A remark on the supposed equivalence between complete markets and perfect foresight hypothesis

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A REMARK ON THE SUPPOSED EQUIVALENCE BETWEEN COMPLETE MARKETS AND PERFECT FORESIGHT HYPOTHESIS

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ABSTRACT
We consider a sequential equilibrium model over two periods, during the first of which agents have perfect information and their expectations are formed as if there were complete future markets. We show that, in the second period, equilibrium prices may well be different from those expected, without any unexpected change having occurred. This result highlights a lack of correspondence between the perfect foresight hypothesis and that of complete markets.

KEYWORDS
Arrow-Debreu equilibrium, Complete markets, Sequential equilibrium, Perfect foresight, Indeterminacy

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I. STRICTURES ON THE DOUBLE INTERPRETATION OF ARROW-DEBREU'S THEORY

Economists unanimously recognise that current economic decisions are widely affected by expectations, among which those about the prices of commodities delivered in the future have certainly to be included. There are, however, three main different ways to treat these prices, with different related notions of equilibrium: i) focusing on normal levels around which the prices are

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expected to fluctuate; ii) trying to guess expected future prices date by date; iii) assuming complete future markets and determining current and future prices simultaneously.

Within the modern versions of the general equilibrium theory only the second and third way of treating expected prices are considered. In particular, following the last one, we have the intertemporal equilibrium, which is usually recognised to be analytically rigorous but completely detached from reality. In fact, as was admitted by Debreu himself (1959: 33), "[o]rganized futures markets concern only a small number of goods, locations and dates", since, as already maintained by Hicks (1939: 137), agents know that "they cannot foretell at all exactly what quantities they will themselves desire to buy or sell at a future period" - or, as Bliss later wrote (1975: 56), "the supposition that everything that concerns an actor is known with certainty by him is, as far as a description of the world is concerned, not an abstraction but a fantasy".

Following the second way we have instead the temporary equilibrium, in which the markets are organized sequentially and at each date, current prices and outputs are determined taking expected future prices as givens, beside of current endowments, preferences and technological possibilities. Here agents have only beliefs about the primitives of the economy (Grandmont: 1970 and 1977; Green, 1973): consequently, incorrect and uncommon expectations about future prices can arise, and with them temporally inconsistent production and consumption plans.

Precisely the weaknesses of temporary equilibrium – such as its need for some extra-assumptions in terms of borrowing constraints and default rules to demonstrate the existence of equilibrium, or its *ad hoc* specification of beliefs about future prices – have given nourishment to the former but unrealistic idea of a complete system of forward markets. More precisely, it has been maintained that the Arrow-Debreu's complete markets hypothesis could be reinterpreted by considering an economy with only a limited number of futures markets but at least a commodity which can be used for loans between every couple of dates. For this case, because of their incompleteness, markets must reopen at each date for spot transactions and for the loans, but the assumption of agents' perfect foresight is considered as a 'perfect substitute' for that of complete markets (Arrow, 1963; Radner, 1968 and 1972).

The transition to the sequential markets model with perfect foresight has not however been without cost, since more stringent informational requirements concerning the participants of the economy must be included in them. In particular, while within the standard Arrow-Debreu framework each agent is assumed to know its own position (endowments, preferences and technological possibilities) but takes prices as exogenous parameters, here each agent is assumed to have a knowledge either about the positions of all the other ones, or directly about the prices that will rule in the future. Otherwise there would be no guarantee of identical and correct beliefs regarding the spot prices which will prevail at each date and in each state. As Bliss (1975: 44, 48) pointed out, the hypothesis of perfect foresight puts an impossible strain upon the assumption of perfect knowledge and “it would be exactly as if there were forward markets”. Not surprisingly, it is often introduced only as a tool for indicative planning, or for the description of a stationary state, or to shape the distance between an ideal situation and one in which people make mistakes.

These strictures on the notions of temporary and intertemporal equilibrium might have been avoided if the theory had followed the first of the ways listed above from the beginning. It corresponds to that normal position or equilibrium we find not only within the analysis of classical economists but also of Walras, Marshall or Wicksell, that is within the initial version of neoclassical theory. In such a normal position, prices the theory refers to are persistent, in the sense that they are consistent with a uniform rate of return on the supply prices of capital goods and the data at their basis change slowly enough (relative to the tendency to equilibrium values) to be able to be at least approximately considered as invariants[1]. Within this framework, it is indeed possible to assume that expectations are not realized perfectly and that equilibrium values emerge on average from the compensation of errors over time. In other words, it is possible to assume that agents have time to grasp and learn the long-period trend, so that no ad hoc hypotheses concerning future changes, unanimity of expectations on equilibrium quantities and prices, perfect knowledge of future tastes, endowments, and technology need to be introduced into the theory.

Usually these problems are not discussed in depth in recent literature on general equilibrium, where these normal positions are wrongly identified with the stationary states[2] (see Garegnani, 1976). On the contrary, the question of how each agent can have the same catalogue of spot commodity prices, together with the extension of sequential equilibrium to an infinite time horizon, has merely further broadened the notions of equilibrium. Moreover, in general no doubt has been cast on the equivalence between the perfect foresight hypothesis and complete markets (see for instance Grandmont 1977: 535; Lucas, 1978; Mas-Colell, Whinston and Green, 1995: 743-744).

According to us this is clearly unsatisfactory. In particular, here we will resume a sequential indeterminacy argument – similar to that in Mandler (1995) and (1999), but simplified and adapted to our aims – in order to show that future prices expectations, which are formulated by the agents as if they were in an Arrow-Debreu model, will generally prove incorrect. This result negates the attempt to justify the heroic assumption of the completeness of markets by arguing its equivalence to that of agents having complete information and making no systematic forecast errors about future

[1] We find the same kind of approximation in the “quasi-stationary states” of chemistry or biology.

[2] In economics the expression “stationary state” has currently a different meaning from that the other sciences give to it. In particular, while in classical mechanics “stationary state” is just a synonym of “equilibrium”, in economics the stationary state is a very special kind of equilibrium, that in which every impulse to net capital accumulation has ceased.

prices. It arises not from unpredictable accidents, but from the fact that expectations formed on the grounds of an Arrow-Debreu equilibrium system can capture only one of the infinite configurations which prices can assume in the second-period equilibrium of a sequential model over two periods.

II. THE COMPLETE MARKETS MODEL

We consider an intertemporal equilibrium model over two periods. In each period the following commodities are delivered: m purely consumption goods; n purely (circulating) capital goods; labour.

We denote by: $p_t = [p_{1,t},...,p_{m,t}]$ the price vector of consumption goods delivered in period t, with t = 1, 2. Similarly, $q_t = [q_{1,t},...,q_{n,t}]$ is the price vector of capital goods delivered in period t.[3] The wage rate in period t is w_t. All these prices are actual values.

With reference to the consumption side, we assume each household h to have intertemporal endowments consisting of: capital goods available at the beginning of period 1, in quantities $\omega_1^h \in R^n$; labour in period 1, in quantity λ_1^h; labour in period 2, in quantity λ_2^h. For every possible price system, the household determines its consumption plan, $(x_1^h, x_2^h) \in R^{2m}$, by maximizing its intertemporal utility function – which is assumed to be continuous, differentiable at least twice and concave on R^{2m} – subject to its intertemporal budget constraints: $q_1' \cdot \omega_1^h + w_1\lambda_1^h + w_2\lambda_2^h = p_1' \cdot x_1^h + p_2' \cdot x_2^h$. In this way we get the functions of demand for consumption goods $x_1^h(.)$ and $x_2^h(.)$ whose properties are well-known.

By aggregating demands and supplies among households we get: $x_t(.) = \sum_h x_t^h(.)$; $\omega_1 = \sum_h \omega_1^h$; $\lambda_t = \sum_h \lambda_t^h$, with t = 1, 2.

As for the production side, we assume linear activities, absence of joint production and absence of alternative methods. In particular we denote by A the n×m matrix having as columns the input coefficients of the n capital goods in the production of the m consumption goods; while a_ℓ is the row vector of the input coefficients of labour in the m activities. Similarly, B and b_ℓ are, respectively, the n×n matrix of capital good coefficients and labour coefficients in the production of the n capital goods. The non-negative vector $y_t \in R_+^m$ represents the quantities produced of

[3] Both p_t and q_t are intended as column vectors, while p_t' and q_t' will denote the corresponding row (or transpose) vectors.

consumption goods delivered in period t, with t = 1, 2, while the non-negative vector $y_c \in R^n_+$ represents the quantities produced of capital goods delivered in period 2. We also assume a free disposal activity for each capital good initially available and denote by f_j the quantity of the j-th capital good sent to the free disposal.

Definition 1: given: a vector of endowments $(\omega_1, \lambda_1, \lambda_2) \in R^{n+2}_{++}$, a couple of demand functions $x_1(.)$ and $x_2(.)$ with the customary prosperities, a n+1×m matrix of technical coefficients $\tilde{A} = \begin{bmatrix} A \\ a_\ell \end{bmatrix}$ and a n+1×n matrix of technical coefficients $\tilde{B} = \begin{bmatrix} B \\ b_\ell \end{bmatrix}$; then $\{\omega_1, \lambda_1, \lambda_2, x_1(.), x_2(.), \tilde{A}, \tilde{B}\}$ is an element of the set of economies E.

Definition 2: a system of prices $\{p_1, q_1, w_1, p_2, q_2, w_2\}$ and a system of quantities $\{y_1, y_c, y_2\}$ are an intertemporal equilibrium for the economy $\{\omega_1, \lambda_1, \lambda_2, x_1(.), x_2(.), \tilde{A}, \tilde{B}\} \in E$ if and only if they satisfy the following conditions:[4]

(1) $x_t(.) = y_t$ for t = 1, 2

(2) $\omega_1 = Ay_1 + By_c + f$ with $q_{j,1} \cdot f_j = 0$ and $f_j \geq 0$ for j = 1, ..., n.

(3) $y_c = Ay_2$

(4) $\lambda_1 \geq a_\ell \cdot y_1 + b_\ell \cdot y_c$ with "=" if $w_1 > 0$

(5) $\lambda_2 \geq a_\ell \cdot y_2$ with "=" if $w_2 > 0$

(6) $p'_t \leq q'_t A + w_t a_\ell$ for t = 1, 2

(7) $q'_2 \leq q'_1 B + w_1 b_\ell$

(8) $\sum_{i=1}^{m} p_{i,2} = 1$

It is a well-known result that the system has at least one solution for every economy in E. Moreover, almost every economy in E has a finite number of equilibria (cf. Mas-Colell 1975). We may also conceive the possibility of economies with one and only one equilibrium.

[4] Note that there is no substantial difference between definition of equilibrium here adopted and that in Mas-Colell, Whinston and Green (1995: 607).

Let $\hat{E} \subset E$ be the set of economies for which the system (1)-(8) has only one solution; we can easily assume $\hat{E}$ to be non empty. Besides, even if this is not necessary, we simplify our argument by concentrating the attention on economies which are elements of $\hat{E}$.

III. INCOMPLETE MARKETS MODEL.

We now consider a model which is identical to the former, but with markets organised sequentially. There are the same commodities; the same individuals; firms have the same technology. We simply remove the complete market hypothesis. In particular, we assume that at the beginnings of period 1 agents cannot trade the commodities delivered in period 2.

As a consequence, at the beginning of period 2 markets will reopen in order to allow agents to trade the commodities delivered in that period. In particular, firms will hire from individuals capital goods and labour to be employed in period 2 by giving in exchange period 2 consumption goods. The equilibrium relative prices of those commodities will actually be determined only at the beginning of period 2; while at the beginning of period 1 – i.e., at the time agents have to take their decision concerning the first period – these prices are only expected prices.

In accordance with the supposed possibility of the dual interpretation of the same formal model, we assume that the expected prices of commodities delivered in 2 are determined simultaneously with the prices of commodities delivered in period 1 by a system of equilibrium conditions which is identical to the system (1)-(8).

Definition 3: a system of prices $\{p_1, q_1, w_1, p^e_2, q^e_2, w^e_2\}$ and a system of quantities $\{y_1, y_c, y^e_2\}$ are a first-period equilibrium for the economy $\{\omega_1, \lambda_1, \lambda_2, x_1(.), x_2(.), \tilde{A}, \tilde{B}\} \in S$ if and only if they satisfy the following conditions:

(9) $x_1(.) = y_1$

(10) $x_2(.) = y^e_2$

(11) $\omega_1 = Ay_1 + By_c + f$ with $q_{j,1} \cdot f_j = 0$ and $f_j \geq 0$ for j = 1, ..., n.

(12) $y_c = Ay^e_2$

(13) $\lambda_1 \geq a_\ell \cdot y_1 + b_\ell \cdot y_c$ with "=" if $w_1 > 0$

(14) $\lambda_2 \geq a_\ell \cdot y^e_2$ with "=" if $w^e_2 > 0$

(15) $p_1' \leq q_1' A + w_1 a_\ell$

(16) $\quad p_2^{e\prime} \leq q_2^{e\prime} A + w_2^e a_\ell$

(17) $\quad q_2^{e\prime} \leq q_1' B + w_1 b_\ell$

(18) $\quad \sum_{i=1}^{m} p_{i,2}^e = 1$

Since we are concentrating our attention on economies which are elements of $\hat{E}$, then the system (9)-(18) – which is formally equivalent to the system (1)-(8) – has a unique solution. Let us denote the solution by $\{\hat{p}_1, \hat{q}_1, \hat{w}_1, \hat{p}_2^e, \hat{q}_2^e, \hat{w}_2^e\}$ and $\{\hat{y}_1, \hat{y}_c, \hat{y}_2^e\}$.

Having determined the quantities $\hat{y}_c$ of capital goods produced during the first period, the endowments of capital goods in the second period will be ω_2, with $\omega_2 = \hat{y}_c$. Therefore, the endowment of the second-period economy will be $(\omega_2, \lambda_2) \in R_{++}^{n+1}$.

Having determined the prices $\hat{p}_1$, $\hat{q}_1$ and $\hat{w}_1$, the demand function for the consumption goods delivered in period 2 will be: $x_2(\hat{p}_1, \hat{q}_1, \hat{w}_1, p_2, q_2, w_2) = \hat{x}_2(p_2, q_2, w_2)$. The continuity and the differentiability immediately extend from $x_2(.)$ to $\hat{x}_2(.)$. As for Walras' law, it can easily be proved that if $x_1(.) = \hat{y}_1$ and $\omega_2 = \hat{y}_c$ then $p_2' \cdot \hat{x}(.) = q_2' \cdot \omega_2 + w_2 \lambda_2$.

Definition 4: given a vector of endowments $(\omega_2, \lambda_2) \in R_{++}^{n+1}$, a demand function $\hat{x}_2(.)$ with the customary prosperities and a n+1×m matrix of technical coefficients $\tilde{A} = \begin{bmatrix} A \\ a_\ell \end{bmatrix}$, then $\{\omega_2, \lambda_2, \hat{x}_2(.), \tilde{A}\}$ is an element of the set of second-period economies S.

Definition 5: a system of prices $\{p_2, q_2, w_2\}$ and a vector of quantities y_2 are a second-period equilibrium for the economy $\{\omega_2, \lambda_2, \hat{x}_2(.), \tilde{A}\} \in S$ if and only if they satisfy the following conditions:

(19) $\quad \hat{x}_2(.) = y_2$

(20) $\quad \omega_2 = A y_2 + d \qquad$ with $q_{j,2} \cdot d_j = 0$ and $d_j \geq 0$ for j = 1, ..., n.

(21) $\quad \lambda_2 \geq a_\ell \cdot y_2 \qquad$ with "=" if $w_2 > 0$

(22) $\quad p_2' \leq q_2' A + w_2 a_\ell$

(23) $\quad \sum_{i=1}^{m} p_{i,2} = 1$

Proposition 1: let $\{\hat{p}_1, \hat{q}_1, \hat{w}_1, \hat{p}^e_2, \hat{q}^e_2, \hat{w}^e_2\}$ and $\{\hat{y}_1, \hat{y}_c, \hat{y}^e_2\}$ be the unique solution of the system (9)-(18), then $\{\hat{p}_2, \hat{q}_2, \hat{w}_2\} = \{\hat{p}^e_2, \hat{q}^e_2, \hat{w}^e_2\}$ and $\hat{y}_2 = \hat{y}^e_2$ is a solution of the system (19)-(23).

Proof: let us posit $d = 0$ and seek a solution of the system (19)-(23), if one exists, under this hypothesis. Note that conditions (19)-(23), with $d = 0$, are identical to conditions (10), (12), (14), (16) and (18). Therefore, since $\{\hat{p}_1, \hat{q}_1, \hat{w}_1, \hat{p}^e_2, \hat{q}^e_2, \hat{w}^e_2\}$ and $\{\hat{y}_1, \hat{y}_c, \hat{y}^e_2\}$ solve the system (9)-(18), then the prices $\{\hat{p}_2, \hat{q}_2, \hat{w}_2\} = \{\hat{p}^e_2, \hat{q}^e_2, \hat{w}^e_2\}$ and the quantities $\hat{y}_2 = \hat{y}^e_2$ must solve the system (19)-(23) with $d = 0$ □

Proposition 2: Let us define the set P as follows:

(24) $\quad P = \{p_2, q_2, w_2 : \hat{x}_2(.) = \hat{y}^e_2 ; p'_2 \cdot \hat{y}^e_2 = q'_2 A \cdot \hat{y}^e_2 + w_2 a_\ell \cdot \hat{y}^e_2 ; \sum p_{i,2} = 1\}$.

a) Every $\{p_2, q_2, w_2\} \in P$ is a solution of the system (19)-(23) with $\hat{y}_2 = \hat{y}^e_2$.

b) The set P is non-empty.

c) If n > m, then the set P has infinite elements.

Proof:

a) Every $\{p_2, q_2, w_2\} \in P$ satisfies, by definition, equilibrium conditions (19), (22) and (23). When $\hat{y}_2 = \hat{y}^e_2$ and $\omega_2 = \hat{y}_c$, condition (20) is satisfied too. Because of Walras' law, conditions (19), (20), (22) and (23) imply condition (21).

b) The proof follows from proposition 1.

c) By definition, a vector $\{p_2, q_2, w_2\}$ is an element of P if and only if it satisfies the following conditions:

(25) $\quad \hat{x}_2(.) - \hat{y}^e_2 = 0$

(26) $\quad (p'_2 - q'_2 A - w_2 a_\ell) \cdot \hat{y}^e_2 = 0$

(27) $\quad \sum p_{j,2} - 1 = 0$

Let $\{p_2, q_2, w_2\}$ be a solution of the system (25)-(27), we define a matrix M as follows:

(28) $$M = \left[\begin{array}{c:c:c} \multicolumn{3}{c}{D\hat{x}_2} \\ \hdashline I & -A' & -a'_\ell \\ \hdashline u & 0 & 0 \end{array}\right]$$

where $D\hat{x}_2$ is the m×(m+n+1) Jacobian matrix at $\{p_2, q_2, w_2\}$ of demand function for consumption goods delivered in period 2, I is the m×m identity matrix and u is an m-dimension row vector of 1s. As is well-known, $\{p_2, q_2, w_2\}$ is a locally unique solution of the system (25)-(27) only if the matrix M is of column full rank. Therefore, since the matrix M has m+m+1 rows and m+n+1 columns, if n > m then it cannot have column full rank □

Because of the continuity of equilibria of the second-period economy in the case where n > m, the unique system of expected prices, which were determined in the first period by an equilibrium system identical to that with complete markets, will consequently not be generically realized. In other words, expectations formed on the grounds of an Arrow-Debreu equilibrium system can capture only one of the infinite configurations which prices can assume in the second-period equilibrium of a sequential model over two periods. Consequently, no equivalence between the perfect foresight hypothesis and complete markets can correctly be advanced.

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