On the Computation of the Hausdorff Dimension of the Walrasian Economy:Further Notes

Dominique, C-Rene

Independent Research

9 August 2009

ON THE COMPUTATION OF THE HAUSDORFF DIMENSION OF THE WALRASIAN ECONOMY: FURTHER NOTES

C-René Dominique*

ABSTRACT: In a recent paper, Dominique (2009) argues that for a Walrasian economy with m consumers and n goods, the equilibrium set of prices becomes a fractal attractor due to continuous destructions and creations of excess demands. The paper also posits that the Hausdorff dimension of the attractor is d = ln (n) / ln (n-1) if there are n copies of sizes (1/(n-1)), but that assumption does not hold. This note revisits the problem, demonstrates that the Walrasian economy is indeed self-similar and recomputes the Hausdorff dimensions of both the attractor and that of a time series of a given market.

KEYWORDS: Fractal Attractor, Contractive Mappings, Self-similarity, Hausdorff Dimensions of the Walrasian Economy, the Hausdorff dimension of a time series of a given market.

INTRODUCTION

Dominique (2009) has shown that the equilibrium price vector of a Walrasian pure exchange economy is a closed invariant set $E^* \varepsilon\ R^{n-1}$ (where R is the set of real numbers and n-1 are the number of independent prices) rather than the conventionally assumed stationary fixed point. And that the Hausdorff dimension of the attractor lies between one and two if n self-similar copies of the economy can be made. This last assumption is not valid due to the fact that $[n / (n - 1)] > 1$. Additionally, one could remark that with n non independent variables, the open set condition (see (5) below) on the sequence of contractions θ_p and θ_q, $\forall p, q\ \varepsilon\ R^n$ is violated as some intersections of similitudes, θ_p, θ_q are not empty. This is true too. But a simpler explanation is that the set was over-covered. This note shows that the Hausdorff dimension coincides with the Euclidean dimensions of the space. In other words, the Walrasian economy resembles a space-filling curve whose Hausdorff dimension is equal to the dimension of the space the attractor fills.

Problems of this sort receive in-dept treatments in mathematics and in physics, but not yet in economics. The purpose of this note, therefore, is to show how the Stefan Banach (1892- 1945) Contractive Mapping Theorem could be brought to bear on the economic problem as well. Mainly for completeness, however, the first part of the note reformulates the economic problem. The second briefly reviews the theory of self-similar sets. The Third Part demonstrates that the Walrasian economy is indeed a self-similar structure and computes its fractal dimension. The last part discusses the implications and lessons to be drawn from that simple exchange model.

I THE WALRASIAN PROBLEM

Consider a simple pure exchange model with $i\ \varepsilon\ m$ consumers and $j\ \varepsilon\ n$ goods. Consumers are driven by two basic axioms. Namely, they are *self-interested* agents, and they are endowed with *monotone preferences*. However, in a competitive economy, they act uncooperatively and they are unable to identify equilibrium prices. Therefore, they continuously adjust their budget shares and /or the quantity of their initial endowments brought to the market according to the conventional rule:

$$(1) \qquad dp_j / dt \rightarrow \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \quad \text{if} \quad \zeta_j(\mathbf{p}) \rightarrow \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0, \quad \forall\, j\ \varepsilon\ (n\text{-}1),$$

where time is represented by t, $\zeta(.)$ stands for excess demand and I here recall that the n^{th} good is the numéraire. Under (1), a number of system matrices $\mathbf{M}_k$ (k = 1, 2, ….) are generated, mapping the k^{th} excess demand vector (ζ_k ($\mathbf{p}$)) into $\mathbf{p}^*$ such that each p_j is the image of $\zeta_j(\mathbf{p})$ under $\mathbf{M}_k$. That is,

$$(2) \qquad \mathbf{M_k} : \zeta(\mathbf{p}) \rightarrow E^*_k = \mathbf{p}^*_k \quad k = 1, 2, \ldots,$$

where $\mathbf{p^*_k} = \{p^*_1, p^*_2, \ldots, p^*_{n-1}, 1\}$. However, $\mathbf{M}_k$ represents a particular T_k , *a member of the open subset of all invertible linear transformations in the endomorphism (End (R^{n-1})* T, where

$$(3) \qquad T_k\ \varepsilon\ T = \{T_k : T_k\ \varepsilon\ \text{End}\ (R^{n-1})\}, \quad k = 1, 2, \ldots.$$

The excess demand functions are homogeneous of degree zero in prices, and prices are non-negative numbers. Hence, the price vector $\mathbf{p}_k$ can be normalized as $p_j = p_k / \Sigma^{n-1}{}_k\, p_k, \forall p_k$. Then, the equilibrium price vector E belongs to the (n-1)-dimensional unit simplex $S^{n-1} = E^* = \{\mathbf{p}^*\ \varepsilon\ R^{n-1}{}_+ : \Sigma^{n-1}{}_1\, p_k = 1\}$. Without loss of generality, I will now and forward, consider n-1dimensional space, reference the discussion and the result to the matrix $\mathbf{M}_k$, and express the end result in terms of (n – 1) = n*.

Equation (3) arises in a natural manner. Each destruction and/or creation in the seemingly random sequence of variations of excess demands is equivalent to the selection of a different member of the set T; process (1) can therefore be assumed continuous. Obviously, if T had only one member, E^*_k would be unique and asymptotically stable. However, that situation does not obtain in the present case. Instead, and under continuous adjustments, the price vector becomes a wobbling vector whose tip traces out the boundary of the attracting set. Before considering the consequences of this, however, the note briefly reviews the theoretical foundations of the notion of self-similarity. This then allows us to move straight to what is of interest without having to provide too many proofs.

II THEORETICAL CONSIDERATIONS

As is well-known, the Hausdorff measure is a generalization of Euclidean dimensions in the sense that it generates non-negative numbers as dimensions of any metric space. This means that it coincides with Euclidean dimensions for regular sets but it additionally measures the dimensions of any irregular sets such fractal attracting sets or, equivalently, strange attractors.

Concisely, the Hausdorff dimension of a set Y is a real number $d \varepsilon (0, \infty)$. To determine d, one considers the number of balls $B(y_j, r_j)$ of radius $r_j > 0$ needed to cover Y. As r_j decreases, B (.) increases, and vice versa. That is, if the number of B (.) grows (shrinks) at the rate $(1/r_j)^d$, as r_j shrinks (grows) to zero (approaches one), then d is the dimension of Y.

There exist various procedures for computing d, namely, 'box-counting' (also known as 'Minkawski-Bouligand'), 'Minkowski', 'Frostman', etc. But, except for some well-documented cases, all of these procedures give the same value. Here, however, it is more convenient to focus on the Hausdorff measure for self-similar sets. Again, for tractability, I will drop the index k, and refer to the attractor as $E^*(= S^{n-1})$.

E* is self-similar if it is a fixed point of a set-valued transformations $\theta = \{\theta_j \varepsilon \theta\}$, $(j \varepsilon n\text{-}1)$ such that $\theta(E^*) = E^*$. This means that there are n-1 contracting similarity maps such that:

$$(4) \qquad |\theta_j (p_j - \theta_{j+1}(p_{j+1})| = r_j \circ |p^*_j - p^*_{j+1}|, \quad \forall j \varepsilon n\text{-}1,$$

where $0 < r_j < 1$ is a contracting factor, and $|.|$ is the Euclidean norm. Then, this results in a unique non-empty compact set A such that $A = \cup^{n-1}{}_{j=1} \theta_j (A)$. The set A is the self-similar set generated by the maps θ_j consisting of identical (but smaller in scale) copies $A_j = \theta_j (A)$, and each A_j consists of even smaller copies $A_{j,h} = \theta_j (\theta_h (A))$, $\forall j, h \varepsilon n\text{-}1$.

Suppose now that the sequence of contractions yields a feasible non-empty open set A*. Then the Hausdorff dimension d satisfies:

$$(5) \qquad \begin{aligned} &a) \quad \cup^{n-1}{}_{j=1} \theta_j (A^*) \subseteq A^* \\ &b) \quad \theta_j (A^*) \cap \theta_{j+1} = 0, \ \forall j, j+1 \varepsilon n\text{-}1. \end{aligned}$$

Condition (5a) is the *open set condition* on the sequence θ_j; (5b) is the *null intersection rule* requiring that the set in the union are pair wise disjoint(1).

Condition (5b) is strong, but it holds on weaker grounds, i., e. when the intersections are points. Hence, if each θ_j is a similitude (2), i. e., a composition of an isometry and a dilatation around some point, then the unique fixed point of θ is a set whose d is the unique solution to (6). That is,

$$(6) \qquad \Sigma^{n-1}{}_{j=1} (r_j)^d = C (r)^d = 1,$$

where $C = a^{n-1}$ if the radius is (1/a), $a \varepsilon R_+$, and C is then the number of self-similar copies induced by a particular (r).

Because of the purpose, this review is rather brief; however, for the reader who needs a more mathematically elaborate discussion, Hutchinson (1990) is an excellent source. Now then, on the assumption that (4), (5) and (6) are satisfied for (n – 1), I will show that the Walrasian economy is self-similar.

III APPLICATION

Above, I examine contractive mappings, but I could very well consider stretching around a point. In that case, I would stretch from the smallest copy with dilatation factors, $\lambda = 1, 2, 4, \ldots$. In economics, there is a valid argument to the effect that the minimum sized market consists of 3 goods or 3 prices, where one good is the numéraire. The reason is that the elements of E* are the prices and prices are relative values. Hence, only with 3 that those definite statements about uniqueness and asymptotic stability can be generalized to n-1 dimensions. At the same time, however, it should be emphasized that such statements presuppose fixed coefficients of the system matrix, but as already noted in the First Part, they are not. We have instead an n-1 dimensional object to analyze. Even though it cannot be visualized, if it contains no hole, then its Hausdorff dimension(3) is most likely n-1. I will now turn to a demonstration of this.

Let the number of copies C of sizes (1/a) be a^{n-1}. Since self-similarity implies invariance with respect to scale, the Hausdorff dim of a^{n-1} copies must be the same for $(a/2)^{n-1}$ copies of sizes (2/a), or for $(2a)^{n-1}$ copies of sizes (1/2a), etc. Using (6) in n-1space, when the number of copies of size (1/a) is $(a)^{n-1}$, we have:

$$a^{n-1}\,(1/a)^{d} = 1, \tag{7}$$

where d is the Hausdorff dimension. Taking natural logarithms of both sides gives:

$$(n-1)\ln a - d\ln a = 0, \quad \rightarrow d_x = n-1.$$

This means that the attractor is what Falconer (1985) calls a *deterministic fractal*, filling the whole space available to it. Such a self-similar object is invariant to scale. Therefore, I can increase the number of copies, a^{n-1}, to $(2a)^{n-1}$ if the radii of balls are reduced by 1/ 2a without affecting the value of d. To wit:

$$(2a)^{n-1}\,(1/2a)^{d} = 1,$$

then:

$$d_{2a} = (n-1)\,[\ln a + \ln 2] \,/\, [\ln a + \ln 2] = n-1.$$

I will now do the reverse operation, i. e. reduce the number of copies by 2 and double the radius, then:

$$d_{a/2} = (n-1)\,(\ln a - \ln 2) \,/\, (\ln a - \ln 2) = n-1.$$

Table 1 below presents the results of the operations of shrinking and stretching around some points. As it can be seen from the left side, shrinking from a = 2 to a = 2^{n-1} leaves the Hausdorff dim invariant[4]. The way the table is constructed makes it easier to see that the smallest radius r = $1/2^{n-1}$ produces the highest number of copies. The right side shows that as r tends to one, the set tends to a countable set whose Haus-

Table 1 The Self-similarity of the Walrasian Economy

Contraction Dilatation

Euclidean dim	a	Radius r	Copies C	Hausdorff dim d*	Euclidean dim	a	Radius r	Copies C	Hausdorff dim d*
n-1	2^1	$1/2^1$	2^{n-1}	n-1	n-1	2^{n-1}	$1/2^{n-1}$	$2^{(n-1)(n-1)}$	n-1
n-1	2^2	$1/2^2$	$2^{2(n-1)}$	n-1	. . .	. . .	. . .	. . .	. . .
n-1	2^3	$1/2^3$	$2^{3(n-1)}$	n-1	n-1	2^4	$1/2^4$	$2^{4(n-1)}$	n-1
n-1	2^4	$1/2^4$	$2^{4(n-1)}$	n-1	n-1	2^3	$1/2^3$	$2^{3(n-1)}$	n-1
n-1	. . .	. . .	. . .	. . .	. . .	2^2	$1/2^2$	$2^{2(n-1)}$	n-1
1n-1	2^{n-1}	$1/2^{n-1}$	$2^{(n-1)(n-1)}$	n-1	n-1	2^1	$1/2^1$	2^{n-1}	n-1
					n-1	2^0	$1/2^0$	$\rightarrow 1$	$\rightarrow 0$**

- Computed from (6) or (7).

** A countable set whose Hausdorff dim is zero.

dorff dim is zero. Either operation, shrinking or dilating, leaves the Walrasian economy invariant to scale. In terms of (7), therefore, the Hausdorff dimension of the economy is:

$$d_e = (n-1) = n^*, \tag{7'}$$

where n* stands for the number of independent variables. This means that *the attractor fills up the whole space available to it as for a space filling curve.*

For well over 130 years, it was assumed that the Walrasian system was a linear system. This result shows that it never was. What was omitted was the wobbliness of the vector. It can be explained both informally and formally.

Less formally, the Walrasian equilibrium price vector could be compared to a particle trapped inside the positive octant of the (n-1)D - unit simplex. The surface of the positive octant has no indentations or protuberances. The particle visits every interior point of the attractor, but never crosses the boundary.

More formally: A* mimics a compact invariant set of equilibrium prices. *It is a fractal or strange attractor*. The motion of the state of equilibrium can be compared to a flow, denoted: ϕ (**p**). If the flow enters a neighborhood N of A* at $t \geq 0$, then ϕ (**p**) $\rightarrow$A* as $t \rightarrow \infty$, for all **p** in A*. The union of all N's of A* is the domain of attraction and is also the stable manifold A* to which all orbits are attracted. In fact, numerical analysis shows that A* is made of an infinite number of branched surfaces which are interleaved and which intersect but trajectories do not. Instead, trajectories move from one branched surface to another. It follows that the attractor contains: i) an uncountable sets of periodic orbits, ii) a ccountable set of aperiodic orbits, and iii) a dense orbit.

It might be worth recalling that Hausdorff dimensions do not contradict Euclidean dimensions. But fractals are ordinarily defined as structures whose Hausdorff dimensions are non-integer; this is not a necessary condition, however. For example, the boundary of the Mandelbrot and Julia sets (perhaps the most fractal of fractals) both have d = 2; the Sierpinski curve, the More curve, the Hilbert curve in 3D, the Peano curve, the H-fractal, etc., all have d = 2. Even the Brownian motion of dim ≥ 2 has a d =2 according to Falconer (1985). In conclusion, therefore, *as the Smith-Volterra-Cantor curve or the Takagi curve, the Walrasian economy's Hausdorff dimension is (n – 1).*

IV CONCLUDING REMARKS

It is shown in Table 1 that the Walrasian market is a dynamic structure that is self-similar or invariant to scale with a Hausdorff dimension of d = (n – 1). To arrive at this conclusion, the attracting set of prices was normalized to a unit- (n*)-D simplex, and only the positive octant was considered. Since n* is the number of independent variables in the system, then the attractor fills up the whole space. This result is well established in the literature; hence, there is nothing surprising there.

There are a number of lessons to be learnt from this simple model, however. That the Walrasian economy turns out to be a fractal structure is both compelling and unsurprising. To see this, let us for one moment perceive the economy as a product of human behavior in the search of wherewithal. Undoubtedly, it is driven by human decisions, which are thought processes. The human brain happens to be a fractal in a 3-D space which, however, does not occupy the whole 3-D space. The best estimates of its Hausdorff dimension, d_B, fall between 2.72 and 2.79. Hence, it is not surprising that the economy (or any of the so-called social sciences for that matter) turns out to be a dynamic fractal as well.

Contemplating this result, the first consequence that comes to mind is that *trade, or exchange, as nature itself, is a perfectly natural process*. To the extent that exchange implies openness, it tends to defeat temporarily the Entropy Law as well. That is, openness allows for positive growth, whereas closeness corresponds to negative growth or degeneracy. Moreover, as we have learnt from Mandelbrot (1982), fractal geometry is the geometry of nature, but nature uses the process of destructive creation to produce novelties. It is not at all surprising therefore that the market would behave in the similar manner in a competitive setting (in obedience to the basic axioms mentioned above). That is to say that the market destroys and creates excess demands to drive itself toward some equilibrium states, but a stationary fixed point equilibrium is never attained [5]; instead continuous adjustments drive the market straight to a fractal attractor.

The question now is why was this not known before? The reason is that in the mid-19th century, the mathematics of fractal were not available, hence Walras could not have used them to probe the workings of his system. But the same could not be said of his successors in the post 1980-period, however. Had they paid heed to fractal geometry, they would not have spent time searching for global asymptotic stability in markets. For, they would have realized that markets (or economies) live in perpetual disequilibrium, which is necessary condition for positive growth.

Now then, how can this result be reconciled with the findings of empirical investigations? Usually, such investigations focus on *one* market, say, the capital market (see, Peters, 1989, 1991). As observed at the outset, it takes 3 variables to model a single market. In the case of capital, Peters found a Hausdorff dimension of 2.3 for the S&P-500, meaning that 3 variables are indeed necessary to model that market. Then is there a discrepancy between his findings and our results? Absolutely not. His result arises from a time series. Time series are images of trajectories around the attractor. I surmise that a time series of a market is a projection ρ: $R^2_+ \times R_+ \rightarrow R_+ \times R_+$, where the frequency of oscillation due to an eigenvalue is equal to its imaginary part. So the Hausdorff dim calculated from a time series is therefore not the Hausdorff dim of the whole system. Since real parts are suppressed, changes in matrix $\mathbf{M}_k$ (hence in the dominant eigenvalue) produce a more jagged structure in 2D, exceeding the topological dim of the projection. It is therefore important to distinguish between the Hausdorff dim of the economy, d_e, and those (d_{TS}) of periodic and aperiodic trajectories around the attractor, where only distances from the origin and time are recorded. In other words,

$$d_e = n^* \rightarrow \text{for n independent variables constituting the economy, and}$$

$$2 < d_{TS} > 3 \rightarrow \text{for economic time series.}$$

If the economy and its markets are fractal structures, we must then abandon all hopes of eliminating volatility, cycles and even tsunamis in the presence of competition. Policy makers should take notes. For, if our findings are accepted (and they are compelling), and if it is desirable to minimize the amplitudes of these inevitable fluctuations, then to the chagrin of free-marketers everywhere, efficient regulation is a must.

These results have implications for econometrics, risk analyses, and forecasting as well. From (7), it can be observed that the ln-ln plot of the number of copies and r gives a straight line with a slope equal to d. In addition, if we now trace the semi-ln plot of copies C and r, we have a curve. Consider the eighth and ninth columns of Table 1. Take ln C as ordinate, and r as abscissa, then at any point on the curve: d = (-) vertical distance / ln horizontal distance. As r tends to one, the slope of the curve tends to zero, the Hausdorff dim falls to zero, and we have just one copy. As r tends to zero, the ln C increases without limit. Under the circumstances, the 'mean' and the 'standard deviation' become useless statistics; and so it is in all fractal distributions. This also means that these systems are sensitive to initial conditions; hence those involved in these areas of research should not be surprised to go astray whenever they venture into the long term.

Finally, among all those theorists who, in the 1960s, have attempted to prove asymptotic stability of Walrasian models, Herbert Scarf (1960) is the only one that stumbled on a fractal attractor to my knowledge. It was not called fractal attractor then, because very few people knew what a fractal attractor was; anyway, such a phenomenon was unknown in economics. For that reason, this note vindicates Scarf's foresight.

NOTES

(1) Eq. (5b) is usually preceded by H^d or the Hausdorff measure which an outer measure that assigns a number $\alpha\ \varepsilon\ (0, \infty)$ to each finite set in a metric space. The relation between H^d and d is:

$$d\,(.) = \inf\{d \geq 0 : H^d\,(.) = 0\} = \sup\,[\{\, d \geq 0 : H^d\,(.) = \infty\} \cup \{0\}],$$

where $\inf\{\varnothing\} = 0$. For the set A*, the 0 dim H^d is the number of points in A*. For a proof, see Hutchinson (1990). Condition (5b) can also admit tangent point intersections as a weaker condition.

(2) A similitude is a *homothety* that leaves the origin fixed while ruling out rotation. If it preserves distances between two topological spaces, it is then an *isometry*.

(3) For example, the human brain is a 3-dimensional object that does not occupy the whole space; consequently its Hausdorff measure is less than 3.

(4) As an example, consider the Sierpinski triangle which is easily visualized. It has 3 copies of sizes 1/2 or 9 copies of sizes 1/4. Hence, applying (7) or (7'), we have

$$3\,(1/2)^d = 9\,(1/4)^d \rightarrow d = \ln 3/ \ln 2 = \ln 9/ \ln 4 = 1.58.$$

The triangle is therefore invariant to scale. When a n-dimensional object is not visualizable, the number of copies of sizes 1/2 is 2^n, or the number of copies of sizes 1/4 is 4^n. In both cases, d = n, because $(4)^n = (2^n.\ 2^n)$. For convenience, we let $2^n = x$. The number of copies doubles when copy sizes are reduced by 2. Writing $(2^n.\ 2^n)\,(1/4)^d = 1 = (x)^2\,(1/4)^d$. Then for x copies,

$$2^n\,(1/2)^d = x\,(1/2)^d = 1 \rightarrow d = \ln x / \ln 2.$$

For twice as many copies, we have: $(x)^2\,(1/4)^d = 1$. Taking ln and recalling that ln 4 = 2 ln 2:

$$2 \ln x = 2d \ln 2 \rightarrow d = \ln x/ \ln 2.$$

Invariance to scale implies self-similarity.

(5) The only way to achieve a stationary fixed point is for agents never to change the distribution of their preference and their supply from period to period. But that would violate the axiom of monotone pre-preference.

REFERENCES

(1) Bernard, S. (2001). *Universalité et Fractals.* Paris: Collection Champs, Flammarion.

(2) Besicovich, A., S. (1929). "On Linear Sets of Points of Fractional Dimensions." *Mathematische Annalen*, 101.

(3) Dominique, C-R. (2009). "Could Markets' Equilibrium Sets Be Fractal Attractors?"MPRA Paper no13624 University of Munich.

(4) Falconer, J. K. (1985). *The Geometry of Fractal Sets*. New York: Cambridge University Press.

(5) Hausdorff, F. (1919). “Dimension und auberes.” *Mathematische Annalen*, 79, 157-179.

(6) Hutchinson, J., E. (1981). “Fractals and Self-Similarity.” *Indiana Univ. Math. Jour*., 30, 713-47.

(7) Mandelbrot, B. (1982). *The Fractal Geometry of Nature*. San Francisco: W. H. Freeman.

(8) Moran, P. A. (1946). “Additive Functions of Intervals and Hausdorff Measure.” *Proc. Cambridge Philos. Soc.* 42, 15-23.

(9) Peters, E. (1991). “A Chaotic Attractor for the S&P-500.” *Financial Analyst Jour*., March/April.

(10)-------- (1989). “Fractal Structure in the Capital Market.” *Financial Analyst Jour*., July/August.

(11) Scarf, H. (1960). “Some Examples of Global Instability of the Competitive Equilibrium.” *Inter’l Econ. Review*, 1, 157-172.

(12) Schief, A. (1994). “Separation Properties for Self-Similar Sets.” *Proc. Amer. Math. Soc*., 122, 111-15.

(13) Szpilrajn, E. (1937). La dimension et la mesure.” *Fundamenta Mathematica*, 28, 81-89.

* Formerly Professor of Economics.