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A selection of maximal elements under non-transitive indifferences

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Abstract

In this work we are concerned with maximality issues under intransitivity of the indifference. Our approach relies on the analysis of "undominated maximals" (cf., Peris and Subiza [7]). Provided that an agent's binary relation is acyclic, this is a selection of its maximal elements that can always be done when the set of alternatives is finite. In the case of semiorders, proceeding in this way is the same as using Luce's selected maximals.

We put forward a sufficient condition for the existence of undominated maximals for interval orders without any cardinality restriction. Its application to certain type of continuous semiorders is very intuitive and accommodates the well-known "sugar example" by Luce.

Key words: Maximal element, Selection of maximals, Acyclicity, Interval order, Semiorder

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1 Introduction

Even though there are arguments to ensure the existence of maximal elements for binary relations in very general settings, this concept does not always explain choice under non-transitive indifference well. Luce [6] argued that in order to account for certain procedural aspects better, some "selection of maximals" helps the researcher. From his Introduction: "... a maximization principle is almost always employed which states in effect that a rational being will respond to any finite difference in utility, however small. It is, of course, false that people behave in this manner". After attaching to intransitivities of the indifference the imperfect response sensitivity to small changes in utility, he proposed the concept of a semiorder as a way to deal with intransitive indifferences without giving up transitivity of the strict preference.

In this work we are concerned with maximality considerations under intransitivity of the indifference. It is based on the analysis of "undominated maximals", a concept introduced and explored in Peris and Subiza [7]. They establish two particularly remarkable facts. For one thing, such selection of maximals can be done when the set of alternatives is finite provided that the binary relation is acyclic. For another, proceeding in this way is the same as using Luce's selected maximals [2] in the case of semiorders.

In light of these two facts it seems interesting to provide conditions for the existence of undominated maximals without any cardinality restriction. We put forward a sufficient condition for interval orders, whose application to certain type of continuous semiorders is very intuitive and accommodates the well-known "sugar example" by Luce.

In Section 2 we establish our notation and preliminary definitions. Then in Section 3 we analyse the existence of undominated maximals in the case of

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[2] [6, Section 3] states that in terms of a semiorder on a set it is possible to define a natural weak ordering of the same set. If the semiorder is a weak order then the induced weak ordering is identical to the given one. Luce's selected maximals are the maximal elements of such induced weak ordering, provided they exist. Example 2 below formalizes this construction.

unrestricted cardinality of the set of alternatives. As an application, a concrete specification leading to Luce's analysis of the "sugar example" is provided. Finally, we investigate the role of the different assumptions in our results. Section 4 contains some conclusions and remarks.

2 Notation and preliminaries

Let us fix a ground set X of alternatives. Unless otherwise stated, henceforth $\succ$ denotes an *acyclic relation*, i.e., $x_1 \succ x_2 \succ ... \succ x_n$ implies $x_1 \neq x_n$ for all $x_1, ..., x_n \in X$. Its *lower* (resp., *upper*) *contour set* associated with $x \in X$ is $\{z \in X : x \succ z\}$ (resp., $\{z \in X : z \succ x\}$). A subset $A \subseteq X$ is a *lower* (resp., *upper*) set of $\succ$ when $a \in A$, $x \in X$, and $a \succ x$ (resp., $x \succ a$) implies $x \in A$.

Denote by $\succsim$ the *complement of the dual* of $\succ$ (i.e., $x \succsim y$ if and only if $y \succ x$ is false), and by $\sim$ the *indifference relation* associated with $\succ$ (i.e., $x \sim y$ if and only if both $x \succsim y$ and $y \succsim x$).

With every acyclic relation $\succ$ on X we associate the *traces* $\succ^*$ and $\succ^{**}$ defined as follows: for all $x, y \in X$,

$$x \succ^* y \Leftrightarrow \exists \xi \in X : x \succ \xi \succsim y,$$

$$x \succ^{**} y \Leftrightarrow \exists \eta \in X : x \succsim \eta \succ y.$$

Therefore, if we denote by $\succsim^*$ and $\succsim^{**}$ the respective complements of the duals of $\succ^*$ and $\succ^{**}$ we have

$$x \succsim^* y \Leftrightarrow (y \succ z \Rightarrow x \succ z),$$

$$x \succsim^{**} y \Leftrightarrow (z \succ x \Rightarrow z \succ y).$$

We recall that a binary relation $\succ$ on X is an *interval order* if it is *irreflexive* and the following condition is verified for all $x, y, z, w \in X$:

$$(x \succ z) \textit{ and } (y \succ w) \Rightarrow (x \succ w) \textit{ or } (y \succ z).$$

Further, a binary relation $\succ$ on X is a *semiorder* if $\succ$ is an interval order and the following condition is verified for all $x, y, z, w \in X$:

$$(x \succ y) \textit{ and } (y \succ z) \Rightarrow (x \succ w) \textit{ or } (w \succ z).$$

If $\succ$ is an interval order then $\succ^*$ and $\succ^{**}$ are *weak orders* (i.e., *asymmetric* and *negatively transitive* binary relations). If $\succ$ is a semiorder then the binary relation $\succ^0 = \succ^* \cup \succ^{**}$ is a weak order (cf., Fishburn [5], Theorem 2 of Section 2) and therefore we have that $x \succ^* y$ implies that $x \succsim^{**} y$ for all $x, y \in X$.

Using the terminology in Peris and Subiza [7], the *weak dominance relation* $\succsim^D$ and the *strict dominance relation* $\succ^D$ associated with an interval order $\succ$ on a set X can be defined as follows: for each $x, y \in X$,

$$x \succsim^D y \Leftrightarrow x \succsim^* y \text{ and } x \succsim^{**} y,$$

$$x \succ^D y \Leftrightarrow x \succsim^D y \text{ and } not(y \succsim^D x).$$

We denote by $M(X, \succ)$ the set of *maximal elements* relative to $\succ$ on X, i.e., $M(X, \succ) = \{x \in X : \forall z \in X,\ z \succ x \text{ is false}\}$.

If τ is a topology on X, $\succ$ is *upper semicontinuous* if its lower contour sets are open. From Alcantud [1], we say that (X, τ) is $\succ$*-upper compact* if for each collection of lower open sets which covers X there exists a finite subcollection that also covers X.

3 Selection of maximal elements for acyclic relations

The set of *Undominated Maximal elements* of X is defined as

$$UM(X, \succ) = M(X, \succ) \cap M(X, \succ^D)$$

It is known that if we restrict ourselves to *finite* sets, maximal elements do exist under acyclicity of $\succ$ (cf., Peris and Subiza [7, Theorem 2]). In Subsection 3.1 we show that even if we focus on semiorders and impose classical (and restrictive) conditions in the vein of the Bergstrom-Walker theorem, when the ground set is infinite the set of undominated maximals may be empty. Then in Subsection 3.2 we produce general conditions for the existence of undominated maximals on topological spaces with arbitrary cardinality. Subsection 3.3 yields a Corollary with an application to a celebrated analysis by R. D. Luce.

3.1 Undominated maximals vs. maximal elements

In the case of binary relations on *finite* sets, the existence of maximal elements for any subset is equivalent to the acyclicity of the relation. In turn that assumption ensures the existence of undominated maximals for any such subset.

When we move to *infinite* sets, maximal elements (and undominated maximals) may not exist. The literature has provided many additional conditions under which maximal elements do exist. There are different tendences in this literature but the most celebrated approach probably is the Bergstrom-Walker theorem and variations of it. Its basic form states that upper semicontinuous acyclic relations on compact topological spaces have maximal elements. Example 1 below shows that even in the case of semiorders on countable sets, this specification does not suffice to ensure that undominated maximals exist.

Example 1 *Let us fix* $A = \mathbb{N} = \{1, 2, 3, ...\}$. *The next expression produces an upper semicontinuous semiorder with respect to the* excluded point *topology associated with* $\{1\}$ *on* A [3], *which is always compact:*

$$m \succ n \text{ if and only if } m \text{ is odd, } n \text{ is even, and } m + 1 \geqslant n$$

Although A *has an infinite number of maximal elements (namely, the odd numbers) there are not undominated maximals because* $(m+2) \succ^D m$ *when* m *is odd.*

If we adhere to the topological approach in our quest for conditions that guarantee that undominated maximals do exist then we need to consider other suitable assumptions. That is the purpose of Subsection 3.2 below.

3.2 Existence of undominated maximals for unrestricted domains

The next Lemma shows that an alternative expression for the set of undominated maximals can be given under only acyclicity of $\succ$.

Lemma 1 *Suppose that* $\succ$ *is an acyclic relation on* X. *Then*

$$UM(X, \succ) = M(M(X, \succ), \succ^D)$$

Proof: Along the proof of Peris and Subiza [7, Theorem 2] the inclusion $M(M(X, \succ), \succ^D) \subseteq UM(X, \succ)$ is proved. The fact that every $x \in UM(X, \succ)$ belongs to $M(M(X, \succ), \succ^D)$ is immediate. $\lhd$

Next we present some technical and useful properties that hold in our setting.

Lemma 2 *Suppose that* $\succ$ *is an acyclic relation on* X. *Then:*

(1) $\succ^D \subseteq \succ^*$ *on* $M(X, \succ)$

[3] The open sets are the subsets of A that do not contain 1 plus A

(2) $M(X, \succ^*) \subseteq M(X, \succ)$

Proof: In order to check *(1)* we notice that the original expression for $\succ^D$, namely

$$x \succ^D y \Leftrightarrow \begin{cases} x \succ^* y \quad \text{and} \quad x \succsim^{**} y & (a) \\ \qquad\qquad \text{or} & \\ x \succ^{**} y \quad \text{and} \quad x \succsim^{*} y & (b) \end{cases}$$

can be simplified because now $x \succ^{**} y$ is impossible since y is maximal for $\succ$. This fact rules out *(b)* and yields the conclusion.

Part *(2)* is direct because $\succ$ includes $\succ^*$: if $x \succ y$ then $x \succ y \succsim y$ due to irreflexivity of $\succ$. $\lhd$

Remark 1 *Besides Lemma 2 (1) we can further note that if $\succ$ is a semiorder then $\succ^D = \succ^*$ on $M(X, \succ)$. Thus $x, y \in UM(X, \succ) = M(M(X, \succ), \succ^D) = M(M(X, \succ), \succ^*)$ now yields $x \sim^* y$. Because $x \sim^{**} y$ is trivial here, we conclude as in Peris and Subiza [7, Proposition 2] that* [4] *$x \approx y$ when $x, y \in UM(X, \succ)$ (but for semiorders only).*

For the reader's convenience we give a proof of a preliminary result stated in Bridges [3, Proposition 2.1]:

Lemma 3 *Suppose that $\succ$ is an irreflexive relation on X. Then $\succ^*$ is asymmetric if and only if $\succ$ is an interval order* [5] *.*

Proof: Necessity is trivial because in fact $\succ^*$ is a weak order provided that $\succ$ is an interval order . For sufficiency assume that $x, y, z, w \in X$ satisfy $x \succ z$ and $y \succ w$. If both $x \succ w$ and $y \succ z$ are false we obtain $y \succ w \succsim x$ and $x \succ z \succsim y$. This means $y \succ^* x \succ^* y$, against asymmetry of $\succ^*$. $\lhd$

We are ready to present our main result.

Theorem 1 *Suppose that $\succ$ is an irreflexive relation on X topological space. If $\succ^*$ is asymmetric and upper semicontinuous and X is $\succ^*$-upper compact then $\succ$ has undominated maximal elements on X.*

Proof: The relation $\succ$ must be an interval order by Lemma 3, therefore it is acyclic. Proposition 2 in Alcantud [1] ensures that $M(X, \succ^*)$ is non-empty

[4] The reader can check that $\approx = \sim^* \cap \sim^{**}$ is Fishburn's equivalence relation as defined in [7, page 3].

[5] We do not use the fact that these conditions are also equivalent to the asymmetry of $\succ^{**}$, which is stated in Bridges' Proposition.

and $\succ^*$-upper compact. Because

$$M(X,\succ^*) \cap M(X,\succ) \subseteq M(M(X,\succ),\succ^*)$$

we can apply Lemma 2 *(2)* to deduce $\varnothing \neq M(X,\succ^*) \subseteq M(M(X,\succ),\succ^*)$. Now we use Lemma 2 *(1)* to produce

$$\varnothing \neq M(M(X,\succ),\succ^*) \subseteq M(M(X,\succ),\succ^D)$$

and then Lemma 1 in order to enforce

$$\varnothing \neq M(M(X,\succ),\succ^D) = UM(X,\succ)$$

$\lhd$

3.3 An application to Luce's maximal elements

Corollary 1 *Suppose that $\succ$ is a continuous interval order with respect to a given topology on X, and that $\succ^* = \succ^{**}$. Therefore X has undominated maximals as long as it is $\succ^*$-upper compact.*

Proof: Because $\succ^* \cup \succ^{**} = \succ^* = \succ^{**}$ is a weak order, $\succ$ is a semiorder. Besides, $\succ^* = \succ^{**}$ is continuous because $\succ^*$ is lower semicontinuous and $\succ^{**}$ is upper semicontinuous (cf., Bosi et al. [2], proof of implication $(ii) \Rightarrow (iii)$ in Theorem 3). Now Theorem 1 applies. $\lhd$

Corollary 1 contains the following widely known specification.

Example 2 *Consider the case where $X = \mathbb{R}$ with the usual topology, and take any $u : X \longrightarrow \mathbb{R}$ continuous and $K > 0$. Then the continuous semiorder defined by $x \succ y$ if and only if $u(x) > u(y) + K$ satisfies $\succ^* = \succ^{**}$ (cf., Campión et al. [4, Theorem 3.5]). Therefore $\succ$ has undominated maximals on any compact set. In particular, let $u = id$ and $K = 2$. It is immediate that $\succ^* = \succ^{**} = \succ^* \cup \succ^{**}$ is the usual order of the real numbers. Then the compact set $C = \{15, 16, 17, 18, 19, 20\}$ has undominated maximal elements. Moreover they coincide with Luce's maximals (namely, $LM(C,\succ) = M(C,\succ^* \cup \succ^{**})$) because for any semiorder P on a set A the equality $LM(A,P) = UM(A,P)$ holds true by Peris and Subiza [7, Theorem 4 (c)]. This is how Luce's maximal set selects $\{20\}$, the "true" maximal element in Luce's "sugar example" (v., e.g., [7, page 4]).*

We proceed to prove that the assumption that $\succ^* = \succ^{**}$ in Corollary 1 is not superfluous. As to the role of the $\succ^*$-upper compactness assumption, we address to Example 4.

Example 3 *Take the semiorder given by Example 1 but now endow the ground set A with the topology specified by the following basis (cf., Willard [8, Section 2.5]):*

$$\{\{m, m+2, m+4, ...\} : m \text{ is odd }\} \bigcup \{\{2, 4, 6, ..., n\} : n \text{ is even }\}$$

Then $\succ$ is continuous and the topology is $\succ^*$-upper compact because $\succ^*$ is the weak order given by

$$... \succ^* 5 \succ^* 3 \succ^* 1, \quad 2 \sim^* 4 \sim^* 6 \sim ..., \quad 1 \succ^* 2$$

thus the only lower (with respect to $\succ^*$) open set that contains 1 *is* $A = \mathbb{N}$. *Also, because $\succ^{**}$ is the weak order given by*

$$1 \sim^{**} 3 \sim^{**} 5 \sim ..., \quad ... \succ^{**} 6 \succ^{**} 4 \succ^{**} 2, \quad 1 \succ^{**} n \text{ for each even } n$$

it is apparent that $\succ^* \neq \succ^{**}$.

3.4 On the assumptions of Theorem 1

Theorem 1 shows that an adequate relationship between the binary relation and the topology on X produces the desired conclusion. Examples 4 and 5 below show that in the precise combination of properties that we have proposed (upper semicontinuity and upper compactness of the topology, both with respect to $\succ^*$) neither of them is superfluous.

Example 4 *Consider $B = [0, 1) \subseteq \mathbb{R}$ in Example 2. Then its usual topology is not $\succ^*$-upper compact, and $\succ$ has not even maximal elements on B.*

Therefore the $\succ^*$-upper compactness assumption is not superfluous in Theorem 1. The same is true for Corollary 1.

Example 5 *Consider $D = [0, +\infty) \subseteq \mathbb{R}$ in Example 2. If we endow it with the excluded point topology associated with* 0, *then $\succ^*$ is not upper semicontinuous. Although D is $\succ^*$-upper compact, $\succ$ has not even maximal elements on D.*

Therefore upper semicontinuity of $\succ^*$ is not superfluous in Theorem 1.

4 Concluding remarks

In trying to fill the gap about lack of general conditions for existence of certain selections of maximals, we have focused on at least acyclic relations because they are the natural setting for maximality purposes. The usual conditions ensuring that maximal elements exist (upper semicontinuity with respect to compact topologies) do not even permit to guarantee that a maximal is undominated when the relation is a semiorder. The characterization of undominated maximality given by Lemma 1 seems to point at making assumptions on the strict dominance relation, because useful topological properties of the set of maximal elements are known (as recalled along the proof of Theorem 1). We have explored an intuitive approach to this possibility, based on making assumptions on a trace of the original relation instead [6]. Because the structure of interval orders is very rich and it is related to that of their traces, our proposal favours an especially intuitive specification for a case where the relation is in the class of semiorders (cf., Corollary 1). In light of Remark 1 we can assure that assumptions on the strict dominance relation are being made in such case. As a consequence, we deduce the existence of Luce's maximal elements in settings like the highly cited "sugar example".

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[6] Our main theorem concerns at least interval orders, as Lemma 3 explains. Nonetheless its statement makes it clear that the assumptions that must be checked in order to apply it are very simple.