Child Benefit and Fiscal Burden with Endogenous Fertility

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April 2009

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This version: April 2009

Abstract

This paper studies a possibility of efficiency improvement by child benefit programs in an overlapping generations economy with endogenous fertility and government debt. We derive conditions for improving an efficiency by child benefit using Representative-Consumer efficiency (RC-efficiency), an efficiency criterion for an endogenous fertility setting developed by Michel and Wigniolle (2007).

It is shown that the result crucially depends on the relative amount of accumulated government debt in the economy. It is likely to hold in an economy of developed countries with a low fertility rate. We provide an implication of the results in the real economy.

Keywords: Endogenous fertility, Pareto-efficiency, child benefit, fiscal burden
JEL Classification Numbers D9, J13, D61

1. Introduction

The purpose of this paper is to analyze the relationship between child benefit and fiscal burden in the setting of an overlapping generation model with endogenous fertility. Lump-sum tax and public debt can be resources of child benefit. Although the tax burden of each generation is concentrated on its respective working period, this period also corresponds to the child-rearing period in some cases. Therefore, implementing child

[1] This paper was presented at the First Workshop on Demography and Public Finance at Hitotsubashi University in March 7, 2009. We are grateful for helpful comments and discussions from Takashi OSHIO (Professor, Graduate School of Economics, Hitotsubashi University (Kobe University until March 2009)), Motohiro SATO (Associate Professor, Graduate School of Economics, Hitotsubashi University), Ryo ISHIDA (University of Michigan, US), and the workshop participants. Takahata was supported by a grant from Global COE program "Research Unit for Statistical and Empirical Analysis in Social Sciences." Finally, the opinions expressed in this paper are those of the authors and do not represent the views of the organizations to which we belong. Any remaining errors are the responsibility of the authors.

benefit programs financed by lump-sum tax in an exogenous fertility setting is a zero-sum game in that it transfers the fiscal burden to the same generation. On the other hand, financing child benefit programs by issuing debt is a zero-sum game in that it transfers the fiscal burden from the current generation to the future generation. In this paper, we focus mainly on child benefit financed by public debt in an endogenous fertility setting. Also, if certain conditions are satisfied, we clarify that the benefit has the potential to improve each generation's utility through the mitigation of per-capita fiscal burden.

In industrialized countries, the fiscal burden has been increasing. In Japan, especially, the debt-GDP ratio is the highest among industrialized countries, even beyond that of Italy. As is well known, the sustainability of the Japanese fiscal and social security system is declining because of its low fertility rate, aging, and decreasing population. This situation is due to the fact that Japan now holds public debt explicitly and implicitly: the explicit debt is about 180% to GDP with regard to government bonds and the implicit debt is about 230% with regard to the social security system, public pension, medical insurance, and elderly assistance. Therefore, Japan holds a total of approximately 410% public debt to GDP.

Table1. Public Debt-GDP ratio and Total Fertility Rate (TFR) of Industrialized Countries

Country	Japan	Italy	France	Germany	UK	US
Public Debt	1.71	1.17	0.71	0.64	0.50	0.66
TFR	1.33	1.32	1.87	1.28	1.66	2.04

Source: United Nations (2006) "Population, Resources, Environment and Development."

Moreover, the baby boomer generation comprising the largest population is now moving over to the benefit side of the social security system. Thus, attempts to reduce the benefit will face political limitations. This means that the 410% public debt must be paid mainly by the current working generation and future generations.

In addition, the fertility rate in Japan has been decreasing since the baby boom in the 1950s. To maintain the population level, it is considered necessary for a woman to have 2.08 children. The total fertility rate in Japan was above 2.08 before the 1970s, but since then, it has fallen below that number. The relationship between (explicit) debt-GDP ratio and fertility rate in developed countries is shown in Table 1.

These demographic factors raise the following question: what is the most economically efficient way for the burden to be shared by each generation? The answer will essentially differ depending on whether the model is exogenous fertility or endogenous fertility.

For this reason recent studies have clarified that the Pareto-efficiency condition of the exogenous fertility model differs from that of endogenous. First, in the case of an exogenous fertility model, we make use of the overlapping generations (OLG) model which was introduced by Diamond (1965). Three types of steady states exist in the model: under-accumulation, golden rule, and over-accumulation. The first two steady states are Pareto-efficient, but the third is not. In addition, an empirical study by Abel, Mankiw, Summers, and Zeckhauser (1989) reports that in industrialized countries dynamic efficiency is satisfied. In a steady state, dynamic efficiency corresponds to under-accumulation (or golden rule). Therefore, the possibility that industrialized countries are in the state of under-accumulation seems high. In an exogenous fertility setting, an allocation is said to be Pareto-efficient if it is impossible to make some individuals better off without making other individuals worse off. For this reason, in an exogenous case, we cannot improve any generation's utility while at the same time sacrificing another generation's utility.

However, recent studies clarify the properties of the competitive equilibrium with an endogenous fertility setting. Raut and Srinivasan (1994) and Charkrabarti (1999) analyze the properties of the inter-temporal equilibrium with endogenous fertility. Conde-Ruiz et al. (2002) and Golosov et al. (2004) present the definition of Pareto-efficiency criteria in an endogenous fertility framework.

As a development of these studies, surprisingly, Michel and Wigniolle (2007)[2] point out the possibility that under-accumulation may not be efficient in an endogenous fertility setting. This implies that there is a possibility of improving one generation's welfare without making another generation's welfare worse off by some policies, even when it is in an under-accumulation state near the steady state. Moreover, the remarkable point of Michel and Wigniolle (2007) is to clarify that the Representative-Consumer efficient (RC-efficient) condition, which is a concept developed in their study, deeply connects with the sign-of-inequality relationship between the child-rearing cost and wage rate. That is, if by some policies we can give some effects to this relationship, we would have a possibility to improve RC-efficiency.

Michel and Wigniolle (2007) provide proof that, by utilizing an OLG model with endogenous population growth, the possibility to improve RC-efficiency also exists in the case of under-accumulation. But they did not analyze an economy model with public debt. Therefore, we have great interest in the possibility of improving RC-efficiency in an economy with huge public debt, low fertility rate, and endogenous population growth.

Therefore, we should focus on the child benefit programs financed by debt. The policy has the possibility to affect the conditions of RC-efficiency through the following path. First, there is a path of reducing the per-capita fiscal burden through fertility rate increase, which is found in even a simple model without capital accumulation. Second, as is shown in a model with capital accumulation, child benefit may affect an individual's expenditure through the current fertility level and interest rate which causes the consumption amount in the second period. The first condition has particular effect when an economy holds huge public debt like that of Japan.

Intuitively, there is a possibility to attain RC-improvement by a child benefit program financed by newly issued debt when the accumulated amount of debt is huge. This is because of the following logic. Suppose we have a child benefit program financed by newly issued debt and it raises the fertility rate to a certain level. An influence of newly issued

[2]Although there have been several approaches that endogenize fertility decisions, Michel and Wigniolle (2007) depend on the benchmark framework, which assumes that children are consumption goods that appear in the utility function of the parents. The basic articles are Becker (1960), Willis (1973), and Eckstein and Wolpin (1985). Other approaches depend on the literature based on the additional assumption of descendant altruism, as in Becker and Barro (1988) or the assumption of ascendant altruism and strategic behavior of parents, as in Nishimura and Zhang (1992).

debt to the accumulated debt is different depending on the size of accumulated debt. If the effect of the rise in fertility is the same regardless of the amount of accumulated debt, then such a policy may lessen per-capita debt without harming any generation. In this scenario, even the initial generation is not made worse off since they do not need to endure the burden. The problem of worsening the situation of the initial generation might occur if a child benefit program were financed by a lump-sum tax. We will show situation using a simple model first intuitively, and derive conditions in a general setting after that.

This paper is organized as follows. In Section 2 we introduce a simple model for grasping an intuitive understanding. In Section 3, we will set the model for our main analysis. In Section 4, we derive the conditions of RC-improvement using the model. In Section 5, we analyze the superiority of public debt and tax with child benefit financing, as continuing discussions of Sections 3 and 4.. Section 6 presents some concluding remarks.

2. Simple analysis

In this section, we analyze a simple model to show the characteristics of child benefit financed by public debt as preparation for analyzing the rigorous model in the next section. As an example, we first make an intuitive analysis of the relationship between child benefit and fiscal burden in a case with intergeneration-selfishness in a simple economy with only two generations: parent generation and child generation. Next, by using an OLG model with only two generations, we show the possibility that the child benefit improves RC-efficiency.

First,, for simplicity, we consider the economy only with two generations, the first and the second generation. Individuals live two periods, young and old, and they have children when they are young. We assume that the second (young) generation does not have children, and that the government expenditure is set to zero in the baseline case. The debt amount at the beginning is set to D, and the government subsidizes δ per child for child-rearing activity, financed by issuing bonds. In this simple model, we let N denote the population of the first generation, nN that of the second generation, r interest rate, z child-rearing cost, X_j and Y_j the consumption when young and old, W the lifetime income, and the fiscal burden in lump-sum tax T_j ($j=1,2$). Using the above, we get the following budget constraints for a representative household:

$$(z-\delta)n + X_1 + \frac{Y_1}{1+r} = W_1 - T_1 \qquad (1)$$

$$X_2 + \frac{Y_2}{1+r} = W_2 - T_2 \qquad (2)$$

The intertemporal government budget constraint is the following:

$$D + \delta nN = T_1 N + \frac{T_2}{1+r} nN \qquad (3)$$

Solving per capita fiscal burden of the second generation from equation (1) to (3), we get the following relationship:

$$T_2 = (1+r)\frac{D + \delta nN - T_1 N}{nN} \qquad (4)$$

If $\partial T_2 / \partial \delta < 0$ is satisfied, enlarging child benefit programs financed by bond will decrease the fiscal burden of the second generation. It is possible to rewrite the condition as in the following:

$$\frac{d}{n} > \frac{\delta}{\eta_{\delta n}} \quad , where \quad d \equiv D/N - T_1 \quad \text{and} \quad \eta_{\delta n} \equiv \frac{\delta}{n}\frac{\partial n}{\partial \delta} \qquad (5)$$

The left-hand side represents the fiscal burden of the second generation. On the other hand, the denominator of the right-hand side represents the elasticity of fertility to child benefit programs. As long as the ratio of child-rearing subsidy to elasticity is less than the per capita fiscal burden of the second generation, the child benefit programs decrease the per capita fiscal burden of the second generation. Specifically, the after-tax lifetime income of the second generation increases, which implies that the lifetime utility rises. The lifetime utility of the first generation also rises by the child benefit programs financed by bonds. In other words, in the case equation (5) holds, child-rearing policy financed by bonds may attain RC-improvement.

3. Model

In this section, we construct the model for considering the condition for having the child benefit financed by bonds to effect RC-improvement. The detailed settings are shown in the following.

3.1. Household

Generation t lives two periods, period t when they are young and period $t+1$ when old, earn lifetime income W, enjoy consumption X_t when young and Y_{t+1} when old, and raise children n_t at cost z, subsidized with δ_t. Generation t has to take over per capita debt d_t from generation $t-1$ by paying lump-sum tax T_t when young, and give their per capita debt d_{t+1} to the following generation $t+1$ when old.

Assumption 1 U is a function from $\mathbb{R}^3_+$ to $\mathbb{R} \cup \{-\infty\}$, and U maps $\mathbb{R}^3_{++}$ to $\mathbb{R}$, with

$$U(\bar{n}, \bar{X}, \bar{Y}) = \lim_{(n,X,Y)\to(\bar{n},\bar{X},\bar{Y})} U(n, X, Y) \quad \text{for every} (\bar{n}, \bar{X}, \bar{Y}) \in \mathbb{R}^3_+ / \mathbb{R}^3_{++}.$$

U is twice continuously differentiable on $\mathbb{R}^3_{++}$, strictly concave, increasing in each argument, homogeneous of degree one, and satisfies the Inada conditions:

$$\lim_{n\to 0} U'_n = \lim_{X\to 0} U'_X = \lim_{Y\to 0} U'_Y = +\infty.$$

In this case, the lifetime utility and the budget constraint of generation t is described in the following:

$$U_t = U(n_t, X_t, Y_{t+1}) \qquad (6)$$

$$(z-\delta_t)n_t + X_t + S_t = W_t - T_t \qquad (7)$$

$$Y_{t+1} = R_{t+1} S_t \qquad (8)$$

The first-order conditions for maximizing the lifetime utility are as follows:

$$U_X = R_{t+1} U_Y = \frac{U_n}{z-\delta_t} \qquad (9)$$

From the above equations, we can derive the following relationships:

$$X_t = X(W_t - T_t, R_{t+1}, \delta_t) \qquad (10)$$

$$Y_{t+1} = Y(W_t - T_t, R_{t+1}, \delta_t) \qquad (11)$$

$$n_t = n(W_t - T_t, R_{t+1}, \delta_t) \qquad (12)$$

$$S_t = S(W_t - T_t, R_{t+1}, \delta_t) \qquad (13)$$

Functions X, Y, n, s are defined on $\mathbb{R}^3_{++}$ and are continuously differentiable.

3.2. Firm

We assume that, in period t, there exists a representative firm producing goods with capital K_t and labor L_t under perfect competition using the following function which is homogeneous of degree one and we can define f as $f(k) \equiv F(k,1)$.

$$Q_t = F(K_t, L_t) \qquad (14)$$

Assumption 2 $f : \mathbb{R}_+ \to \mathbb{R}_+$, and for all $k > 0$, $f'(k) > 0$ and $f''(k) < 0$.

Then we get the following condition from profit maximization:

$$W_t = f\left(\frac{K_t}{L_t}\right) - \frac{K_t}{L_t} f'\left(\frac{K_t}{L_t}\right) \equiv W\left(\frac{K_t}{L_t}\right) \qquad (15)$$

$$R_t = f'\left(\frac{K_t}{L_t}\right) \equiv R\left(\frac{K_t}{L_t}\right) \qquad (16)$$

3.3. Government

Suppose that the population of generation t is expressed as $N_t = n_{t-1}N_{t-1}$, and that the government subsidizes child-rearing under the following budget constraint: the reimbursement of per capita debt of generation $t-1$ and child-rearing subsidy δ_t are financed by lump-sum tax T_t and newly issued bond d_t:

$$T_t N_t + d_t N_t = R_t (d_{t-1} N_{t-1} + n_{t-1}\delta_{t-1} N_{t-1})$$

$$\Leftrightarrow \quad T_t = \frac{R_t d_{t-1}}{n_{t-1}} + R_t \delta_{t-1} - d_t \qquad (17)$$

3.4. Market equilibrium

Suppose that the labor market is balanced as $L_t = N_t$, and that the capital and the saving in the capital market are balanced. Then, with $k_t \equiv K_t / N_t$, we have the following:

$$W_t = W(k_t) \quad \text{and} \quad R_t = R(k_t) \qquad (18)$$

$$K_{t+1} = N_t (S_t - d_t) \qquad (19)$$

The equation (19) is verified to be equivalent to the following commodity market clearing condition by simple operation:

$$f(k_t) = n_t k_{t+1} + X_t + \frac{Y_t}{n_{t-1}} + n_t z + R_t \delta_{t-1} - n_t \delta_t \qquad (20)$$

Definition 1 Starting from initial conditions N_{-1}, N_0, K_0, and $Y_0 = R_0(d_{-1} + (K_0 / N_{-1}))$, given debt management policies and child-rearing subsidies $\{(d_t, \delta_t)_{t=0}^{\infty}\}$, an inter-temporal equilibrium is a sequence $\{(K_t, N_t, X_t, Y_t, n_t)_{t=0}^{\infty}\}$, which satisfies (7)-(9) and (17)-(19).

4. The inter-temporal equilibrium

In this section, based on the model constructed above, we will examine the condition for the child benefit financed by issuing bonds to effect RC-improvement. Then an example with a simple function is considered.

First, we will derive the condition of child benefit programs to improve lifetime utility of all generations without sacrificing welfare of any generation. It is difficult to derive such a condition rigorously in an analytical sense. In this paper, for simplicity, we assume that $U = U_t(\cdot)$ is homogeneous of degree one. In addition, we define the variables as the following:

$$\begin{aligned}
\tilde{n}_t &\equiv \frac{n_t}{W_t - T_t} \\
\tilde{X}_t &\equiv \frac{X_t}{W_t - T_t} \\
\tilde{Y}_{t+1} &\equiv \frac{Y_{t+1}}{W_t - T_t} \qquad (21) \\
\tilde{S}_t &\equiv \frac{S_t}{W_t - T_t} \\
\tilde{d}_t &\equiv \frac{d_t}{W_t - T_t}. \\
\tilde{U}_t &\equiv \frac{U_t}{W_t - T_t}.
\end{aligned}$$

The government budget constraint can be rewritten as

$$T_t = \frac{\frac{R_t \tilde{d}_{t-1}}{\tilde{n}_{t-1}} + R_t \delta_{t-1} - W_t \tilde{d}_t}{1 - \tilde{d}_t}$$

And the budget constraints (7) and (8) can be transformed as the following:

$$(z - \delta_t)\tilde{n}_t + \tilde{X}_t + \frac{\tilde{Y}_t}{R_{t+1}} = 1 \qquad (22)$$

From equation (9) and (22), we can solve the variables of equation (21) as a function of $(z - \delta_t, R_{t+1})$. Substituting these variables into (19), we obtain the following:

$$\tilde{n}_t(z - \delta_t, R_{t+1})k_{t+1} = \tilde{S}_t(z - \delta_t, R_{t+1}) - \tilde{d}_t \qquad (23)$$
$$\Leftrightarrow k_{t+1} = k_{t+1}(\delta_t)$$

Proposition 1 Given $k_0 = K_0 / N_0$ and $\{(\tilde{d}_t)_{t=0}^{\infty}\}$, an inter-temporal equilibrium is characterized by the sequence $\{(\delta_t)_{t=0}^{\infty}\}$ such that $\forall t \geq 0$,

$$\tilde{n}_t[z-\delta_t, R(k_{t+1})]k_{t+1} = \tilde{S}_t[z-\delta_t, R(k_{t+1})] - \tilde{d}_t \qquad (24)$$

The proof is straightforward. A sequence $(k_t)_{t\geq 0}$ is characterized by a sequence $(\delta_t)_{t\geq 0}$ in this setting, while an inter-temporal equilibrium is characterized by a sequence $(k_t)_{t\geq 0}$ in Michel and Wigniolle (2007). Hence, an inter-temporal equilibrium is characterized by a sequence $(\delta_t)_{t\geq 0}$.

We define the function Γ as:

$$\Gamma(z-\delta, k) \equiv \tilde{n}[z-\delta, f'(k)]k - \tilde{s}[z-\delta, f'(k)] - \tilde{d}$$

Γ is defined on $\mathbb{R}^2_{++}$ and continuously differentiable. The equation (24) can be written as:

$$\Gamma(z-\delta_t, k_{t+1}(\delta_t)) = 0$$

In this setting, the equation is no more dynamic since the function is only of k_{t+1} but not of k_t. Since k_{t+1} is a function of δ_t, once we have a sequence of $(\delta_t)_{t\geq 0}$, a unique inter-temporal equilibrium $(\delta_t)_{t\geq 0}$ may exist. We would like to show this in the following. Before that, we need to make some assumptions on saving and fertility to the change of interest rate and child benefit.

In the following we will deal with several cases in terms of preferences. In the first case, we will consider the case with preferences with which $\partial k_{t+1}/\partial \delta_t > 0$ is satisfied. Next, we consider the opposite case. Before that, we assume the following in advance.

Assumption 3 To the change of interest rate, assume that savings and fertility rates change as in the following:

$$\tilde{s}_R \geq 0, \tilde{n}_R \leq 0$$

Proposition 2 Under Assumptions 1, 2, and 3, $\forall k_0 > 0$, for any sequence $\forall\{\delta_t\}_{t=0}^{\infty}$, there exists a unique inter-temporal equilibrium $\{k_{t+1}\}_{t=0}^{\infty}$ starting from a given initial condition $k_0 > 0$.

Proof See Appendix A.

The difference from Michel and Wigniolle (2007) is that there are no dynamics in k in our model since we have assumed homogeneity with household preferences, which drops the effect of wage rate determined by the capital level in the same period.

Our next interest is in the relationship between k and δ. In the following, we will consider two cases about this sign: one is the case when $\partial k/\partial \delta > 0$, and the other is when $\partial k/\partial \delta < 0$. For this, we need to make additional assumptions about s and n to a change of δ.

Definition 2 An inter-temporal equilibrium $(K_t, N_t, X_t, Y_t, n_t)_{t\geq 0}$ is said to be converging if the sequence $k_t = K_t / N_t$ converges to a limit $\bar{k} > 0$ when t goes to infinity. If k_t converges to a limit $\bar{k}$, it is straightforward to show that $R_t = R(k_t)$, $W_t = W(k_t)$, X_t, Y_t and n_t are converging to constant values $\bar{R}$, $\bar{W}$, $\bar{X}$, $\bar{Y}$ and $\bar{n}$.

Definition 3 A converging inter-temporal equilibrium $(K_t, N_t, X_t, Y_t, n_t)_{t\geq 0}$ is said to converge in under-accumulation if $\bar{R} > \bar{n}$. It is said to converge in over-accumulation if $\bar{R} < \bar{n}$.

Definition 4 (RC-allocation) A feasible allocation with representative consumers (or RC-allocation) is a sequence $(K_t^i, N_t^i, X_t^i, Y_t^i, n_t^i)_{t\geq 0}$ of positive variables that satisfies $\forall t \geq 0$:

$$F(K_t, N_t) = K_{t+1} + N_t X_t + N_{t-1} Y_t + N_{t+1} z$$
$$N_{t+1} = n_t N_t.$$

Definition 5 (RC-dominance) Let $(K_t^i, N_t^i, X_t^i, Y_t^i, n_t^i)_{t\geq 0}$ for $i = 1,2$ be two feasible RC-allocations. Allocation 1 is said to RC-dominate allocation 2 if it leads to a higher level of utility for all generations, with a strict improvement for (at least) one generation. Formally,

$$\forall t \geq 0, U(X_t^1, Y_{t+1}^1, n_t^1) \geq U(X_t^2, Y_{t+1}^2, n_t^2),$$
$$\exists t_0 \geq 0, \quad \text{such that} \quad U(X_{t0}^1, Y_{t0+1}^1, n_{t0}^1) > U(X_{t0}^2, Y_{t0+1}^2, n_{t0}^2).$$

If Allocation 2 were changed to Allocation 1 by using child-benefit programs, RC-improvement would be achieved.

4.1. *Case 1: if* $\eta_{\delta s} > \eta_{\delta n}$ *where*

$$\eta_{\delta s} \equiv \frac{\delta_t}{s_t}\frac{\partial s_t}{\partial \delta_t}, \; \eta_{\delta n} \equiv \frac{\delta_t}{n_t}\frac{\partial n_t}{\partial \delta_t}$$

First we will consider the sign of $\partial k_{t+1} / \partial \delta_t$. Taking the derivative of (23) with respect to δ_t in this case, we have:

$$-\frac{\partial \tilde{n}_t}{\partial \delta_t} k_{t+1} - \frac{\partial \tilde{n}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial \delta_t} k_{t+1} - \tilde{n}_t \frac{\partial \tilde{k}_{t+1}}{\partial \delta_t} = -\frac{\partial \tilde{S}_t}{\partial \delta_t} - \frac{\partial \tilde{S}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial \delta_t}$$

$$\Leftrightarrow \quad \frac{\partial k_{t+1}}{\partial \delta_t} = \frac{-\frac{\partial \tilde{n}_t}{\partial \delta_t} k_{t+1} + \frac{\partial \tilde{S}_t}{\partial \delta_t}}{\left(\tilde{n}_t + \frac{\partial \tilde{n}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}} k_{t+1} - \frac{\partial \tilde{S}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}}\right)} > 0 \qquad (25)$$

Provided Assumption 3, we can derive the following proposition.

Proposition 3 RC-improvement is achieved by child benefit with public debt resources when the following is true:

$$\delta_t n_t > \eta_{\delta R} \frac{Y_{t+1}}{R_{t+1}} \qquad (26)$$

$$\eta_{\delta n} > \frac{n_t \delta_t}{d_t} \qquad (27)$$

where

$$\eta_{\delta n} \equiv \frac{\delta_t}{n_t} \frac{\partial n_t}{\partial \delta_t}, \; \eta_{\delta R} \equiv -\frac{\delta_t}{R_{t+1}} \frac{\partial R_{t+1}}{\partial \delta_t}.$$

Proof See Appendix B.

The first condition implies that the income from child benefit programs must exceed the loss of the second period consumption from a decrease in interest rate. If individuals originally plan to consume much in their second period, an interest rate decrease might make them worse off because of negative income effects. On the other hand, if individuals originally have children, child benefit may bring positive income effects. For satisfying the condition, it is necessary for the second effect to dominate the first. It is possible to consider that if the elasticity is small enough, the condition is satisfied as long as there exists a certain level of child benefit programs.

The second condition requires that the elasticity of fertility rate to child benefit be bigger than the ratio of per-capita amount of child benefit to that of accumulated debt. In countries which have a huge amount of accumulated debt with a relatively small level of child benefit, the second condition is likely to hold; meanwhile it requires that the child benefit programs be bigger for the first condition to be true.

It is interesting that equation (27) and (5) are exactly the same. This implies that this condition (27) more likely holds as the amount of per capita debt is larger, which is true for the financial situation of the government sector in Japan, which we have already examined in the simple analysis in Section 2. However, in the model of this section with capital accumulation, we need the additional condition (26). It would be possible to examine the possibility of RC-improvement by child-rearing subsidies financed by bonds if we can confirm that those equations (26) and (27) may actually hold in an empirical analysis. An interesting point is whether it is possible to implement RC-improvement in the real economy.

We will briefly discuss that possibility in the following with particular focus on the case of the Japanese economy which has a huge amount of per-capita debt and low fertility rate. From (26) and (27), we can derive the condition of δ_t as in the following:

$$\frac{\eta_{\delta n} d_t}{n_t} > \delta_t > \eta_{\delta R} \frac{Y_{t+1}}{R_{t+1} n_t}$$

Although it is hard to estimate the actual elasticities of the Japanese economy, $\eta_{\delta R}$ is not considered too high, and it might be almost zero. The Japanese economy is large enough that the interest rate is almost unaffected by such a policy. On the other hand, $\eta_{\delta n}$ could be assumed to be 0.05, which implies that the 100% increase of child benefit would entail 10% of increase in fertility rate. In this case, when the child benefit increases from

10,000 yen to 20,000 yen every month, the fertility rate might increase from 1.34 to 1.39. If the debt amount d_t is 20 million yen, then as long as child benefit per child is below 1 million yen, this condition satisfies.

Though the population has not been decreasing dramatically at present, in future it will decline much more rapidly. In such a case, the per capita amount of debt will increase dramatically and there would be a greater possibility for RC-improvement.

4.2. *Case 2: if* $\tilde{s}_\delta \le 0$ *and* $\tilde{n}_\delta > 0$ *are satisfied*

First we will consider the sign of $\partial k_{t+1} / \partial \delta_t$.

$$\frac{\partial k_{t+1}}{\partial \delta_t} = \frac{-\frac{\partial \tilde{n}_t}{\partial \delta_t} k_{t+1} + \frac{\partial \tilde{s}_t}{\partial \delta_t}}{\left(\tilde{n}_t + \frac{\partial \tilde{n}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}} k_{t+1} - \frac{\partial \tilde{s}_t}{\partial R_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}}\right)} < 0 \qquad (25')$$

In this case, we need the conditions for improving efficiency as discussed above.

Proposition 4 RC-improvement is achieved by child benefit with public debt resources when the following is true:

$$\eta_{\delta n} > \varepsilon_{\delta R} - \eta_{\delta W} \frac{W_t \tilde{n}_{t-1}}{R_t \tilde{d}_{t-1}} + (1 + \varepsilon_{\delta R}) \frac{\tilde{n}_{t-1} \delta_{t-1}}{\tilde{d}_{t-1}} \qquad (28)$$

where

$$\varepsilon_{\delta R} \equiv \frac{\delta_t}{R_{t+1}} \frac{\partial R_{t+1}}{\partial \delta_t}, \quad \eta_{\delta w} \equiv \frac{\delta_t}{W_{t+1}} \frac{\partial W_{t+1}}{\partial \delta_t}$$

Proof See Appendix C.

This condition implies that the elasticity of fertility rate to child benefit should be high enough to dominate the right-hand-side effects. We will consider a small change of δ_t. First, when the elasticity of k to δ is considered to be not too high, it is likely for the condition to be satisfied. In that case, both $|\varepsilon_{\delta R}|$ and $|\eta_{\delta W}|$ are close to zero, which renders the right-hand side of (28) also close to zero and facilitates satisfying the condition.

Second, when d_{t-1} is increases, the condition becomes more likely to be satisfied, as shown in Proposition 3. It shows that the possibility to improve efficiency by child benefit is higher in both cases when the amount of existing debt is huge.

Example 1 We assume specific forms for preferences and production technology in the above model. Suppose we have preferences:

$$U = n_t^\alpha X_t^\beta Y_{t+1}^{1-\alpha-\beta}$$

and production technology:

$$f(k_t) = A_t k_t^\rho .$$

Then the sufficient conditions for RC-improvement are

$$\delta_t / z < \Phi_t \equiv 1 - \frac{1-\rho}{\tilde{d}_t - \left((1-\rho)(1-\alpha-\beta) + \rho\alpha\right)} \alpha \qquad (29)$$

and

$$d_1 \equiv (1-\rho)(1-\alpha-\beta)+\alpha < \tilde{d}_t < d_2 \equiv 1-\alpha-\beta \qquad (30)$$

See Appendix D for the detailed calculation to derive the above results.

In this example, it is possible to examine a quantitative exercise about RC-improvement in the real economy. Since our concern is whether RC-improvement occurs through adjusting the child benefit amount, providing the parameters of the above function from the real economy, we can derive the exact condition for RC-improvement.

In a typical economy in the world, capital income ratio ρ is 0.3. Suppose that the preference α over children sets 0.025 or 0.005, and the preference β over consumption during the young period sets 0.6 or 0.7, then we can calculate the parameter Φ of (29) as Table 2. This parameter Φ is the upper limit of the child benefit amount under the constraint that $\tilde{d}_t$ satisfies (30).

Namely, as long as the economy satisfies the condition of $\delta / z < \Phi$, we can implement RC-improvement in the economy by increasing the child benefit. In such a case, only the amount of debt matters while the fertility rate does not.

5. Debt financing vs tax financing

In this section, we analyze mainly the superiority of public debt or lump-sum tax with child benefit financing as a discussion continuing from previous sections. To this end, we change the budget constraint of the government sector in our model, as follows:

$$T_t = \frac{R_t d_{t-1}}{n_{t-1}} + R_t(1-\theta)\delta_{t-1} - d_t + \theta\delta_t n_t \qquad (31)$$

where θ stands for the ratio of tax resource to child-rearing support: if $\theta = 1$, then child benefit is financed by tax only, and if $\theta = 0$, this is financed by public debt only.

Then, from (21) and (31), we can derive the indirect utility function, as follows:

$$U = \left(W(\delta_t, \tilde{d}_{t-1}) - T(\delta_{t-1}, \delta_t, \tilde{d}_{t-1}, \tilde{d}_t, \theta)\right)\tilde{U}\left[\tilde{n}(\delta_t, \tilde{d}_{t-1}), \tilde{X}(\delta_t, \tilde{d}_{t-1}), \tilde{Y}(\delta_t, \tilde{d}_{t-1})\right] \qquad (32)$$

where

$$T(\delta_{t-1}, \delta_t, \tilde{d}_{t-1}, \tilde{d}_t, \theta) \equiv \frac{\dfrac{R_t \tilde{d}_{t-1}}{\tilde{n}_{t-1}} + R_t(1-\theta)\delta_{t-1} - W_t \tilde{d}_t + \theta\delta_t W_t \tilde{n}_t}{(1-\tilde{d}_t + \theta\delta_t \tilde{n}_t)}$$

In this setting, we can derive the following proposition under the assumption for the preferences and production technology of Example 1.

Proposition 5 In the case of Example 1 with $\delta_{t-1} = \delta_t$ and $\tilde{d}_{t-1} = \tilde{d}_t$, when $\tilde{d}_t > d^*$, all tax financing ($\theta = 1$) is optimal. On the other hand, when $\tilde{d}_t < d^*$, all debt financing ($\theta = 0$) is optimal if $0 \le \delta_t \le \delta^*$ and all tax financing ($\theta = 1$) is optimal if $\delta^* < \delta_t < z$ .Where d^* and δ^* denotes as follows:

$$d^* \equiv 1-\alpha-\beta-\frac{\rho}{1-\rho}$$

$$\delta^* \equiv \left[1-\frac{\alpha\rho}{(1-\rho)(1-\alpha-\beta-\tilde{d})-(1-\alpha)\rho}\right]z$$

Proof See Appendix E.

According to Proposition 5, debt financing is optimal only in the case of $\tilde{d}_t < d^*$. In addition, the condition that the sign of d^* is positive as follows:

$$1-\alpha-\beta-\frac{\rho}{1-\rho}>0$$

$$\Leftrightarrow \quad \alpha<1-\beta-\frac{\rho}{1-\rho} \qquad (33)$$

And, to the consumption smoothing during young and old periods, we assume the following.

Assumption 4 The preference parameters of Example 1 satisfy the following relationship.

$$\beta>1-\alpha-\beta$$

$$\Leftrightarrow \quad \alpha>1-2\beta \qquad (34)$$

Then, from the constraint of $\alpha>0$, (33) and (34), we can derive the following corollary, as the necessary conditions with regards to debt financing.

Corollary 1 Under Proposition 4 and Assumption 4, the necessary condition that debt financing is optimal is $\rho<1/3$.

Proof First, $\beta<1-\rho/(1-\rho)$ is derived from (33) and $\alpha>0$. Next, $\beta>\rho/(1-\rho)$ is derived from (33) and (34). Hence, from these relationships, $2\rho/(1-\rho)<1$ holds.

Moreover, we can derive the following proposition and example in the case of tax financing, as in previous sections.

Proposition 6 RC-improvement is achieved by child benefit with lump-sum tax resources when the following is true:

Case 1: if $\eta_{\delta s}>\eta_{\delta n}$

$$n_t>\eta_{\delta R}\frac{1}{R_{t+1}}\frac{Y_{t+1}}{\delta_t} \text{ and } \tilde{d}_{t-1}>\tilde{n}_{t-1}W_t/R_t \qquad (35)$$

Case 2: if $\tilde{s}_\delta \le 0$ *and* $\tilde{n}_\delta>0$

$$\eta_{\delta n}>\varepsilon_{\delta R}+\eta_{\delta W}\frac{\tilde{n}_{t-1}W_t}{R_t\tilde{d}_{t-1}} \text{ and } \tilde{d}_{t-1}>\tilde{n}_{t-1}W_t/R_t \qquad (36)$$

Proof See Appendix F.

Example 2 We assume the preferences and production technology with Example 1. Then the sufficient condition for RC-improvement of Proposition 6 is

$$\tilde{d}_t > \frac{1-\alpha-\beta}{1+\rho/(1-\rho)} \qquad (37)$$

See Appendix G for the detailed calculation to derive the above results.

6. Conclusion

In this paper, we derive the condition of RC-efficiency in an endogenous population growth setting. According to this, when the elasticity of interest rate to child benefit policy is close to zero and there exists a huge amount of accumulated debt, having a certain level of child benefit programs financed by issuing debt and lump-sum tax is RC-improving.

The weakness of this study is the assumptions we made on the preferences, such as homogeneity. This study report would be more worthwhile if it were possible to show those results more generally. We will take over this assignment in a following study.

Table 2. Range of Child Benefit with RC-improvement

1) *Case 1*: $\rho = 0.3$, $\alpha = 0.025$ and $\beta = 0.6$ or 0.7.

Preference Parameters of Utility			Debt Parameters			Upper Limit of Child Benefit
α	β	$1-\alpha-\beta$	$\tilde{d}$	d_1	d_2	Φ
0.025	0.700	0.275	0.220	0.218	0.275	0.125
0.025	0.700	0.275	0.230	0.218	0.275	0.417
0.025	0.700	0.275	0.240	0.218	0.275	0.563
0.025	0.700	0.275	0.250	0.218	0.275	0.650
0.025	0.700	0.275	0.260	0.218	0.275	0.708
0.025	0.700	0.275	0.270	0.218	0.275	0.750
0.025	0.600	0.375	0.290	0.288	0.375	0.125
0.025	0.600	0.375	0.300	0.288	0.375	0.417
0.025	0.600	0.375	0.310	0.288	0.375	0.563
0.025	0.600	0.375	0.320	0.288	0.375	0.650
0.025	0.600	0.375	0.330	0.288	0.375	0.708
0.025	0.600	0.375	0.340	0.288	0.375	0.750
0.025	0.600	0.375	0.350	0.288	0.375	0.781
0.025	0.600	0.375	0.360	0.288	0.375	0.806
0.025	0.600	0.375	0.370	0.288	0.375	0.825

2) *Case 2*: $\rho = 0.3$, $\alpha = 0.05$ and $\beta = 0.6$ or 0.7.

Preference Parameters of Utility			Debt Parameters			Upper Limit of Child Benefit
α	β	$1-\alpha-\beta$	$\tilde{d}$	d_1	d_2	Φ
0.050	0.700	0.250	0.230	0.225	0.250	0.125
0.050	0.700	0.250	0.240	0.225	0.250	0.300
0.050	0.600	0.350	0.300	0.295	0.350	0.125
0.050	0.600	0.350	0.310	0.295	0.350	0.300
0.050	0.600	0.350	0.320	0.295	0.350	0.417
0.050	0.600	0.350	0.330	0.295	0.350	0.500
0.050	0.600	0.350	0.340	0.295	0.350	0.563

Appendix A: Proof of Proposition 2

It is possible to show this with the same logic as Michel and Wigniolle (2007). We will follow their proof for the most part and change the different points. In equilibrium, the market adjusts only the capital level, and we do not need to consider a change of δ. Hence, for a given sequence of $\{\delta_t\}_{t=0}^{\infty}$, Γ is represented in the following way:

$$\Gamma(z-\bar{\delta},k) \equiv \tilde{n}[z-\bar{\delta},f'(k)]k - \tilde{s}[z-\bar{\delta},f'(k)] - \tilde{d} \ .$$

We will show that $\Gamma = 0$ has a unique solution. In order to show this, we will check the property of Γ. First, we will check monotonicity of this function. The derivative of the first term $\partial(\tilde{n}[z-\bar{\delta},f'(k)]k)/\partial k$ is positive, since $R = f'(k)$ is monotonically decreasing in k and $\tilde{n}_R \leq 0$. The derivative of the second term $\partial\tilde{s}[z-\bar{\delta},f'(k)]/\partial k$ is negative, since $\tilde{s}_R \geq 0$. Hence, Γ is increasing in k.

Next, suppose we have a certain level of child benefit δ. At this time, when k goes to 0, it can be bounded in such a way that $k < 1 \Rightarrow f'(k) > f'(1)$. Then we obtain the following inequalities:

$$\tilde{n}[z-\delta,f'(k)] \leq \tilde{n}[z-\delta,f'(1)]$$

$$\tilde{s}[z-\delta,f'(k)] \geq \tilde{s}[z-\delta,f'(1)]$$

and thus

$$\Gamma(z-\delta,k) \leq \tilde{n}[z-\delta,f'(1)]k - \tilde{s}[z-\delta,f'(1)] - \tilde{d} \ .$$

Finally, we have

$$\lim_{k\to 0}\Gamma(z-\delta,k) \leq -\tilde{s}[z-\delta,f'(1)] - \tilde{d} < 0 \ .$$

When k goes to $+\infty$, we can prove the following by using the contrary thought as $k > 1 \Rightarrow f'(k) < f'(1)$. We then obtain the following inequality:

$$\Gamma(z-\delta,k) \geq \tilde{n}[z-\delta,f'(1)]k - \tilde{s}[z-\delta,f'(1)] - \tilde{d} \ .$$

Thus

$$\lim_{k\to\infty}\Gamma(z-\delta,k) = +\infty \ .$$

Hence, for any given sequence $\forall\{\delta_t\}_{t=0}^{\infty}$, a unique inter-temporal equilibrium $\{k_{t+1}\}_{t=0}^{\infty}$ exists and the proposition has been proven.

Appendix B: Proof of Proposition 3

We will use (25) in the proof. Calculate the change of lifetime utility δU_t when the amount of child benefit δ_t is raised:

$$\Delta U_t = (W_t - T_t)\left(\frac{\partial \tilde{U}_t}{\partial \tilde{n}_t}\frac{\partial \tilde{n}_t}{\partial \delta_t} + \frac{\partial \tilde{U}_t}{\partial \tilde{X}_t}\frac{\partial \tilde{X}_t}{\partial \delta_t} + \frac{\partial \tilde{U}_t}{\partial \tilde{Y}_{t+1}}\frac{\partial \tilde{Y}_{t+1}}{\partial \delta_t}\right)\Delta\delta_t + \tilde{U}_t \Delta(W_t - T_t)$$

$$\Delta U_t = (W_t - T_t)\left(\frac{\partial \tilde{U}_t}{\partial \tilde{n}_t}\frac{\partial \tilde{n}_t}{\partial \delta_t} + \frac{\partial \tilde{U}_t}{\partial \tilde{X}_t}\frac{\partial \tilde{X}_t}{\partial \delta_t} + \frac{\partial \tilde{U}_t}{\partial \tilde{Y}_{t+1}}\frac{\partial \tilde{Y}_{t+1}}{\partial \delta_t}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \Delta T_t\right).$$

It is possible to transform this equation using the household first-order conditions

$$\Delta U_t = (W_t - T_t)\lambda\left((z-\delta_t)\frac{\partial \tilde{n}_t}{\partial \delta_t} + \frac{\partial \tilde{X}_t}{\partial \delta_t} + \frac{1}{R_{t+1}}\frac{\partial \tilde{Y}_{t+1}}{\partial \delta_t}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \Delta T_t\right).$$

Moreover, taking derivative of equation (22) with respect to δ_t, we obtain

$$\left((z-\delta_t)\frac{\partial \tilde{n}_t}{\partial \delta_t} + \frac{\partial \tilde{X}_t}{\partial \delta_t} + \frac{1}{R_{t+1}}\frac{\partial \tilde{Y}_{t+1}}{\partial \delta_t}\right) = \tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_t$$

then, by substituting this into the above equation, we have the following equation:

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \Delta T_t\right) \qquad \text{(A-1)}$$

where

$$\Delta T_t = \frac{\partial T(\delta_{t-1})}{\partial \delta_{t-1}}\Delta\delta_{t-1} = \frac{\partial\left(\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}_t\right)}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t}\Delta\delta_{t-1}$$

Using this, we can rewrite as

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}} - \frac{\partial\left(\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}_t\right)}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t}\right)\Delta\delta_{t-1}.$$

This equation represents the effect of the change in child-rearing subsidies $\{(\Delta\delta_t)_{t=0}^{\infty}\}$ on lifetime utility of generation t, given a set of debt policies $\{(d_t)_{t=0}^{\infty}\}$. If the sign of the big parenthesis of the first term and the coefficient of the second term are both positive, it is possible to bring welfare improvement to all generations by enlarging child benefit programs since the sign of $\tilde{U}_t$ is positive from homogeneity. Moreover, since $\partial W_t / \partial \delta_{t-1} > 0$ from equation (15) and (25), the latter is always true in the case that the coefficient sign of the first term is positive. Namely, the sufficient condition of RC-improvement is as follows:

$$n_t > \eta_{\delta R}\frac{1}{R_{t+1}}\frac{Y_{t+1}}{\delta_t} \qquad \text{(A-2)}$$

where

$$\eta_{\delta R} \equiv -\frac{\delta_t}{R_{t+1}}\frac{\partial R_{t+1}}{\partial \delta_t}$$

and

$$\frac{\partial T_t}{\partial \delta_{t-1}} = \frac{\partial\left[\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}_t\right]}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t} < 0 \qquad \text{(A-3)}$$

This form (A-3) holds when

$$\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}}[-\eta_{\delta n} - \eta_{\delta R}] + R_t(1-\eta_{\delta R}) - \eta_{\delta W}\frac{W_t\tilde{d}_t}{\delta_{t-1}} < 0$$

$$\Leftrightarrow \quad -\eta_{\delta n} - \eta_{\delta R} + \frac{\tilde{n}_t\delta_{t-1}}{\tilde{d}_{t-1}}(1-\eta_{\delta R}) - \eta_{\delta W}\frac{W_t}{R_t}\frac{\tilde{d}_t\tilde{n}_{t-1}}{\tilde{d}_{t-1}} < 0$$

$$\Leftrightarrow \quad \eta_{\delta n} > -\eta_{\delta R} + \frac{\tilde{n}_t\delta_{t-1}}{\tilde{d}_{t-1}}(1-\eta_{\delta R}) - \eta_{\delta W}\frac{W_t}{R_t}\frac{\tilde{d}_t\tilde{n}_{t-1}}{\tilde{d}_{t-1}} \qquad \text{(A-4)}$$

where

$$\eta_{\delta w} \equiv \frac{\delta_t}{W_{t+1}}\frac{\partial W_{t+1}}{\partial \delta_t}$$

Since $\eta_{\delta R} > 0$ and $\eta_{\delta W} > 0$, we can express in the following form:

$$\frac{\tilde{n}_{t-1}\delta_{t-1}}{\tilde{d}_{t-1}} > -\eta_{\delta R} + \frac{\tilde{n}_t\delta_{t-1}}{\tilde{d}_{t-1}}(1-\eta_{\delta R}) - \eta_{\delta W}\frac{W_t}{R_t}\frac{\tilde{d}_t\tilde{n}_t}{\tilde{d}_{t-1}} \qquad \text{(A-5)}$$

As a result of (A-2), (A-4) and (A-5), we can get the following sufficient conditions for RC-improvement:

$$n_t > \eta_{\delta R}\frac{1}{R_{t+1}}\frac{Y_{t+1}}{\delta_t}$$

$$\eta_{\delta n} > \frac{\tilde{n}_{t-1}\delta_{t-1}}{\tilde{d}_{t-1}}$$

Appendix C: Proof of Proposition 4

We will use (25') in the proof. Calculate the change of lifetime utility δU_t when the amount of child benefit δ_t is raised, we can use the result already obtained from the previous section:

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \Delta T_t\right) \qquad \text{(B-1)}$$

where

$$\Delta T_t = \frac{\partial T(\delta_{t-1})}{\partial \delta_{t-1}}\Delta\delta_{t-1} = \frac{\partial\left(\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}_t\right)}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t}\Delta\delta_{t-1}$$

Using ΔT_t in the original equation,

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}} - \frac{\partial\left(\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}_t\right)}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t}\right)\Delta\delta_{t-1}$$

The first term is always positive in this case. We are interested in the sign of the second term. To have sufficient conditions for improving the utility level, the term should be positive. Hence,

$$\frac{\partial W_t}{\partial \delta_{t-1}} - \frac{\partial\left(\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} + R_t\delta_{t-1} - W_t\tilde{d}\right)}{\partial \delta_{t-1}}\frac{1}{1-\tilde{d}_t} > 0.$$

The condition is calculated as in the following:

$$\frac{\partial W_t}{\partial \delta_{t-1}} - \left(\frac{\tilde{d}_{t-1}}{\tilde{n}_{t-1}}\frac{\partial R_t}{\partial \delta_{t-1}} - \frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}^2}\frac{\partial \tilde{n}_{t-1}}{\partial \delta_{t-1}} + \frac{\partial R_t}{\partial \delta_{t-1}}\delta_{t-1} + R_t - \frac{\partial W_t}{\partial \delta_{t-1}}\tilde{d}_t\right)\frac{1}{1-\tilde{d}_t} > 0$$

Cancelling out the term $1/(1-\tilde{d}_t)$, we can transform the condition as

$$\frac{\partial W_t}{\partial \delta_{t-1}} - \left(\frac{\tilde{d}_{t-1}}{\tilde{n}_{t-1}}\frac{\partial R_t}{\partial \delta_{t-1}} - \frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}^2}\frac{\partial \tilde{n}_{t-1}}{\partial \delta_{t-1}} + \frac{\partial R_t}{\partial \delta_{t-1}}\delta_{t-1} + R_t\right) > 0$$

$$\eta_{\delta W}\frac{W_t}{\delta_{t-1}} - \varepsilon_{\delta R}\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}} + \frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}}\eta_{\delta n} - \varepsilon_{\delta R}R_t - R_t > 0$$

$$\eta_{\delta W}\frac{W_t}{\delta_{t-1}} + \frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}}(\eta_{\delta n} - \varepsilon_{\delta R}) - (1+\varepsilon_{\delta R})R_t > 0$$

$$\eta_{\delta n} > \varepsilon_{\delta R} - \eta_{\delta W}\frac{W_t\tilde{n}_{t-1}}{R_t\tilde{d}_{t-1}} + (1+\varepsilon_{\delta R})\frac{\tilde{n}_{t-1}\delta_{t-1}}{\tilde{d}_{t-1}}$$

Hence, the sufficient condition is shown in the following:

$$\eta_{\delta n} > \varepsilon_{\delta R} - \eta_{\delta W}\frac{W_t\tilde{n}_{t-1}}{R_t\tilde{d}_{t-1}} + (1+\varepsilon_{\delta R})\frac{\tilde{n}_{t-1}\delta_{t-1}}{\tilde{d}_{t-1}}$$

Appendix D: Calculation of Example 1

The first-order conditions from household optimization are

$$n_t = \frac{\alpha(W_t - T_t)}{z - \delta_t}$$
$$X_t = \beta(W_t - T_t)$$
$$Y_t = R_{t+1}(1-\alpha-\beta)(W_t - T_t).$$

We can normalize for $W_t - T_t$:

$$\tilde{n}_t = \frac{\alpha}{z-\delta}$$
$$\tilde{X}_t = \beta$$
$$\tilde{Y}_t = R_{t+1}(1-\alpha-\beta).$$

Deriving the saving amount as

$$s_t = (1-\alpha-\beta)(W_t - T_t)$$
$$\Rightarrow \quad \tilde{s}_t = (1-\alpha-\beta).$$

Factor prices are

$$W_t = (1-\rho)A_t k_t^{\rho}$$
$$R_t = \rho A_t k_t^{\rho-1}.$$

Obtaining the dynamics of k,

$$\frac{\alpha k_{t+1}}{z-\delta_t} = 1-\alpha-\beta-\tilde{d}_t$$
$$\Leftrightarrow \quad k_{t+1} = \frac{1-\alpha-\beta-\tilde{d}_t}{\alpha}(z-\delta_t)$$

where

$$\tilde{d}_t < 1-\alpha-\beta. \qquad \text{(D-1)}$$

Therefore,

$$\frac{\partial k_{t+1}}{\partial \delta_t} = -\frac{1-\alpha-\beta-\tilde{d}_t}{\alpha}.$$

Calculate the elasticities

$$\varepsilon_{\delta R} = \frac{\delta_t}{R_{t+1}}\frac{\partial R_{t+1}}{\partial \delta_t} = \frac{\delta_t}{A_{t+1}k_{t+1}^{\rho-1}}\frac{\partial (A_{t+1}k_{t+1}^{\rho-1})}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial \delta_t} = \frac{\delta_t}{k_{t+1}}\frac{(1-\rho)(1-\alpha-\beta-\tilde{d}_t)}{\alpha} = \frac{(1-\rho)\delta_t}{(z-\delta_t)}$$

$$\eta_{\delta W} = \frac{\delta_t}{W_{t+1}}\frac{\partial W_{t+1}}{\partial \delta_t} = \frac{\delta_t}{A_{t+1}k_{t+1}^{\rho}}\frac{\partial (A_{t+1}k_{t+1}^{\rho})}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial \delta_t} = -\frac{\delta_t}{k_{t+1}}\frac{\rho(1-\alpha-\beta-\tilde{d}_t)}{\alpha} = -\frac{\rho\delta_t}{(z-\delta_t)}$$

$$\eta_{\delta n} = \frac{\delta_t}{n_t}\alpha(W_t - T_t)\frac{\partial(1/(z-\delta_t))}{\delta_t} = \frac{\delta_t}{n_t}\frac{\alpha(W_t - T_t)}{(z-\delta_t)^2} = \frac{\delta_t}{z-\delta_t}.$$

We can derive sufficient conditions for RC-improvement

$$\eta_{\delta n} > \varepsilon_{\delta R} - \eta_{\delta W}\frac{W_t \tilde{n}_{t-1}}{R_t \tilde{d}_{t-1}} + (1+\varepsilon_{\delta R})\frac{\tilde{n}_{t-1}\delta_{t-1}}{\tilde{d}_{t-1}}$$

$$\Leftrightarrow \quad \frac{\delta}{z-\delta} > \frac{(1-\rho)\delta}{(z-\delta)} + \frac{\rho\delta}{(z-\delta)}\frac{1-\rho}{\rho}\frac{1-\alpha-\beta-\tilde{d}}{\alpha}(z-\delta)\frac{\tilde{n}}{\tilde{d}} + (1+\frac{(1-\rho)\delta}{(z-\delta)})\frac{\tilde{n}\delta}{\tilde{d}}$$

$$\Leftrightarrow \quad 1 > (1-\rho) + (1-\rho)\frac{1-\alpha-\beta-\tilde{d}}{\alpha}(z-\delta)\frac{\tilde{n}}{\tilde{d}} + (z-\rho\delta)\frac{\tilde{n}}{\tilde{d}}$$

$$\Leftrightarrow \quad \rho\tilde{d} > (1-\rho)(1-\alpha-\beta-\tilde{d}) + \alpha\frac{(z-\rho\delta)}{(z-\delta)}$$

$$\Leftrightarrow \quad \tilde{d} > (1-\rho)(1-\alpha-\beta) + \alpha\frac{(z-\rho\delta)}{(z-\delta)}$$

$$\Leftrightarrow \quad \tilde{d} > (1-\rho)(1-\alpha-\beta) + \rho\alpha + \frac{1-\rho}{1-\delta/z}\alpha$$

$$\Leftrightarrow \quad \tilde{d} - \big((1-\rho)(1-\alpha-\beta) + \rho\alpha\big) > \frac{1-\rho}{1-\delta/z}\alpha \qquad \text{(D-2)}$$

Incidentally, since $\delta \geq 0$, $\tilde{d} \geq d_1$ is derived from (D-2), where d_1 denotes as follows:

$$d_1 \equiv (1-\rho)(1-\alpha-\beta) + \alpha \qquad \text{(D-3)}$$

Therefore, from (D-1) to (D-3),

$$\delta / z < 1 - \frac{1-\rho}{\tilde{d} - \big((1-\rho)(1-\alpha-\beta) + \rho\alpha\big)}\alpha$$

and

$$d_1 \equiv (1-\rho)(1-\alpha-\beta) + \alpha < \tilde{d} < d_2 \equiv 1-\alpha-\beta$$

Appendix E: Proof of Proposition 5

In the indirect utility function (32), $T(\delta_{t-1},\delta_t,\tilde{d}_{t-1},\tilde{d}_t,\theta)$ depends on the parameter θ. Hence, under the assumption that δ and $\tilde{d}$ is fixed, we can derive the maximum condition of (32) as follows:

$$\min_{\theta} T(\delta_{t-1},\delta_t,\tilde{d}_{t-1},\tilde{d}_t,\theta) = \min_{\theta}\left[\frac{\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}} + R_t(1-\theta)\delta_{t-1} - W_t\tilde{d}_t + \theta\delta_t W_t\tilde{n}_t}{(1-\tilde{d}_t+\theta\delta_t\tilde{n}_t)}\right] \qquad \text{(D-1)}$$

To search for the optimal value θ of (D-1), we analyze the sign of the function $\partial T/\partial\theta$ using the preferences and production technology of Example 1.

$$\text{The sign of } \partial T/\partial\theta = \text{The sign of } \frac{\partial\left\langle \log\left[\frac{\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}} + R_t(1-\theta)\delta_{t-1} - W_t\tilde{d}_t + \theta\delta_t W_t\tilde{n}_t}{(1-\tilde{d}_t+\theta\delta_t\tilde{n}_t)}\right]\right\rangle}{\partial\theta}$$

$$= \text{The sign of } \left[\frac{-R_t\delta_{t-1}+\delta_t W_t\tilde{n}_t}{\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_t} + R_t(1-\theta)\delta_{t-1} - W_t\tilde{d}_t + \delta_t\theta W\tilde{n}_t} - \frac{\delta_t\tilde{n}_t}{(1-\tilde{d}_t+\theta\delta_t\tilde{n}_t)}\right]$$

$$= \text{The sign of } \left[(1-\tilde{d}_t+\theta\delta_t\tilde{n}_t)(-R_t\delta_{t-1}+\delta_t W_t\tilde{n}_t) - \delta_t\tilde{n}_t\left(\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_t} + R_t(1-\theta)\delta_{t-1} - W_t\tilde{d}_t + \delta_t\theta W\tilde{n}_t\right)\right]$$

$$= \text{The sign of } \left[W_t - \left[\frac{\delta_{t-1}}{\delta_t} + \tilde{n}_t\,\delta_{t-1} + \tilde{d}_{t-1} - \tilde{d}_t\,\frac{\delta_{t-1}}{\delta_t}\right]R_t/\tilde{n}_t\right]$$

$$= \text{The sign of } \Xi$$

In addition, we can denote Ξ as follows in the case with $\delta_{t-1}=\delta_t$ and $\tilde{d}_{t-1}=\tilde{d}_t$.

$$\Xi = W_t - \left[\frac{1}{\tilde{n}_t} + \delta_{t-1}\right]R_t \qquad \text{(D-2)}$$

$$\Leftrightarrow \quad \Xi(\delta) = (1-\rho)Ak(\delta)^{\rho} - \left(\frac{1}{\tilde{n}(\delta)} + \delta\right)\rho Ak(\delta)^{\rho-1}$$

$$= (1-\rho)A\left[\frac{1-\alpha-\beta-\tilde{d}}{\alpha}(z-\delta)\right]^{\rho} - \left(\frac{1}{\frac{\alpha}{z-\delta}} + \delta\right)\rho A\left[\frac{1-\alpha-\beta-\tilde{d}}{\alpha}(z-\delta)\right]^{\rho-1}$$

$$= \left\{ \left[(1-\rho)(1-\alpha-\beta-\tilde{d})-\rho \right] z - \left[(1-\rho)(1-\alpha-\beta-\tilde{d})-(1-\alpha)\rho \right] \delta \right\} \frac{A}{\alpha} \left[\frac{1-\alpha-\beta-\tilde{d}}{\alpha}(z-\delta) \right]^{\rho-1}$$

(D-3)

Because (D-2) doesn't depend on the parameter θ and $\lim_{\delta \to z} \Xi(\delta) = -\infty$, the sign of (D-2) is determined by the sign of the following value on (D-3).

$$\Xi(0) = \left\{ \left[(1-\rho)(1-\alpha-\beta-\tilde{d})-\rho \right] z \right\} \frac{A}{\alpha} \left[\frac{1-\alpha-\beta-\tilde{d}}{\alpha} z \right]^{\rho-1} \qquad \text{(D-4)}$$

The sign of (D-4) is subject to the following rules:

1) When $\tilde{d} > d^* \equiv 1-\alpha-\beta-\rho/(1-\rho)$, $\Xi(\delta) < 0$ ($for \quad \forall \delta \in [0, z)$).
2) When $\tilde{d} > d^* \equiv 1-\alpha-\beta-\rho/(1-\rho)$, there exists δ^* from (D-3) , then $\Xi(\delta) > 0$ ($for \quad \forall \delta \in [0, \delta^*)$) or $\Xi(\delta) < 0$ ($for \quad \forall \delta \in (\delta^*, z)$)

   Where δ^* denotes as follows:

$$\delta^* \equiv \left[1 - \frac{\alpha\rho}{(1-\rho)(1-\alpha-\beta-\tilde{d})-(1-\alpha)\rho} \right] z$$

Therefore, from (D-2), (D-3), and the above rules, we can derive the following rules.

1) When $\tilde{d} > d^* \equiv 1-\alpha-\beta-\rho/(1-\rho)$, $\theta = 1$ is optimal.
2) When $\tilde{d} < d^*$, $\theta = 0$ is optimal if $0 \le \delta \le \delta^*$ and $\theta = 1$ is optimal if $\delta^* < \delta < z$.

Appendix F: Proof of Proposition 6

We will use (25) and (25') in the proof. Calculate the change of lifetime utility δU_t when the amount of child benefit δ_t is raised, we can use (31) with $\theta = 1$ and the results already obtained from the previous section:

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \Delta T_t\right) \qquad \text{(F-1)}$$

where

$$\Delta T_t = \frac{\partial T(\delta_{t-1},\delta_t)}{\partial \delta_{t-1}}\Delta\delta_{t-1} + \frac{\partial T(\delta_{t-1},\delta_t)}{\partial \delta_t}\Delta\delta_t$$

$$= \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_{t-1}}\Delta\delta_{t-1} + \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_t}\Delta\delta_t$$

$$= \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_{t-1}}\Delta\delta_{t-1} + \frac{\partial\left(W_t + \frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t - W_t(1-\tilde{d}_t)}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_t}\Delta\delta_t$$

$$= \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_{t-1}}\Delta\delta_{t-1} - \frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t}{(1-\tilde{d}_t+\delta_t\tilde{n}_t)^2}\delta_t\tilde{n}_t\Delta\delta_t$$

Using ΔT_t in the original equation,

$$\Delta U_t = \lambda(W_t - T_t)\left(\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1}\right)\Delta\delta_t + \tilde{U}_t\left(\frac{\partial W_t}{\partial \delta_{t-1}} - \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_{t-1}}\right)\Delta\delta_{t-1}$$

$$+\tilde{U}_t\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t}{(1-\tilde{d}_t+\delta_t\tilde{n}_t)^2}\delta_t\tilde{n}_t\Delta\delta_t \qquad \text{(F-2)}$$

Case 1: if $\eta_{\delta s} > \eta_{\delta n}$

In this case, we are interested in the sign of all terms of (F-2). For having sufficient conditions for improving utility level, the terms should be positive. Hence,

$$\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1} > 0 \quad , \quad \frac{\partial W_t}{\partial \delta_{t-1}} - \frac{\partial\left(\frac{\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t\tilde{d}_t + \delta_t W_t \tilde{n}_t}{1-\tilde{d}_t+\delta_t\tilde{n}_t}\right)}{\partial \delta_{t-1}} > 0.$$

and

$$\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t > 0$$

The condition is calculated in the following:

$$\tilde{n}_t + \frac{1}{R_{t+1}^2}\frac{\partial R_{t+1}}{\partial \delta_t}\tilde{Y}_{t+1} > 0\,, \quad \frac{\partial W_t}{\partial \delta_{t-1}} - \frac{-\frac{R_t}{\tilde{n}_{t-1}^{\ 2}}\frac{\partial \tilde{n}_{t-1}}{\partial \delta_{t-1}}\tilde{d}_{t-1} + \frac{1}{\tilde{n}_{t-1}}\frac{\partial R_t}{\partial \delta_{t-1}}\tilde{d}_{t-1} - \frac{\partial W_t}{\partial \delta_{t-1}}\tilde{d}_t + \delta_t\frac{\partial W_t}{\partial \delta_{t-1}}\tilde{n}_t}{1 - \tilde{d}_t + \delta_t \tilde{n}_t} > 0.$$

and

$$\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t > 0$$

Cancelling out the term $1/(1-\tilde{d}_t)$, we can transform the condition as

$$n_t > \eta_{\delta R}\frac{1}{R_{t+1}}\frac{Y_{t+1}}{\delta_t}\,, \quad -\eta_{\delta W}\frac{W_t}{\delta_{t-1}} + \frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}}\eta_{\delta n} + \eta_{\delta R}\frac{R_t\tilde{d}_{t-1}}{\tilde{n}_{t-1}\delta_{t-1}} > 0 \qquad \text{(F-3)}$$

and

$$\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} - W_t > 0$$

Since $\eta_{\delta R} > 0$ and $\eta_{\delta W} < 0$, the sufficient condition is shown in the following:

$$n_t > \eta_{\delta R}\frac{1}{R_{t+1}}\frac{Y_{t+1}}{\delta_t}$$

and

$$\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} > W_t$$

Case 2: if $\tilde{s}_\delta \le 0$ *and* $\tilde{n}_\delta > 0$

In this case, the first term of (F-2) is always positive. We are interested only in the sign of the second term and the third term of (F-2). Hence, from (F-3), the sufficient condition is shown in the following:

$$\eta_{\delta n} > \varepsilon_{\delta R} + \eta_{\delta W}\frac{\tilde{n}_{t-1}W_t}{R_t\tilde{d}_{t-1}}$$

and

$$\frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} > W_t$$

Appendix G: Proof of Example 2

It is possible to transform (36) using the equations of Appendix D.

$$\eta_{\delta n} > \varepsilon_{\delta R} + \eta_{\delta W}\frac{\tilde{n}_{t-1}W_t}{R_t\tilde{d}_{t-1}} \quad \text{and} \quad \frac{R_t}{\tilde{n}_{t-1}}\tilde{d}_{t-1} > W_t$$

$$\Leftrightarrow \qquad \frac{\delta_t}{z-\delta_t} > \frac{(1-\rho)\delta_t}{(z-\delta_t)} + \frac{\rho\delta_t}{(z-\delta_t)}\frac{1}{\tilde{d}_{t-1}}\frac{\alpha}{(z-\delta_{t-1})}\frac{1-\rho}{\rho}\frac{1-\alpha-\beta-\tilde{d}_{t-1}}{\alpha}(z-\delta_{t-1})$$

$$\text{and}$$

$$1 > \frac{1}{\tilde{d}_{t-1}}\frac{\alpha}{(z-\delta_{t-1})}\frac{1-\rho}{\rho}\frac{1-\alpha-\beta-\tilde{d}_{t-1}}{\alpha}(z-\delta_{t-1})$$

$$\Leftrightarrow \qquad \frac{\rho}{1-\rho}\tilde{d}_{t-1} > (1-\alpha-\beta-\tilde{d}_{t-1}) \quad \text{and} \quad \frac{\rho}{1-\rho}\tilde{d}_{t-1} > (1-\alpha-\beta-\tilde{d}_{t-1})$$

$$\Leftrightarrow \qquad \tilde{d}_{t-1} > \frac{1-\alpha-\beta}{1+\rho/(1-\rho)}$$

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