Comment: The Identification Power of Equilibrium in Simple Games

Victor Aguirregabiria

Department of Economics. University of Toronto

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Victor Aguirregabiria
Department of Economics. Unversity of Toronto.
150 St. George Street, Toronto, Ontario M5S 3G7, Canada.
E-mail: victor.aguirregabiria@utoronto.ca

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Abstract

This paper studies the identification of structural parameters in dynamic games when we replace the assumption of Markov Perfect Equilibrium (MPE) with weaker conditions such as rational behavior and *rationalizability*. The identification of players' time discount factors is of especial interest. I present identification results for a simple two-periods/two-players dynamic game of market entry-exit. Under the assumption of level-2 rationality (i.e., players are rational and they know that they are rational), a exclusion restriction and a large-support condition on one of the exogenous explanatory variables are sufficient for point-identification of all the structural parameters.

Keywords: Identification, Empirical dynamic discrete games, Rational behavior, Rationalizability.

1 Introduction

Structural econometric models of individual or firm behavior typically assume that agents are rational in the sense that they maximize expected payoffs given their subjective beliefs about uncertain events. Empirical applications of game theoretic models have used stronger assumptions than rationality. Most of these studies apply the Nash equilibrium solution, or some of its refinements, to explain agents' strategic behavior. The Nash equilibrium (NE) concept is based on assumptions on players' knowledge and beliefs which are more restrictive than rationality. Though there is not a set of necessary conditions to generate the NE outcome, the set of sufficient conditions typically includes the assumption that players' actions are common knowledge. For instance, Aumann and Brandenburger (1995) show that mutual knowledge of payoff functions and of rationality, and common knowledge of the conjectures (actions), imply that the conjectures form a NE. This assumption on players' knowledge and beliefs may be unrealistic in some applications. Therefore, it is relevant to study whether the *principle of revealed preference* can identify the parameters in players' payoffs under weaker conditions than NE. For instance, we would like to know if rationality is sufficient for identification. It is also relevant to study the identification power of other assumptions which are stronger than rationality but weaker than NE, such as common knowledge rationality: i.e., everybody knows that players are rational; everybody knows that everybody knows that players are rational, etc. Common knowledge rationality is closely related to the solution concepts *iterated strict dominance* and *rationalizability* (see chapter 2 in Fudenberg and Tirole, 1991).

The paper by Andres Aradillas-Lopez and Elie Tamer (2008) is the first study that deals with these interesting identification issues. The authors study the identification power of rational behavior and rationalizability in three classes of static games which have received significant attention in empirical applications: binary choice games, with complete and with incomplete information, and auction games with independent private values. Their paper contributes to the literature on identification of incomplete econometric models, i.e., models that do not provide unique predictions on the distribution of endogenous variables (see also Tamer, 2003, and Haile and Tamer, 2003). Aradillas-Lopez and Tamer's paper shows that standard exclusion restrictions and large-support conditions are sufficient to identify structural parameters despite the non-uniqueness of the model predictions. Though structural parameters can be point-identified, the researcher still faces an identification issue when he uses the estimated model to perform counterfactual experiments. Players' behavior under the counterfactual scenario is not point-identified. This problem also appears in models with multiple equilibria. However, a nice feature of Aradillas-Lopez and Tamer's approach is that, at least for the class of models that they consider, it is quite simple to obtain bounds of the model predictions under the counterfactual scenario.

The main purpose of this paper is to study the identification power of rational behavior and rationalizability in a class of empirical games that has not been analyzed in Aradillas-Lopez and Tamer's paper: dynamic discrete games. Dynamic discrete games are of interest in economic applications where agents interact over several periods of time and make decisions that affect their future payoffs. In static games of incomplete information, players form beliefs on the probability distribution of their opponents' actions. In dynamic games, players should also form beliefs on the probability distribution of players' future actions, including their own future actions, and on the distribution of future exogenous state variables. The most common equilibrium concept in applications of dynamic games is Markov Perfect Equilibrium (MPE). As in the case of NE, the concept of MPE is based on strong assumptions on players' knowledge and believes. MPE assumes that players maximize expected intertemporal payoffs and have rational expectations, and that players' strategies are common knowledge. In this paper, I maintain the assumption that every player knows his own strategy function and has rational expectations on his own future actions. However, I relax the assumption that players' strategies are common knowledge. I study the identification of structural parameters, including players' time discount factors, when we replace the assumption of common knowledge strategies with weaker conditions such as rational behavior.

I present identification results for a simple two-periods/two-players dynamic game of market entry-exit. Under the assumption of level-2 rationalizability (i.e., players are rational and they know that they are rational), a exclusion restriction and a large-support condition on one of the exogenous explanatory variables are sufficient for point-identification of all the structural parameters, including time discount factors.

2 Dynamic discrete games

2.1 Model and basic assumptions

There are two firms which decide whether to operate or not in a market. I use the indexes $i \in \{1,2\}$ and $j \in \{1,2\}$ to represent a firm and its opponent, respectively. Time is discrete an indexed by $t \in \{1,2,...,T\}$, where T is the time horizon. Let $Y_{it} \in \{0,1\}$ be the indicator of the event "firm i is active in the market at period t". Every period t the two firms decide simultaneously whether to be active in the market or not. A firm makes this decision to maximize its expected and discounted profits $E_t\left(\sum_{s=0}^{T-t} \delta_i^s \Pi_{i,t+s}\right)$, where $\delta_i \in (0,1)$ is the firm's discount factor and Π_{it} is its profit at period t. The decision to be active in the market has implications not only on a firm's current profits but also on its expected future profits. More specifically, there is an entry cost that should be paid only if a currently active firm was not active at previous period. Therefore, a firm's incumbent status (or lagged entry decision) affects current profits. The one-period profit function is:

$$\Pi_{it} = \begin{cases} \mathbf{Z}_i \ \boldsymbol{\eta}_{it} + \gamma_{it} \ Y_{i,t-1} + \alpha_{it} \ Y_{jt} - \varepsilon_{it} & if \quad Y_{it} = 1 \\ 0 & if \quad Y_{it} = 0 \end{cases} \tag{1}$$

Y_{jt} represents the opponent's entry decision. $\mathbf{Z}_i$ is a vector of time-invariant, exogenous market and firm characteristics that affect firm i's profits. $\boldsymbol{\eta}_{it}$, γ_{it} and α_{it} are parameters. The parameter $\gamma_{it} \geq 0$ represents firm i's entry cost at period t. The parameter $\alpha_{it} \leq 0$ captures the competitive effect. At period t, firms know the variables $\{Y_{1,t-1}, Y_{2,t-1}, \mathbf{Z}_1, \mathbf{Z}_2\}$ and the parameters $\{\boldsymbol{\eta}_{1t}, \boldsymbol{\eta}_{2t}, \gamma_{1t}, \gamma_{2t}, \alpha_{1t}, \alpha_{2t}\}$. For the sake of simplicity, I also assume that firms know without any uncertainty future values of the parameters $\{\boldsymbol{\eta},\gamma,\alpha\}$. The vector $\boldsymbol{\theta}$ represents the whole sequence of parameters form period 1 to T. The variable ε_{it} is private information of firm i at period t. A firm has uncertainty on the current value of his opponent's ε, and on future values of both his own and his opponent's $\varepsilon' s$. The variables ε_{1t} and ε_{2t} are independent of $(\mathbf{Z}_1, \mathbf{Z}_2)$, independent of each other, and independently and identically distributed over time. Their distribution functions, H_1 and H_2, are absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}$.

2.2 Rational forward-looking behavior

The literature on estimation of dynamic discrete games has applied the concept of Markov Perfect Equilibrium (MPE). This equilibrium concept assumes that: (1) players' strategy functions depend only on payoff relevant state variables; (2) players are forward looking, maximize expected intertemporal payoffs, have rational expectations, and know their own strategy functions; and (3) players' strategy functions are common knowledge. The concept of rational behavior that I consider here maintains assumptions (1) and (2), but it relaxes condition (3).

Let $\mathbf{X}_t$ be the vector with all the payoff-relevant and common knowledge state variables at period t: $\mathbf{X}_t \equiv (Y_{i,t-1}, Y_{j,t-1}, \mathbf{Z}_i, \mathbf{Z}_j)$. The information set of player i is $\{\mathbf{X}_t, \varepsilon_{it}\}$. Let $\sigma_{it}(\mathbf{X}_t, \varepsilon_{it})$ be a strategy function for player i at period t. This is a function from the support of $(\mathbf{X}_t, \varepsilon_{it})$ into the binary set $\{0, 1\}$. Associated with any strategy function σ_{it} we can define a probability function $P_{it}(\mathbf{X}_t)$ that represents the probability of $Y_{it} = 1$ conditional on $\mathbf{X}_t$ and on player i following strategy σ_{it}. That is, $P_{it}(\mathbf{X}_t) \equiv \int I\left\{\sigma_{it}\left(\mathbf{X}_t, \varepsilon_{it}\right) = 1\right\} dH_i\left(\varepsilon_{it}\right)$, where $I\{.\}$ is the indicator function. It will be convenient to represent players' behavior and beliefs using these *conditional choice probability* (CCP) functions. The CCP function $P_{jt}(\mathbf{X}_t)$ represents firm i's beliefs on the probability that firm j will be active in the market at period t if current state is $\mathbf{X}_t$. I use $\mathbf{P}_j$ to represent the sequence of CCPs $\{P_{jt}(.) : t = 1, 2, ..., T\}$. Therefore, $\mathbf{P}_j$ contains firm i's beliefs on his opponent's current and future behavior.

A strategy function $\sigma_{it}(\mathbf{X}_t, \varepsilon_{it})$ is *rational* if for every possible value of $(\mathbf{X}_t, \varepsilon_{it})$ the action $\sigma_{it}(\mathbf{X}_t, \varepsilon_{it})$ maximizes player i's expected and discounted sum of current and future payoffs, given his beliefs on the opponent's strategies.

For the rest of the paper, I concentrate on a two-period version of this game: $T = 2$. Let $\mathbf{P}_j \equiv \{P_{j1}(.), P_{j2}(.)\}$ be firm i's beliefs on the probabilities that firm j will be active at periods 1 and 2. At the last period, firms play a static game, and the definition of a rational strategy is the same as in a static game. Therefore, $\sigma_{i2}(\mathbf{X}_2, \varepsilon_{i2})$ is a *rational strategy function* for firm i at period 2 if $\sigma_{i2}(\mathbf{X}_2, \varepsilon_{i2}) = I\left\{\varepsilon_{i2} \leq \Delta_{i2}^{\mathbf{P}_j}(\mathbf{X}_2)\right\}$, where the threshold function $\Delta_{i2}^{\mathbf{P}_j}(\mathbf{X}_2)$ is the difference between the expected payoff of being in the market and the payoff of not being in the market at period 2. That is,

$$\Delta_{i2}^{\mathbf{P}_j}(\mathbf{X}_2) \equiv \mathbf{Z}_i\ \boldsymbol{\eta}_{i2} + \gamma_{i2}\ Y_{i1} + \alpha_{i2}\ P_{j2}(\mathbf{X}_2) \tag{2}$$

Now, consider the game at period 1. The strategy function $\sigma_{i1}(\mathbf{X}_1, \varepsilon_{i1})$ is rational if $\sigma_{i1}(\mathbf{X}_1, \varepsilon_{i1}) = I\left\{\varepsilon_{i1} \leq \Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)\right\}$, where the threshold function $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ represents the difference between the expected value of firm i if he is active at period 1 minus its value if it is not active, given that firm i behaves optimally in the future and that he believes that his opponent's CCP function is $\mathbf{P}_j$. That is,

$$\begin{aligned}\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1) &\equiv \mathbf{Z}_i\boldsymbol{\eta}_{i1} + \gamma_{i1}Y_{i0} + \alpha_{i1}\ P_{j1}(\mathbf{X}_1) + \delta_i\ P_{j1}(\mathbf{X}_1)\left[V_{i2}^{\mathbf{P}_j}(1,1) - V_{i2}^{\mathbf{P}_j}(0,1)\right] \\ &+ \delta_i\left(1 - P_{j1}(\mathbf{X}_1)\right)\left[V_{i2}^{\mathbf{P}_j}(1,0) - V_{i2}^{\mathbf{P}_j}(0,0)\right]\end{aligned} \tag{3}$$

where $V_{i2}^{\mathbf{P}_j}(\mathbf{X}_2)$ is firm i's value function at period 2 averaged over ε_{i2}, i.e., $V_{i2}^{\mathbf{P}_j}(\mathbf{X}_2) \equiv \int \max\{0\ ;\ \mathbf{Z}_i\boldsymbol{\eta}_{i2} + \gamma_{i2}Y_{i1} + \alpha_{i2}\ P_{j2}(\mathbf{X}_2) - \varepsilon_{i2}\}\, dH_i(\varepsilon_{i2})$. According to this definition of rational strategy function, we say that the CCP functions $P_{i1}(.)$ and $P_{i2}(.)$ are rational for firm i if, given beliefs $\mathbf{P}_j$, we have that:

$$P_{it}(\mathbf{X}_t) \quad = \quad H_i\left(\Delta_{it}^{\mathbf{P}_j}(\mathbf{X}_t)\right) \qquad \text{for } t = 1, 2 \tag{4}$$

At the last period, the game is static and it has the same structure as in Aradillas-Lopez and Tamer (2008). Therefore, the derivation of rationalizability bounds on $P_{i2}(\mathbf{X}_2)$, and the conditions for set- and point-identification of $\{\boldsymbol{\eta}_{i2}, \gamma_{i2}, \alpha_{i2}\}$ are the same as in that paper. Section 2.3 discusses two important properties of the threshold functions $\Delta_{it}^{\mathbf{P}_j}(\mathbf{X}_t)$. Section 2.4 derives rationalizability bounds on $P_{i1}(\mathbf{X}_1)$. Section 3 shows how these bounds can be used to identify the parameters $\{\delta_i, \boldsymbol{\eta}_{i1}, \gamma_{i1}, \alpha_{i1}\}$.

2.3 Two important properties of the threshold functions

The assumption of rationality (or of level-k rationality) implies informative bounds on players' behavior only if the effect of beliefs $\mathbf{P}_j$ on the threshold function $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ is bounded with probability one. Otherwise, the best response probability of an arbitrarily pessimistic (optimistic) rational player would be zero (one) with probability one. In Aradillas-Lopez and Tamer's static game this condition holds if the parameters take finite values. In our finite horizon dynamic model this condition is also necessary and sufficient. If the parameters $\{\delta_i, \boldsymbol{\eta}_{i1}, \boldsymbol{\eta}_{i2}, \gamma_{i1}, \gamma_{i2}, \alpha_{i1}, \alpha_{i2}\}$ take finite values, then there are two finite constants, c_i^{low} and c_i^{high}, such that for any belief $\mathbf{P}_j$ and any finite value of $\mathbf{X}_1$ the threshold function $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ is bounded by these constants: $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1) \in \left[c_i^{low}, c_i^{high}\right]$. For an infinite horizon dynamic game (i.e., $T = \infty$), we also need the discount factor δ_i to be smaller than one.

The recursive derivation of rationality bounds in Aradillas-Lopez and Tamer's static game is particularly simple because the expected payoff function is strictly monotonic in beliefs $\mathbf{P}_j$. This monotonicity condition is not really needed for identification, but it simplifies the analysis and, likely, the estimation procedure. In our two-period game, $\Delta_{i2}^{\mathbf{P}_j}(\mathbf{X}_2)$ is a non-increasing function of $P_{j2}(\mathbf{X}_t)$ if and only if $\alpha_{i2} \leq 0$. However, the monotonicity of $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ with respect to $P_{j1}(\mathbf{X}_1)$ does not follow simply from the restrictions $\alpha_{i1} \leq 0$ and $\alpha_{i2} \leq 0$. Restrictions on other parameters, or on beliefs, are needed to satisfy this monotonicity condition. At period 1 we have that:

$$\frac{\partial \Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)}{\partial P_{j1}(\mathbf{X}_1)} = \alpha_{i1} + \delta_i \left(V_{i2}^{\mathbf{P}_j}(1,1) - V_{i2}^{\mathbf{P}_j}(0,1) - V_{i2}^{\mathbf{P}_j}(1,0) + V_{i2}^{\mathbf{P}_j}(0,0) \right) \tag{5}$$

It is clear that $\alpha_{i1} \leq 0$ is not sufficient for $\Delta_{i1}^{\mathbf{P}_j}$ to be a non-increasing function of $P_{j1}(\mathbf{X}_1)$. We also need the value function $V_{i2}^{\mathbf{P}_j}(Y_{i1}, Y_{j1})$ to be not "too" super-modular. That is,

$V_{i2}^{\mathbf{P}_j}(1,1) - V_{i2}^{\mathbf{P}_j}(0,1) - V_{i2}^{\mathbf{P}_j}(1,0) + V_{i2}^{\mathbf{P}_j}(0,0)$ should be either negative (i.e., $V_{i2}^{\mathbf{P}_j}$ is sub-modular) or positive but not larger than $-\alpha_{i1}/\delta_i$ (i.e., $V_{i2}^{\mathbf{P}_j}$ is super-modular but not "too much"). In order to derive sufficient conditions, it is important to take into account that $V_{i2}^{\mathbf{P}_j}(\mathbf{X}_2) \equiv G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2} + \gamma_{i2}Y_{i1} + \alpha_{i2}\ P_{j2}(\mathbf{X}_2))$ where the function $G_i(a)$ is $E_{\varepsilon_i}(\max\{0; a - \varepsilon_i\})$. This function has the following properties: it is continuously differentiable; its first derivative is $H_i(a) \in (0,1)$; it is convex; $\lim_{a \to -\infty} G_i(a) = 0$; $\lim_{a \to +\infty} G_i(a) - a = 0$; and for any positive constant b, we have that $G_i(a+b) - G_i(a) < b$. There are different sets of sufficient conditions for $\partial \Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1) / \partial P_{j1}(\mathbf{X}_1) \leq 0$. For instance, a simple set of conditions is $\alpha_{i1} \leq 0$, $\alpha_{i2} \leq 0$, and $\alpha_{i1} - 2\delta_i\alpha_{i2} \leq 0$. Other set of conditions is $\alpha_{i1} \leq 0$, $\alpha_{i2} \leq 0$, firm i believes that *ceteris paribus* it is more likely that the opponent's will be active at period 2 if it was active at period 1 (i.e., $P_{j2}(Y_{i1}, 1) \geq P_{j2}(Y_{i1}, 0)$ for $Y_{i1} = 0, 1$), and $\alpha_{i1} - \delta_i\alpha_{i2} \leq 0$. For the rest of the paper, I assume that $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ is non-increasing in $P_{j1}(\mathbf{X}_1)$.

2.4 Bounds with forward-looking rationality

Let $k \in \{0, 1, 2, ...\}$ be the index of the level of rationality of both players. I define $P_{it}^{L,k}(\mathbf{X}_t)$ and $P_{it}^{U,k}(\mathbf{X}_t)$ as the lower and the upper bound, respectively, for player i's CCP at period t under level-k rationality. Level-0 rationality does not impose any restriction and therefore $P_{it}^{L,0}(\mathbf{X}_t) = 0$ and $P_{it}^{U,0}(\mathbf{X}_t) = 1$ for any state $\mathbf{X}_t$. For the last period, $t = 2$, the derivation of the probability bounds is exactly the same as in the static model. Therefore, for $k \geq 1$:

$$
\begin{aligned}
P_{i2}^{L,k}(\mathbf{X}_2) &= H_i\left(\mathbf{Z}_i\ \boldsymbol{\eta}_{i2} + \gamma_{i2}\ Y_{i1} + \alpha_{i2}\ P_{j2}^{U,k-1}(\mathbf{X}_2)\right) \\
P_{i2}^{U,k}(\mathbf{X}_2) &= H_i\left(\mathbf{Z}_i\ \boldsymbol{\eta}_{i2} + \gamma_{i2}\ Y_{i1} + \alpha_{i2}\ P_{j2}^{L,k-1}(\mathbf{X}_2)\right)
\end{aligned}
\tag{6}
$$

The rest of this subsection derives a recursive formula for the probability bounds at period 1. Let Π_j^k be the set of player j's CCPs (at periods 1 and 2) which are consistent with level-k rationality. By definition, level-k rationality bounds at period 1 are $P_{i1}^{L,k}(\mathbf{X}_1) =$

$H_i\left(\Delta_{i1}^{L,k}(\mathbf{X}_1)\right)$ and $P_{i1}^{U,k}(\mathbf{X}_1) = H_i\left(\Delta_{i1}^{U,k}(\mathbf{X}_1)\right)$, where:

$$\begin{aligned}\Delta_{i1}^{L,k}(\mathbf{X}_1) &\equiv \min_{\mathbf{P}_j \in \Pi_j^{k-1}} \left\{\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)\right\} \\ \Delta_{i1}^{U,k}(\mathbf{X}_1) &= \max_{\mathbf{P}_j \in \Pi_j^{k-1}} \left\{\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)\right\}\end{aligned} \tag{7}$$

Given the monotonicity of $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ with respect to $\mathbf{P}_j$, the minimum and the maximum of $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ are reached at the boundaries of the set Π_j^k. More specifically, it is possible to show that the value of (P_{j1}, P_{j2}) that minimizes $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ is:

$$\left\{P_{j1}^{U,k-1}(\mathbf{X}_1)\ ;\ P_{j2}^{U,k-1}(1,1)\ ;\ P_{j2}^{U,k-1}(1,0)\ ;\ P_{j2}^{L,k-1}(0,1)\ ;\ P_{j2}^{L,k-1}(0,0)\right\} \tag{8}$$

That is, the most pessimistic belief for firm i (i.e., the one that minimizes $\Delta_{i1}^{\mathbf{P}_j}$) is such that the probability that the opponent is active at period 1 takes its maximum value, and when firm i decides to be active (inactive) at period 1, the probability that the opponent is active at period 2 takes its maximum (minimum) value. Similarly, the value of (P_{j1}, P_{j2}) that maximizes $\Delta_{i1}^{\mathbf{P}_j}(\mathbf{X}_1)$ is:

$$\left\{P_{j1}^{L,k-1}(\mathbf{X}_1)\ ;\ P_{j2}^{L,k-1}(1,1)\ ;\ P_{j2}^{L,k-1}(1,0)\ ;\ P_{j2}^{U,k-1}(0,1)\ ;\ P_{j2}^{U,k-1}(0,0)\right\} \tag{9}$$

Firm i's most optimistic belief (i.e., the one that maximizes $\Delta_{i1}^{\mathbf{P}_j}$) is such that the probability that the opponent is active at period 1 takes its minimum value, and when firm i decides to be active (inactive) at period 1, the probability that the opponent is active at period 2 takes its minimum (maximum) value.

Therefore, we have the following recursive formulas for the bounds $\Delta_{i1}^{L,k}(\mathbf{X}_1)$ and $\Delta_{i1}^{U,k}(\mathbf{X}_1)$. For $k \geq 1$:

$$\begin{aligned}\Delta_{i1}^{L,k}(\mathbf{X}_1) &= \mathbf{Z}_i\,\boldsymbol{\eta}_{i1} + \gamma_{i1}\,Y_{i0} + \alpha_{i1}\,P_{j1}^{U,k-1}(\mathbf{X}_1) \\ &+ \delta_i\left[P_{j1}^{U,k-1}(\mathbf{X}_1)\,W_{i2}^{L,k}(1) + \left(1 - P_{j1}^{U,k-1}(\mathbf{X}_1)\right)W_{i2}^{L,k}(0)\right] \\ \Delta_{i1}^{U,k}(\mathbf{X}_1) &= \mathbf{Z}_i\,\boldsymbol{\eta}_{i1} + \gamma_{i1}\,Y_{i0} + \alpha_{i1}\,P_{j1}^{L,k-1}(\mathbf{X}_1) \\ &+ \delta_i\left[P_{j1}^{L,k-1}(\mathbf{X}_1)\,W_{i2}^{U,k}(1) + \left(1 - P_{j1}^{L,k-1}(\mathbf{X}_1)\right)W_{i2}^{U,k}(0)\right]\end{aligned} \tag{10}$$

where,

$$\begin{aligned}
W_{i2}^{L,k}(1) &\equiv G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}+\alpha_{i2}P_{j2}^{U,k-1}(1,1)\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\alpha_{i2}P_{j2}^{L,k-1}(0,1)\right)\\
W_{i2}^{L,k}(0) &\equiv G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}+\alpha_{i2}P_{j2}^{U,k-1}(1,0)\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\alpha_{i2}P_{j2}^{L,k-1}(0,0)\right)\\
W_{i2}^{U,k}(1) &\equiv G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}+\alpha_{i2}P_{j2}^{L,k-1}(1,1)\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\alpha_{i2}P_{j2}^{U,k-1}(0,1)\right)\\
W_{i2}^{U,k}(0) &\equiv G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}+\alpha_{i2}P_{j2}^{L,k-1}(1,0)\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\alpha_{i2}P_{j2}^{U,k-1}(0,0)\right)
\end{aligned} \tag{11}$$

For instance, for level-1 rationality we have:

$$\begin{aligned}
\Delta_{i1}^{L,1}(\mathbf{X}_1) &= \mathbf{Z}_i\ \boldsymbol{\eta}_{i1}+\gamma_{i1}\ Y_{i0}+\alpha_{i1}+\delta_i\left[G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}+\alpha_{i2}\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}\right)\right]\\
\Delta_{i1}^{U,1}(\mathbf{X}_1) &= \mathbf{Z}_i\ \boldsymbol{\eta}_{i1}+\gamma_{i1}\ Y_{i0}+\delta_i\left[G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\gamma_{i2}\right)-G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}+\alpha_{i2}\right)\right]
\end{aligned} \tag{12}$$

An important implication of the monotonicity in $\mathbf{P}_j$ of the threshold function $\Delta_{i1}^{\mathbf{P}_j}$ is that the sequence of lower bounds $\{\Delta_{i1}^{L,k}(\mathbf{X}_1) : k \geq 1\}$ is non-decreasing and the sequence of upper bounds $\{\Delta_{i1}^{U,k}(\mathbf{X}_1) : k \geq 1\}$ is non-increasing. That is, for any value of $\mathbf{X}_1$ and any $k \geq 1$:

$$\begin{aligned}
\Delta_{i1}^{L,k+1}(\mathbf{X}_1) &\geq \Delta_{i1}^{L,k}(\mathbf{X}_1)\\
\Delta_{i1}^{U,k+1}(\mathbf{X}_1) &\leq \Delta_{i1}^{U,k}(\mathbf{X}_1)
\end{aligned} \tag{13}$$

The bounds become sharper when we increase the level of rationality.

3 Identification

Suppose that we have a random sample of many (infinite) independent markets at periods 1 and 2. For each market in the sample we observe a realization of the variables $\{Y_{i0}, Y_{i1}, Y_{i2}, \mathbf{Z}_i : i = 1,2\}$. The realizations of the unobservable variables $\{\varepsilon_{it}\}$ are independent across markets. We are interested in using this sample to estimate the vector of structural parameters $\boldsymbol{\theta} \equiv \{\delta_i, \boldsymbol{\eta}_{it}, \gamma_{it}, \alpha_{it} : i = 1,2; t = 1,2\}$.

Let $P_{it}^0(\mathbf{X}_t)$ be the true conditional probability function $\Pr(Y_{it} = 1|\mathbf{X}_t)$ in the population. And let $\boldsymbol{\theta}^0$ be the true value of $\boldsymbol{\theta}$ in the population. I consider the following assumptions on the DGP. For any player $i \in \{1,2\}$ and any period $t \in \{1,2\}$:

(A1) *the reduced-form probability* $P_{it}^{0}(\mathbf{X}_t)$ *is identified at any point in the support of* $\mathbf{X}_t$*;*

(A2) *the variance-covariance matrix* $Var(\mathbf{Z}_i, Y_{i,t-1})$ *has full rank;*

(A3) *the distribution function* H_i *is known to the researcher;*

(A4) $\alpha_{it}^{0} \leq 0$*, and* $\boldsymbol{\theta}^{0}$ *belongs to a compact set* Θ*.*

Assumptions (A1) and (A3) imply that the population threshold function $\Delta_{it}^{0}(\mathbf{X}_t) \equiv H_i^{-1}\left(P_{it}^{0}(\mathbf{X}_t)\right)$ is identified at any point in the support of $\mathbf{X}_t$. I use $\Delta_{it}^{0}(\mathbf{X}_t)$ instead of $P_{it}^{0}(\mathbf{X}_t)$ in the analysis below.

Level-k rationality implies the following restrictions on the threshold functions evaluated at the true $\boldsymbol{\theta}^{0}$:

$$\Delta_{it}^{L,k}\left(\mathbf{X}_t, \boldsymbol{\theta}^{0}\right) \;\; \leq \Delta_{it}^{0}(\mathbf{X}_t) \leq \;\; \Delta_{it}^{U,k}\left(\mathbf{X}_t, \boldsymbol{\theta}^{0}\right) \tag{14}$$

Note that, by the monotonicity in k of the rationalizability bounds, if a value of $\boldsymbol{\theta}$ satisfies the restrictions for level-k rationality, then it also satisfies the restrictions for any level k' smaller than k. Let Θ^{k} be the identified set of parameters for level-k rational players. By definition:

$$\Theta^{k} = \left\{\boldsymbol{\theta} \in \Theta : \Delta_{it}^{L,k}\left(\mathbf{X}_t, \boldsymbol{\theta}\right) \leq \Delta_{it}^{0}(\mathbf{X}_t) \leq \Delta_{it}^{U,k}\left(\mathbf{X}_t, \boldsymbol{\theta}\right) \quad \text{for any } (i, t, \mathbf{X}_t)\right\} \tag{15}$$

In the context of dynamic games, the discount factor δ_i is a particularly interesting parameter. Does the identified set Θ^{k} include the whole interval $(0,1)$ for the discount factor, or can we rule out some values for that parameter? For instance, can we rule out that players are myopic (i.e., $\delta_i = 0$)? Consider the case of level-1 rationality. Given the restriction $\Delta_{i1}^{0}(\mathbf{X}_1) \leq \Delta_{i1}^{U,1}\left(\mathbf{X}_1, \boldsymbol{\theta}^{0}\right)$, and assuming that $\gamma_{i2}^{0} - \alpha_{i2}^{0} \geq 0$, it is straightforward to show that:

$$\delta_i^{0} \;\; \geq \;\; \sup_{\mathbf{X}_1} \left\{ \frac{\Delta_{i1}^{0}(\mathbf{X}_1) - \mathbf{Z}_i \boldsymbol{\eta}_{i1}^{0} - \gamma_{i1}^{0} Y_{i0}}{G_i\left(\mathbf{Z}_i \boldsymbol{\eta}_{i2}^{0} + \gamma_{i2}^{0}\right) - G_i\left(\mathbf{Z}_i \boldsymbol{\eta}_{i2}^{0} + \alpha_{i2}^{0}\right)} \right\} \tag{16}$$

This expression illustrates several aspects on the identification of δ_i^{0}. Level-1 rationality implies informative restrictions on the set of parameters, such that Θ^{1} does not contain the whole parameter space. In particular, given some values of the other parameters, we can guarantee that the lower bound on δ_i^0 (the RHS of the inequality) is strictly positive. Expression (16) also illustrates that we can rule out some values of the discount factor in the interval $(0,1)$ only if we impose further restrictions: either restrictions on the other parameters, or exclusion and support restrictions on the observable explanatory variables.

The rest of the paper presents sufficient conditions for point identification of the parameters in $\boldsymbol{\theta}^0$. To prove point identification one should establish that for any vector $\boldsymbol{\theta} \neq \boldsymbol{\theta}^0$ there are values of $\mathbf{X}_t$ with positive probability mass such that the inequality $\Delta_{it}^{L,k}(\mathbf{X}_t, \boldsymbol{\theta}) \leq \Delta_{it}^0(\mathbf{X}_t) \leq \Delta_{it}^{U,k}(\mathbf{X}_t, \boldsymbol{\theta})$ does not hold: i.e., either $\Delta_{it}^{L,k}(\mathbf{X}_t, \boldsymbol{\theta}) > \Delta_{it}^0(\mathbf{X}_t)$ or $\Delta_{it}^{U,k}(\mathbf{X}_t, \boldsymbol{\theta}) < \Delta_{it}^0(\mathbf{X}_t)$. The following exclusion restriction and large-support assumption is key for the point identification results that I present below.

(A5) *There is a variable $Z_{i\ell} \subset \mathbf{Z}_i$ such that $\eta_{i1\ell}^0 \neq 0$, $\eta_{i2\ell}^0 \neq 0$, and conditional on any value of the other variables in $(\mathbf{Z}_i, \mathbf{Z}_j)$, denoted by $\mathbf{Z}_{(-i\ell)}$, the random variable $\{Z_{i\ell}|\mathbf{Z}_{(-i\ell)}\}$ has unbounded support.*

THEOREM 1 (Point identification under level-1 rationalizability). Suppose that players are level-1 rational and assumptions (A1)-(A5) hold. Let $\eta_{i1\ell}^0$ and $\eta_{i2\ell}^0$ be the parameters associated with the exclusion restrictions in assumption (A5). Then, $\eta_{i1\ell}^0$ and $\eta_{i2\ell}^0$ are point-identified.

PROOF: For notational simplicity, I omit in this proof the subindex i, but it should be understood that all variables and parameters are player i's. First, I prove the identification of $\eta_{2\ell}^0$. Suppose that $\boldsymbol{\theta}$ is such that $\eta_{2\ell} \neq \eta_{2\ell}^0$. Given $\boldsymbol{\theta}$ and an arbitrary value of $(\mathbf{Z}_{(-\ell)}, Y_1)$, let Z_ℓ^* be the value of Z_ℓ that makes the lower bound function evaluated at $\boldsymbol{\theta}$ equal to the upper bound function evaluated at $\boldsymbol{\theta}^0$, i.e., $\Delta_2^{L,1}(Z_\ell^*, \mathbf{Z}_{(-\ell)}, Y_1; \boldsymbol{\theta}) = \Delta_2^{U,1}(Z_\ell^*, \mathbf{Z}_{(-\ell)}, Y_1; \boldsymbol{\theta}^0)$. Given the form of these functions, this value is:

$$Z_\ell^* \equiv \left(\eta_{2\ell} - \eta_{2\ell}^0\right)^{-1} \left(\mathbf{Z}_{(-\ell)} \left[\boldsymbol{\eta}_{2(-\ell)}^0 - \boldsymbol{\eta}_{2(-\ell)}\right] + Y_1 \left[\gamma_2^0 - \gamma_2\right] - \alpha_2\right) \tag{17}$$

Z_ℓ^* is a finite value that belongs to the support of Z_ℓ. Suppose that $\eta_{2\ell} > \eta_{2\ell}^0$. Then, for values of Z_ℓ greater than Z_ℓ^* we have that:

$$\Delta_2^{L,1}(\mathbf{X}_2, \boldsymbol{\theta}) = \mathbf{Z}\boldsymbol{\eta}_2 + \gamma_2 Y_1 + \alpha_2 > \mathbf{Z}\boldsymbol{\eta}_2^0 + \gamma_2^0 Y_1 = \Delta_2^{U,1}(\mathbf{X}_2, \boldsymbol{\theta}^0) \tag{18}$$

what contradicts the restrictions imposed by level-1 rationality. By assumption (A5), the probability $\Pr(Z_\ell > Z_\ell^* | \mathbf{Z}_{(-\ell)}, Y_1)$ is strictly positive. Since the previous argument can be applied for any possible value of $(\mathbf{Z}_{(-\ell)}, Y_1)$, the result holds with a positive probability mass $\Pr(Z_\ell > Z_\ell^*)$. Therefore, we can reject any value of $\eta_{2\ell}$ strictly greater than $\eta_{2\ell}^0$. Similarly, if $\eta_{2\ell} < \eta_{2\ell}^0$, then for values of Z_ℓ smaller than Z_ℓ^* we have that $\Delta_2^{L,1}(\mathbf{X}_2, \boldsymbol{\theta}) > \Delta_2^{U,1}(\mathbf{X}_2, \boldsymbol{\theta}^0)$. We can reject any value of $\eta_{2\ell}$ strictly smaller than $\eta_{2\ell}^0$. Hence, $\eta_{2\ell}^0$ is identified.

Now, consider the identification of $\eta_{1\ell}^0$. Note that the proof below does not assume that $\eta_{2\ell}^0$ is known. Identification of $\eta_{1\ell}^0$ does not require $\eta_{2\ell}^0$ to be identified. Given the form of the functions $\Delta_1^{L,1}$ and $\Delta_1^{U,1}$, we have that:

$$\begin{array}{rcl} \Delta_1^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}) - \Delta_1^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}^0) & = & \mathbf{Z}\left(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_1^0\right) + Y_0\left(\gamma_1 - \gamma_1^0\right) + \alpha_1 \\ & + & \delta\left[G\left(\mathbf{Z}\boldsymbol{\eta}_2 + \gamma_2 + \alpha_2\right) - G\left(\mathbf{Z}\boldsymbol{\eta}_2\right)\right] \\ & - & \delta^0\left[G\left(\mathbf{Z}\boldsymbol{\eta}_2^0 + \gamma_2^0\right) - G\left(\mathbf{Z}\boldsymbol{\eta}_2^0 + \alpha_2^0\right)\right] \end{array} \tag{19}$$

Suppose that $\boldsymbol{\theta}$ is such that $\eta_{1\ell} > \eta_{1\ell}^0$. By the properties of function $G(.)$, the values $\delta\left[G\left(\mathbf{Z}\boldsymbol{\eta}_2 + \gamma_2 + \alpha_2\right) - G\left(\mathbf{Z}\boldsymbol{\eta}_2\right)\right]$ and $\delta^0\left[G\left(\mathbf{Z}\boldsymbol{\eta}_2^0 + \gamma_2^0\right) - G\left(\mathbf{Z}\boldsymbol{\eta}_2^0 + \alpha_2^0\right)\right]$ are bounded within the intervals $[0, \delta\left(\gamma_2 + \alpha_2\right)]$ and $[0, \delta^0\left(\gamma_2^0 - \alpha_2^0\right)]$, respectively. Since the parameter space Θ is a compact set, it is clear that both $\delta\left(\gamma_2 + \alpha_2\right)$ and $\delta^0\left(\gamma_2^0 - \alpha_2^0\right)$ are finite values. This implies that, for any arbitrary value of $(\mathbf{Z}_{(-\ell)}, Y_1)$, we can always find a finite value of Z_ℓ, say $\bar{Z}_\ell$, such that for $Z_\ell > \bar{Z}_\ell$ we have that $\Delta_1^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}) - \Delta_1^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}^0) > 0$, what contradicts the restrictions imposed by level-1 rationality. By assumption (A5), the probability $\Pr(Z_\ell > \bar{Z}_\ell | \mathbf{Z}_{(-\ell)}, Y_0)$ is strictly positive. Therefore, we can reject any value of $\eta_{1\ell}$ strictly greater than $\eta_{1\ell}^0$. We can apply a similar argument to show that we can reject any value of $\eta_{1\ell}$ strictly smaller than $\eta_{1\ell}^0$. Hence, $\eta_{1\ell}^0$ is identified. Q.E.D. ■

Point identification of all the parameters of the model requires at least level-2 rationality. Furthermore, in this dynamic game, at least two additional conditions are needed. First, the identification the discount factor requires the last period entry cost, γ_{i2}^0, to be strictly positive. If this parameter is zero, the dynamic game becomes static at period 1, and the discount factor does not play any role in the decisions of rational players. Second, the parameters $\eta^0_{i1\ell}$ and $\eta^0_{i2\ell}$, in assumption (A5), should have the same sign.

THEOREM 2 (Point identification under level-2 rationalizability). Suppose that: assumptions (A1)-(A5) hold; players are level-2 rational; the parameters $\eta^0_{i1\ell}$ and $\eta^0_{i2\ell}$, in assumption (A5), have the same sign; and $\gamma^0_{i2} > 0$. Then, all the structural parameters in $\boldsymbol{\theta}^0$ are point-identified.

PROOF: Aradillas-Lopez and Tamer (2008) show that, under the conditions of this Theorem, all the parameters in the static game are identified. Therefore, this proof considers that the vector $(\boldsymbol{\eta}^0_{i2}, \gamma^0_{i2}, \alpha^0_{i2})$ is known and it concentrates on the identification of $(\delta^0_i, \boldsymbol{\eta}^0_{i1}, \gamma^0_{i1}, \alpha^0_{i1})$. The proof goes through four cases which cover all the possible values of $\boldsymbol{\theta} \neq \boldsymbol{\theta}^0$.

Case (i): Suppose that $\boldsymbol{\theta}$ is such that $\eta_{i1\ell} \neq \eta^0_{i1\ell}$. Theorem 1 shows that we can reject this value of $\boldsymbol{\theta}$.

Case (ii): Suppose that $\boldsymbol{\theta}$ is such that $\eta_{i1\ell} = \eta^0_{i1\ell}$, but $\boldsymbol{\eta}_{i1(-\ell)} \neq \boldsymbol{\eta}^0_{i1(-\ell)}$ or/and $\gamma_{i1} \neq \gamma^0_{i1}$. I prove here that, given that $\boldsymbol{\theta}$, there is a set of values of $\mathbf{X}_1$, with positive probability mass, such that $\Delta^{L,2}_{i1}(\mathbf{X}_1, \boldsymbol{\theta}) > \Delta^{U,2}_{i2}\left(\mathbf{X}_1, \boldsymbol{\theta}^0\right)$, what contradicts the restrictions of level-2 rationality. By definition:

$$
\begin{array}{rcl}
\Delta^{L,2}_{i1}(\boldsymbol{\theta}) - \Delta^{U,2}_{i2}\left(\boldsymbol{\theta}^0\right) & = & \mathbf{Z}_i\left(\boldsymbol{\eta}_{i1} - \boldsymbol{\eta}^0_{i1}\right) + Y_{i0}\left(\gamma_{i1} - \gamma^0_{i1}\right) + \alpha_{i1} P^{U,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta}) - \alpha^0_{i1} P^{L,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta}^0) \\
& + & \delta_i \left[P^{U,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta})\ W^{L,2}_{i2}(1) + \left(1 - P^{U,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta})\right) W^{L,2}_{i2}(0) \right] \\
& - & \delta^0_i \left[P^{L,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\ W^{U,2}_{i2}(1) + \left(1 - P^{L,1}_{j1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right) W^{U,2}_{i2}(0) \right]
\end{array}
\tag{20}
$$

Given $\boldsymbol{\theta}$, let $(\mathbf{Z}_{i(-\ell)}, Y_{i0})$ be a vector such that $\mathbf{Z}_{i(-\ell)}\left(\boldsymbol{\eta}_{i1} - \boldsymbol{\eta}^0_{i1}\right) + Y_{i0}\left(\gamma_{i1} - \gamma^0_{i1}\right) > 0$. By the non-collinearity assumption in (A2) and the exclusion restriction in (A5), for any pair $(Z_{i\ell}, Z_{j\ell})$ the set of values $(\mathbf{Z}_{i(-\ell)}, Y_{i0})$ satisfying the previous inequality has positive probability mass. Now, given the monotonicity of the probabilities $P^{L,1}_{j1}$, $P^{U,1}_{j1}$, $P^{L,1}_{j2}$ and $P^{U,1}_{j2}$ with respect to $Z_{j\ell}$, and given that $sign\left(\eta^0_{j1\ell}\right) = sign\left(\eta^0_{j2\ell}\right)$, we can find values of $Z_{j\ell}$ large enough (or small enough, depending on the sign of the parameter) such that these probabilities are arbitrarily close to zero. That is the case both for the probabilities evaluated at $\boldsymbol{\theta}$ and for those evaluated at $\boldsymbol{\theta}^0$ because in both cases the values of $\eta_{j1\ell}$ and $\eta_{j2\ell}$ are the true ones, $\eta_{j1\ell}^0$ and $\eta_{j2\ell}^0$. Therefore, for these values of $Z_{j\ell}$ we have that:

$$\begin{array}{rcl}\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) & \simeq & \mathbf{Z}_{i(-\ell)}(\boldsymbol{\eta}_{i1} - \boldsymbol{\eta}_{i1}^0) + Y_{i0}(\gamma_{i1} - \gamma_{i1}^0) \\ & + & (\delta_i - \delta_i^0)\left[G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0) - G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0)\right]\end{array} \tag{21}$$

By the definition of the function $G_i(.)$, as $Z_{i\ell}\eta_{i\ell 2}^0$ goes to $-\infty$, both $G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0)$ and $G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0)$ go to zero. Therefore, for these pairs of $(Z_{i\ell}, Z_{j\ell})$ we have that $\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) \simeq \mathbf{Z}_{i(-\ell)}(\boldsymbol{\eta}_{i1} - \boldsymbol{\eta}_{i1}^0) + Y_{i0}(\gamma_{i1} - \gamma_{i1}^0) > 0,$, what contradicts the restrictions of level-2 rationality. Thus, $\boldsymbol{\eta}_{i1(-\ell)}^0$ and γ_{i1}^0 are identified.

Case (iii): Suppose that $\boldsymbol{\theta}$ is such that $\boldsymbol{\eta}_{i1} = \boldsymbol{\eta}_{i1}^0$ and $\gamma_{i1} = \gamma_{i1}^0$ but $\alpha_{i1} \neq \alpha_{i1}^0$. Now,

$$\begin{array}{rcl}\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) & = & \alpha_{i1}P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}) - \alpha_{i1}^0 P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0) \\ & + & \delta_i\left[P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta})\, W_{i2}^{L,2}(1) + \left(1 - P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta})\right) W_{i2}^{L,2}(0)\right] \\ & - & \delta_i^0\left[P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\, W_{i2}^{U,2}(1) + \left(1 - P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right) W_{i2}^{U,2}(0)\right]\end{array} \tag{22}$$

Suppose that $\alpha_{i1} > \alpha_{i1}^0$. There are values of $Z_{j\ell}$ large enough (or small enough) such that the probabilities $P_{j1}^{L,1}$, $P_{j1}^{U,1}$, $P_{j2}^{L,1}$ and $P_{j2}^{U,1}$ are arbitrarily close to one. For these values:

$$\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) \simeq \alpha_{i1} - \alpha_{i1}^0 + (\delta_i - \delta_i^0)\left[G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0 + \alpha_{i2}^0) - G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \alpha_{i2}^0)\right] \tag{23}$$

As $Z_{i\ell}\eta_{i\ell 2}^0$ goes to $-\infty$, $G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0 + \alpha_{i2}^0)$ and $G_i(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \alpha_{i2}^0)$ go to zero. Therefore, for these pairs of $(Z_{i\ell}, Z_{j\ell})$ we have that $\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) \simeq \alpha_{i1} - \alpha_{i1}^0 > 0$, what contradicts the restrictions of level-2 rationality. Similarly, when $\alpha_{i1} < \alpha_{i1}^0$ we can show that there is a set of values of $\mathbf{X}_1$, with positive probability mass, such that $\Delta_{i1}^{U,2}(\mathbf{X}_1, \boldsymbol{\theta}) < \Delta_{i2}^{L,2}(\mathbf{X}_1, \boldsymbol{\theta}^0)$, what also contradicts the restrictions of level-2 rationality. Thus, α_{i1}^0 is identified.

Case (iv): Suppose that $\boldsymbol{\theta}$ is such that $\boldsymbol{\eta}_{i1} = \boldsymbol{\eta}_{i1}^0$, $\gamma_{i1} = \gamma_{i1}^0$, and $\alpha_{i1} = \alpha_{i1}^0$, but $\delta_i \neq \delta_i^0$. Then,

$$\begin{array}{rcl}\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}(\boldsymbol{\theta}^0) & = & \alpha_{i1}^0\left[P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}) - P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right] \\ & + & \delta_i\left[P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta})\, W_{i2}^{L,2}(1) + \left(1 - P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta})\right) W_{i2}^{L,2}(0)\right] \\ & - & \delta_i^0\left[P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\, W_{i2}^{U,2}(1) + \left(1 - P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right) W_{i2}^{U,2}(0)\right]\end{array} \tag{24}$$

Suppose that $\delta_i > \delta_i^0$. There are values of $Z_{j\ell}$ large enough (or small enough) such that the probabilities $P_{j1}^{L,1}$, $P_{j1}^{U,1}$, $P_{j2}^{L,1}$ and $P_{j2}^{U,1}$ are arbitrarily close to zero. For these values:

$$\Delta_{i1}^{L,2}(\boldsymbol{\theta}) - \Delta_{i2}^{U,2}\left(\boldsymbol{\theta}^0\right) \simeq \left(\delta_i - \delta_i^0\right)\left[G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0\right) - G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0\right)\right] > 0 \tag{25}$$

what contradicts the restrictions of level-2 rationality. Now, consider the difference between $\Delta_{i1}^{U,2}(\boldsymbol{\theta})$ and $\Delta_{i2}^{L,2}\left(\boldsymbol{\theta}^0\right)$. We have that:

$$\begin{aligned} \Delta_{i1}^{U,2}(\boldsymbol{\theta}) - \Delta_{i2}^{L,2}\left(\boldsymbol{\theta}^0\right) &= \alpha_{i1}^0\left[P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta}) - P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right] \\ &+ \delta_i\left[P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta})\; W_{i2}^{U,2}(1) + \left(1 - P_{j1}^{L,1}(\mathbf{X}_1, \boldsymbol{\theta})\right) W_{i2}^{U,2}(0)\right] \\ &- \delta_i^0\left[P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\; W_{i2}^{L,2}(1) + \left(1 - P_{j1}^{U,1}(\mathbf{X}_1, \boldsymbol{\theta}^0)\right) W_{i2}^{L,2}(0)\right] \end{aligned} \tag{26}$$

Suppose that $\delta_i < \delta_i^0$. There are values of $Z_{j\ell}$ large enough (or small enough) such that the probabilities $P_{j1}^{L,1}$, $P_{j1}^{U,1}$, $P_{j2}^{L,1}$ and $P_{j2}^{U,1}$ are arbitrarily close to zero. For these values:

$$\Delta_{i1}^{U,2}(\boldsymbol{\theta}) - \Delta_{i2}^{L,2}\left(\boldsymbol{\theta}^0\right) \simeq \left(\delta_i - \delta_i^0\right)\left[G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0 + \gamma_{i2}^0\right) - G_i\left(\mathbf{Z}_i\boldsymbol{\eta}_{i2}^0\right)\right] < 0 \tag{27}$$

what contradicts the restrictions of level-2 rationality. Thus, δ_i^0 is identified. Q.E.D. ■

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