted by

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0\leq s\leq T}$ the short rate given by $r_t = f(t,t)$. The equations of the model in the numeraire measure $\mathbb{N}$ associated to N_t are

$$df(t,u) = \sigma(t,u)\nu(t,u)dt + \sigma(t,u)\cdot dW_t$$

or

$$dP^N(t,u) = -P^N(t,u)\nu(t,u)\cdot dW_t$$

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. The expected value in that measure is denoted $\mathrm{E}_{\mathbb{N}}$.

The prices of zero-coupons and cash-account value can formally be written explicitely as function of the volatility ν and the Brownian motion. Those technical results are given in Appendix A.

2.1. **Libor Market Model.** The idea behind the Libor Market model is to embed different Black-like equation for the forward (Libor) rate between standard dates $t_0 < \cdots < t_n$ into a unique HJM model. The Libor rates L_s^i are defined by

$$1 + \delta_i L_s^i = \frac{P(s,t_i)}{P(s,t_{i+1})}.$$

The equations underlying the Libor Market Model are

$$dL_t^j = \gamma_j(L_t^j, t).dW_t^{j+1} \tag{4}$$

in the probability space with numeraire $P(t,t_{j+1})$. The γ_j $(0 \leq j \leq n-1)$ are m-dimensional functions. For fundamental reasons explained in Appendix B.1 not all such models are well-defined. In this note the shifted log-normal model is analysed. For that model

$$\gamma_j(L_t^j, t) = (L_t^j + a_j)\gamma_j(t)$$

for some constants a_j and positive functions $\gamma_j(t)$.

The Brownian motion change between the N_t and the $P(t,t_{j+1})$ numeraires is given by

$$dW_t^{j+1} = dW_t + \nu(t,t_{j+1})dt.$$

The difference $\nu(t,t_{j+1}) - \nu(t,t_j)$ can be written as

$$\nu(t,t_{j+1}) - \nu(t,t_j) = \frac{1}{L_t^j + \frac{1}{\delta_j}}\gamma_j(L_t^j, t)$$

Recursively the change between the different numeraires $P(t,t_{j+1})$ and $P(t,t_n)$ is given by

$$dW_t^{j+1} = -\sum_{i=j+1}^{n-1} \frac{1}{L_t^i + \frac{1}{\delta_i}}\gamma_i(L_t^i, t)dt + dW_t^n.$$

All the rates can be written with respect to the same (last) numeraire

$$dL_t^j = -\left(\sum_{i=j+1}^{n-1} \frac{1}{L_t^i + \frac{1}{\delta_i}}\gamma_i(L_t^i, t).\gamma_j(L_t^j, t)\right) dt + \gamma_j(L_t^j, t).dW_t^n. \tag{5}$$

3. FIReD valuation

The models introduced in the previous section are used to price the floored instrument. The first step is to obtain a generic theoretical value independent of the exact model. The generic formula is specialized to the different models to obtain explicit results.

The generic value is specialized to the Gaussian HJM and the BMM. Based on this a crude approximated price is obtained for the general shifted log-normal LMM through a initial freeze approximation. This approximation corresponds to a flat smile.

In a second stage the approximated result is improved using a strike dependent predictor-corrector like approach that creates a smile. The approximation that may look simplistic at first glance is very efficient. Almost all the volatility skew present in the shifted log-normal LMM is preserved by the approximation.

3.1. **Generic valuation.** The payment related to the instrument happen in t_n. Generically the value of the instrument is

$$F_0 = N_0 \, \mathrm{E}_{\mathbb{N}} \left[\max\left(\prod_{i=1}^{n-1} \frac{P(s_i,t_i)}{P(s_i,t_{i+1})}, K \right) N_{t_n}^{-1} \right].$$

Using Lemma 1, in the cash account numeraire,

$$\begin{aligned}\prod_{i=1}^{n-1} \frac{P(s_i,t_i)}{P(s_i,t_{i+1})} &= \frac{P(0,t_1)}{P(0,t_n)} \exp\left(\frac{1}{2} \sum_{i=1}^{n-1} \int_0^{s_i} (|\nu(s,t_{i+1})|^2 - |\nu(s,t_i)|^2) ds \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} \int_0^{s_i} (\nu(s,t_{i+1}) - \nu(s,t_i)) \cdot dW_s \right)\end{aligned}$$

By Lemma 2,

$$N_{t_n}^{-1} = P(0,t_n) \exp\left(-\int_0^{t_n} \nu(s,t_n) dW_s - \frac{1}{2} \int_0^{t_n} \nu^2(s,t_n) ds \right).$$

The numeraire is changed to $P(0,t_n)$ and the new numeraire Brownian motion is related to the cash account one by $W_s^n = W_s + \int_0^s \nu(\tau,t_n) d\tau$. The expected value is E^n. The sum of the integrals in the exponential can be written as

$$\begin{aligned}&\sum_{i=1}^{n-1} \int_0^{s_i} (\nu(s,t_{i+1}) - \nu(s,t_i)) \cdot dW_s + \frac{1}{2} \sum_{i=1}^{n-1} \int_0^{s_i} (|\nu(s,t_{i+1})|^2 - |\nu(s,t_i)|^2) ds \\ =\ & \sum_{i=1}^{n-1} \int_0^{s_i} (\nu(s,t_{i+1}) - \nu(s,t_i)) \cdot dW_s^n \\ & -\frac{1}{2} \sum_{i=1}^{n-1} \int_0^{s_i} (\nu(s,t_{i+1}) - \nu(s,t_i)) \cdot (2\nu(s,t_n) - \nu(s,t_{i+1}) - \nu(s,t_i)) ds\end{aligned}$$

Using the identity $\nu(s,t_n) - \nu(s,t_i) = \sum_{j=i}^{n-1} (\nu(s,t_{j+1}) - \nu(s,t_j))$ and rearranging the terms the last sum can be written as

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \int_0^{\min(s_i,s_j)} (\nu(s,t_{i+1}) - \nu(s,t_i)).(\nu(s,t_{j+1}) - \nu(s,t_j)) ds.$$

The random variable involved in the valuation is thus

(6)
$$Z = \sum_{i=1}^{n-1} \int_0^{s_i} \nu(s,t_{i+1}) - \nu(s,t_i).dW_s^n - \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \int_0^{\min(s_i,s_j)} (\nu(s,t_{i+1}) - \nu(s,t_i)).(\nu(s,t_{j+1}) - \nu(s,t_j)) ds.$$

and the option value is

$$F_0 = \mathrm{E}^n \left[\max\left(P(0,t_1) \exp(Z), P(0,t_n) K \right) \right].$$

3.2. **Gaussian HJM.** The pricing is specialized in the case of the Gaussian HJM model. In this case the volatility ν is deterministic. The zero-coupon bond and cash-account value given by Lemma 1 and 2 are log-normally distributed.

The model version most used in practice is the extended Vasicek (or Hull-White) model. The volatility is $\nu(s,t) = \sigma/a(1 - \exp(-a(t-s)))$.

Theorem 1. *In the HJM model, the price of an instrument paying in t_n the maximum of a fixed amount $K > 0$ and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is given in 0 by*

$$F_0 = P(0,t_1) N(\kappa + \sigma) + K P(0,t_n) N(-\kappa)$$

where

$$\sigma^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\int_0^{\min(s_i,s_j)} (\nu(s,t_{i+1}) - \nu(s,t_i)).(\nu(s,t_{j+1}) - \nu(s,t_j))ds.$$

and

$$\kappa = \frac{1}{\sigma}\left(\ln\left(\frac{P(0,t_1)}{KP(0,t_n)}\right) - \frac{1}{2}\sigma^2\right).$$

The price of an instrument paying in t_n the minimum of a fixed amount K and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is given in 0 by

$$C_0 = P(0,t_1)N(-\kappa - \sigma) + KP(0,t_n)N(\kappa)$$

Proof. The stochastic integrals of deterministic functions in the first term are normally distributed. If their sum is denoted $-\sigma X$, the value of the instrument is

$$F_0 = \mathrm{E}^n\left[\max\left(P(0,t_1)\exp\left(-\frac{1}{2}\sigma^2 - \sigma X\right), P(0,t_n)K\right)\right]$$

where X is a random variable with a standard normal distribution with respect to $\mathbb{P}^n$ [18, Theorem 3.1]. The first term of the maximum operator is the actual maximum when $X < \kappa$. So we obtain

$$F_0 = P(0,t_1)\,\mathrm{E}^n\left[\exp\left(-\sigma X - \frac{1}{2}\sigma^2\right)\mathbb{1}(X<\kappa)\right] + KP(0,t_n)\mathbb{P}^n\left(X \geq \kappa\right)$$

which, by standard manipulation on the expectation and on the normal distribution, leads to the result.

The price of the capped instrument can be obtained by put-call parity. □

3.3. **Bond Market Model.** In this section the LMM is a shifted log-normal model with

$$dL_t^j = \left(L_t^j + 1/\delta_j\right)\gamma_j(s).dW_t^{j+1}.$$

With this very specific displacement, the model is equivalent to the Bond Market Model described in [1]. The difference in volatility becomes

$$\nu(t,t_{j+1}) - \nu(t,t_j) = \frac{L_t^j + \frac{1}{\delta_j}}{L_t^j + \frac{1}{\delta_j}}\gamma_j(t) = \gamma_j(t).$$

The volatility is deterministic.

Theorem 2. *In the BMM, the price in 0 of an instrument paying in t_n the maximum of a fixed amount $K > 0$ and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is*

$$F_0 = P(0,t_1)N(\kappa + \sigma) + KP(0,t_n)N(-\kappa)$$

where

$$\sigma^2 = \sum_{i=1}^{n-1}\sum_{i=1}^{n-1}\tau_{i,j}$$

with $T = (\tau_{i,j})$ $(1 \leq i, j \leq n-1)$,

$$\tau_{i,j} = \int_0^{\min(s_i,s_j)} \gamma_i(s)\cdot\gamma_j(s)ds,$$

and

$$\kappa = \frac{1}{\sigma}\left(\ln\left(\frac{P(0,t_1)}{KP(0,t_n)}\right) - \frac{1}{2}\sigma^2\right).$$

The price in 0 of an instrument paying in t_n the minimum of a fixed amount K and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is

$$C_0 = P(0,t_1)N(-\kappa-\sigma) + KP(0,t_n)N(\kappa)$$

Proof. The stochastic integrals $X_i = \int_0^{s_i} \gamma_i(s) \cdot dW_s^n$ are normally distributed with mean 0 and covariance T. The integrals in the generic price can be written as

$$-\sum_{i=1}^{n-1} X_i - \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \tau_{i,j}.$$

The composition becomes

$$\prod_{i=1}^{n-1} \frac{P(s_i,t_i)}{P(s_i,t_{i+1})} = \frac{P(0,t_1)}{P(0,t_n)} \exp\left(-\sum_{i=1}^{n-1} X_i - \frac{1}{2}\sigma^2\right).$$

The sum of the random variables X_i is a normally distributed variable

From there the result is obtained as in the previous therorem. □

3.4. **Initial freeze approximation.** For the next two section a general shifted log-normal LMM is used ($a_j \leq 1/\delta_j$)

$$dL_t^j = \left(L_t^j + a_j\right)\gamma_j(s).dW_t^{j+1}.$$

The difference in volatility is approximated by

$$\nu(t,t_{j+1}) - \nu(t,t_j) = \frac{L_t^j + a_j}{L_t^j + \frac{1}{\delta_j}}\gamma_j(t) \simeq \frac{L_0^j + a_j}{L_0^j + \frac{1}{\delta_j}}\gamma_j(t).$$

The approximation is done by freezing the rates at their inital value to compute the volatility.

This can be related to the *nomarlized* volatility procedure in the caplet valuation in the Black model. The mormalized volatility is such that $\sigma_{\text{Normal}} = \sigma_{\text{Black}}.L_0$. With such a volatility the log-normal and normal equations have the same initial coefficients.

When the actual shift a_j is close to $1/\delta_j$, the volatility depend only slightly of the rate and using a constant approximation is acceptable.

Theorem 3. *In the LMM, the price in 0 of an instrument paying in t_n the maximum of a fixed amount K and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is (approximately)*

$$F_0 = P(0,t_1)N(\kappa+\sigma) + KP(0,t_n)N(-\kappa)$$

where

$$\sigma^2 = \lambda^T T \lambda$$

with $T = (\tau_{i,j})$ $(1 \leq i,j \leq n-1)$,

$$\tau_{i,j} = \int_0^{\min(s_i,s_j)} \gamma_i(s)\cdot\gamma_j(s)ds, \quad \lambda_i = \frac{L_0^i + a_i}{L_0^i + 1/\delta_i},$$

and

$$\kappa = \frac{1}{\sigma}\left(\ln\left(\frac{P(0,t_1)}{KP(0,t_n)}\right) - \frac{1}{2}\sigma^2\right).$$

The price in 0 of an instrument paying in t_n the minimum of a fixed amount K and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is (approximately)

$$C_0 = P(0,t_1)N(-\kappa-\sigma) + KP(0,t_n)N(\kappa)$$

The proof is similar to the previous one, replacing $\gamma_j(t)$ by $\lambda_j\gamma_j(t)$.

3.5. **Skew preserving approximation.** The previous approximation is equivalent to having a model with the skew $a_i = 1/\delta_i$ but the volatility changed. In particular the implied smile using this approximation will be flat with repesct to the Bond Market valuation.

In the Runge-Kutta or predicator-corrector approach to the Monte-Carlo simulation, the drift of the equation is approximated by the average of the initial and final drift. This appraoch can appear simplistic as the path can be a lot more complex than simply moving regularly to its end value; the average value over the path can be very different from the average between the starting and end points. The approaximation should not be viewed as representing a single path but representing the average over all the paths with the same start and end. On average the paths that start and end like that particular path have a drift equal to the average between the initial and end value. Stricto sensus the above sentence is not correct and the *equal to* should be replace by *approximated by*.

Here we don't approximate the drift of the equation but directly the solution, neithertheless a similar approach to this insight can be applied.

The approach proposed is developped with in mind *out-of-the-money* options. By put/call parity a in-the-money call can be transform into a out-of-the-money put and inversely. So this is not a real restriction.

The paths that enter into the valuation are path that ends in-the-money. The other are not taken account and replaced by K. The interesting paths have to travel from the current *at-the-money* level to at least the strike. The average level of those path that enter into the expected value can be approximated by the average between the starting points and the strike point. As those paths end in the money the value at the end point will be at least the strike.

Let $\alpha_i = \sqrt{\tau_{i,i}}$. In the previous section, the composition is approximated with

$$\prod_{i=1}^{n-1} \frac{P(s_i,t_i)}{P(s_i,t_{i+1})} \simeq \frac{P(0,t_i)}{P(0,t_{i+1})} \exp\left(-\sum \lambda_i \alpha_i X_i - \frac{1}{2}\sigma^2\right)$$

Let $\tilde{\sigma}^2 = \sigma^2 / \sum \lambda_i^2 \alpha_i^2$. The composition is at the strike with all the X_i at the same level $\bar{x}$ when

$$\frac{P(0,t_1)}{P(0,t_n)} \exp\left(-\sum_{i=1}^{n-1} \lambda_i \alpha_i \bar{x} - \frac{1}{2}\sigma^2\right) = K. \tag{7}$$

The rates L_i^K for which the strike is achieved can be defined by

$$1 + \delta_i L_K^i = (1 + \delta_i L_0^i) \exp\left(-\lambda_i \alpha_i \bar{x} - \frac{1}{2}\lambda_i^2 \alpha_i^2 \tilde{\sigma}^2\right).$$

The volatility difference is approximated by

$$\nu(t,t_j) - \nu(t,t_{j+1}) = \frac{L_t^j + a_j}{L_t^j + \frac{1}{\delta_j}} \gamma_j(t) \simeq \frac{1}{2}\left(\frac{L_0^j + a_j}{L_0^j + \frac{1}{\delta_j}} + \frac{L_K^j + a_j}{L_K^j + \frac{1}{\delta_j}}\right)\gamma_j(t).$$

Theorem 4. *In the LMM, the price in 0 of an instrument paying in t_n the maximum of a fixed amount $K > 0$ and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is (approximately)*

$$F_0 = P(0,t_1)N(\kappa + \sigma) + KP(0,t_n)N(-\kappa)$$

where

$$\sigma_K^2 = \lambda_K^T T \lambda_K$$

with $T = (\tau_{i,j})$ $(1 \le i,j \le n-1)$,

$$\tau_{i,j} = \int_0^{\min(s_i,s_j)} \gamma_i(s)\cdot\gamma_j(s)ds, \quad \lambda_K^i = \frac{1}{2}\left(\frac{L_0^j + a_j}{L_0^j + \frac{1}{\delta_j}} + \frac{L_K^j + a_j}{L_K^j + \frac{1}{\delta_j}}\right)$$

and

$$\kappa = \frac{1}{\sigma_K}\left(\ln\left(\frac{P(0,t_1)}{KP(0,t_n)}\right) - \frac{1}{2}\sigma_K^2\right).$$

The price in 0 of an instrument paying in t_n the minimum of a fixed amount K and of a deposit rolled over the periods $[t_i, t_{i+1}]$ fixed in s_i is (approximately)

$$C_0 = P(0,t_1)N(-\kappa-\sigma) + KP(0,t_n)N(\kappa)$$

The above formula is exact for $a_i = 1/\delta_i$ and equal to the one of Theorem 2.

To better analyse the skew associated to the approximation a notion of *implied volatility* with respect to the BMM is created. This is the flat volatility in Theorem 2 that would give the same price. It is given by

$$\gamma_{\text{Implied}} = \frac{\sigma_K}{\sqrt{\sum\sum \min(s_i, s_j)}}.$$

This is a one dimensional volatility. Even if for all i, $|\gamma_i| = \gamma$ when the model is multi-factor, it is possible to have $\gamma_{\text{Implied}} \neq \gamma$ (actually $\gamma_{\text{Implied}} < \gamma$).

4. Approximation goodness and volatility skew

4.1. **Caplet.** The first review of the approximation is done with a standard caplet. For the caplet an exact formula exists for the displaced difussion model. For each caplet there will be four values reviewed: one exact, one inital value approximation (Theorem 3), one average strike value approximation (Theorem 4) and one Monte Carlo value.

The tests are done for different displacement a_i ranging from 0 (log-normal model, standard BGM) to $1/\delta_i$ (BMM). All the displacements are identical for the different rates. The rate environement of this test is a flat 5% yield curve. The strikes range from 5% to 7%. A floored out-of-the-money instrument is priced. The end date is 2y and a quaterly caplet is priced. The Monte Carlo simulation is run with 50,000 paths and a predictor-corrector approach as described in [19]. The results are displayed in Figure 1.

The inital value approximation gives, as expected a flat smile. The three other approaches give very similar results. The implied volatilities are almost undistinguisable on the figure and equivalent for all practical purposes. Note that a different random seed was used in the Monte Carlo simulation for each displacement.

Of course as an exact result exist for the caplets all the approximations developped here are not really relevant. But this a good way to test the approximation and the Monte Carlo approach used.

The extreme cases where a_i is small are the one for which the approximation could be the least efficient. For that reason some extra figures are proposed for those cases. The difference between the exact smile and the different approached is proposed in Figure 2(a).

As a comparison, for $a_i = 0$ the caplet smile effect is around 5 bps at the 7% strike. The maximum difference between the exact value and the approximation is 0.4 bps. For very large strike, the Monte Carlo simulation is usually more precise, close to ATM strike the approximation is more precise. This is a relatively standard observation. The Monte Carlo approaches perform better for out-of-money options. It can be consider as a variance reduction technique in Monte-Carlo simulations to price only out-of-the-money options and uses put/call parity. This was described in [9].

Figure 2(b) represents the full smile, including in-the-money options. All the curves represent the difference with the exact formula. The thick continuous curve is the approximation. The approximation is sligtly less good for the in-the-money options. The Monte Carlo simulation is run ten times. Generaly the simulation gives beter results for far out-of-the-money strikes but worst for at-the-money or in-the-money options. The thick dotted line is the average smile of the ten simulation.

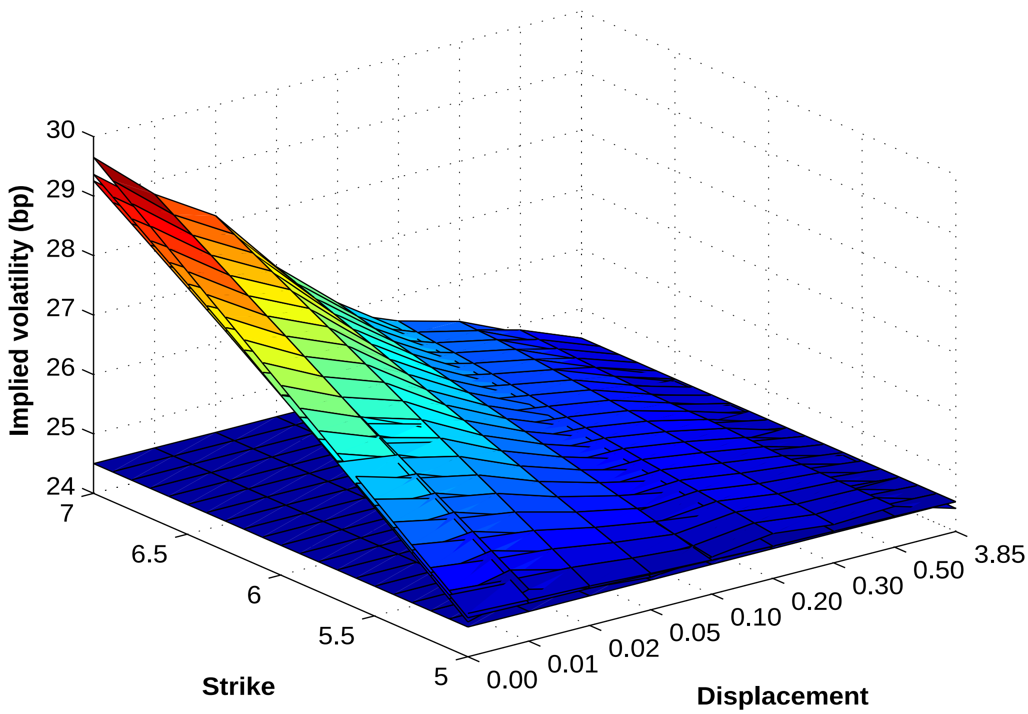


FIGURE 1. Caplet smile, one factor

4.2. **FIReD.** For the second test a quarterly FIReD product with a start date three month from today and a two year maturity is priced. The product is tested with the three methods available: two approximations and Monte-Carlo simulations. The implied volatility results are displayed in Figure 3.

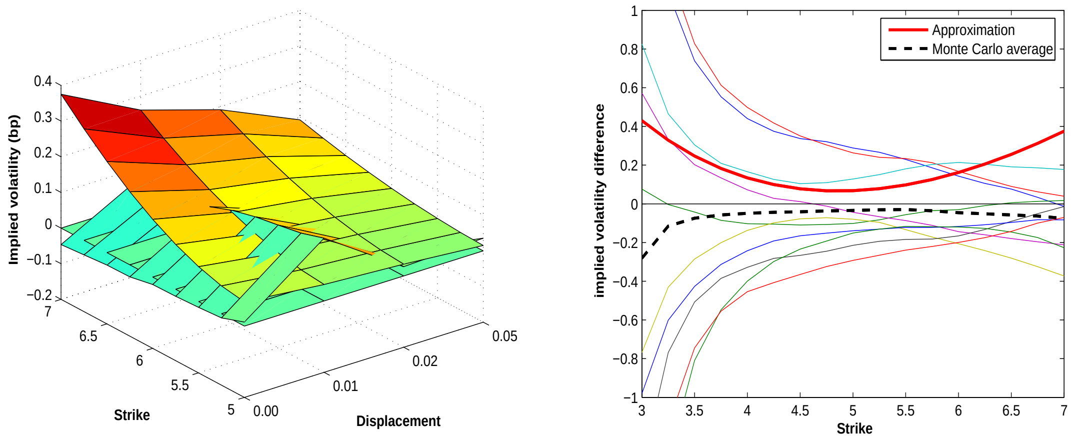
The inital value approximation gives, as expected a flat smile. The two other approaches give very similar results. The implied volatilities are almost undistinguisable on the figure and equivalent for all practical purposes.

Of course the explicit solution proposed is a lot faster that the Monte Carlo appraoch. It requires only to solve a the one dimension equation (7).

Like in the previous section several Monte Carlo simulations are run with the same conditions to compare the approximation to the standard Monte Carlo simulation (Figure 4(a)). Here there is no reference smile has there is no exact formula. As a reference, the average between 20 Monte Carlo simulation with 200,000 paths is used.

In this case the very low strikes have been removed from the figure. For the composition instrument the strike is an average level that the rate should reach. This average include the very short term rate that start in three or six month. The probability to go to very low levels in a short period of time is extremely low. There the error due to the Monte Carlo simulation is very large and the results are (almost) meaningless.

The results proposed here are not restricted to a flat curve like the previous exemples. Figure 4(b) is similar to the previous one but with a curve upward sloping with a short rate at 3.5%



(a) Relative smile

(b) Several Monte Carlo simulations, $a_i = 0$

FIGURE 2. Caplet smiles, one factor.

and a two year rate at 5%. In this case the asymmetry between the in-the-money and out-of-the-money option is larger. Both the Mont-Carlo and approximation are less good.

The approximation error described above are for the worst case where $a_i = 0$. For larger value of a_i the error is reduced, up to reach 0 at $a_i = 1/\delta_i$. A displacement $a_i = 0.1$ gives a skew roughtly in the middle of 0 and $1/\delta_i$. Already for this value the error is significanlty lower than in the worst case. Figure 5(a) is the equivalent of Figure 4(a) but with $a_i = 0.1$. The error is significanlty lower for the approximation with a maximum of 0.08 bps while the Monte Carlo simulation exhibit a similar error profile.

4.3. **FIReD pricing with two factors.** The formulas developped here are valid for multifactor models. The caplets depend of only one rate and consequently contains no information about the correlation between the rates. An example of two factors models is analysed. The structure used is the one suggested by Rebonato in [19, Section 7.3.1]. The volatilities are

$$\gamma_i = |\gamma_i|(\sin(\theta_i), \cos(\theta_i)).$$

When the θ are all equal, this is equivalent to a one factor model. More they are differences between the angles, more decorrelation is introduced. In the examples analysed, the θ_i are equally spaced between $\pi/8$ and $\pi/2$ in the first version and between $\pi/4$ and $\pi/2$ in the second version.

The implied volatilities relative to the one factor approximation is represented in Figure 5(b). Some decorrelation between the rates is introduced in the two factor models. It is expected that the total volatility of the composition will be lower with the two factors models. This is exactly what is observed. The model with more decorrelation ($\theta_1 = \pi/8$) has an implied volatility around 1 bps below the one of the one factor. The approximate implied volatility, ten Monte Carlo simulations and their average are represented. The model with less decorelation has an implied volatility around 0.4 bps below the reference volatility. The smile error for the FIReD product is similar to the one for the caplet. It is possible to calibrate the caplet with the approximated formula and then price with the same approximation. This procedure reduces further the error.

Even if the instrument is similar to a cap, its pricing requires some correlation information not available in the caplet market. Even if the product is not very exotic, the *exoticity* is more

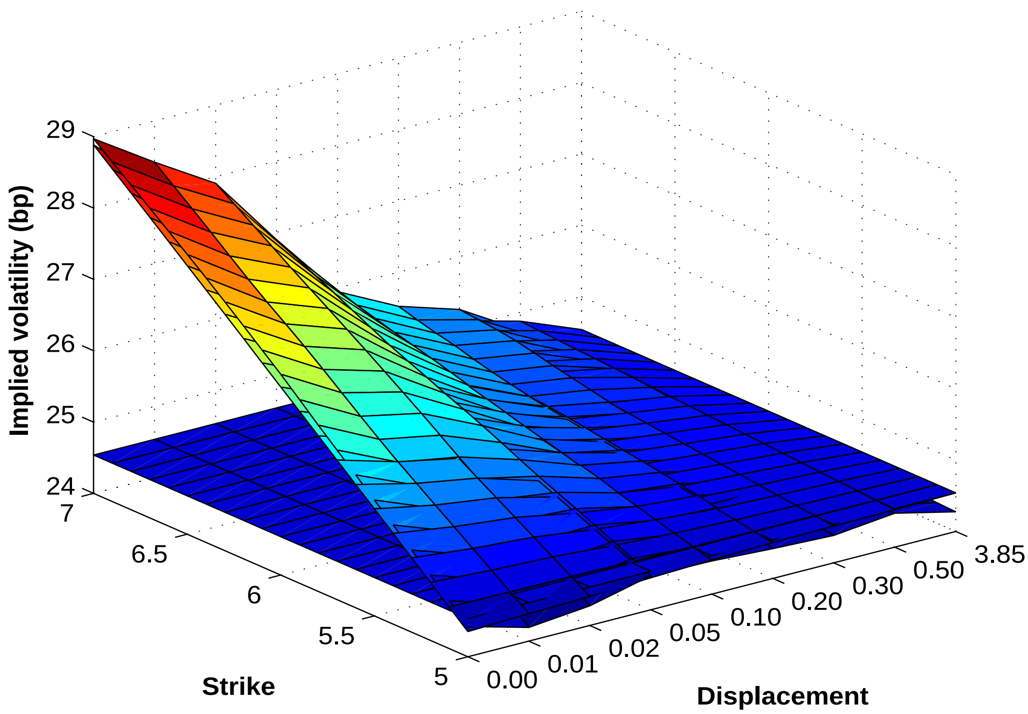


FIGURE 3. FIReD smile, one factor

important than the error introduced by the approximation or the Monte Carlo simulation. The correct model choice has more importance than the approximation.

5. Conclusion

A simple interest rate exotic product is studied in the shifted log-normal LMM and Gaussian HJM multi-factors models. An explicit valuation formula is proposed for the product, by opposition to the standard LMM approach involving Monte-Carlo simulations. The approach proposed preserves the shifted log-normal smile. The solution is obtained through approximations related in spirit to the predictor-corrector approach of Monte-Carlo simulations.

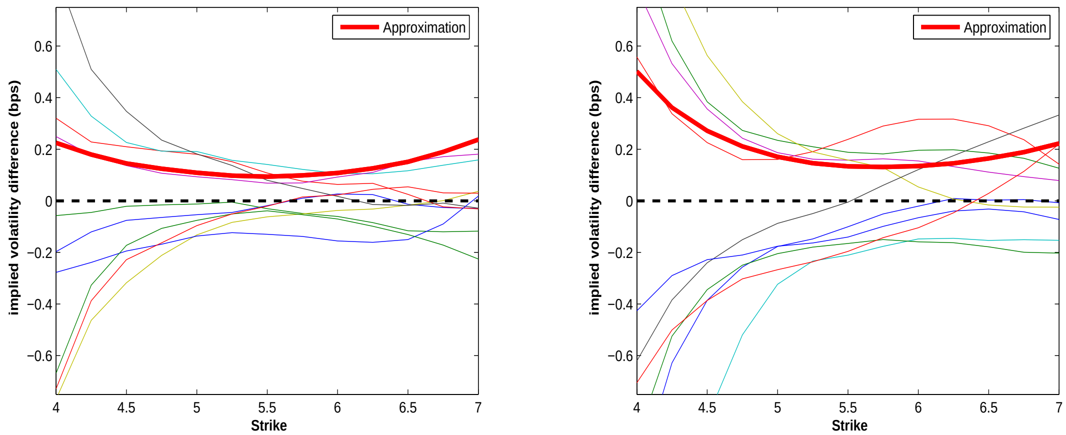
In the worst case, when there is no shift, the approximation is still within the bid-offer of the exact solution. The small error to pay in the approximation is a very small price with respect to the immense advantage of the method in term of speed and robustness.

The method is valid for multi-factors models. The correlation between the different Libor rates are taken into account.

The approach proposed is explicit and includes multi-factor and smile effects.

Appendix A. Tecnical lemmas

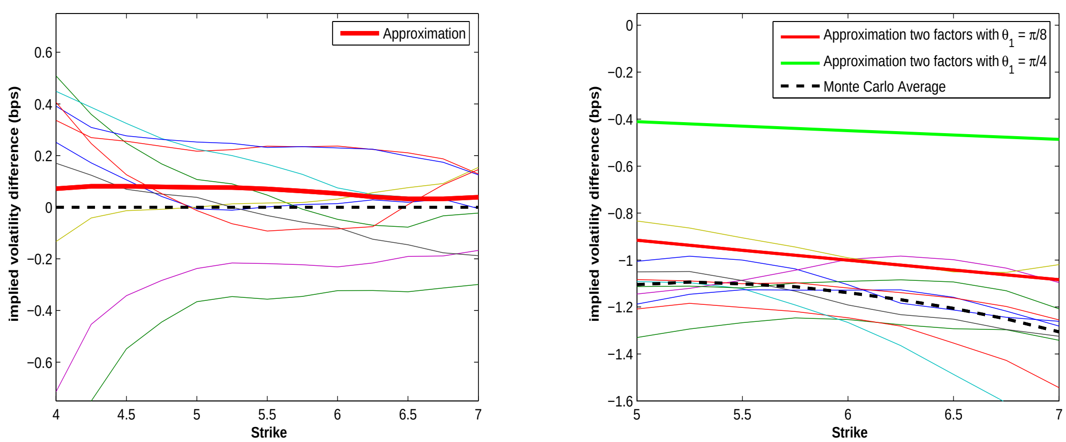
The two following technical lemmas were presented in [10] for the Gaussian one-factor HJM. Similar formulas can be found in [3, (3.3),(3.4)] in the framework of coherent interest-rate models.



(a) Flat curve, $a_i = 0$

(b) Upward sloping curve, $a_i = 0$

FIGURE 4. FIReD smiles with several Monte Carlo simulations



(a) Flat curve, $a_i = 0.1$

(b) Two factors, $a_i = 0$

FIGURE 5. FIReD smiles with several LMM Monte Carlo simulations

Lemma 1. *Let $0 \leq t \leq u \leq v$. In HJM framework the price of the zero coupon bond is*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\int_t^u (\nu(s,v) - \nu(s,u)) \cdot dW_s - \frac{1}{2}\int_t^u \left(|\nu(s,v)|^2 - |\nu(s,u)|^2\right) ds\right).$$

Lemma 2. *Let $0 \leq u \leq v$. In the HJM framework*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds\right) = P(u,v) \exp\left(-\int_u^v \nu(s,v) \cdot dW_s - \frac{1}{2}\int_u^v |\nu(s,v)|^2 ds\right).$$

Appendix B. LMM: Non-arbitrage free, existence and Ho-Lee comparison

B.1. **Non-arbitrage free.** The possibility to embed pure arithmetic Brownian equations (4) with $\gamma_i(L,t)$ constant into a HJM framework is analysed. Unfortunately this is not possible and this can be seen easily.

In the HJM framework, the bond prices are given by Equation (3) and are always positive. On the other side the link between forward rate and prices is

$$1+\delta_i L_s^i = \frac{P(s,t_i)}{P(s,t_{i+1})}.$$

If L is modelled by a pure arithmetic Brownian motion, it can become very negative (with a positive probability). When $L_s^i < -1/\delta_i$, the ratio of the prices is negative. A contradiction with the previous assertion.

The dynamic of the forward rate has to be modified (artificially) to ensure that the model can be embedded in an well-behaved HJM framework. Essentially the same non-existence proof holds for hte displaced diffusion model with $a_i > 1/\delta_i$. Conditions sufficient to ensure the existence of such a framework are presented in Appendix B.2.

The impact of the function modification far away from the current rate level is very small. For a three months rate starting in one year, a current rate of 5% and a volatility of 1%, the probability to have a rate below $1/\delta_i$ is $N(-(L_0+4)/\sigma\sqrt{\theta})$ $N(-400)$!.

B.2. **Existence.** Let $\gamma_i(L,t) = p_i(L)\gamma_i(t)$. Suppose the functions p_i are globally Lipschitz, $p_i(L_0^i) > 0$[1] and p_i have zeros z_i with $-1/\delta_i \leq z_i < L(0,t_i)$. With those conditions it is possible to prove the existence of a HJM model that contains the Equations (4). The argument is the same as in [12, Section 18.2.2]. The first steps is to prove the existence of the solution of (4). This follows from the global Lipschitz condition with Itô's theorem [12, Theorem 6.27]. Note that because of the condition on the zero of p_i and the Lipschitz property, the solutions have a lower non-attainable barrier at $z_i \geq \delta_i$.

The second main step is to prove that the $P(.,t_n)$ numeraire rebased assets are martingales. For this it is useful to prove that the integrals

$$\int_0^t \frac{\delta_j p_j(L_s^j)}{1+\delta_j L_s^j}\gamma_j(s).dW_s^j$$

are of bounded quadratic variation. Or by the identity between the quadratic variation of an Itô integral and a Lebesgue integral [12, Theorem 4.18] it is sufficient to prove that

$$\int_0^t \left(\frac{\delta_j p_j(L_s^j)}{1+\delta_j L_s^j}\right)^2 |\gamma_j(s)|^2 ds$$

is bounded. The boundedness result comes from the global Lipschitz condition (in particular at z_i and at infinity), the lower barrier on the solution L in z_i and the fact that $z_i \geq -1/\delta_i$.

LMM with displaced log-normal or normal rates have been described in other places, in particular in [19, Chapter 11] and [5], but the question of existence of such a model was not discussed and the condition on the displacement not mentioned.

[1]The choice of p_i being positive at the current level of rate is arbitrary and without loss of generality. If $p_i(L_0^i) < 0$, by changing the Brownian motion to $-W^j$, the condition is satisfied. The standardization to a positive volatility is more a tradition than a mathematical constraint.

B.3. **Ho-Lee comparison.** Let

$$p_i(L) = (1 + \delta_i L_t^i)\gamma_i(t).$$

With that choice, the volatility differences simplify to

$$\nu(t, t_{j+1}) - \nu(t, t_j) = \frac{1 + \delta_i L_t^i}{L_t^j + 1/\delta_j}\gamma_j(t) = \delta_j \gamma_j(t).$$

This framework can be linked to the continuous version of the Ho and Lee model [11]. For this take the simplified version of the shifted log-normal model were all volatilities are constant and such that $\gamma_j(t) = \bar{\sigma}$ for all t.

Then the bond volatility function is given by

$$\nu(t, t_j) - \nu(t, t_{j+1}) = \delta_j \bar{\sigma} = \bar{\sigma}(t_{j+1} - t_j)$$

which is exactly the Ho and Lee volatility structure with short rate volatility $\bar{\sigma}$.

The (old fashioned) Ho and Lee model can now be renamed with the more fashionable name of one-factor shifted log-normal Libor market model. This emphasize once more the brotherhood between LMM and HJM models quoted at the beginning of this note, a designation borrowed from Gątarek [6].

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