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Abstract

For a correspondence from a partially ordered set to a lattice, three sets of sufficient conditions for the existence of a monotone selection are obtained. (1) The correspondence is weakly ascending while every value satisfies a completeness condition, e.g., is chain-complete. (2) The correspondence is ascending while the target is a sublattice of the Cartesian product of a finite number of chains. (3) Both source and target are chains while the correspondence is generated by the maximization of a strongly acyclic interval order with the single crossing property. The theorems give new sufficient conditions for the existence of (epsilon) Nash equilibria. *JEL* Classification Number: C 72.

Key words: Monotone selection; (weakly) ascending correspondence; interval order; single crossing; (epsilon) Nash equilibrium

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1 Introduction

The existence of monotone selections is a fascinating topic from a purely mathematical viewpoint. It is also important for economic theory, especially, game theory.

To derive the existence of a Nash equilibrium from Tarski's (1955) fixed point theorem, or its immediate generalization by Abian and Brown (1961), we need monotone selections from the best response correspondences. (In contrast, Kakutani's theorem, while being a generalization of Brouwer's, is usually applied in the absence of continuous selections.) Admittedly, there are fixed point theorems for increasing correspondences without monotone selections, see e.g., Smithson (1971, Theorem 1.1) or Roddy and Schröder (2005, Lemma 2.8), but so far they have found no application in game theory. When it comes to *decreasing* best responses (or monotonicity in a more elaborate sense), every equilibrium existence result in the literature hinges on the availability of monotone (in an appropriate sense) selections (McManus, 1962, 1964; Novshek, 1985; Kukushkin, 1994, 2000, 2004, 2007; Dubey et al., 2006).

The "standard" theory of games with strategic complementarities as developed in Topkis (1979), Veinott (1989), Vives (1990), Milgrom and Roberts (1990), Milgrom and Shannon (1994), and summarized in Topkis (1998), imposes assumptions ensuring that the best responses to any profile of other players' strategies form a complete sublattice of the player's strategy set, hence the maximal, or minimal, best response provides a selection needed.

Those assumptions hold in a wide variety of important situations. However, if, in the spirit of Milgrom and Shannon (1994), one asks whether they are, indeed, indispensable for the best responses to exist and be increasing, the answer will be an unequivocal "No." Smith's (1974) Theorems 4.1 and 4.2 show a clear difference between the two properties of a preference ordering – "to attain a maximum on every compact subset" and "to have a nonempty, closed set of maximizers on every compact subset"; the difference does not disappear when an order structure replaces, or is added to, topology. Similarly, quasisupermodularity is necessary for monotone comparative statics (Milgrom and Shannon, 1994, Theorem 4), but not for increasing best responses when a strategy set is fixed; and if the preferences are not quasisupermodular, the set of best responses need not be a sublattice.

It even happens that no assumption ensuring the existence of the best responses can be justified. Although a Nash equilibrium may still exist (Dasgupta and Maskin, 1986; Reny, 1999), a more robust approach is to consider ε-best responses and ε-equilibrium. Moreover, the description of preferences by, say, semiorders may be suggested for its own sake rather than for technical convenience: it is sometimes argued that the "complete rationality" of the players' preferences is unrealistic.

Monotone selections from the best, or ε-best, response correspondences would greatly help in every situation described above. To the best of my knowledge, there is no result in the previous literature wherefrom their existence could be derived. This paper strives to fill in the lacuna although some interesting questions remain open.

There is another objective, also related to the existence of Nash equilibria. The Appendix to Milgrom and Shannon (1994) contains an "almost topology-free" sufficient condition for a utility function on a complete lattice to attain its maximum. Unfortunately, their proof relies on "Theorem A2," actually, a misquotation from Veinott (1989), which is just wrong. The mistake seems to have never been corrected in economics literature, and Veinott's result to have never been published at all. Theorem 1 of this paper is a valid version of "Theorem A2," whose assumptions are even weaker than Veinott's. A correct proof of Milgrom and Shannon's Theorem A4 easily follows.

Section 2 reproduces some standard notions. In Section 3, we consider correspondences increasing w.r.t. extensions of the basic order from points to subsets. In Theorem 1, it is the weak Veinott "order," which need not even be transitive; in Theorem 2, the usual Veinott order. For the existence of a monotone selection, we have to assume that every value satisfies a completeness condition in Theorem 1, or that the target is a sublattice of the Cartesian product of a finite number of chains in Theorem 2.

In Section 4, the correspondence is generated by the maximization of a binary relation on a chain; the relation depends on a parameter from a partially ordered set, and the single crossing condition holds. Theorem 3 shows the existence of a monotone selection if the preference relation is always a strongly acyclic interval order and the set of parameters is a chain. If the relation is only strongly acyclic and transitive, then there may be no monotone selection, but Theorem 4 shows that a Nash equilibrium exists anyway. The main proofs are deferred to Section 5; concluding remarks are collected in Section 6.

2 Preliminaries

Notions such as a *partially ordered set* (a *poset*), a *chain*, and a (*complete*) *lattice* are assumed commonly known. We denote $\mathfrak{B}_X$ the set of all nonempty subsets of a given set X. A poset X is called *Zorn bounded above/below* if every chain $Y \in \mathfrak{B}_X$ admits an upper/lower bound. A poset X is called *chain-complete upwards/downwards* if $\sup Y/\inf Y$ exists in X for every chain $Y \in \mathfrak{B}_X$; in this case, $X' \in \mathfrak{B}_X$ is *chain-subcomplete upwards/downwards* if $\sup Y \in X'/\inf Y \in X'$ for every chain $Y \in \mathfrak{B}_{X'} \subseteq \mathfrak{B}_X$. A poset X is called *chain-complete* if it is chain-complete both upwards and downwards; $X' \in \mathfrak{B}_X$ is then *chain-subcomplete* if it is chain-subcomplete both upwards and downwards.

A *correspondence* from T to X is a mapping $R\colon T \to \mathfrak{B}_X$; a mapping $r\colon T \to X$ is a *selection* from R if $r(t) \in R(t)$ for all $t \in T$. Given a poset X and a correspondence $R\colon T \to \mathfrak{B}_X$, we denote $R^+(t) := \{x \in R(t) \mid \nexists y \in R(t)\,[y > x]\}$ and $R^-(t) := \{x \in R(t) \mid \nexists y \in R(t)\,[y < x]\}$ for every $t \in T$. If T and X are posets, a mapping $r\colon T \to X$ is *increasing* if $r(t') \geq r(t)$ whenever $t' > t$; r is *decreasing* if $r(t') \leq r(t)$ whenever $t' > t$. A *monotone selection* from a correspondence $R\colon T \to \mathfrak{B}_X$ is a selection from R which is increasing. Since the reversal of the order on T (or, equivalently, on X) makes an increasing mapping decreasing and vice versa, there is no need for a separate study of "anti-monotone" selections.

The framework of the "naive" set theory with the unrestricted Axiom of Choice is adopted throughout. In practice, we invoke either Zorn's Lemma (if a poset is Zorn bounded above/below, then it contains a maximal/minimal point) or Zermelo's Theorem (every set can be well-ordered).

3 Selections from increasing correspondences

We start with a few ways to extend an order given on X to $\mathfrak{B}_X$. Let X be a poset and $Y, Z \in \mathfrak{B}_X$. The following relations are, at least, transitive:

$$Y \geq^{\mathrm{Inf}} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, \exists z' \in Z \, [y \geq z' \; \& \; z \geq z'];\tag{1a}$$

$$Y \geq^{\mathrm{Sup}} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, \exists y' \in Y \, [y' \geq z \; \& \; y' \geq y].\tag{1b}$$

A wider variety of similar relations is used in the literature (Smithson, 1971; Echenique, 2002; Roddy and Schröder, 2005; Heikkilä and Reffett, 2006; Quah, 2007), but we do not need them here.

When X is a lattice, Veinott's order (Topkis, 1998) seems most popular. Following Veinott (1989), we define it as a conjunction of the "lower and upper halves," and also define a weak version:

$$Y \geq^{\wedge} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, [y \wedge z \in Z];\tag{1c}$$

$$Y \geq^{\vee} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, [y \vee z \in Y];\tag{1d}$$

$$Y \geq^{\mathrm{Vt}} Z \rightleftharpoons [Y \geq^{\wedge} Z \; \& \; Y \geq^{\vee} Z];\tag{1e}$$

$$Y \geq^{\mathrm{wV}} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, [y \vee z \in Y \;\text{ or }\; y \wedge z \in Z].\tag{1f}$$

The relation $\geq^{\mathrm{Vt}}$ is antisymmetric and transitive on $\mathfrak{B}_X$; it is reflexive on sublattices. Neither $\geq^{\mathrm{wV}}$, nor $\geq^{\wedge}$ or $\geq^{\vee}$ need even be transitive. Clearly, $Y \geq^{\wedge} Z \Rightarrow Y \geq^{\mathrm{Inf}} Z$ and $Y \geq^{\vee} Z \Rightarrow Y \geq^{\mathrm{Sup}} Z$.

Let $\geq^{*}$ denote one of the relations (1) and T be a poset. A correspondences $R\colon T \to \mathfrak{B}_X$ is *increasing w.r.t.* $\geq^{*}$ if $R(t') \geq^{*} R(t)$ whenever $t' > t$. Note the strong inequality in the last condition; it is natural when dealing with binary relations that need not be reflexive. Veinott (1989) called correspondences increasing w.r.t. $\geq^{\mathrm{Vt}}$ ($\geq^{\mathrm{wV}}$) in this sense (weakly) ascending.

The best-known results on the existence of monotone selections (Topkis, 1998) are applicable to ascending correspondences to complete lattices. Similar results hold under weaker assumptions. One example is given by Smithson (1971, Theorem 1.7); the following result assumes monotonicity in a stronger sense, but demands less of the values $R(t)$.

Proposition 3.1. *A correspondence R from a poset T to a poset X admits a monotone selection if it is increasing w.r.t. $\geq^{\mathrm{Inf}}$ while $R^-(t) \neq \emptyset$ for every $t \in T$.*

Proof. For every $t \in T$, we pick $r(t) \in R^-(t)$ arbitrarily. If $t' > t$, then, by (1a), there is $x \in R(t)$ such that $x \leq r(t)$ and $x \leq r(t')$. Since $r(t) \in R^-(t)$, we must have $x = r(t)$. $\square$

Corollary. *A correspondence R from a poset T to a poset X admits a monotone selection if it is increasing w.r.t. $\geq^{\mathrm{Inf}}$ while every $R(t)$ $(t \in T)$ is Zorn bounded below.*

Dually, the existence of a monotone selection follows from monotonicity w.r.t. $\geq^{\mathrm{Sup}}$ and $R^+(t) \neq \emptyset$ for every t. Without any restriction on $R(t)$, Proposition 3.1 is wrong, as was shown in Smithson (1971, p. 307); that example simultaneously shows that Milgrom and Shannon's (1994) "Theorem A2" is also wrong.

Theorem 1. *Let X be a lattice, T be a poset, $R\colon T \to \mathfrak{B}_X$ be increasing w.r.t. $\geq^{\mathrm{wV}}$, and such that, for every $t \in T$, $R(t)$ is chain-complete upwards and $R^-(t) \neq \emptyset$ (e.g., $R(t)$ is Zorn bounded below). Then there exists a monotone selection from R.*

The proof is deferred to Section 5.1. Naturally, the dual version is valid as well. Note that the upward chain-completness of every $R(t)$ in Theorem 1 cannot be replaced with just Zorn boundedness above.

Example 3.2. Let $X := [-1, 1]$ and $R\colon X \to \mathfrak{B}_X$ be this: $R(0) := X \setminus \{0\}$; $R(x) := \{x/2\}$ for $x \in X \setminus \{0\}$. Clearly, every $R(x)$ is Zorn bounded both above and below, whereas R is increasing w.r.t. $\geq^{\mathrm{wV}}$; however, R admits neither monotone selection, nor fixed point.

Referring to Theorem 1 instead of "Theorem A2," we obtain a correct proof of Milgrom and Shannon's (1994) Theorem A4.

Theorem A4 (Milgrom and Shannon, 1994). *Let X be a complete lattice; let $f\colon X \to \mathbb{R}$ be quasisupermodular and such that for every chain $C \subseteq X$, there hold*

$$\limsup_{x \in C,\, x \uparrow \sup C} f(x) \leq f(\sup C) \tag{2a}$$

and

$$\limsup_{x \in C,\, x \downarrow \inf C} f(x) \leq f(\inf C). \tag{2b}$$

Then $\operatorname{argmax}_{x \in X} f(x)$ *is a nonempty complete sublattice of X.*

Remark. The original formulation of Milgrom and Shannon is about $\operatorname{argmax}_{x \in S} f(x)$, where S is a complete sublattice of X; however, this extension is straightforward because all assumptions about f are inherited by its restriction to S. The following proof essentially belongs to Veinott (1989) and Shannon (1990, Proposition 2). The existence proof remains valid under either "half" of quasisupermodularity, but then the set of maximizers need not be a sublattice.

Proof. We consider the mapping $R\colon f(X) \to \mathfrak{B}_X$ defined by $R(a) := \{x \in X \mid f(x) \geq a\}$. Conditions (2) ensure that every $R(a)$ is chain-subcomplete in X. If $x \in R(a)$ and $y \in R(b)$, but $y \wedge x \notin R(a)$, then $f(x) > f(y \wedge x)$, hence $f(y \vee x) > f(y)$ by the quasisupermodularity of f, hence $y \vee x \in R(b)$; therefore, $R(b) \geq^{\mathrm{wV}} R(a)$. We see that R is increasing w.r.t. $\geq^{\mathrm{wV}}$, whatever order is chosen on $f(X)$ (this funny thing is possible because $\geq^{\mathrm{wV}}$ is *not* an order). Let us consider the order on $f(X)$ induced from $\mathbb{R}$; by Theorem 1, there exists a monotone selection r from R. Clearly, $C := \{r(a)\}_{a \in f(X)}$ is a chain in X; denoting $x^* := \sup C$, we have $f(x^*) \geq f(x)$ for every $x \in X$ by (2a), i.e., $x^* \in \operatorname{argmax}_{x \in X} f(x) \neq \emptyset$. Now, the quasisupermodularity of f ensures, in a standard way, that $\operatorname{argmax}_{x \in X} f(x)$ is a sublattice of X. Being chain-subcomplete, it is a complete sublattice of X. $\square$

Without topological restrictions on values $R(t)$, the existence of a monotone selection can be obtained either for a weakly ascending correspondence from a finite poset, or for an ascending correspondence.

Proposition 3.3. *Let X be a lattice, T be a finite poset, and $R\colon T \to \mathfrak{B}_X$ be increasing w.r.t. $\geq^{\mathrm{wV}}$. Then there exists a monotone selection from R.*

Theorem 2. *Let X be a sublattice of the Cartesian product of a finite number of chains. Let T be a poset and $R\colon T \to \mathfrak{B}_X$ be increasing w.r.t. $\geq^{\mathrm{Vt}}$. Then there exists a monotone selection from R.*

The proofs are deferred to Sections 5.2 and 5.3 respectively. The validity of Proposition 3.3 was mentioned in Veinott (1989), but a different proof was suggested.

4 ε-Best responses

In this section, we study correspondences generated by the maximization of a binary relation. Formally, we consider a *parametric family* $\langle \succ^t \rangle_{t \in T}$ of binary relations on X; the parameter t may be interpreted as (an aggregate of) the choice(s) of other agent(s). We define

$$R(t) := \{x \in X \mid \nexists y \in X\, [y \succ^t x]\}. \tag{3}$$

A relation $\succ$ is *strongly acyclic* if there exists no *infinite improvement path*, i.e., no sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for all k. A relation $\succ$ is an *interval order* if it is irreflexive and transitive, and satisfies the condition

$$\forall x, y, a, b \in X\, \big[[y \succ x \;\&\; b \succ a] \Rightarrow [y \succ a \;\text{ or }\; b \succ x] \big].$$

The importance of those properties of preference relations is shown by these straightforward characterization results.

Proposition 4.1. *Let $\succ$ be a binary relation on a set X. For every $Y \subseteq X$, we denote $M(Y) := \{x \in Y \mid \nexists y \in Y\, [y \succ x]\}$. Then these two statements are equivalent.*

1. *$\succ$ is strongly acyclic and transitive.*
2. *Whenever $Y \subseteq X$ and $x \in Y \setminus M(Y)$, there is $y \in M(Y)$ such that $y \succ x$.*

Proposition 4.2. *Let $\succ$ be a binary relation on a set X, and $M(Y)$, for every $Y \subseteq X$, be the same as in Proposition 4.1. Then these two statements are equivalent.*

1. *$\succ$ is a strongly acyclic interval order.*
2. *Whenever $Y \subseteq X$ and $\{x^0, \dots, x^k\} \subseteq Y \setminus M(Y)$, there is $y \in M(Y)$ such that $y \succ x^m$ for each $m = 0, \dots, k$.*

Routine proofs are omitted.

A parametric family $\langle \succ^t \rangle_{t \in T}$ has the *single crossing* property if these conditions hold:

$$\forall x, y \in X\, \forall t, t' \in T\, \big[[t' > t \,\&\, y \succ^t x \,\&\, y > x] \Rightarrow y \succ^{t'} x\big]; \tag{4a}$$

$$\forall x, y \in X\, \forall t, t' \in T\, \big[[t' > t \,\&\, y \succ^{t'} x \,\&\, y < x] \Rightarrow y \succ^{t} x\big]. \tag{4b}$$

The property (equivalent to Milgrom and Shannon's when every $\succ^t$ is an ordering represented by a numeric function) can be perceived as monotonicity w.r.t. a partial order on the set of binary relations on X. The fact was noticed (in somewhat narrower contexts) by Quah and Strulovici (2007) and Alexei Savvateev (a seminar presentation, 2007).

Theorem 3. *Let X and T be chains such that both $\min T$ and $\max T$ exist. Let a parametric family $\langle \succ^t \rangle_{t \in T}$ of binary relations on X satisfy both conditions (4). Let every $\succ^t$ be a strongly acyclic interval order. Then there exists a monotone selection from R.*

Remark. The restriction on T is obviously satisfied if it is complete,

The proof is deferred to Section 5.4. As an example, let $u \colon X \times T \to \mathbb{R}$ be bounded above and $\varepsilon > 0$; let the preference relation be

$$y \succ^t x \rightleftharpoons u(y, t) > u(x, t) + \varepsilon.$$

It is easily seen that every $\succ^t$ is a strongly acyclic interval order (actually, a *semiorder*). $R(t)$ consists of all ε-maxima of $u(\cdot, t)$. Both conditions (4) hold if u satisfies Topkis's (1979) *increasing differences* condition:

$$[t' > t \,\&\, y > x] \Rightarrow u(y, t') - u(x, t') \geq u(y, t) - u(x, t);$$

in this context, the property is equivalent to the *supermodularity* of u (as a function on the lattice $X \times T$). Neither Theorem 1, nor Theorem 2 are applicable here: R is increasing w.r.t. $\geq^{\mathrm{wV}}$, but not necessarily w.r.t. $\geq^{\mathrm{Vt}}$; $R(t)$ need not be complete, nor even Zorn bounded.

The assumption in Theorem 3 that the preferences are described by interval orders cannot just be dropped, even for finite sets X and T.

Example 4.3. Let $X := \{0, 1, 2, 3, 4\}$, $T := \{0, 1\}$ (both with natural orders), and relations $\succ^t$ be defined by: $2 \succ^0 4 \succ^0 0 \succ^0 1 \succ^0 3$; $1 \succ^1 3 \succ^1 2 \succ^1 4 \succ^1 0$. Clearly, $R(0) = \{2\}$ while $R(1) = \{1\}$, so there is no monotone selection. On the other hand, conditions (4) are easy to check: (4a) is nontrivial only for $4 \succ^0 0$; (4b), only for $1 \succ^1 3$ and $2 \succ^1 4$.

Under the assumption that every $\succ^t$ is strongly acyclic and transitive, Theorem 3 is valid for finite X or T, but not generally.

Example 4.4. Let $X := [-2, 2]$, $T := [-1, 1]$ (both with natural orders), and relations $\succ^t$ be defined by

$$y \succ^t x \rightleftharpoons [u_1(y, t) > u_1(x, t) \;\&\; u_2(y, t) > u_2(x, t)], \tag{5}$$

where $u\colon X \times T \to \mathbb{R}^2$ is this: $u(1, t) := \langle 5, 2\rangle$ and $u(-1, t) := \langle 2, 5\rangle$ for all $t \in T$; $u(2, t) := u(-2, t) := u(x, t) := \langle 0, 0\rangle$ for all $x \in]-1, 1[$ and $t \in T$; whenever $x \in]1, 2[$ and $t \geq 0$,

$$u_1(x, t) := \begin{cases} x + t - 1, & \text{if } x + t \leq 2, \\ x + t + 4, & \text{if } x + t > 2, \end{cases} \quad \text{while } u_2(x, t) := 6 - x - t;$$

whenever $x \in]-2, -1[$ and $t \geq 0$, $u(x, t) := \langle 6 + x, -1 - x\rangle$; finally, $u_i(x, t)$ for $t < 0$, $i = 1, 2$, and $x \in]-2, -1[\cup]1, 2[$ is such that the equality

$$u_i(x, t) = u_{3-i}(-x, -t) \tag{6}$$

holds for all $t \in T$, $i = 1, 2$, and $x \in X$.

The very form of (5) ensures that every $\succ^t$ is irreflexive and transitive. Whenever $x \in \{-2\}\cup]-1, 1[\cup\{2\}$ and $y \in]-2, -1] \cup [1, 2[$, $y \succ^t x$ for every $t \in T$. Whenever $x, y \in]-2, -1[$ or $x, y \in]1, 2[$, $y \succ^t x$ does not hold for any $t \in T$. Let $t \geq 0$; if $-2 < x < -1$, then $u_1(x) < 5$ and $u_2(x) \leq 1$, hence $1 \succ^t x$; if $1 < x \leq 2 - t < 2$, then $u_1(x) \leq 1$ and $u_2(x) < 5$, hence $-1 \succ^t x$; if $2 - t < y < 2$, then $u_1(y) > 6$ and $u_2(y) > 3$, hence $y \succ^t 1$. "Dually," by (6), $y \succ^t -1 \succ^t x$ whenever $t < 0$, $-2 < y < -2 - t$, and $1 < x < 2$; $1 \succ^t x$ whenever $t < 0$ and $-2 - t \leq x < -1$. Thus, we see that every relation $\succ^t$ is strongly acyclic: no more than three consecutive improvements can be made from any starting point (e.g., $2 - t/2 \succ^t 1 \succ^t -1.5 \succ^t -2$ when $t > 0$). Conditions (4) are easy to check.

It is also easily checked that $R(0) = \{-1, 1\}$, $R(t) =]-2, -2 - t[\cup\{1\}$ for $t < 0$, and $R(t) = \{-1\}\cup]2 - t, 2[$ for $t > 0$. Suppose there is a monotone selection r from R. If $r(t) > -1$ for some $t > 0$, then $2 > r(t) > 2 - t$; defining $t' := 2 - r(t) > 0$, we have $t' < t$, hence $r(t') \leq r(t)$, hence $r(t') < 2 - t'$, hence $r(t') \in R(t')$ is only possible if $r(t') = -1$. Therefore, $r(t) = -1$ for some $t > 0$; dually, $r(t) = 1$ for some $t < 0$. We have a contradiction, i.e., there is no monotone selection: Theorem 3 cannot be extended to strongly acyclic and transitive preference relations.

The point of Examples 4.3 and 4.4 is that one cannot expect a connection between the single crossing property (4) and behavior of "best" responses without Statement 2 of Proposition 4.2 or, at least, of Proposition 4.1.

Let us consider a modification of the standard notion of a strategic game. There is a finite set N of *players* and a poset X_i of *strategies* for each $i \in N$. We denote $X_N := \prod_{i\in N} X_i$ and $X_{-i} := \prod_{j\neq i} X_j$; both are posets with the Cartesian product of the orders on components. Each player i's *preferences* are described by a parametric family of binary relations $\succ_i^{x_{-i}}$ ($x_{-i} \in X_{-i}$) on X_i; the player's *"best" response correspondence* R_i is defined by (3) with $T := X_{-i}$. A *Nash equilibrium* is $x_N \in X_N$ such that $x_i \in R_i(x_{-i})$ for each $i \in N$.

Theorem 4. *Let Γ be a strategic game where each X_i is a complete chain. Let the parametric family of preference relations of each player satisfy both conditions* (4)*. Let every relation $\succ_i^{x_{-i}}$ be strongly acyclic and transitive. Then Γ possesses a Nash equilibrium.*

The proof is deferred to Section 5.5. Example 4.4 shows that the assumptions of the theorem do not ensure the existence of monotone selections from the "best" response correspondences.

5 Proofs

5.1 Proof of Theorem 1

We define $\mathfrak{R}$ as the set of mappings $F\colon T \to \mathfrak{B}_X$ such that:

$$\forall t \in T\, [R^-(t) \subseteq F(t) \subseteq R(t)]; \tag{7a}$$

$$\forall t \in T\, \forall y, x \in R(t)\, [y > x \,\&\, y \in F(t) \Rightarrow x \in F(t)]; \tag{7b}$$

$$\forall t \in T\, [F(t) \text{ is chain-subcomplete upwards in } R(t)]; \tag{7c}$$

$$F \text{ is increasing w.r.t. } \geq^{\mathrm{wV}} . \tag{7d}$$

Clearly, $R \in \mathfrak{R} \neq \emptyset$. We define $\bar{F}\colon T \to \mathfrak{B}_X$ by

$$\bar{F}(t) := \bigcap_{F\in\mathfrak{R}} F(t) \tag{8}$$

for every $t \in T$.

Lemma 5.1.1. $\bar{F} \in \mathfrak{R}$.

Proof. Conditions (7a), (7b), and (7c) are satisfied trivially; only (7d) deserves some attention. Let $t' > t$, $y \in \bar{F}(t')$, and $x \in \bar{F}(t)$; then $y \in F(t')$ and $x \in F(t)$ for every $F \in \mathfrak{R}$. If $y \wedge x \in R(t)$, then $y \wedge x \in F(t)$ for every $F \in \mathfrak{R}$ by (7b) for F, hence $y \wedge x \in \bar{F}(t)$. Otherwise, $y \vee x \in F(t')$ for every $F \in \mathfrak{R}$ by (7d) for F, hence $y \vee x \in \bar{F}(t')$. Since y and x were arbitrary, $\bar{F}(t') \geq^{\mathrm{wV}} \bar{F}(t)$, hence $\bar{F} \in \mathfrak{R}$ indeed. $\square$

For every $F \in \mathfrak{R}$ and $t \in T$, we have $F^+(t) \neq \emptyset$ since $F(t)$ is chain-complete upwards.

Lemma 5.1.2. *If $t' > t$, $y \in F^+(t')$ and $x \in R^-(t)$, then $y \geq x$.*

Proof. Otherwise, we would have $y \vee x > y$ and $y \wedge x < x$; therefore, $y \vee x \notin F(t')$ and $y \wedge x \notin R(t)$, contradicting $F(t') \geq^{\mathrm{wV}} F(t) \subseteq R(t)$. □

For any $F \in \mathfrak{R}$, we define its *transformation* $F^\tau \colon T \to \mathfrak{B}_X$ by

$$F^\tau(t) := \{x \in F(t) \mid \forall t' > t\, \forall y \in F^+(t')\, [y \geq x]\}. \tag{9}$$

Lemma 5.1.3. *For every $F \in \mathfrak{R}$, there holds $F^\tau \in \mathfrak{R}$.*

Proof. Condition (7a) immediately follows from Lemma 5.1.2 and (9); (7b) is obvious. To check (7c), we fix $t \in T$ and consider a chain $Z \subseteq F^\tau(t)$; we have to show that $\sup Z \in F^\tau(t)$ (sup here means the least upper bound in $R(t)$; it may depend on t). Suppose $x := \sup Z \notin F^\tau(t)$, i.e., there are $t' > t$ and $y \in F^+(t')$ such that $y \not\geq x$; then we have $x > y \wedge x$. On the other hand, $y \wedge x \in F(t) \subseteq R(t)$ because $F(t') \geq^{\mathrm{wV}} F(t)$ and $y \in F^+(t')$; for every $z \in Z$, we have $x \geq z$ by the definition of x and $y \geq z$ because $z \in F^\tau(t)$, hence $y \wedge x \geq z$ as well. Clearly, we have a contradiction with the definition of x.

Finally, let us check (7d); let $t' > t$, $y \in F^\tau(t')$, and $x \in F^\tau(t)$. If $y \wedge x \in F(t)$, then $y \wedge x \in F^\tau(t)$ by (7b) for F^τ. Otherwise, $y \vee x \in F(t')$. For every $t'' > t$ and $z \in F^+(t'')$, we have $z \geq x$ because $x \in F^\tau(t)$, and $z \geq y$ because $y \in F^\tau(t')$. Therefore, $z \geq y \vee x$; since t'' and z were arbitrary, $y \vee x \in F^\tau(t')$. □

Lemma 5.1.4. *$\bar{F} = \bar{F}^\tau$.*

Immediately follows from (8), (9), and Lemma 5.1.3.

Finally, let $r \colon T \to X$ be an arbitrary selection from $\bar{F}^+$. Lemma 5.1.4 and (9) immediately imply that r is increasing.

5.2 Proof of Proposition 3.3

We denote $T^+ := \{t \in T \mid \nexists t' \in T\, [t' > t]\}$; for every $t \in T$, we denote $T^\downarrow(t) := \{t' \in T \mid t' < t\}$; for every $t \in T$ and $x^* \in X$, $R^\downarrow(t; x^*) := \{x \in R(t) \mid x \leq x^*\}$.

Lemma 5.2.1. *For every $t^* \in T^+$, there exists $x^* \in R(t^*)$ such that $R^\downarrow(t; x^*) \neq \emptyset$ for every $t \in T^\downarrow(t^*)$.*

Proof. For every $x \in R(t^*)$, we denote $Z^+(x) := \{t \in T^\downarrow(t^*) \mid R^\downarrow(t; x^*) \neq \emptyset\}$ and $Z^-(x) := T^- \setminus Z^+(x)$. Then we pick $x^0 \in R(t^*)$ arbitrarily and define a sequence $x^0, x^1, \dots$ by recursion. Let $x^k \in R(t^*)$ have been defined; if $Z^+(x^k) = T^\downarrow(t^*)$, we take x^k as x^* and finish the process. Otherwise, we pick $t \in Z^-(x^k)$ arbitrarily; we have $R(t^*) \geq^{\mathrm{wV}} R(t)$.

Picking $x \in R(t)$ arbitrarily, we apply (1f) with $y = x^k$. Since $R^{\downarrow}(t; x^k) = \emptyset$, we have $x \wedge x^k \notin R(t)$, hence $x \vee x^k \in R(t^*)$. Defining $x^{k+1} := x \vee x^k > x^k$, we obtain $Z^+(x^k) \subset Z^+(x^{k+1})$. Since T is finite, the sequence must stabilize at some stage, i.e., reach the situation $Z^+(x^k) = T^{\downarrow}(t^*)$. $\square$

Now the proposition is proven with straightforward induction in $\#T$. Picking $t^* \in T^+$ arbitrarily and x^* as in Lemma 5.2.1, we define $\bar{R}\colon T \setminus \{t^*\} \to X$ by $\bar{R}(t) := R^{\downarrow}(t; x^*)$ for $t \in T^{\downarrow}(t^*)$ and $\bar{R}(t) := R(t)$ otherwise. Obviously, $\bar{R}(t)$ is increasing w.r.t. $\geq^{\mathrm{wV}}$, hence it admits a monotone selection r by the induction hypothesis. Adding $r(t^*) := x^*$, we obtain a monotone selection from R on T.

5.3 Proof of Theorem 2

Let $X \subseteq \prod_{m \in M} C_m$, where each C_m is a chain. We assume M totally ordered, say, $M = \{0, 1, \ldots, \bar{m}\}$; then, invoking Zermelo's Theorem, we assume each C_m well ordered with an order $\gg_m$ (generally, having nothing to do with the basic order on C_m). Given $y \neq x$, we define $D(y, x) := \{m \in M \mid y_m \neq x_m\}$, $d := \min D(y, x)$, and $y \gg x \rightleftharpoons y_d \gg_d x_d$;

Lemma 5.3.1. *X is well ordered by $\gg$.*

Proof. Being a lexicographic combination, $\gg$ is clearly an order. For every $Y \in \mathfrak{B}_X$, we define $c_1 := \min\{y_1 \mid y \in Y\}$, $c_2 := \min\{y_2 \mid y \in Y \ \&\ y_1 = c_1\}$, etc. (the minima are w.r.t. $\gg_1$, $\gg_2$, etc.); then $(c_1, c_2, \ldots, c_{\bar{m}})$ is the minimum of Y w.r.t. $\gg$. $\square$

Lemma 5.3.2. *Let $x, y \in X$ and $y \ngeq x$. Then $x \gg y \wedge x \iff y \vee x \gg y$.*

Proof. Denoting $D^- := \{m \in M \mid y_m < x_m\}$, we immediately see that $D^- = D(y, y \vee x) = D(x, y \wedge x)$; let $d := \min D^-$. If $x_d \gg_d y_d$, then $x \gg y \wedge x$ and $y \vee x \gg y$. If $y_d \gg_d x_d$, then $y \gg y \vee x$ and $y \wedge x \gg x$. $\square$

Remark. We might say that $\gg$ is "*quasimodular.*"

Now we define $r(t) := \min R(t)$ (w.r.t. $\gg$); it exists and is unique. r is a selection from R by definition; let us show it is increasing. Let $t' > t$, $y = r(t')$ and $x = r(t)$; since R is increasing w.r.t. $\geq^{\mathrm{Vt}}$, we have $y \wedge x \in R(t)$ and $y \vee x \in R(t')$. If $y \ngeq x$, then $x \neq y \wedge x$ and $y \neq y \vee x$, hence $y \wedge x \gg x$ and $y \vee x \gg y$ by the definition of r. Thus, we have a contradiction with Lemma 5.3.2.

5.4 Proof of Theorem 3

We call a subset $T' \subseteq T$ an *interval* if $t \in T'$ whenever $t' < t < t''$ and $t', t'' \in T'$. The intersection of any number of intervals is an interval too. Till the end of the proof, we denote $[t', t'']$ the least interval containing both t' and t'' (thus $[t', t''] = [t'', t']$).

Lemma 5.4.1. *For every $x \in X$, the set $\{t \in T \mid x \in R(t)\}$ is an interval.*

Proof. Suppose the contrary: $t' < t < t''$ and $x \in R(t') \cap R(t'')$, but $x \notin R(t)$. By Proposition 4.1, we can pick $x^* \in R(t)$ such that $x^* \succ^t x$. If $x^* > x$, we have $x^* \succ^{t''} x$ by (4a), contradicting the assumed $x \in R(t'')$. If $x^* < x$, we have $x^* \succ^{t'} x$ by (4b) with the same contradiction. $\square$

The key role is played by the following recursive definition of sequences $x^k \in X$, $t^k \in T$, and $T^k \subseteq T$ $(k \in \mathbb{N})$ such that:

$$t^k \in T^k; \tag{10a}$$

$$T^k \text{ is an interval;} \tag{10b}$$

$$\forall t \in T^k \left[x^k \in R(t)\right]; \tag{10c}$$

$$\forall m < k \left[T^k \cap T^m = \emptyset\right]; \tag{10d}$$

$$\forall m < k \left[[t^k < t^m \Rightarrow x^k < x^m] \;\&\; [t^k > t^m \Rightarrow x^k > x^m]\right]; \tag{10e}$$

$$\forall m < k \left[x^k \succ^{t^k} x^m \;\text{ or }\; x^m \in R(t^k)\right]; \tag{10f}$$

$$\forall t \in T \left[[x^k \in R(t) \;\&\; t \notin T^k] \Rightarrow \exists m < k \left(t \in T^m \;\text{ or }\; t^m \in [t, t^k]\,\right)\right]. \tag{10g}$$

We start with an arbitrary $t^0 \in T$, pick $x^0 \in R(t^0)$, and set $T^0 := \{t \in T \mid x^0 \in R(t)\}$. Now (10a), (10c), and (10g) for $k = 0$ immediately follow from the definitions; (10b), from Lemma 5.4.1; (10d), (10e), and (10f) hold by default.

Let $k \in \mathbb{N} \setminus \{0\}$, and let x^m, t^m, T^m satisfying (10) have been defined for all $m < k$. We define $\bar{T}^k := \bigcup_{m<k} T^m$. For every $t \in \bar{T}^k$, there is a unique, by (10d), $\mu(t) < k$ such that $t \in T^{\mu(t)}$. By (10c), $r(t) := x^{\mu(t)}$ is a selection from R on $\bar{T}^k$. The conditions (10b) and (10e) imply that r is increasing. If $\bar{T}^k = T$, then we already have a monotone selection, so we stop the process.

Otherwise, we pick $t^k \in T \setminus \bar{T}^k$ arbitrarily and denote $K^- := \{m < k \mid t^m < t^k\}$, $K^+ := \{m < k \mid t^m > t^k\}$ $K^* := \{m < k \mid x^m \notin R(t^k)\}$, $m^- := \operatorname{argmax}_{m \in K^-} t^m$, $m^+ := \operatorname{argmin}_{m \in K^+} t^m$, and $I := [t^{m^-}, t^{m^+}]$. If one of $K^\pm$ is empty (both cannot be), the respective $m^\pm$ is left undefined, in which case $I := \{t \in T \mid t^{m^-} < t\}$ or $I := \{t \in T \mid t < t^{m^+}\}$.

By Proposition 4.2, we can pick $x^k \in R(t^k)$ such that $x^k \succ^{t^k} x^m$ for each $m \in K^*$, hence (10f) holds. Finally, we define $T^k := \{t \in T \setminus \bar{T}^k \mid x^k \in R(t)\} \cap I$. Now the conditions (10a), (10c), and (10d) immediately follow from the definitions; (10b) and (10g), from Lemma 5.4.1.

Checking (10e) needs a bit more effort. If we assume that $x^{m^-} \in R(t^k)$, then the condition (10g) for m^- and t^k implies the existence of $m < m^-$ such that $t^{m^-} < t^m < t^k$, contradicting the definition of m^-; therefore, $x^k \succ^{t^k} x^{m^-}$ by (10f). If $x^k < x^{m^-}$ then $x^k \succ^{t^{m^-}} x^{m^-}$ by (4b), contradicting (10c) for m^-. Therefore, $x^k > x^{m^-} \geq x^m$ for all

$m \in K^-$. A dual argument shows that $x^k < x^{m^+} \leq x^m$ for all $m \in K^+$. Thus, (10e) holds.

To summarize, either we obtain a monotone selection on some step, or our sequences are defined [and satisfy (10)] for all $k \in \mathbb{N}$.

Lemma 5.4.2. *If conditions* (10) *hold for all* $k \in \mathbb{N}$, *then there exists an increasing sequence* $\langle k_h \rangle_{h \in \mathbb{N}}$ *such that* t^{k_h} *is either monotone increasing or monotone decreasing in* h, *and* $x^{k_{h+1}} \succ^{t^{k_{h+1}}} x^{k_h}$ *for each* $h \in \mathbb{N}$.

Proof. We denote $\mathbb{N}^{\downarrow}$, respectively, $\mathbb{N}^{\uparrow}$, the set of $k \in \mathbb{N}$ such that $t^m < t^k$, or $t^m > t^k$, holds for an infinite number of $m \in \mathbb{N}$. Clearly, $\mathbb{N} = \mathbb{N}^{\downarrow} \cup \mathbb{N}^{\uparrow}$; without restricting generality, $\mathbb{N}^{\downarrow} \neq \emptyset$. We consider two alternatives.

Let there exist $\min\{t^k \mid k \in \mathbb{N}^{\downarrow}\} = t^*$; then the set $\{m \in \mathbb{N} \mid t^m < t^k\}$ is finite for every $t^k < t^*$, hence the set $\{m \in \mathbb{N} \mid t^k < t^m < t^*\}$ is infinite. We define $k_0 := \min\{k \in \mathbb{N} \mid t^k < t^*\}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}$ for which $t^{k_h} < t^k < t^*$. The minimality of k_h ensures that $k_{h+1} > k_h$. Whenever $t^{k_h} < t^m < t^{k_{h+1}}$, we have $m > k_{h+1}$ by the same minimality; therefore, $x^{k_h} \notin R(t^{k_{h+1}})$ by (10g), hence $x^{k_{h+1}} \succ^{t^{k_{h+1}}} x^{k_h}$ by (10f).

Let $\min\{t^k \mid k \in \mathbb{N}^{\downarrow}\}$ not exist; then the set $\{m \in \mathbb{N}^{\downarrow} \mid t^m < t^k\}$ is nonempty (actually, infinite) for every $k \in \mathbb{N}^{\downarrow}$. We set $k_0 := \min \mathbb{N}^{\downarrow}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}^{\downarrow}$ for which $t^k < t^{k_h}$. The minimality of k_h again ensures that $k_{h+1} > k_h$. Whenever $t^{k_{h+1}} < t^m < t^{k_h}$, we have $m \in \mathbb{N}^{\downarrow}$, hence $m > k_{h+1}$; therefore, $x^{k_h} \notin R(t^{k_{h+1}})$ by (10g), hence $x^{k_{h+1}} \succ^{t^{k_{h+1}}} x^{k_h}$ by (10f). □

The final step of the proof consists in showing that the existence of a sequence described in Lemma 5.4.2 contradicts the strong acyclicity assumption. We denote $t^- := \min T$ and $t^+ := \max T$. If t^{k_h} is increasing, the relations $x^{k_{h+1}} \succ^{t^{k_{h+1}}} x^{k_h}$ "translate," by (4a), to $x^{k_{h+1}} \succ^{t^+} x^{k_h}$ for each $h \in \mathbb{N}$. If t^{k_h} is decreasing, we obtain $x^{k_{h+1}} \succ^{t^-} x^{k_h}$ for each $h \in \mathbb{N}$ by (4b).

5.5 Proof of Theorem 4

The key role is played by the following recursive definition of a sequence $x_N^k \in X_N$ ($k \in \mathbb{N}$) such that $x_N^{k+1} \geq x_N^k$ and $x_i^{k+1} \in R_i(x_{-i}^k)$ for all $k \in \mathbb{N}$ and $i \in N$. By the latter condition, x_N^k is a Nash equilibrium if $x_N^{k+1} = x_N^k$. On the other hand, the sequence must stabilize at some stage because of the strong acyclicity assumption.

We define $x_i^0 := \min X_i$ for each $i \in N$. Given x_N^k, we, for each $i \in N$ independently, check whether $x_i^k \in R_i(x_{-i}^k)$ holds. If it does, we define $x_i^{k+1} := x_i^k$; otherwise, we pick $x_i^{k+1} \in R_i(x_{-i}^k)$ such that $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ (it exists by Proposition 4.1). Supposing $x_i^{k+1} < x_i^k$ (hence $k > 0$), we obtain $x_i^{k+1} \succ_i^{x_{-i}^{k-1}} x_i^k$ by (4b), contradicting the induction hypothesis $x_i^k \in R_i(x_{-i}^{k-1})$. Therefore, $x_i^{k+1} > x_i^k$, hence $x_N^{k+1} \geq x_N^k$.

Supposing that $x_N^{k+1} > x_N^k$ for all $k \in \mathbb{N}$, we denote $x_i^\omega := \sup_k x_i^k$ for each $i \in N$; the completeness of X_i is essential here. Whenever $x_i^{k+1} \neq x_i^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ and $x_i^{k+1} > x_i^k$ as was shown in the previous paragraph; since $x_{-i}^\omega \geq x_{-i}^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^\omega} x_i^k$ by (4a). Since N is finite, there must be $i \in N$ such that $x_i^{k+1} > x_i^k$ for an infinite number of k. Clearly, the elimination of repetitions in the sequence $\langle x_i^k \rangle_k$ makes it an infinite improvement path for the relation $\succ_i^{x_{-i}^\omega}$, which contradicts the supposed strong acyclicity.

Remark. There is an obvious similarity with the Algorithm II of Topkis (1979); X_i need not be chains there because the assumptions on preferences are much stronger. An analog of Topkis's Algorithm I could work here as well, but then the proof would be a bit longer.

6 Concluding remarks

6.1. We may say that a correspondence $R\colon T \to \mathfrak{B}_X$ *admits enough monotone selections* if, whenever $t \in T$ and $x \in R(t)$, there is a monotone selection r from R such that $r(t) = x$. It is easily seen from the proofs that a correspondence R satisfying the assumptions of Theorem 2 or Theorem 3 admits enough monotone selections; a similar assertion about Theorem 1 would be wrong as simple finite examples show.

6.2. It is not necessary in Theorem 1 for $R(t)$ to be chain-subcomplete in X as was assumed in Theorem 3.2 of Veinott (1989); X itself need not be complete. This gain in generality is hardly needed in any context, but it comes at no cost.

6.3. The function f in Theorem A4 can be replaced with a quasisupermodular preference ordering without any significant change in the proof; only a straightforward modification of conditions (2) is needed. Actually, both quasisupermodularity and (2) can be weakened considerably at the price of a longer proof. A plausible conjecture that a quasisupermodular ordering on a complete lattice attains its maximum if it attains a maximum on every subcomplete chain remains neither proven nor disproved.

6.4. To the best of my knowledge, there is no example of an ascending correspondence without a monotone selection. However, the current proof of Theorem 2 is useless even if X is a sublattice of the Cartesian product of an infinite number of chains, to say nothing of a non-distributive lattice. Vladimir Danilov (personal communication, 2007) proved the theorem for an arbitrary lattice X, but a countable poset T. On the other hand, strategy sets in economic models are often sublattices of $\mathbb{R}^m$, so Theorem 2 is sufficient to prove the existence of a Nash equilibrium under strategic complementarity; only the existence of the best responses is needed, and not the completeness of $R_i(x_{-i})$.

6.5. It is immediately seen from the proof of Theorem 3 that there exists a monotone selection r from R with a finite range $r(T)$. Actually, the theorem remains valid without the existence of $\min T$ or $\max T$, but then the finiteness of $r(T)$ can no longer be asserted.

This extension is not included here because it is useless for game-theoretic applications (there is virtually no fixed point theorem without completeness), while requiring a significantly longer proof.

6.6. It remains unclear whether the assumption that both X and T are chains can be dropped or weakened in Theorem 3. From the game-theoretic viewpoint, however, the question does not seem pressing. The existence of an ε-Nash equilibrium in a game with increasing best responses may hold in the absence of monotone selections as Theorem 4 and Example 4.4 demonstrate. If the best responses are, say, decreasing, then, indeed, all existence results in the literature need monotone selections, but they also need the strategies effectively be scalar and each player be only affected by a scalar aggregate of the partners/rivals' choices.

6.7. The assumption in Theorem 4 that each X_i is a chain is strong enough to be extremely irritating; however, I have no idea at the moment whether and how it could be dispensed with.

References

Abian, S., and A.B. Brown, 1961. A theorem on partially ordered sets with applications to fixed point theorems. Canadian Journal of Mathematics 13, 78–83.

Dasgupta, P., and E. Maskin, 1986. The existence of equilibrium in discontinuous games, I: Theory. Review of Economic Studies 53, 1–26.

Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

Echenique, F., 2002. Comparative statics by adaptive dynamics and the correspondence principle. Econometrica 70, 833–844.

Heikkilä, S., and K. Reffett, 2006. Fixed point theorems and their applications to the theory of Nash equilibria. Nonlinear Analysis 64, 1415–1436.

Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Economics Letters 46, 23–26.

Kukushkin, N.S., 2000. Potentials for binary relations and systems of monotonic reactions. Doklady Akademii Nauk 373(1), 23–25. (in Russian; an English translation in Doklady. Mathematics 2000, 62(1), 16–18).

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

Kukushkin, N.S., 2007. Monotonicity-based conditions for the acyclicity of Cournot tatonnement. Doklady Akademii Nauk 413(4), 457–460. (in Russian; an English translation in Doklady. Mathematics 2007, 75(2), 273–276).

McManus, M., 1962. Numbers and size in Cournot oligopoly, Bulletin of Economic and Social Research 14, 14–22.

McManus, M., 1964. Equilibrium, number and size in Cournot oligopoly. Bulletin of Economic and Social Research 16, 68–75.

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

Quah, J., 2007. The comparative statics of constrained optimization problems. Econometrica 75, 401–431.

Quah, J., and B. Strulovici, 2007. Comparative statics, informativeness, and the interval dominance order. Oxford University, Nuffield College. Working Paper 2007-W04. Available at http://www.nuffield.ox.ac.uk/economics/papers/2007/w4/hlp27.pdf

Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.

Roddy, M.S., and B.S.W. Schröder, 2005. Isotone relations revisited. Discrete Mathematics 290, 229–248.

Shannon, C., 1990. An ordinal theory of games with strategic complementarities. Department of Economics, Stanford University. Available at http://elsa.berkeley.edu/users/cshannon/wp/qsm-org.pdf

Smith, T.E., 1974. On the existence of most-preferred alternatives. International Economic Review 15, 184–194.

Smithson, R.E., 1971. Fixed points of order preserving multifunctions. Proceedings of the American Mathematical Society 28, 304–310.

Tarski, A., 1955. A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics 5, 285–309.

Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787.

Topkis, D.M., 1998. Supermodularity and Complementarity. Princeton University Press, Princeton.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.