Pattern recognition and subjective belief learning in repeated mixed strategy games

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10 July 2009

PATTERN RECOGNITION AND SUBJECTIVE BELIEF LEARNING IN REPEATED MIXED STRATEGY GAMES[1]

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Abstract

This paper aspires to fill a conspicuous gap in the existing literature on learning in games, namely the absence of any empirical verification of learning rules involving pattern recognition. An extension of weighted fictitious play is proposed both obeying cognitive laws of subjective perception, and allowing for two-period pattern detection of opponents' behavior. The unconditional prior probability of a subject employing a pattern detecting belief model is 0.34, as estimated by a mixture (latent-class) model of the elicited belief and action data series from Nyarko and Schotter (2002), or 0.551 using only action data. The conditional prior probability of using pattern recognition was found to depend positively on a measure of the exploitable two-period patterns in an opponent's action choices, in stark contrast to the minimax hypothesis. Also, standard weighted fictitious play models are found to significantly bias memory parameter estimates upwards, compared to the proposed subjective fictitious play models. Finally, simulations of learning models reveal that the simple win-stay/lose-shift heuristic may be effective even against more complex pattern detecting models.

Keywords: Behavioral game theory; Learning; Fictitious play; Pattern detection; Simulations; Beliefs; Repeated games; Mixed Strategy Nash equilibria; Economics and psychology; Agent based computational economics.

1. introduction

Repeated mixed strategy games have been one of the foci of both the experimental game theory literature and its theoretical counterpart - Camerer (2003) and Kagel and Roth (1995) are excellent introductions to the field of behavioral and experimental game theory. The literature is rife with experimental studies investigating whether human play is well described by theoretical solutions such as the mixed strategy Nash equilibrium (MSNE), the Quantal Response Equilibrium (McKelvey and Palfrey, 1995), or other equilibrium concepts and refinements[4].

These theoretical solutions implicitly assume instantaneous equilibration, and therefore remain silent on the learning dynamics of the off-equilibrium path. In response to this, researchers resorted to postulating theories of learning originally inspired by the psychology and artificial intelligence literature which already had a strong history of grappling with such issues. Most learning rules employed in the literature are derivatives of two basic models, belief learning (Cheung and Friedman, 1997) and reinforcement learning (Roth and Erev, 1995), or a mixture of both as in the EWA model

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[1]This is the first of three papers towards a PhD degree, conferred by the University of Sydney in May 2008.

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[3]The author would like to thank Kunal Sengupta for constructive input and thesis supervision, Yaw Nyarko and Andrew Schotter for kindly providing the experimental data, Editor David Levine and three anonymous referees for valuable observations, Ananish Chaudhuri and participants at the First Summer Workshop in Experimental Economics and Social Sciences (ExESS), University of Auckland Business School. This research was made possible by funding provided by the Government Department of Education, Science and Training (Australian Government), the College of Humanities and Social Sciences and the Faculty of Economics and Business at the University of Sydney.

[4]Examples of specific papers are Palacios-Huerta (2003), Chiappori et al. (2002), Walker and Wooders (2001), Binmore et al. (2001), Bloomfield (1994), Brown and Rosenthal (1990), Rapoport and Budescu (1997) and Levitt et al. (2008).

(Camerer and Ho, 1999). Weighted fictitious play belief learning, henceforth abbreviated to *wfp*, is the basis of the approach that will be employed in this paper and assumes that players form beliefs about their opponent's future play by observing the historical frequency of their opponent's actions. These beliefs are then translated into final actions through the use of a stochastic decision rule.

This study will re-interpret the data from the innovative experiment by Nyarko and Schotter (2002), which directly elicited players' beliefs about their opponents' actions. The main contributions of this paper to the existing literature are:

1. The specification of a pattern-detecting weighted fictitious play model and empirical validation of its use by human subjects.
2. The specification of an extension to *wfp* that incorporates common psychophysical laws of subjective perception, and its validation as a more accurate model of belief formation.
3. An alternative explanation for the existence of negative individually estimated responsiveness/sensitivity parameters in decision rules, a result which is normally ascribed either to anticipatory learning (Selten, 1991) or simply irrational behavior. It will be contended that this could occur due to model misspecification if a non-pattern detecting learning model is used to fit the behavior of a pattern detecting player. The same argument will be put forth to explain the occurrence of negative individually estimated memory parameters in weighted fictitious play belief models.
4. The presentation of a taxonomy of heterogeneity which makes important distinctions between the causes of observed heterogeneity in subjects' behavior. Between-subjects heterogeneity that is due to innate differences in behavior or ability, and within-subjects heterogeneity that is a consequence of subjects conditioning their behavior on characteristics of their opponent's action choices.
5. Empirical validation of the existence of significant within-subjects heterogeneity as subjects are more likely to using a pattern-detecting belief model the more their opponents deviate from independently distributed (or serially uncorrelated) actions, as prescribed by a mixed strategy Nash equilibrium.
6. Investigation through agent based simulations of the evolutionary fitness of various types of belief learning rules, with and without pattern recognition. The win-stay/lose-shift heuristic, despite its simplicity, will be shown to perform very well against more complex pattern detecting models of behavior.

The layout of this paper is as follows. Section 2 is a literature review drawing both from the behavioral game theory literature as well as from the psychology literature with regards to human ability at randomizing and detecting sequential patterns. Section 3 proposes modifying the parametric form of standard *wfp* models to allow for pattern detection, and to obey principles of psychophysics and subjective perception. Section 4 presents a taxonomy of heterogeneity and is followed by Section 5 that introduces the original experiment by Nyarko and Schotter (2002). Section 6 presents the results of the estimation of stated belief models using only the elicited beliefs, whereas Section 7 estimates models of action choice using a two-stage procedure that makes use of action data and the fitted stated beliefs from the previous section. Section 8 employs a joint estimation procedure that indirectly estimates the underlying beliefs only from the action data. Section 9 represents a change in methodology as agent based computational models are used to evaluate the evolutionary fitness of various belief learning models. Finally, Section 10 summarizes the main conclusions of this paper.

2. LITERATURE REVIEW

The majority of experimental game theory studies collect data on the observable actions of players and then attempt to fit a model of off-equilibrium behavior. This entails the simultaneous estimation of both the belief generating mechanism, which is not directly observable, as well as the decision rule. Salmon (2001) finds that simultaneous estimation, and therefore indirect estimation of the belief model, often performs poorly in recovering true underlying parameter values. Nyarko and Schotter (2002) made an important contribution to the literature by implementing an experimental setup that made beliefs observable, thereby effectively avoiding the econometric problems of joint estimation. In their paper, they not only collect data on the actions of players in a repeated game with a unique mixed strategy Nash equilibrium (MSNE), but also elicit beliefs by asking players to state the probability with which they thought their opponents would play their pure strategies before each round.

There exist very few theoretical studies of learning models incorporating pattern recognition in the game theory literature, and to the best of our knowledge, no relevant empirical studies. Fudenberg and Levine (1998) in their authoritative book on learning in games discuss some of the theoretical implications of what they refer to as conditional fictitious play. This is defined as a broad class of fictitious play learning algorithms where fictitious play frequencies are calculated for each of a number of predefined disjoint subsets of the history of play, instead of the standard case where these frequencies are calculated over a single set of the history. Beliefs are then calculated by conditioning on the last realized subset of play i.e. using the associated fictitious play frequencies corresponding to that particular subset. A special case of this broad range of learning rules will be employed in this paper, with the subsets of the history defined as the strategies consisting of all two-period temporally consecutive combinations of an opponent's actions.

Aoyagi (1996) proves that in zero-sum games with a unique Nash equilibrium, if both players follow conditional fictitious play rules that asymptotically recognize patterns of the same length, then players' beliefs converge to the Nash equilibrium action profile with probability one. Sonsino (1997) examines the convergence properties of game play when players can recognize cyclical strategic patterns in opponents' behavior, proving that convergence to fixed patterns of pure strategy Nash equilibria occurs with probability one for a large class of games. Finally, it was proven that a necessary condition for convergence to a mixed strategy Nash equilibrium is the use of arbitrarily long histories of play.

2.1. *Literature review of humans' (in)ability to randomize*

Studies in the psychology literature, such as Bar-Hillel and Wagenaar (1991) and Rapoport and Budescu (1997), find that people have difficulty in creating truly random sequences of variables. They tend to produce over-alternating sequences (with too many runs) and regress towards the prescribed frequencies as they find such sequences more representative of the distributions they are emulating.

Game theorists have been interested in these documented inefficiencies of human randomization as they imply that one should expect deviations from the MSNE prescription of independently distributed actions. Palacios-Huerta (2003) and Chiappori et al. (2002) investigate penalty kicks in professional soccer, concluding that the minimax hypothesis cannot be rejected. Palacios-Huerta and Volij (2008) find that professional soccer players exhibited transfer of learning as they continued to play according to minimax even in laboratory settings. On the other hand, Levitt et al. (2008) compare college students and professional soccer, bridge, poker players in laboratory settings, finding significant deviations from MSNE behavior for all groups indicating that the professional players have not transferred their learning from the field. Walker and Wooders (2001) examine tennis serves and find that there is evidence of professional players conditioning on past actions, but behavior is closer to the MSNE for professional players or experts than for inexperienced subjects.

In conclusion, the verdict is still out with regards professionals playing the MSNE when outside of their field of expertise. However, the literature is more consistent in its findings that non-professionals that do not have a long history of playing a specific game with large enough monetary incentives to fine-tune their strategies will not conform closely to the MSNE prescription of serially uncorrelated action choices.

2.2. *Literature review of pattern detection or sequence learning in humans*

The question of whether humans have the ability to detect patterns is a well established research topic in the psychology literature referred to as sequence learning, Clegg et al. (1998) provides a concise introduction. Explicit learning is the result of conscious and intentional cognitive processes that subjects are able to report, whereas implicit learning occurs subconsciously rendering the subject unaware and therefore unable to directly acknowledge this type of learning (Cleeremans et al., 1998). The seminal paper by Nissen and Bullemer (1987) advanced the view that sequence learning is primarily an implicit, rather than explicit, form of learning. The current state of the literature has accepted that humans engage in sequence learning and the latest research is primarily directed at using different experimental methodologies to indirectly reveal implicit learning, with the purpose of determining the relative importance of explicit versus implicit learning.

Before proceeding further it is necessary to define sequence learning and a measure of the depth of such learning. The definition of nth order probability information is the use of information at time $t - n + 1$ to infer behavior at time t. If information from all periods between $t - n + 1$ and $t - 1$ are used then this is referred to as nth order adjacent dependency, alternatively if not all of the periods are relevant then it is referred to as non-adjacent dependency[5]. Sequence learning involves pattern detection because adjacent nth order probability information essentially involves recognizing n consecutive time period strategies or patterns. For example, second-order probability information involves calculating the probability of an action conditional on the action played in the previous

[5]The numbering of order probabilities in this paper differs from that in the psychology literature for ease of exposition later in the paper. In the psychology literature an nth order probability refers to the use of information at the $t - n$th period instead of the definition given in the main text which refers to the $t - n + 1$st period. All references to order probabilities will henceforth refer to the definition adopted in this paper.

period.

The existence of sequence learning is well documented by studies such as Remillard (2007) and Remillard and Clark (2001), who find evidence of implicit sequence learning of second- through to fifth-order adjacent and non-adjacent probabilities. Other studies in the experimental psychology field, such as Gomez (1997), have found evidence of explicit knowledge of second-order probabilities in which the subjects were consciously aware of their learning. Whether sequence learning is predominantly implicit or explicit has important implications for experimental game theoretic models of pattern recognition. If such learning is explicit then it should be detectable in the elicited belief series, whereas if it is implicit then its effect would only be indirectly apparent when using action data to estimate the learning models.

In conclusion, the aforementioned research justifies investigating pattern recognition models of learning in game theory as they have documented that pattern detection of sufficient depth is possible in the human brain, both explicitly and implicitly.

3. PROPOSED EXTENSIONS TO STANDARD WEIGHTED FICTITIOUS PLAY

This section will propose two significant extensions of the standard weighted fictitious play model used in the literature. The first extension will directly model pattern recognition in the belief formation process. The second extension, referred to as subjective fictitious play, *sfp* henceforth, will incorporate common principles of subjective perception, whilst also nesting the standard *wfp* model for specific parameter values. Finally, the consequences of allowing negative estimates of key parameters will be discussed, in particular how this may allow non-pattern detecting learning models to indirectly capture two-period patterns.

3.1. *Modification of learning rules to include cases of pattern recognition*

In standard weighted fictitious play (Cheung and Friedman, 1997), or equivalently single-period fictitious play *fp1,* the beliefs, $fp1_{i,t}(a_j)$, of player i regarding the probability of his opponent playing action a_j are equal to the count $C_{i,t}(a_j)$, presented in equation 1. Let a_i represent the action taken by player i and for any action a_j, define player i's count of a_j at time $t \geq 2$ as:

$$C_{i,t}(a_j) = \frac{I_{t-1}(a_j) + \sum_{u=1}^{t-2} \gamma_i^u \cdot I_{t-u-1}(a_j)}{1 + \sum_{u=1}^{t-2} \gamma_i^u} \tag{1}$$

The indicator function $I_t(a_j)$, takes the value of one if player j chose action a_j in time period t or the value of zero otherwise. The memory decay parameter for each player is γ_i and memory loss (or weighting of past observations) is assumed to be exponential in discrete time. Let the nth-order probabilities of play refer to the probability of playing an action conditional on the actions chosen in the previous $n-1$ periods, with the special case $n = 1$ referring to the probability of playing an action irregardless of prior history. It is clear that the *fp1* learning algorithm keeps track of an opponent's first-order probabilities of action choices and will therefore be able to detect deviations from the MSNE predictions of observed frequencies of single period actions[6].

The *fp1* algorithm however is not designed to detect second- and higher-order deviations from the MSNE because it does not keep track of consecutive temporal sequences of actions. Fictitious play can be generalized to what shall be referred to as n-period (weighted) fictitious play (*fpn*) where n is an integer greater than or equal to one and refers to the complexity and depth of pattern detection[7]. *Fp2* is more sophisticated than *fp1* because it tracks how many times two temporally consecutive sequences of actions have been observed and then conditions the probability of an action being played on the previous action chosen by the opponent. In this case it is assumed that a player is making use of second-order probability information[8]. For example, suppose our opponent's play is r-r-g-g-r-r-r[9], *fp2* will evaluate how often the following sequences have turned up in past play: r followed by r, g followed by g, r followed by g and g followed by r.

Let the subscripts i and j denote two different players, then given actions a_j and a'_j, $I_t(a_j|a'_j)$ is an indicator function that takes a value of one if a_j was the action played at time t and a'_j was the action played at time $t-1$ and a value of zero otherwise. Define for player i at time $t \geq 3$, the count of a_j given action a'_j with memory parameter γ_i as:

$$C_{i,t}(a_j|a'_j) = \frac{I_{t-1}(a_j|a'_j) + \sum_{u=1}^{t-2} \gamma_i^u \cdot I_{t-u-1}(a_j|a'_j)}{1 + \sum_{u=1}^{t-2} \gamma_i^u} \tag{2}$$

The *fp2* beliefs of player i regarding action a_j given action a'_j from the discrete strategy set S_j of player j is[10]:

$$\mathit{fp2}_{i,t}(a_j|a'_j) = \frac{C_{i,t}(a_j|a'_j)}{\sum_{a_j \in S_j} C_{i,t}(a_j|a'_j)} \tag{3}$$

Subjects' decisions as to what depth of pattern recognition to employ will depend not only on the likely depth of an opponent's behavioral patterns but also on the cost of detecting these patterns. The number of frequency variables or counts an *fpn* belief model must keep track of is the number of actions available raised to the power of n. Also, further cognitive resources are needed to remember the last $n-1$ periods of play in order to be able to condition, so that the cognitive requirements increase with n at a faster than exponential rate. Increasing the size of the action space or the depth of pattern recognition leads to drastically higher computational cost and therefore pattern detection

[6] Shachat and Swarthout (2004) empirically verify that humans better respond to deviations in first-order probabilities of play as long as they are relatively far away from the MSNE.

[7] This proposed learning rule is a special case of the class of learning rules that Fudenberg and Levine (1998) refer to as conditional fictitious play.

[8] In general any *fpn* model uses nth order adjacent probability information.

[9] Counts are created by allowing for overlapping sequences so that each action is counted twice, once as the last action in one 2-period sequence and one as the first action in another 2-period sequence. This is because there is an inherent problem in that two very different sequences can be obtained by changing when the counting starts. Also if overlapping sequences are not used then conditioning on the previous action will be problematic.

[10] This definition assumes that the denominator is not zero i.e. that the action a'_j has been played at least once in the past. In cases where a'_j has not been observed beliefs are assumed to be given by a uniform distribution over $a_j \in S_j$.

will likely be restricted to relatively low depth.

3.2. *On a subjective variant of weighted fictitious play incorporating psychophysical principles of perception*

Aside from introducing a pattern detecting variant of weighted fictitious play, this paper advances the existing literature by allowing players' perceptions to follow commonly ascribed rules of psychophysics pertaining to the translation of physical, or objective, stimuli to their subjective correlates in the human mind. Specifically, equation 4 presents the class of belief learning rules henceforth referred to as subjective fictitious play, denoted by $sfpn_{i,j,t}$ (of order n, at time t, for individual i and action j - the latter subscript will henceforth be dropped for simplicity). The encapsulating non-linear function in equation 4 transforms the objective *fpn* variables presented above to player i's final subjective beliefs.

$$sfpn_{i,t} = \frac{\delta_i (fpn_{i,t})^{\lambda_i}}{\delta_i (fpn_{i,t})^{\lambda_i} + (1 - fpn_{i,t})^{\lambda_i}} \tag{4}$$

This specific functional form has been used in the choice under uncertainty literature (Goldstein and Einhorn, 1987; Kilka and Weber, 2001; Lattimore et al., 1995) as a behavioral model of probability weighting functions, transforming objective probabilities to subjective probabilities. It exhibits the following two desirable properties and advantages over standard *wfp*.

Firstly, it incorporates principles of subjective perception since the curvature of the function is controlled by the discriminability parameter λ_i, allowing the subjective sensitivity to $fpn_{i,t}$ to vary over its domain. Conveniently, if $\lambda_i = 1$ a simple linear relationship ensues implying that subjective beliefs are equivalent to objective beliefs. For values of $0 < \lambda_i < 1$ the function is concave over domain values from zero to some critical value less than one, and convex thereafter till a value of one. Alternatively, if $\lambda_i > 1$ this is reversed as the function changes from convex to concave. The attractiveness parameter, δ_i controls the elevation of the function thereby allowing for a prior inclination in beliefs that an opponent is more likely to play certain actions than others. For example, a player may exhibit such an inclination based on the payoff structure of the game.

Secondly, this parametric form nests many other important models, in particular the constraints $\lambda_i = \delta_i = 1$ reduce the model to the standard weighted fictitious play models *fp1* and *fp2*. Another important baseline model is constant beliefs with random fluctuations, which is captured by this model when $\lambda_i = 0$, with the mean of the stated beliefs controlled by parameter δ_i.

3.3. *Stochastic decision rules*

Players are assumed to stochastically best respond to the expected payoffs of actions given their beliefs. The decision rule in equation 5 defines the probability of subject i playing action a_i, $Pr_{i,t}(a_i)$, as a logit function where S_i is the discrete strategy set of player i, and $E(\pi(a_i))$ is equal to the expected payoffs of playing action a_i given beliefs over all the opponent's actions $a_j \in S_j$. The degree of responsiveness to expected payoffs is controlled by the parameter β_i of the decision rule and is assumed to be the same for all actions. As $\beta_i \to 0$ the probability distribution over actions tends to the uniform distribution where all actions are played with equal probability. However, as $\beta_i \to \infty$ the decision rule approaches deterministic best response, where the action with the highest expected payoff will be played with certainty. Finally, players' action choices may be affected by a judgment that is independent of the evolution of play, captured by the constant α_{a_i}, which is different for each action a_i.

$$Pr_{i,t}(a_i) = \frac{e^{\alpha_{a_i} + \beta_i \cdot E(\pi(a_i))}}{\sum_{a_i \in S_i} e^{\alpha_{a_i} + \beta_i \cdot E(\pi(a_i))}} \tag{5}$$

3.4. *Indirect detection of patterns by unrestricted non-pattern detecting belief models*

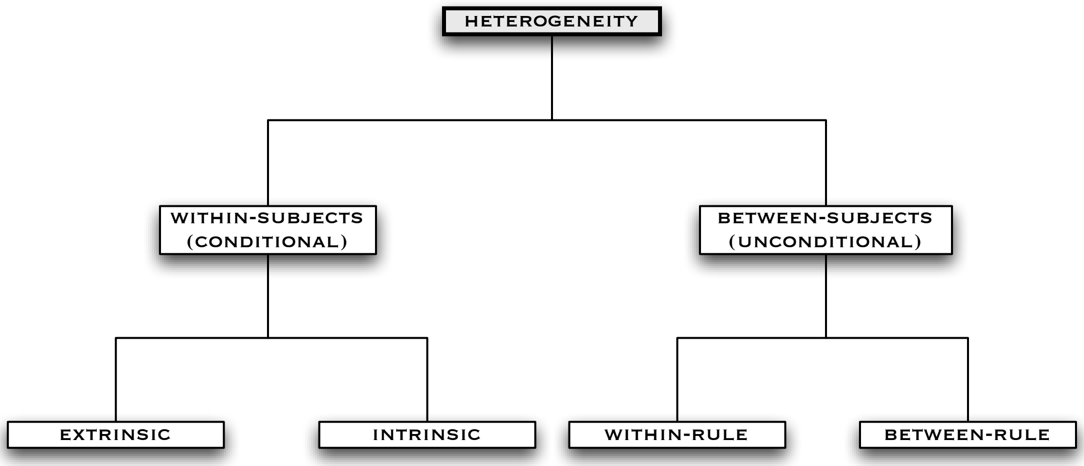
Despite the expectation that the memory parameter in a belief model should be positive, previous studies have not imposed this on the econometric models, and in many cases individual estimates are found to be less than zero (Cheung and Friedman, 1997; Nyarko and Schotter, 2002). However, as will be elaborated below, negative γ_i values in conjunction with a non-pattern detecting fictitious play belief model may be capable of indirectly capturing patterns.

Assume that there exists negative serial correlation in an opponent's action data so that actions alternate more often than would be expected with independent draws. This will necessarily lead to negatively correlated fictitious play beliefs regardless of the sign of γ_i, however whether these beliefs are in phase with the patterns will depend on the sign. If $\gamma_i > 0$ the fictitious play beliefs at time t for the action played at time $t-1$ will be higher than the beliefs at $t-1$. This result is clearly inconsistent with the negative correlation in the action sequence, and therefore pattern recognition is impossible as the belief model will on average be predicting the wrong action. However, if $\gamma_i < 0$, the opposite will hold so that the fictitious play beliefs are moving in the correct direction as if they were anticipating the negative correlation in actions.

Given $\gamma_i < 0$ then a decision rule will best respond to the patterns if the expected payoff difference between the two actions alternates on either side of zero. One way of accomplishing this is to ensure that for consecutive rounds beliefs alternate in different regions of the best response probability space i.e. if the MSNE is to play an action with probability 0.6, the beliefs will have to alternate on either side of 0.6. This can always be accomplished by an *fp1* model by setting γ_i to be negative and arbitrarily close to zero, as a reduction in memory depth increases the variability of fitted beliefs ultimately leading them to be arbitrarily close to 0 and 1. Econometric estimation of *sfp1* can also accomplish this another way as there necessarily exists a value of δ that will guarantee this behavior. Similarly, this requirement could be achieved directly by the decision rule through the manipulation of α. Hence, misspecification due to the modeling of a pattern detecting subject with a non-pattern detecting belief model could lead to identification problems for these three parameters as they are all associated with indirectly capturing patterns. Given that previous studies employed standard non-pattern detecting fictitious play models, it is possible that the negative estimates of individual memory parameters are a result of this misspecification.

Likewise, the existing literature has not restricted β_i estimates to non-negative values in the

FIGURE 1.— A taxonomy of heterogeneity



econometric models, even though in the absence of anticipatory learning (Selten, 1991) this is a reasonable conjecture. Although anticipatory learning is certainly a possibility, it requires extreme sophistication on behalf of experimental subjects, whereas there exists a much simpler explanation. A non-pattern detecting model, whether it be *fp1* or *sfp1*, is still capable of best responding to two-period patterns in an opponents' action sequence when $\gamma_i > 0$ if $\beta_i < 0$. The argument is identical to that just made above, replacing γ_i with β_i. Therefore it is likely that the negative estimates of individual sensitivity/responsiveness parameters of prior studies are indirectly capturing pattern detection on behalf of subjects.

In order to avoid these issues which would blur the distinction between pattern detecting and non-pattern detecting models, all estimated models in this paper will restrict $\hat{\gamma}_i$ and $\hat{\beta}_i$ to be necessarily non-negative.

4. ON A TAXONOMY OF HETEROGENEITY

Despite the empirical confirmation of subject heterogeneity in experimental games, analysis is pursued without regards to the different possible sources of heterogeneity in subjects' behavior. This paper will proceed in defining a taxonomy of heterogeneity and providing empirical evidence of the relative importance of the various taxa. It will become clear in the ensuing discussion that identifying the sources and types of heterogeneity exhibited by subjects is of paramount importance in truly understanding strategic decision making.

Following Figure 1, at the first level of classification we propose a distinction of heterogeneity into two taxa, within-subjects (conditional) heterogeneity and between-subjects (unconditional) heterogeneity. Let each player have a set of learning rules at their disposal. Within-subjects heterogeneity is defined as the heterogeneity that arises if the learning rules employed by players were elements of all the players' sets, from which it can be inferred that any observed heterogeneity arose because each player chose to employ a different learning rule by conditioning on some information. Between-subjects (unconditional) heterogeneity is defined as the heterogeneity that arises if the observed learning rules were not elements in the sets of all the players so that the observed heterogeneity occurred not due to choice but due to an inherent inability i.e. cognitive bounds that differ amongst subjects. With regards to this study between-subjects heterogeneity assumes that not all subjects are privy to the pattern detecting model *sfp2*, perhaps due to different cognitive bounds. Within-subjects heterogeneity, on the other hand, implies that each player has the ability to use the *sfp1* and *sfp2* learning models but chooses which learning rule to employ by observing whether their opponent exhibits serially correlated behavior. If not, then a player may employ the *sfp1* rule which has lower computational costs, otherwise the player may switch to the *sfp2* rule instead.

Spiliopoulos (2008) specifically designed an experiment to discriminate within- and between-subjects heterogeneity by observing how each subject behaved against three different predetermined computer algorithm opponents. Within-subjects heterogeneity was found to be a significantly more important source of behavioral heterogeneity than between-subjects, demonstrating the necessity of acknowledging this in behavioral modeling.

Having defined the first hierarchical level of taxa, within-subjects heterogeneity can be further subdivided into two further taxa, coined as extrinsic and intrinsic within-subjects heterogeneity. The latter refers to heterogeneity that is conditioned on variables pertaining to game play, essentially referring to whether subjects adapt their strategy to the opportunities for exploitation presented by their opponent. Extrinsic within-subjects heterogeneity is essentially heterogeneity that cannot be attributed to the game play variables that a researcher would include in an econometric model, and therefore would be captured in the error term. Note, that this taxa includes both the cases where subjects may be conditioning on extrinsic but theoretically observable random variables i.e. sunspots, and the effects of extrinsic but non-observable variables such as the levels of various neurotransmitters in the brain and how they affect behavior.

Finally, for the purposes of this study we will distinguish between two other taxa of between-subjects heterogeneity: between subjects, within-rules and between-subjects, between-rules heterogeneity. The latter refers to the availability of different learning rule models, in this case *sfp1* and *sfp2*. The former refers to heterogeneity in the parameter estimates of rules that are available to players, as is the case if two subjects both use the *sfp2* rule, but the estimated *sfp2* model for both of them exhibits significantly different parameter values.

5. DATASET AND METHODOLOGY

The N&S game is given in Table I, and the experimental data used was the treatment where each subject repeatedly playing the same game 60 times against the same opponent. The mixed strategy Nash equilibrium for both players was to play red 60% of the time and green 40%. Subjects would receive monetary compensation both according to their payoffs from playing the game and from the accuracy of their stated beliefs compared to opponents' realized actions[11].

Three different approaches will be employed to obtain the necessary empirical results. Section 6 will directly estimate models of subjects' elicited beliefs without resorting to their action choices. Section

[11] A quadratic scoring rule was used as an incentive mechanism for truthful revelation of beliefs.

TABLE I
N&S Payoff matrix

		Column player	
		Green	Red
Row player	Green	6,2	3,5
	Red	3,5	5,3

7 will model subjects' action choices using a two-stage procedure where the fitted belief estimates from the previous section are used as regressors and the decision rule parameters are estimated using a concomitant, mixed effects, latent class logit regression. Section 8 will simultaneously estimate both the underlying belief formation model and the decision rule using subjects' action data, without utilizing the elicited belief series. Finally, Section 9 will simulate play based on the interactions of different types of learning rules and parameter values in order to ascertain their evolutionary fitness.

6. ESTIMATION OF STATED BELIEF MODELS

Four different equations will be estimated using the competing belief models *fp1*, *fp2*, *sfp1* and *sfp2*. The estimation problem at time t is represented by equation 6, where $sb_{i,t}$ is equal to player i's stated belief regarding action j and $sfpn_{i,t}$ (or alternatively $fpn_{i,t}$) is the fitted belief. To allow for beliefs to settle, in particular for the *fp2* and *sfp2* models which require a longer initial history to form beliefs regarding all possible two-period combinations, the first ten rounds are not included in the measure of fit, however these values are used in the formation of beliefs. Optimization is performed in two steps, first a randomly generated population of parameter estimates is subjected to a genetic algorithm, and the best candidate is used as a starting point for a quasi-Newton Sequential Quadratic Programming algorithm, implemented in Matlab as the *fmincon* function.

$$\min_{\gamma_i,\delta_i,\lambda_i} \sum_{t=11}^{60} [sb_{i,t} - sfpn_{i,t}]^2 \qquad 0 \le \gamma_i \le 1, \delta_i \ge 0, \lambda_i \ge 0 \tag{6}$$

Comparisons between different belief models and their performance will be exacted by 10-fold cross-validation. The dataset is divided into ten non-overlapping folds of five rounds, and the models are estimated each time by training on nine of the folds and measuring the out of sample error performance on the remaining fold. The cross-validation MSE allows comparisons amongst models with different degrees of freedom, as it bypasses the problem of overfitting to the training data. Tables II and III present the results of the optimization of the class of models represented by equation 6.

6.1. *Do the subjective belief models, sfp1 and sfp2, predict stated beliefs more accurately than the standard or objective fictitious play models?*

The cross-validation MSE falls from 0.090 to 0.076 comparing *fp1* to *sfp1*, and from 0.082 to 0.070 for *fp2* and *sfp2* respectively. The models are compared according to Wilcoxon signed-rank tests performed using the ten paired cross-validation observations of the average (or equivalently, total) CV-MSE of all players. The null hypothesis of zero median difference between *fp1* and *sfp1*

TABLE II

Mean square error (MSE) of individually estimated models

	In sample				10-fold CV			
Player	*fp1*	*fp2*	*sfp1*	*sfp2*	*fp1*	*fp2*	*sfp1*	*sfp2*
1	0.062	0.102	0.038	0.077	0.063	0.103	0.046	0.087
2	0.020	0.026	0.011	0.011	0.020	0.026	0.011	0.011
3	0.143	0.135	0.141	0.131	0.148	0.140	0.152	0.154
4	0.061	0.107	0.051	0.049	0.061	0.111	0.054	0.054
5	0.162	0.028	0.139	0.024	0.163	0.029	0.141	0.024
6	0.125	0.135	0.106	0.105	0.133	0.136	0.118	0.117
7	0.081	0.080	0.036	0.036	0.111	0.091	0.042	0.044
8	0.063	0.065	0.051	0.052	0.063	0.064	0.058	0.064
9	0.047	0.042	0.030	0.021	0.051	0.044	0.032	0.023
10	0.087	0.116	0.069	0.069	0.091	0.118	0.073	0.087
11	0.225	0.004	0.217	0.003	0.225	0.005	0.220	0.007
12	0.102	0.104	0.078	0.080	0.102	0.104	0.084	0.090
13	0.026	0.028	0.024	0.023	0.026	0.028	0.025	0.024
14	0.029	0.034	0.027	0.026	0.029	0.034	0.033	0.029
15	0.086	0.089	0.086	0.086	0.088	0.089	0.095	0.093
16	0.130	0.162	0.099	0.099	0.129	0.163	0.105	0.103
17	0.171	0.169	0.138	0.157	0.200	0.169	0.146	0.189
18	0.059	0.055	0.057	0.055	0.059	0.055	0.059	0.063
19	0.008	0.016	0.004	0.006	0.008	0.016	0.005	0.007
20	0.020	0.031	0.019	0.020	0.020	0.031	0.021	0.021
21	0.039	0.044	0.012	0.012	0.039	0.044	0.013	0.014
22	0.077	0.090	0.065	0.085	0.079	0.090	0.067	0.096
23	0.016	0.014	0.011	0.009	0.016	0.014	0.012	0.010
24	0.051	0.061	0.033	0.045	0.055	0.063	0.035	0.056
25	0.061	0.061	0.045	0.045	0.061	0.061	0.051	0.050
26	0.071	0.077	0.069	0.072	0.080	0.077	0.081	0.080
27	0.219	0.228	0.181	0.182	0.219	0.229	0.194	0.195
28	0.158	0.164	0.120	0.148	0.174	0.166	0.161	0.179
Average	0.086	0.081	0.070	0.062	0.090	0.082	0.076	0.070

is strongly rejected ($z = 2.701, p = 0.0069$), likewise when comparing *fp2* and *sfp2* ($z = 2.803, p = 0.0051$). These two results provide significant evidence that beliefs are not modeled accurately by the standard fictitious play functional form, and fitting beliefs to these models may lead to significant misspecification bias in the estimates of parameters, especially for γ as will be argued in the next section.

The *sfp2* model exhibits lower CV-MSE than the *sfp1* model, 0.070 and 0.076 respectively, and a Wilcoxon signed-rank test ($z = 1.886, p = 0.0593$), indicates that the null hypothesis of zero difference in medians can be rejected at roughly the 6% level of significance. In conclusion, the elicited belief data series provides serious evidence in favor of pattern detection by experimental subjects, a result that will be corroborated by the results from the estimation of action data, where allowing some players to use *sfp2* models will significantly improve fit.

6.2. *Estimates of the memory parameter, $\hat{\gamma}$*

Estimates of the memory parameters from the standard *wfp* models in N&S were centered on one with little dispersion, implying that individuals weighted all past information equally. However, as they point out this is likely the result of the restriction of a one parameter *wfp* model in estimating a highly variable stated belief series. The best such a model can accomplish is to approximately fit the mean of the elicited belief series with a relatively smooth, stable empirical data series. This can only be accomplished by a high value of $\hat{\gamma}$ which minimizes the variability in the empirical beliefs series. However, the subjective belief models can be calibrated to the mean of the stated beliefs by controlling the elevation of the subjective beliefs through the δ parameter, thereby alleviating this possible misspecification problem. The strong evidence for the subjective belief models supports the contention that the standard *wfp* model is indeed misspecified and therefore parameter estimates may be biased.

The results presented in Table III verify this intuition as $\hat{\gamma}$ falls from 0.950 to 0.501 for the *fp1* and *sfp1* models respectively, and from 0.937 to 0.814 for *fp2* and *sfp2* respectively. Hence, the finding of memory parameter estimates near a value of one for *fp1* belief models is most likely due to the inherent model misspecification in standard fictitious play models which does not permit the calibration of the elevation of stated beliefs. Finally, a two-tailed sign test strongly rejects the null hypothesis ($p = 0.002$) that the median of the paired differences in estimates of $\hat{\gamma}$ from *sfp1* and *sfp2* for each player is equal to zero. The significantly higher value of $\hat{\gamma}$ for pattern detecting models is reasonable as greater memory depth is required to effectively detect patterns in opponents' behavior.

Another interesting observation is that $\hat{\gamma}$ for the *sfp1* models in often equal to or close to zero, in particular $\hat{\gamma} < 0.1$ in 10 cases. Hence, the *sfp1* model is often capturing a special case of weighted fictitious play behavior, namely Cournot beliefs. Note that only in two cases is $\hat{\gamma} < 0.1$ in the estimated *sfp2* models, in concurrence with the assumption that pattern detecting models should exhibit high memory parameters.

6.3. *Estimates of the discriminability coefficient, $\hat{\lambda}$*

The mean estimates of the discriminability coefficient for *sfp1* and *sfp2* respectively are 0.581 and 0.426[12] (or excluding zero estimates 0.74 and 0.477 respectively), so that for most players beliefs are concave for values of the *fpn* variables between zero and an interior inflection point, and convex thereafter. These results are in accord with common principles of psychophysics as small deviations from the MSNE should be less likely to be detected than large deviations. Also, larger deviations are more likely to be due to true deviations in the underlying data generating process rather than noise and therefore players should be more likely to respond to larger deviations.

6.4. *Estimates of the attractiveness coefficient, $\hat{\delta}$*

The elevation of the subjective function is controlled by the estimate of $\hat{\delta}$, with mean estimates for *sfp1* and *sfp2*, 0.995 and 1.037 respectively. A null hypothesis that the median of the distribution

[12]These values are in the same range as median values obtained from experimental probability weighting functions in Gonzalez and Wu (1999) and Tversky and Fox (1995), 0.44 and 0.69 respectively.

TABLE III

Comparison of parameter estimates of individually estimated *fpn* and *sfpn* models

	fp1	*fp2*	*sfp1*			*sfp2*		
Pl.	$\hat{\gamma}$	$\hat{\gamma}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\lambda}$
1	0.756	0.000	0.897	0.954	3.423	0.641	2.368	0.109
2	1.000	1.000	-	0.894	0.000	-	0.894	0.000
3	0.885	0.957	0.896	0.825	1.160	0.877	0.910	0.522
4	1.000	1.000	-	0.773	0.000	-	0.886	0.000
5	0.953	0.835	-	0.938	0.000	0.996	0.241	0.869
6	0.955	0.999	1.000	0.847	0.127	1.000	0.791	0.031
7	0.869	0.922	0.070	0.690	0.066	0.767	0.610	0.413
8	1.000	1.000	0.017	1.133	0.026	0.856	1.103	0.151
9	0.985	0.885	0.036	1.075	0.056	0.739	1.085	0.322
10	0.923	0.932	0.922	0.508	0.189	0.009	0.521	0.007
11	1.000	0.786	-	1.023	0.000	1.000	0.324	2.304
12	1.000	1.000	0.719	1.322	0.211	1.000	1.279	0.275
13	1.000	1.000	0.031	0.939	0.015	1.000	0.987	0.304
14	1.000	1.000	1.000	1.385	0.245	1.000	1.336	0.251
15	0.997	1.000	0.005	0.840	0.016	1.000	0.853	0.342
16	1.000	0.969	1.000	0.541	0.790	0.956	0.635	0.242
17	0.516	1.000	0.003	0.750	0.055	1.000	0.669	0.920
18	1.000	1.000	-	1.070	0.000	1.000	1.000	0.749
19	1.000	0.997	0.016	1.072	0.020	0.573	1.097	0.021
20	1.000	0.998	0.042	1.003	0.020	0.790	1.014	0.032
21	1.000	1.000	0.006	0.903	0.003	0.271	0.914	0.004
22	1.000	1.000	0.044	2.000	0.096	0.995	2.447	0.329
23	1.000	1.000	0.696	1.033	0.136	0.833	1.088	0.270
24	0.933	0.958	0.852	1.036	0.236	-	1.075	0.000
25	0.985	1.000	-	0.982	0.000	0.050	0.977	0.006
26	0.845	1.000	0.782	0.957	0.624	1.000	0.796	0.840
27	1.000	1.000	0.998	2.354	1.225	1.000	2.892	0.490
28	1.000	1.000	1.000	0.006	7.529	1.000	0.256	2.126
Average	0.950	0.937	0.501	0.995	0.581	0.814	1.037	0.426

TABLE IV

POOLED ESTIMATION OF *sfp1* AND *sfp2* MODELS

	In sample MSE	Cross-validation MSE	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\lambda}$
sfp1	0.0837	0.0839	1	0.999	0.428
sfp2	0.0780	0.0782	0.682	1.063	0.112

of individual estimates is equal to one can not be rejected in both cases by two-sided sign tests, $p = 0.572$ and $p = 0.442$ respectively. Although the medians are not significantly different from the value of δ implied by the standard *fpn* models, this does not exclude the possibility that there exists considerable subject heterogeneity in the individual estimates that must be modeled using *sfpn* models rather than the standard objective belief models. The large dispersion of estimates indicates that a significant part of the variance is indeed due to heterogeneity and not just imprecise estimates.

6.5. *Does there exist significant between-subjects, within-rules heterogeneity in parameter estimates of the sfpn models?*

Pooling the estimation of *sfp1* and *sfp2* models leads to significantly worse cross-validation performance as documented in Table IV. In particular, CV-MSE worsens from 0.076 to 0.0837 for *sfp1* ($z = 2.395, p = 0.0166$), and from 0.07 to 0.078 for the *sfp2* model ($z = 2.701, p = 0.0069$), both results highly significant according to Wilcoxon signed-rank tests performed on the ten paired cross-validation observations. Misspecifying the models by assuming player homogeneity and pooling subjects leads to estimated values of $\hat{\gamma}$ and $\hat{\lambda}$ that are very different from the majority of the individually estimated parameters presented in the previous sections. This is in accord with Cabrales and Garcia-Fontes (2000) who simulate the econometric properties of learning model estimation, concluding that ignoring subject heterogeneity can seriously bias parameter estimates.

7. TWO-STAGE ESTIMATION OF BELIEFS AND DECISION RULES

This section proceeds with analyses of subjects' behavior based on both the action data and the elicited belief series. The first stage is the estimation of the individual belief model parameters as executed in the previous section by fitting directly to elicited beliefs, whilst the second stage will estimate the remaining decision rule parameters by simulated maximum likelihood.

Model 1 is a latent class, mixed effects, concomitant discrete choice model represented by equations 7-10, and nests all the other models that will be estimated as special cases.

Latent class or mixture models assign a prior probability that each subject belongs to a particular class, in this case whether they are using an *sfp1* or *sfp2* model with prior probabilities denoted by $1 - p$ and p respectively. Concomitant latent class regression models are an extension where prior class probabilities may be influenced by other covariates or concomitant variables. In this model the covariate is r_i, the absolute percentage difference between the expected number of runs, assuming actions are independently distributed, and the observed number of runs. This variable quantifies the degree to which players' behavior has exploitable two-period patterns. The parametrization of the prior class probability is given by equation 9 including the necessary parameter constraints to ensure that both the minimum and maximum lie between zero and one.

The concomitant specification captures within-subjects heterogeneity, allowing for the possibility that players have access to both *sfp1* and *sfp2* belief models, but choose which one to use depending on whether an opponent is displaying two-period patterns that could be exploited. Hence, if $\psi > 0$ this implies that there exists intrinsic within-subjects heterogeneity. The existence of between-subjects heterogeneity can be confirmed if $0 < p < 1$ when $r_i = 1$. Assuming that the existence of perfectly correlated actions is detected with certainty by all subjects, then any subject capable of using *sfp2* should also do so with certainty, with the caveat that the additional computational cost of pattern recognition is less than the resulting gains. Similarly, if $r_i = 0$ the lower computational cost of *sfp1* should ensure[13] that it is used with certainty if there does not exist any between-subjects heterogeneity. Finally, the unconditional prior probability of a subject employing the *sfp2* belief rule is obtained by integrating out r_i from the conditional probability and is given by $\widetilde{p} = \frac{1}{27}\sum_{i=1}^{27} p$.

A logit link function q_t is employed to transform the attractions into probabilities, where the attraction for the green action is normalized to a value of one, as shown in equation 8, and $E\Delta\pi_t(sfpn)$ represents the difference in expected payoffs for choosing the green action over the red action at time t according to a specific belief model.

Finally, a mixed effects specification was chosen as a solution to the necessity of modeling individual parameter heterogeneity in the decision rule whilst keeping the degrees of freedom at reasonable levels. The decision rule parameters $\theta = [\alpha, \beta]$ are assumed to be jointly normally distributed with full covariance matrix as given in equation 10.

All other models that will be estimated will be nested within Model 1, arising from specific parameter restrictions. Model 2 restricts Model 1 by assuming that prior class probabilities do not depend on the variable r_i i.e. that there exists no within-subjects heterogeneity. Whereas these two models allow for two classes within the population, models 3 and 4 assume that either all subjects are *sfp2* or *sfp1* players respectively. Finally, model 1* is identical to model 1 with the exception that fitted *fpn* beliefs are used in the place of *sfpn* beliefs to allow an analysis of the possible consequences of this type of misspecification.

Estimation of all models is performed by simulated maximum likelihood with 500 antithetic draws from the multivariate normal distribution for θ using the Cholesky transformation. The first step in optimizing this function was to randomly select a population of parameter estimates and subject them to a genetic algorithm selection process. The best fitting parameter estimates were then used as the starting point for a quasi-Newton Sequential Quadratic Programming optimization technique implemented as the *fmincon* function in Matlab. Finally, the whole process was repeated 10 times and the resulting best fitting parameters are reported.

[13] An *sfp2* model may even be inferior to an *sfp1* model if one's opponent is not exhibiting patterns since it will be prone to overreacting to temporary random deviations in second-order probabilities.

$$ll = \sum_{i=1}^{27} ln\Bigg[\int (1-p) \prod_{t=11}^{60} q_t\,(\mathit{sfp1})^{a_t}\,[1 - q_t\,(\mathit{sfp1})]^{1-a_t} + p \prod_{t=11}^{60} q_t\,(\mathit{sfp2})^{a_t}\,[1 - q_t\,(\mathit{sfp2})]^{1-a_t}\, d\theta\Bigg] \tag{7}$$

$$q_t\,(\mathit{sfpn}) = \left[1 + e^{\alpha + e^{\beta} E\Delta\pi_t(\mathit{sfpn})}\right]^{-1} \qquad n = 1, 2 \tag{8}$$

$$p = 1 - e^{-(\phi + \psi \cdot r_i)} \qquad \phi \geq 0, \phi + \psi \geq 0 \tag{9}$$

$$\theta = [\alpha, \beta] \sim N\left([\mu_\alpha, \mu_\beta], \begin{bmatrix} \sigma_\alpha & \sigma_{\alpha\beta} \\ \sigma_{\alpha\beta} & \sigma_\beta \end{bmatrix}\right) \tag{10}$$

Standard errors will be estimated using a jackknife procedure which involves dropping one individual at a time from the training dataset and re-estimating the model parameters. The log-likelihood of the subject excluded from each run $ll^i_{cv}$is a cross-validation measure of fit that allows comparisons between models with different degrees of freedom. The sum of the individual cross-validation log-likelihoods ll_{cv}, will be used for model comparison, as will the Akaike and Bayesian information criteria.

The N&S dataset consists of 14 pairs of players, however this analysis will proceed by dropping player 11 on the basis that it is a severe outlier exhibiting particularly perplexing/irrational behavior. Player 11's opponent exhibits severe negative serial correlation in action choices, leading to easily detectable two-period patterns. Player 11's elicited beliefs show that he/she has indeed detected the patterns, but instead of playing a best response to these beliefs the action data in every single case is consistent with the worst response. Hence, from the elicited belief series player 11 is clearly employing a *sfp2* belief model, but estimation using the observable action data concludes that this player uses an *sfp1* model. The only plausible explanation for this clearly irrational behavior is that this player may have simply misunderstood the payoff matrix, and have erred in concluding what the best response action is. The extremeness of this internally inconsistent behavior justifies labelling player 11 as an outlier and excluding him/her from the following analysis.

The econometric results of these models are presented in Table V and the following subsections examine specific hypotheses of behavior.

TABLE V

MAXIMUM LIKELIHOOD ACTION DATA MODELS

Model Restriction	1	2 $\psi = 0$	3 $p = 1$	4 $p = 0$	1*
$\widetilde{p}$	0.337 (0.036)	0.362 (0.045)	-	-	0.347 (0.047)
μ_α	0.031 (0.016)	0.042 (0.017)	0.04 (0.015)	0.048 (0.016)	0.064 (0.012)
σ_α	0.609 (0.032)	0.579 (0.028)	0.505 (0.028)	0.55 (0.035)	0.441 (0.029)
μ_β	-0.623 (0.062)	-0.499 (0.07)	-0.85 (0.059)	-0.7 (0.064)	-1.1 (0.098)
σ_β	1.033 (0.041)	0.812 (0.06)	1.09 (0.052)	1.009 (0.058)	0.987 (0.136)
ϕ	0 (0)	0.449 (0.069)	-	-	0 (0.049)
ψ	2.685 (0.383)	-	-	-	2.778 (0.637)
ll (df)	853.901 (7)	855.625 (6)	873.519 (5)	888.737 (5)	872.48 (7)
AIC	1721.8	1723.2	1757.0	1787.5	1759
BIC	1758.3	1754.5	1783.1	1813.5	1795.4
ll_{cv}	857.89	860.682	876.524	892.486	883.584

7.1. *Does the data support the hypothesis of belief model heterogeneity with respect to whether subjects use pattern recognition?*

Models 1 and 2 assume the existence of belief model heterogeneity through a mixture specification which allows each subject to use any of the two belief models, whereas models 3 and 4 assume that all players are using the same model. Comparisons of models with a different number of latent classes can not be performed using standard LR tests even though the models are nested, because the null hypothesis exists on the boundary of the parameter space and therefore the ratio of likelihoods does not follow a χ^2 distribution (Titterington, 1990). The literature instead proposes information criteria tests or cross-validation as a means of selecting the appropriate number of latent classes. Since the AIC, BIC and ll_{cv} for models 1 and 2 are lower than those of models 3 and 4, these criteria conclude that there exists significant heterogeneity that must be modeled by the latent class/mixture approach of models 1 and 2.

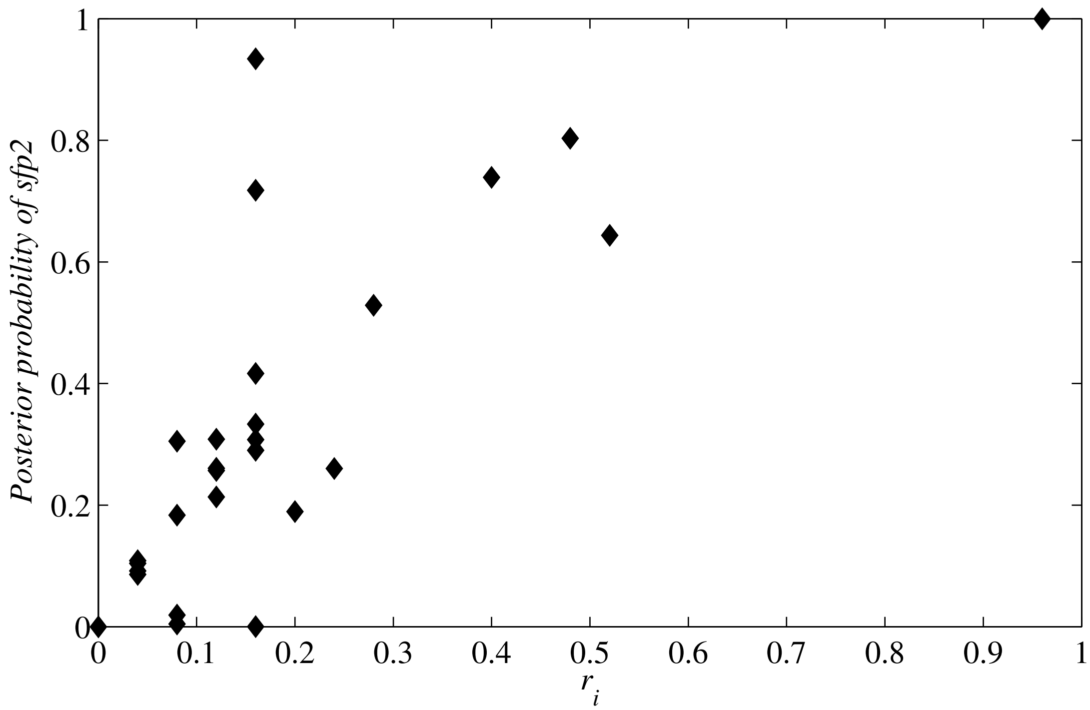
7.2. *Does the data support the hypothesis of intrinsic within-subjects heterogeneity conditioned on opponents' deviations from serially uncorrelated action choices?*

In the context of this experiment we define within-subjects heterogeneity as arising from conditioning on the behavior of a player's opponent, namely the presence of exploitable two-period patterns. The null hypothesis that there exists no within-subjects heterogeneity is tested by comparing models 1 and 2, the latter incorporating the restriction $\psi = 0$.

Given that the two models are nested, preference is given to nested tests over non-nested, as the former should exhibit more desirable econometric properties. An asymptotic LR test provides strong evidence of within-subjects heterogeneity with significance at $p = 0.0634$ ($\chi^2 = 3.447$). Also, using the jackknife standard error, the 95% confidence interval[14] for ψ is $[1.99, 3.63]$ indicating clearly that ψ is significantly different from zero. Turning to non-nested tests, this conclusion is confirmed as

[14]Since $\phi + \psi \geq 0$ and ϕ was estimated to be zero in all the jackknife samples, this requires that $\psi \geq 0$. Therefore the confidence interval is constructed on the assumption that $ln(\psi)$ is normally distributed.

FIGURE 2.— Posterior probability of *sfp2* versus the degree of exploitable patterns



the cross-validation log-likelihood for model 1 is smaller than that of model 2 indicating that the addition of the ψ parameter increased out of sample predictive accuracy. A two-sided sign test of the median difference between paired individual ll^i_{cv} rejects the null hypothesis of no difference at $p = 0.122$. The AIC and BIC give conflicting evidence, as the latter penalizes degrees of freedom more heavily, and therefore choosing between the two measures requires a subjective decision about how strict this penalty should be. Comparison using the cross-validation log likelihood is superior in this sense because by examining out of sample performance any model's advantage from having more degrees of freedom is automatically adjusted for, without resort to a researcher's subjective beliefs with regards the magnitude of the penalty.

Concluding, these results signify that there exists significant within-subjects heterogeneity in the subject pool. This is intuitively confirmed in Figure 2 where there is a clear positive relationship between the posterior probability of each subject employing the *sfp2* model and their opponent's degree of deviation from independently distributed action choices, r_i. The existence of within-subjects heterogeneity is particularly damaging to the minimax hypothesis as not only do two period patterns exist in subjects' action choices (a null of hypothesis of randomly ordered actions was rejected by an exact runs tests for 5 subjects at the 5% significance level), but these patterns are not eliminated even when opponents exploit them with an *sfp2* belief model.

7.3. *Does the data support the hypothesis of between-subjects, between-rules heterogeneity?*

This type of heterogeneity, as mentioned earlier, can be detected by observing the value of the conditional prior class probability at the bounds where $r_i = 0$ and $r_i = 1$. The estimated value in the latter case is 0.932, a value high enough to support the contention that between-subjects, between-rules heterogeneity in belief models is not particularly important. Corroborating this result is the fact that all players are employing *sfp1* ($p = 0$) when $r_i = 0$, as would be expected given that *sfp2* has a much higher computational cost than *sfp1*. In conjunction with the evidence from the previous section, intrinsic within-subjects heterogeneity is significantly more important in this experiment than between-subjects, between-rules heterogeneity.

7.4. *What are the effects of misspecification by using fpn beliefs instead of sfpn beliefs?*

Model 1* is identical to Model 1, with the exception that the estimated beliefs are those based on the *fpn* belief models rather than the *sfpn* models. All the non-nested model selection measures AIC, BIC and ll_{cv} confirm that *fpn* beliefs are inferior not only at directly fitting elicited beliefs, as was shown in Section 6, but also at indirectly fitting action choices. A two-sided sign test rejects the null hypothesis of zero median difference between subjects' paired ll^i_{cv} for the two models at $p = 0.0522$.

A comparison between the two illustrates the possible biases in parameter estimates associated with this type of misspecification. The estimates for the unconditional prior $\tilde{p}$ and ψ are extremely close in both models and there does not seem to be a significant adverse effect of misspecifying the model by using the standard *fpn* belief model. However, estimates of the means for α and β diverge significantly with the estimates for Model 1* equal to roughly twice the estimates from Model 1. This occurs because using an *fpn* model implicitly sets $\lambda = 1$ and $\delta = 1$ in the *sfpn* belief model. Since the mean estimate for $\lambda < 1$, the value of μ_β in model 1* must be smaller to reflect the indirect effect of fixing $\lambda = 1$. Likewise, in Model 1* α must now also implicitly include the impact of δ from the *sfpn* models. Finally, the variance of the two random effects are less in Model 1* implying less heterogeneity in the decision rule parameters.

In conclusion, misspecifying learning models with the standard *fpn* belief rule indicates that there may be some effect on decision rule parameters, however no evidence was found of serious bias in the remaining estimates. The evidence presented here is enough to warrant caution with regards to this, however a more rigorous examination of the effects of misspecification using simulated data is necessary before drawing final conclusions.

8. JOINT ESTIMATION OF BELIEFS AND DECISION RULES USING ACTION DATA

This section models behavior through the simultaneous estimation of the belief and decision rule parameters using only the action data. Simultaneously estimating individual belief model parameters is problematic as it would result in a small observation to parameter ratio, leading to significant overfitting and more importantly convergence problems as the number of local minima is increasing in the number of parameters. For a given sample size, a reduction in the number of parameters in the model can be achieved either by assuming homogeneity of some of the parameters, or by parametrizing the heterogeneity as draws from specific distributions. The latter approach is chosen as the two-stage procedure clearly indicated that there exists significant heterogeneity both in the belief and decision rule. Therefore any simplifying assumption of parameter homogeneity will lead to misspecification, with significant negative impact on the reliability of the parameter estimates (Cabrales and Garcia-Fontes, 2000).

The model to be estimated is the fully specified model in equations 7-10 with the difference that beliefs are estimated concurrently, and belief model parameter heterogeneity is now modeled as draws from specific distributions instead of estimating each parameter individually:

$$sfpn_{i,t} = \frac{\delta[fpn_{i,t}(\gamma_n)]^{\lambda_n}}{\delta[fpn_{i,t}(\gamma_n)]^{\lambda_n} + [1 - fpn_{i,t}(\gamma_n)]^{\lambda_n}}$$

The $\delta, \gamma_n, \lambda_n$ values[15] are drawn from lognormal, beta and gamma marginal distributions respectively, whilst α and β are assumed to follow normal marginal distributions as in the two-stage estimation. The choice of marginal distributions has been partly guided by the stated belief estimation results, so that the qualitative features and shape of the empirically estimated distribution of individual parameters are supported. The joint distribution of these parameters is modeled using a Gaussian copula with a fully specified correlation matrix allowing non-zero partial correlation between all the parameters (Sklar, 1973).

Salmon (2001) finds that concurrent estimation of the underlying belief model and decision rule often may not be particularly efficient at recovering the true data generating models and parameter values. In particular, two pairs of parameters that may not be properly identified in the simultaneous estimation of this model are δ and α, and λ and β. The δ parameter controls the level of beliefs which indirectly also affects the tendency to play a particular action, the latter being what parameter α is directly estimating[16]. Likewise, λ controls the sensitivity to the *fpn* variables and indirectly also affects the sensitivity to expected payoffs, which is directly influenced by the β parameter. This is not a problem with the two-stage procedure as the belief data can be used to isolate the direct effects of these parameters, however their relative contributions will likely be obfuscated by simultaneous estimation. The existence of identification problems will be investigated by calculating the pairwise correlations between jackknifed parameter estimates.

The results of the jointly estimated model and decision rule parameters are provided in Table VI, alongside the previous results for the two-stage estimation procedure for ease of comparison. A sign test fails to reject the null hypothesis that the median of the paired differences of each subject's ll_{cv}^{i} is different from zero[17] ($p = 0.701$). As expected, the standard errors are higher for all parameters in the joint model, reflecting greater uncertainty about parameter estimates due to the inefficiency

[15]The parameter δ has been assumed to be the same for both learning rules in an effort to keep the number of parameters at a minimum. This restriction is credible as δ represents a prior tendency to believe certain actions are more likely to be played by an opponent than others, which should be the same regardless of which belief model is employed. Furthermore, estimating the model without this restriction did not result in any significant difference.

[16]This explains why N&S find significantly smaller values of $\hat{\gamma}$ with simultaneous estimation compared to the values from belief rule estimation, as the inclusion of α in the decision rule essentially makes up for the lack of a mean-capturing parameter in standard weighted fictitious play.

[17]The large difference in ll_{cv} between the two models is mainly attributable to a single player whose ll_{cv}^{i} differs by 14.4.

TABLE VI
Comparison of joint and two-stage estimation

Estimation method	Joint	Two-stage
$\tilde{p}$	0.551 (0.047)	0.337 (0.036)
μ_α	0.131 (0.043)	0.031 (0.016)
σ_α	0.421 (0.037)	0.609 (0.032)
μ_β	-2.195 (0.351)	-0.623 (0.062)
σ_β	1.474 (0.257)	1.033 (0.041)
ϕ	0.042 (0.118)	0 (0)
ψ	5.745 (0.595)	2.685 (0.383)
ll	857.518	853.901
ll_{cv}	876.304	857.89

of joint estimation, which ignores the elicited belief data.

The highest Spearman pairwise correlation coefficients between all the pairs of parameters were 0.61 between μ_α and σ_α, 0.69 between μ_α and μ_β, and 0.72 between σ_α and the first parameter of γ_1. The discovered interactions between the μ_α, σ_α and γ_1 parameters lend additional support to the argument presented in Section 6.2 that γ parameter estimates depend on whether a model includes a parameter that can independently capture the mean of the stated beliefs, in this case the α parameter in the decision rule[18]. This possible indeterminacy appears to be borne out empirically, as the means of α and β are significantly different between the two models and their standard errors are much greater for the joint model.

Notably, the estimates for the unconditional prior $\tilde{p}$, and ψ are significantly higher in the jointly estimated model, possibly the result of the existence of significant implicit sequence learning. Subjects' stated beliefs, by definition, reflect only explicit learning as implicit learning will not be cognitively accessible to them to report. Hence, the two-stage estimation procedure will not be able to adequately reflect subjects' implicit learning. Although implicit sequence learning is unobservable through the elicited beliefs, it should be indirectly observable through its effect on the action data and therefore the jointly estimated model will be able to detect both explicit and implicit learning. The difference in the magnitude of the parameter estimates $\tilde{p}$ and ψ between the two models is the indirect effect of implicit learning. Given the relative magnitudes of the estimated parameters, implicit and explicit learning seem to be of approximately equal importance.

It is encouraging that qualitatively, the two approaches come to the same conclusions and that pattern detecting behavior was found to be significant in both cases. However, the increase in the imprecision of parameter estimates for the joint model along with the differences in the decision rule parameters, echo the warnings of N&S and Salmon (2001) about solely relying on empirical learning models estimated on action data only. Concluding in favor of one of the models is more difficult as although the two-stage model exhibits better econometric properties, it suffers from the inability to fit implicit sequence learning. Given that the psychology literature has already confirmed the existence of significant implicit sequence learning it is highly likely that the differences in $\tilde{p}$ and

[18]This explains why N&S find significantly lower values of $\hat{\gamma}$ with simultaneous estimation compared to the values from belief rule estimation, as the inclusion of α in the decision rule essentially makes up for the lack of a mean-capturing parameter in standard weighted fictitious play.

TABLE VII

Relationship between payoffs and use of *sfp2*

	constant	P_i	D_{role}
Coefficient (s.e.)	3.561 (0.14)	0.256 (0.217)	0.666 (0.137)
$t - statistic$ (p)	4.87 (0.000)	1.18 (0.249)	4.87 (0.000)
R^2	0.2876		
$F(2, 25)$	11.9 (0.0002)		

ψ are at least partially due to this[19], and cannot be solely attributed to parameter bias and/or imprecision of the joint model. However, further research needs to be directed towards resolving this issue with greater confidence.

9. EVOLUTIONARY FITNESS OF LEARNING RULES AND AGENT BASED SIMULATIONS OF BEHAVIOR

The evolutionary fitness and value of learning models can be measured by the payoffs they attain when in competition. A simple investigation of the value of employing the *sfp2* model instead of the *sfp1* model can be performed by regressing average payoffs on the posterior probability calculated earlier that each subject was using the *sfp2* model, denoted by P_i and a dummy variable D_{role}, indicating a row or column player. The results of such a linear regression are given in Table VII and although there is a positive relationship between the posterior probability P_i, and average payoffs, the null hypothesis of no relationship can be rejected only at the 24.9% significance level. This test however may not have enough power to reject the null hypothesis of no differences in payoffs as the payoff surface of this game is relatively flat around the MSNE.

Short of running a new experiment with a larger sample size or higher curvature in the payoff function, this problem can be overcome by examining simulations of computer agents employing different learning rules and parameter values. Such simulations are also more realistic in investigating best response and long run equilibration, because they drop the implicit assumption in the previous analysis that a player's actions are independent of changes in his opponent's behavior. A further advantage of the agent based approach is that analyses of experimental data are necessarily restricted only to the learning models, and the associated parameter values, observed in the subject pool, whereas simulations are unrestricted in this sense.

Simulations were conducted of two agents programmed to play according to either *fp1* or *fp2* with a memory parameter of one, henceforth denoted as *fp1(1)* and *fp2(1)* (the number in the brackets denotes the value of the memory parameter) or *fp1* with a memory parameter of zero, *fp1(0)*, for 100 rounds in each game. In 2×2 games *fp1(0)* is equivalent to the win-stay/lose-shift heuristic (*ws/ls*) because *fp1(0)* assumes that an opponent's action in the current period will be the same as the previous period action and then best responds.

The following variables were tracked during 1,000 simulations of each game: payoffs to each agent,

[19]The differences in parameter estimates between the two models appear to be quite robust as they persisted even when the model was re-estimated with δ allowed to be different for the two belief models, or imposing the restriction that λ_n necessarily lie between zero and one, or restricting some of the elements of the copula correlation matrix to zero.

TABLE VIII
Statistics from simulations of *fp1(1)* versus *fp2(1)*

	fp1(1)		*fp2(1)*	
	Column	Row	Column	Row
Mean payoffs	3.492	4.12	3.88	4.5
$p(r)$	0.613	0.6	0.441	0.597
$p(g)$	0.387	0.4	0.559	0.403
$p(g\|g)$	0.215	0.236	0.382	0.223
$p(g\|r)$	0.168	0.161	0.162	0.166
$p(r\|g)$	0.163	0.157	0.167	0.172
$p(r\|r)$	0.434	0.43	0.27	0.42

first- and second-order probabilities of play[20]. The MSNE payoff for row players is 4.2 and for column players it is 3.8, with both row and column players expected to play red with probability 0.6.

9.1. *Simulation of fp1(1) versus fp2(1)*

In simulations of these two agents, the *fp2(1)* agent had average payoffs higher than the MSNE payoffs (both when the *fp2(1)* agent was a column player and a row player), thereby necessarily imposing lower than MSNE payoffs upon the *fp1(1)* opponent, as can be seen in Table VIII. In both cases the *fp1(1)* player exhibits strong serial correlation as identified by $p(g|g)$[21] and $p(r|r)$ which are both greater than the MSNE prediction of 0.16 and 0.36 respectively. These deviations can then be detected and exploited by the *fp2(1)* agent explaining why this player can attain superior payoffs compared to the MSNE prediction at the expense of the *fp1(1)* player.

9.2. *Simulation of fp1(0) versus fp2(1)*

The results differ significantly when the *fp1* player has a memory parameter of zero instead of one, as shown in Table IX. The *fp1(0)* player now manages to attain better than MSNE payoffs both as a column player as well as a row player to the detriment of the *fp2(1)* player. Both players' first- and second-order probabilities deviate from the MSNE prescription, in particular green is chosen twice in a row more often than the MSNE prescribes, whilst the other two period combinations are chosen less often.

9.3. *Simulation of fp1(1) versus fp1(0)*

Table X shows that an *fp1(0)* agent does significantly better than the MSNE payoffs, both when playing as row and as column player. When the *fp1(1)* agent is a row player green is played twice in a row with probability 0.414 which is much higher than the MSNE prediction of 0.16. Hence, whenever the *fp1(0)* agent plays green and wins he will play green again which will now have a high

[20]Small amounts of error were injected into a best response decision rule so as to generate some variability in actions and to avoid becoming mired in a single deterministic action profile.

[21]For simplicity, the convention used is that the first letter represents the action at time t and the second letter the action at $t-1$.

TABLE IX
Statistics from simulations of *fp1(0)* versus *fp2(1)*

	fp1(0)		*fp2(1)*	
	Column	Row	Column	Row
Mean payoffs	3.9	4.333	3.667	4.1
$p(r)$	0.563	0.563	0.564	0.436
$p(g)$	0.437	0.437	0.436	0.564
$p(g\|g)$	0.211	0.211	0.211	0.337
$p(g\|r)$	0.216	0.216	0.215	0.216
$p(r\|g)$	0.217	0.216	0.216	0.216
$p(r\|r)$	0.336	0.337	0.338	0.211

TABLE X
Statistics from simulations of *fp1(1)* versus *fp1(0)*

	fp1(0)		*fp1(1)*	
	Column	Row	Column	Row
Mean payoffs	4.118	4.517	3.483	3.882
$p(r)$	0.597	0.597	0.598	0.401
$p(g)$	0.403	0.403	0.402	0.599
$p(g\|g)$	0.222	0.221	0.221	0.414
$p(g\|r)$	0.172	0.173	0.172	0.172
$p(r\|g)$	0.173	0.174	0.173	0.171
$p(r\|r)$	0.413	0.412	0.414	0.222

probability of being his best response. When the perfect memory agent is a column player both red and green are repeated more often than they should be thereby again allowing the *fp1(0)* agent to have a higher success rate at playing his best response. The case where the row player is *fp1(0)* and the column player is *fp1(1)* is particularly interesting as the first-order probabilities are equal to the MSNE prediction of playing the red action 60% of the time. However, second-order play deviates from MSNE predictions as the probability of playing red twice in a row is 0.221 instead of 0.36, leading to higher than MSNE payoffs for the row player.

9.4. *General observations from the agent simulations*

From the strategies studied above, an *fp1(0)* agent, equivalent to the *ws/ls* heuristic, outperforms both of the other postulated models, *fp1(1)* and *fp2(1)*. Another interesting result is that whether agents play first-order probabilities less than or greater than the MSNE probabilities may depend on whether an algorithm is playing as a row or column player. Furthermore, the payoff incentives for adopting the best possible learning model (out of the ones considered here) are not very large as payoffs do not really increase much i.e. the curvature of the payoff function is relatively flat around the MSNE. The changes in payoffs are larger when the row player is *fp2(1)* versus *fp1(1)* and in the two possible cases where an *fp1(0)* agent is playing an *fp1(1)* agent.

The performance of a simple heuristic such as *ws/ls* may appear surprising however there exists well documented evidence from the psychology literature that such heuristics can in fact perform well compared to other complex rational models, in some cases even outperforming them. Martignon and Laskey (1999) argue that simple heuristics may perform better than complex models because the latter are vulnerable to overfitting on account of the large number of parameters that they have, a problem that is especially acute in very noisy environments. Also, the reduced number of free parameters of simple heuristics makes them more robust to variations in the environment. Another reason that heuristics may be effective is that they are tailored by evolutionary pressure to exploit inherent structures in the environment. For example, the *ws/ls* heuristic is a very simple way of exploiting positively correlated events in the environment. For an extensive discussion of simple heuristics and their effectiveness/robustness the reader is referred to Gigerenzer and Selten (2001) and Gigerenzer (2000).

10. Conclusion

This paper proposed two extensions to standard fictitious play belief models incorporating pattern recognition and psychophysical principles of subjective perception. These models were empirically fitted to the data from Nyarko and Schotter (2002) as this innovative experiment collected both action data and also elicited subjects' beliefs, allowing for better econometric estimation.

The first extension embeds standard fictitious play beliefs in a non-linear psychophysical function that resulted in significantly better fit, demonstrating that players were more likely to adjust beliefs in the face of larger deviations from the mixed strategy Nash equilibrium than smaller deviations. Standard weighted fictitious play estimates of the memory parameter, γ, are centered on one, whereas the mean estimate for the psychophysical/subjective models was found to be equal to 0.501, with many individuals exhibiting zero (or near zero) memory parameter estimates corresponding to Cournot beliefs.

The second extension permitted the detection of consecutive two-period patterns, and assuming all players were using this model instead of a non-pattern detecting model led to significantly improved fit to subjects' stated beliefs. Latent class models, that allowed for subject heterogeneity in terms of whether individual players used pattern detection or not, were used to estimate the unconditional prior probability of a subject employing a pattern detecting fictitious play model. This probability was estimated at 0.337 when using both elicited beliefs and action data in a two-stage estimation procedure, whereas simultaneous estimation using only the action data led to an estimate of 0.551. The higher estimate for the latter model is consistent with conclusions from the psychology literature that pattern detection may be both a conscious and subconscious mechanism of the human mind.

The finding of player heterogeneity as regards the use of pattern detection prompted the question of whether some subjects were incapable of pattern detection. This explanation is defined as between-subjects heterogeneity, it assumes players do not have the same models of behavior at their disposal, perhaps due to different levels of bounded rationality. Alternatively, within-subjects heterogeneity occurs when agents have the ability to employ different models of behavior but choose which model to apply conditional on their opponents' behavior. Within-subjects heterogeneity was empirically tested by allowing the prior probability of employing a pattern detecting model to depend on the magnitude of an opponent's observable deviation from independently distributed action choices. The conditional prior of using a pattern detecting belief model was found to be significantly higher the more an opponents' action data exhibited exploitable two-period patterns and vice versa. This result is particularly detrimental to the minimax hypothesis as not only do two-period patterns exist in subjects' behavior, but they persist even when players are exploiting them.

Finally, this paper reverted to agent based simulations to examine behavior such as the evolutionary fitness of various belief learning models. Surprisingly, it was found that the simple win-stay/lose-shift heuristic outperformed standard and pattern-detecting fictitious play models.

Further directions for research in this field include eliciting beliefs for other types of strategic games with repeated interactions and allowing for larger action spaces. Pitting subjects against computer algorithms designed to deviate from i.i.d. behavior to various degrees could allow for a more comprehensive analysis as to the type and depth of patterns that subjects are able to detect and exploit. A change in methodology to include neuroeconomic experiments would also be of great interest. Examination of neuronal activity when second- and higher-order probabilities are manipulated could provide direct evidence and details of the encoding process of this type of information.

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