Statistical modelling of financial crashes: Rapid growth, illusion of certainty and contagion

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Abstract

We develop a rational expectations model of financial bubbles and study ways in which a generic risk-return interplay is incorporated into prices. We retain the interpretation of the leading Johansen-Ledoit-Sornette model, namely, that the price must rise prior to a crash in order to compensate a representative investor for the level of risk. This is accompanied, in our stochastic model, by an illusion of certainty as described by a decreasing volatility function. The basic model is then extended to incorporate multivariate bubbles and contagion, non-Gaussian models and models based on stochastic volatility. Only in a stochastic volatility model where the mean of the log-returns is considered fixed does volatility increase prior to a crash.

Keywords: financial crashes, super-exponential growth, illusion of certainty, contagion.

1 Introduction

In this paper we discuss rational expectations models for bubbles and market crashes – a stochastic version of the model in [3]. We derive a number of significant theoretical and empirical implications and the potential relevance to recent events is striking.

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Rational expectations models were introduced with the work of Blanchard and Watson to account for the possibility that prices may deviate from fundamental levels [1]. We take as our main starting point the somewhat controversial subject of log-periodic precursors to financial crashes [2]-[11]. For additional background on log-periodicity and complex exponents see [12]. A first-order approach in [3] and subsequent extensions in [13] state that prior to a crash the price must exhibit a super-exponential growth in order to compensate a representative investor for the level of risk. However, this approach concentrates solely on the drift function and ignores the underlying volatility fluctuations which typically dominate financial time series [14]. Similar in spirit to [3], we derive a second-order condition which incorporates volatility fluctuations and enables us to combine insights from a rational expectations model with a stochastic model [15]-[16]. Our model gives two important characterisations of bubbles in economics: firstly, a rapid super-exponential growth; secondly, an illusion of certainty as described by a decreasing volatility function prior to the crash.

The layout of this paper is as follows. In Section 2 we introduce the basic model and derive the crash-size distribution, the post-crash dynamics and simple estimates of fundamental value. The model is then further extended to incorporate multivariate bubbles (Section 3), contagion across different assets and sectors (Section 4), non-Gaussian models (Section 5) and stochastic volatility (Section 6). Section 7 describes an empirical application to the UK housing bubble of the early to late 2000s [17]. Section 8 is a brief conclusion.

2 The model

In this section we give an alternative formulation of the model solution in [3]. This leads naturally to a stochastic generalisation of the original model, which is then solved in full to give empirical predictions for the distribution of crash-sizes, post-crash dynamics, and simple estimates of fundamental value.

The basic model can be described as follows. Let X_t denote the log-price of an asset at time t. As in [18] the starting point is the equation

$$dX_t = \mu(t)dt + \sigma(t)dW_t - \kappa dj(t), \tag{1}$$

where W_t is a Wiener process and $j(t)$ is a jump process satisfying

$$j(t) = \begin{cases} 0 & \text{before the crash} \\ 1 & \text{after the crash.} \end{cases}$$

κ measures the relative size of the crash since if the crash occurs at time C

$$X_{C_-} = X_C \quad \text{and} \quad X_{C_+} = X_C - \kappa.$$

The corresponding fall in price is

$$[e^{X_C} - e^{X_C - \kappa}] = e^{X_C}[1 - e^{-\kappa}]. \tag{2}$$

Let $h(t)$ be the hazard rate. Suppose a crash has not occurred by time t'. To first order, $j(t)|t' \sim \text{Bernoulli}(p_{t',t})$. where $\int_{t'}^{t} h(u)du = p_{t',t}$. It follows that $E[j(t)|t'] = p_{t',t}$ and $\text{Var}[j(t)|t'] = p_{t',t}(1 - p_{t',t})$. Under the model (1)

$$\begin{aligned} E[X_t - X_{t'}|t'] &= \int_{t'}^{t} \mu(u) - \kappa h(u) du, & (3) \\ \text{Var}[X_t - X_{t'}|t'] &= \int_{t'}^{t} \sigma^2(u) + \kappa^2(h(u) - 2h(u)H(u))du, & (4) \end{aligned}$$

where $H'(t) = h(t)$. We compare (1) with the prototypical Black-Scholes model for a stock price:

$$dX_t = rdt + \sigma dW_t. \tag{5}$$

Under the model (5)

$$\begin{aligned} E[X_t - X_{t'}|t'] &= \int_{t'}^{t} rdu, & (6) \\ \text{Var}[X_t - X_{t'}|t'] &= \int_{t'}^{t} \sigma^2 du. & (7) \end{aligned}$$

The first-order condition, in [3], has the interpretation that $\mu(t)$ in (1) grows in order to compensate a representative investor for the risk associated with a crash. This first-order condition can also be retrieved by equating conditional means (equations (3) and (6)) giving $\mu(t) = r + \kappa h(t)$. If we ignore volatility fluctuations by setting $\sigma(t) = \sigma$, then our pre-crash model for the asset price becomes

$$dX_t = (r + \kappa h(t))dt + \sigma dW_t. \tag{8}$$

However, this is actually a rather poor empirical model [19], failing to account for the volatility fluctuations in (1). Under a Markowitz interpretation, means represent returns and variances/standard deviations represent risk. Suppose that in (1) $\sigma(t)$ adapts in an analogous way to $\mu(t)$ in order to compensate a representative investor for bearing additional amounts of risk. Equating conditional variances, equations (4) and (7) gives

$$\sigma^2(t) = \sigma^2 - \kappa^2 h(t) + 2\kappa^2 h(t)H(t). \tag{9}$$

Note that integrating (9) leads to the following expression for the conditional variance:

$$\sigma^2(t-t') - \kappa^2 \left[H(u) - H^2(u)\right]_{t'}^{t}. \tag{10}$$

The second term in (10) is negative and illustrates an *illusion of certainty* – a decrease in the conditional variance brought about by the bubble process. Intuitively, in order for a bubble to occur not only must the returns increase but the volatility must also decrease, otherwise the fundamental model (5) represents a safer and hence more attractive investment than (1). We use (5) as a model of a 'fundamental' or purely stochastic regime, as in Black-Scholes theory. From (9), our model for a bubble becomes

$$dX_t = [r + \kappa h(t)]dt + \sqrt{\sigma^2 - \kappa^2 h(t)(1-2H(t))}dW_t. \tag{11}$$

The simplest $h(t)$ considered in [3] is

$$h(t) = B(t_c - t)^{-\alpha}, \tag{12}$$

where it is assumed that $\alpha \in (0,1)$ and t_c is a critical time when the hazard function becomes singular, by analogy with phase transitions in statistical mechanical systems [20]. Here, we choose on purely statistical grounds

$$h(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta + t^\beta}, \tag{13}$$

which is the form corresponding to a log-logistic distribution and is intended to capture the essence of the previous approach. The log-logistic distribution is commonly used in survival analysis, see e.g. [21], as the hazard rate has both a relatively simple form and, for $\beta > 1$, has a non-trivial mode at $t = \alpha(\beta-1)^{\frac{1}{\beta}}$. This distribution has probability density

$$f(x) = \frac{\beta\alpha^\beta x^{\beta-1}}{(\alpha^\beta + x^\beta)^2},$$

on the positive half-line. The cumulative density function is

$$F(x) = 1 - \frac{\alpha^\beta}{\alpha^\beta + x^\beta}.$$

The model (11) with $h(t)$ given by (13) has the solution

$$X_t = X_0 + rt + \kappa \ln\left(1 + \frac{t^\beta}{\alpha^\beta}\right) + \int_0^t \sqrt{\sigma^2 - \kappa^2 \frac{\beta u^{\beta-1}}{\alpha^\beta + u^\beta}\left(1 - 2\ln\left(1 + \frac{u^\beta}{\alpha^\beta}\right)\right)}\, dW_u. \quad (14)$$

From (14) the conditional densities can be written as

$$X_t | X_s \sim N(\mu_{t|s}, \sigma^2_{t|s}), \quad (15)$$

where

$$\begin{aligned} \mu_{t|s} &= X_s + r(t-s) + \kappa \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right), \\ \sigma^2_{t|s} &= \sigma^2(t-s) - \kappa^2 \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right) + \kappa^2\left(\ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right)\right)^2. \end{aligned}$$

Under the fundamental equation (5) these expressions are simply $\mu_{t|s} = X_s + r(t-s)$ and $\sigma^2_{t|s} = \sigma^2(t-s)$. Given empirical data, the likelihood function can be simply calculated as a product of normal densities and the likelihood-ratio test can be used to test for bubbles. However, rather than the usual simple form, the appropriate limiting density of this ratio is non-standard and depends on the geometry of the underlying parameter space [22]. The hypothesis of no bubble is the hypothesis that $\kappa = 0$. Since $\kappa \in [0, 1]$, $\kappa = 0$ is a boundary point of the parameter space, and the distribution of the likelihood ratio statistic becomes non-standard. Using the method of [22], we can see that the distribution of the likelihood ratio statistic is approximately

$$\frac{1}{2}\chi^2_2 + \frac{1}{2}\chi^2_3, \quad (16)$$

where the distribution in (16) is obtained by sampling with probability 1/2, from a χ^2_2 and sampling from a χ^2_3 with probability 1/2. [1]

Crash-size distribution. Suppose a crash has not occurred by time t. The crash-size distribution resists an analytical description. However, a Monte Carlo algorithm to simulate the crash-size C is straightforward and reads as follows:

1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.

2. Generate $C \sim (1 - e^{-\kappa})e^Z$,

[1] Under the null hypothesis of no bubble α and β are assumed to lie in the interior of the admissible parameter space, i.e. in the interior of $[0, \infty)$.

where

$$Z \sim N\left(X_t + ru + \kappa \ln\left(\frac{\alpha^\beta + u^\beta}{t^\beta + \alpha^\beta}\right), \sigma^2 u - \kappa^2 \ln\left(\frac{\alpha^\beta + u^\beta}{t^\beta + \alpha^\beta}\right) + \kappa^2 \left(\ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)\right)^2\right)$$

We note that simulating u from the log-logistic distribution is straight-forward and possible via inversion using

$$F^{-1}(x) = \alpha\left(\frac{x}{1-x}\right)^{\frac{1}{\beta}} \quad \text{or} \quad F^{-1}(x) = \left(\frac{\alpha^\beta + t^\beta}{1-x} - \alpha^\beta\right)^{\frac{1}{\beta}} \quad \text{under the constraint } u \geq t.$$

Post-crash dynamics. Before a crash equation (11) applies. After a crash, the price reverts to the fundamental price dynamics (5). Suppose the crash occurs at time C. At $t \geq C$ we have that

$$X_{t+h}|X_t \sim N(rh - \kappa, \sigma^2 h), \tag{17}$$

but for $t < C$

$$\mathrm{Var}(X_{t+h}|X_t) = \sigma^2 h - \kappa^2 \ln\left(\frac{\alpha^\beta + (t+h)^\beta}{\alpha^\beta + (t)^\beta}\right) + \kappa^2 \left(\ln\left(\frac{\alpha^\beta + (t+h)^\beta}{\alpha^\beta + (t)^\beta}\right)\right)^2 \tag{18}$$

(17) predicts a linear-in-time increase in the mean of the log-price in the aftermath of the crash. The final two terms in (18) indicate a rise in volatility in the immediate aftermath of the crash – a rise which may dominate the drift r for some time afterwards.

Fundamental values. The above model suggests a simple approach to estimate fundamental value. Under the fundamental dynamics (5)

$$E(P(t)) = P(0)e^{(r+\frac{\sigma^2}{2})t}, \tag{19}$$

and we use (19) to provide simple estimates of fundamental value in our empirical application in Section 7. This approach recreates the widespread phenomenology of approximate exponential growth in economic time series (see e.g. Chapter 7 in [23]).

3 Multivariate bubbles

In this section we consider a multivariate extension of the basic model in Section 2 and consider models for bubbles in a multivariate portfolio of n assets. The results of this section also motivate the study of contagion in Section 4.

The equation describing fundamental or purely stochastic behaviour becomes

$$dX_t = rdt + \Sigma^{1/2}dW_t, \tag{20}$$

where r is a $n\times 1$ vector, W_t is standard n-dimensional Brownian motion and Σ is a $(n\times n)$ covariance matrix. As a model for multivariate bubbles we replace (1) with

$$dX_t = \mu(t)dt + \Sigma^{1/2}(t)dW_t - \kappa dj(t), \tag{21}$$

where κ is a $n\times 1$ vector of known crash sizes and $\mu(t)$ is $n\times 1$. The instantaneous drift corresponding to (21) is

$$\mu(t) - \kappa h(t). \tag{22}$$

Setting (22) equal to r gives

$$\begin{pmatrix} \mu_1(t) \\ \dots \\ \mu_n(t) \end{pmatrix} = \begin{pmatrix} r_1 + \kappa_1 h(t) \\ \dots \\ r_n + \kappa_n h(t) \end{pmatrix}.$$

The instantaneous variance associated with (20) is Σ. The instantaneous variance associated with (21) is

$$\Sigma(t) + \kappa\kappa^T(h(t) - 2h(t)H(t)),$$

and the second-order condition gives

$$\Sigma(t) = \Sigma - \kappa\kappa^T(h(t) - 2h(t)H(t)).$$

Prior to the crash, we have that $X_t|X_s \sim N(\mu_{t|s}, \Sigma_{t|s})$ where

$$\begin{aligned} \mu_{t|s} &= X_s + r(t-s) + \kappa \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right), \\ \Sigma_{t|s} &= \Sigma(t-s) - \kappa\kappa^T \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right)\left(1 - \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right)\right). \end{aligned}$$

Suppose we have a portfolio $(\omega_1, \dots, \omega_n)^T$ in stocks $(X_1, \dots, X_n)$ with the ω_i non-negative and satisfying $\sum_{i=1}^n \omega_i = 1$. Suppose a crash has not occurred by time t. The portfolio-wide crash-size distribution can be obtained by simulation using the following algorithm:
1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.
2. Generate $C \sim (1 - e^{-\kappa})e^Z$,

where the distribution of Z is normal with mean

$$\omega^T(X_t + ru + \kappa \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)),$$

and variance

$$\omega^T \left(\Sigma u - \kappa\kappa^T \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)\left(1 - \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)\right)\right)\omega.$$

Statistical tests for multivariate bubbles. One can test for the presence of at least one bubble in a portfolio of n assets. Alternatively, one may also test for the presence of a bubble in m of the n assets in the portfolio where $m < n$. Testing for a bubble in at least one of the assets in an n-dimensional portfolio corresponds to testing the null hypothesis that $\kappa = (0, \ldots, 0)^T$. In this case the likelihood-ratio statistic has the null distribution

$$\left(\frac{1}{2}\chi_2^2 + \frac{1}{2}\chi_3^2\right)^n, \tag{23}$$

where the distribution in (23) is an n-fold convolution of the mixture distribution in (16). We can also test the hypothesis of an m-dimensional bubble, where $m < n$. We may test the hypothesis that $\kappa = (\kappa_1, \ldots, \kappa_m, \kappa_{m+1}, \ldots, \kappa_n)^T$ against the alternative $\kappa = (0, \ldots, 0, \kappa_{m+1}, \ldots, \kappa_n)^T$. In this case the approximate distribution of the likelihood ratio statistic is an m-fold convolution of the chi-squared mixture distribution with itself. If $\kappa_{m+1}, \ldots, \kappa_n$ are all non-zero this is a test for bubbles in the remaining m assets assuming bubbles in assets $m + 1$–n. If $\kappa_{m+1}, \ldots, \kappa_n$ all equal zero this is a test that bubbles occur in the subset of m assets only.

4 Contagion

Based on the model of the previous section, we discuss a simple model of the contagious effects directly brought about by the bubble process. First however, we give a brief overview.

Assessing contagion is a delicate theoretical and empirical issue in economics. A distinction needs to be made between genuine contagion and simple co-dependence, with much of the literature failing to make an adequate distinction between the two [24]. Asset prices are assumed to exhibit non-zero correlations in normal times. Contagion occurs when there is a genuine change in the market's correlation or cross-linkage structure brought about by specific events or crises. Anything else is simply co-dependence.

As an illustration, consider the following. Suppose the prices of two assets are correlated. Following exposure to a commmon shock the price of both assets falls. Simplistic empirical analysis may suggest enhanced correlation in these periods *without* a genuine change in market cross-linkages having actually occurred. Empirical approaches based on copulae, see e.g. [25] and the interesting economic interpretation therein, assess contagion on the basis of "intrinsic" copulae properties and do allow for some headway. However, this approach remains largely empirical and, as we show, our basic framework explicitly allows us to model contagion as defined by a change in the cross-linkage or correlation structure directly brought about by the bubble process.

Analogously to (17), our model for post-crash dynamics becomes

$$X_{C+h}|X_C \sim N(rh-\kappa, \Sigma h).$$

We consider two assets X and Y and let $h=1$. Under our model for post-crash dynamics

$$\begin{pmatrix} X_{C+1}-X_C \\ Y_{C+1}-Y_C \end{pmatrix} \sim N\left(\begin{pmatrix} r_X-\kappa_X \\ r_Y-\kappa_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right).$$

We have that

$$E(Y_{C+1}-Y_C|X_{C+1}-X_C=x) = r_Y-\kappa_Y+\frac{\sigma_{XY}}{\sigma_X^2}(x-r_X+\kappa_X). \quad (24)$$

(24) suggests that contagion from X to Y occurs if

$$\frac{\kappa_X\sigma_{XY}}{\sigma_X^2} > \kappa_Y,$$

i.e. if the relative effect of a shock of size κ_X on X has a larger impact on Y than κ_Y. Alternatively, we can model shocks across whole sectors rather than simply across single assets. Let $x \sim N(\mu, \Sigma)$ and write

$$x=(x_A, x_B)^T, \mu=(\mu_A, \mu_B)^T, \Sigma = \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{pmatrix}.$$

The conditional mean of A given B is given by

$$\mu_{A|B} = \mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(x_B-\mu_B),$$

see e.g. Chapter 2 in [26]. Let X and Y be vectors representing different sectors of the economy. Our model for post-crash dynamics becomes

$$\begin{pmatrix} X_{C+1}-X_C \\ Y_{C+1}-Y_C \end{pmatrix} \sim N\left(\begin{pmatrix} r_X-\kappa_X \\ r_Y-\kappa_Y \end{pmatrix}, \begin{pmatrix} \Sigma_X^2 & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y^2 \end{pmatrix}\right).$$

We have that

$$E(Y_{C+1}-Y_C|X_{C+1}-X_C=x)=r_Y-\kappa_Y+\Sigma_{YX}\Sigma_{XX}^{-1}(x-r_X+\kappa_X), \tag{25}$$

and contagion from X to Y occurs if

$$\Sigma_{YX}\Sigma_{XX}^{-1}\kappa_X>\kappa_Y.$$

5 Non-Gaussian models

In this section, motivated by the stylised empirical facts of financial markets (see e.g. [14], Chapter 7), we modify (5) in order to accommodate non-Gaussian behaviour. We use the NIG Lévy process [27]-[28] as our benchmark. The NIG process has the integral representation

$$X_t=X_0+\int_0^t rdu+\int_0^{S_t}\sigma dW_u, \tag{26}$$

where S_t is an inverse Gaussian Lévy process satisfying $E(S_t)=t$ ([14], Chapter 4). Here, we restrict to univariate models, though multivariate models are possible with an equivalent representation using multivariate Brownian motion. For an application of multivariate NIG models to financial data see e.g. Chapter 3 in [29]. Since (26) represents purely stochastic behaviour in the absence of a bubble, our bubble model becomes

$$X_t=\int_0^t r(u)du+\int_0^{S_t}\sigma(u)dW_u-\kappa dj(t), \tag{27}$$

where $j(t)$ is a jump process with hazard function $h(t)$. Prior to a crash we have $\mu(t)=r+\kappa h(t)$ and $\sigma^2(t)=\sigma^2-\kappa^2h(t)+2\kappa^2h(t)H(t)$ as before. The transition densities for the models in (26-27) are symmetric NIG. Adapting a non-standard parameterisation in [30], we write the symmetric NIG density as

$$f(x)=\frac{(\sigma^2)^{-1/2}\sqrt{\chi}e^{\chi}}{\pi}\sqrt{\frac{\chi}{\chi+\frac{(x-\mu)^2}{\sigma^2}}}K_1\left(\sqrt{\chi^2+\frac{\chi(x-\mu)^2}{\sigma^2}}\right).$$

Suppose the price is observed at a sequence of regularly spaced price increments $t_1, \dots, t_n$. We have the following model for log-returns:

$$\log(P_{t+1}) - \log(P_t) \sim NIG(\mu_t, \sigma_t^2, \chi).$$

Under the null hypothesis of no bubble in (26) $\mu_t = r$ and $\sigma_t^2 = \sigma^2$. Under the bubble model (27) we have that

$$\begin{aligned}
\mu_t &= r + \kappa \ln\left(\frac{\alpha^\beta + (t+1)^\beta}{\alpha^\beta + t^\beta}\right), \\
\sigma_t^2 &= \sigma^2 - \kappa^2 \ln\left(\frac{\alpha^\beta + (t+1)^\beta}{\alpha^\beta + t^\beta}\right)\left(1 - \ln\left(\frac{\alpha^\beta + (t+1)^\beta}{\alpha^\beta + t^\beta}\right)\right).
\end{aligned}$$

The likelihood ratio test in (16) can again be used to test for bubbles. Suppose a crash has not occurred by time t. The crash-size distribution can be simulated using:
1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.
2. Generate $C \sim (1 - e^{-\kappa})e^Z$,
where

$$Z \sim NIG\left(X_t + ru + \kappa \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right), \sigma^2 u - \kappa^2 \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)\left(1 - \ln\left(\frac{u^\beta + \alpha^\beta}{t^\beta + \alpha^\beta}\right)\right), \chi\right).$$

We note that under this non-Gaussian model more extreme crash-sizes are more likely.

6 Models for bubbles via a Garch model of stochastic volatility

Large-scale empirical study suggests that on real markets volatility is non-constant, volatility clustering occurs, and a stochastic model for volatility is appropriate [14], Chapter 7. In this section we review a Garch(1, 1) model of stochastic volatility, see e.g. Chapter 12 in [23], and show how this model can be used as an alternative benchmark.

The Garch(1, 1) model can be written as

$$\begin{aligned}
X_{t+1} &= \mu + X_t + \epsilon_{t+1}, \qquad (28) \\
\epsilon_t &\sim N(0, h_t), \\
h_t &= \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1},
\end{aligned}$$

where ϵ_t i.i.d $N(0, 1)$. Suppose a crash has not occurred by time t. As a model for a bubble replace (28) with

$$X_{t+1} = \mu(t) + X_t + \epsilon_{t+1} - \kappa j(t+1). \tag{29}$$

Under the regular model given by (28)

$$E(X_{t+1} - X_t | t) = \mu.$$

Under the bubble model (29)

$$E(X_{t+1} - X_t | t) = \mu(t) - \kappa[H(t+1) - H(t)].$$

Retaining indifference by equating conditional expectations gives $\mu(t) = \mu + \kappa[H(t+1) - H(t)]$, as before. Some further simplification is possible if we make the additional assumption, as in the empirical literature see e.g. [31], that $\mu(t) = \mu$ is constant. In this case the result of the bubble process is volatility-induced financial growth.

Suppose that co-existence of the bubble and fundamental models is explained by equating the conditional expectation of the underlying price process $P(t)$. Under the regular model (28)

$$E[P_{t+1}|t] = P(t)e^{\mu + \frac{(\beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1})}{2}}. \tag{30}$$

Under the bubble model (29)

$$E[P_{t+1}|t] = P(t)e^{-\kappa}[H(t+1) - H(t)]e^{\mu + \frac{h_t}{2}}. \tag{31}$$

Equating (30) and (31) gives

$$h_t = (\beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}) + 2\kappa - 2\ln\left(H(t+1) - H(t)\right).$$

7 Empirical analysis

As an empirical application we consider the UK housing bubble from 2002-2007 by modelling a monthly time series of average house prices. Without high-frequency effects, we restrict attention to the Gaussian model in Section 2. The likelihood for the Gaussian random walk model is 222.366 and for the model in (14) is 234.663. The likelihood ratio in (16) is 24.595 giving a p-value of 0.000 and strong evidence in favour of a speculative bubble in the UK house-price series. A plot of UK house prices and estimated fundamental values (19) is shown in Figure 1. Notable differences can be observed between the two series with prices well in excess of fundamental levels. Out-of-sample historical values and out-of-sample estimated fundamental values are listed in Table 1 and show prices reverting to fundamental values over time. The probability density of the crash-sizes is shown in Figure 2 and suggests a crash in the range £3.5-9,000. However, this appears to underestimate the scale of the likely falls and the estimated fundamental values are perhaps a better guide.

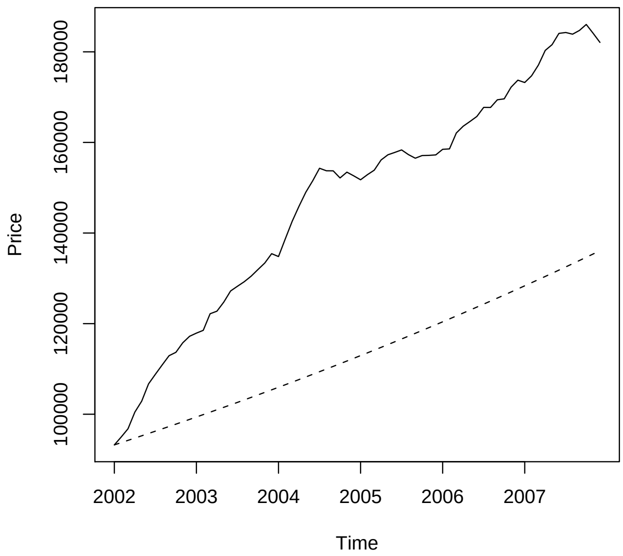

Figure 1: Plot of average UK house-prices and estimated fundamental value.

Date	Historical	Est. Fundamental	Date	Historical	Est. Fundamental
Jan 08	180,473	136,822	Jan 09	150,501	145,855
Feb 08	179,358	133,553	Feb 09	147,746	146,634
Mar 08	179,110	138,288	Mar 09	150,946	147,418
Apr 08	178,555	139,027	Apr 09	151,861	148,205
May 08	173,583	139,769	May 09	154,016	148,997
Jun 08	172,415	140,516	Jun 09	156,442	149,793
Jul 08	169,316	141,266	Jul 09		150,593
Aug 08	164,654	142,021	Aug 09		151,398
Sep 08	161,797	142,780	Sep 09		152,206
Oct 08	158,444	143,543	Oct 09		153,019
Nov 08	158,442	144,309	Nov 09		153,837
Dec 08	153,048	145,080	Dec 09		154,659

Table 1: Out-of-sample house prices and comparison with estimated fundamental value

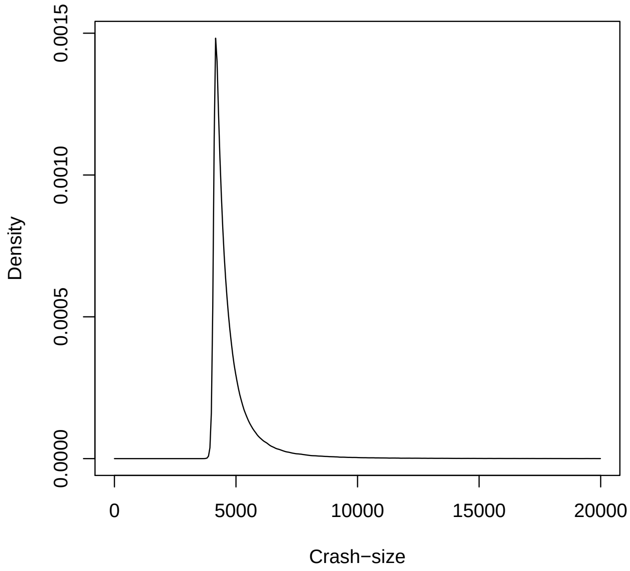

Figure 2: Simulated crash-size distribution.

8 Conclusions

This paper has provided a stochastic version of a the model in [3]. Crash precursors are a super-exponential growth accompanied by an illusion of certainty, characterised by a decrease in the volatility function prior to the crash. Using a benchmark Gaussian model a myriad of potential applications to economics were discussed including statistical tests for bubbles, crash-size distributions and post-crash dynamics, multivariate bubbles and contagion. This framework was further extended to include both non-Gaussian models and stochastic volatility. As a brief empirical application we consider the UK housing bubble in the early to mid 2000s. Prices appear to be in excess of estimated fundamental levels but seem to revert towards estimates of fundamental value out of sample. Further work will include large-scale empirical application of the model and more in-depth explorations of the non-Gaussian and stochastic volatility models in Sections 5-6.

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