On a games theory of random coalitions and on a coalition imputation

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Abstract

The main theorem of the games theory of random coalitions is reformulated in the random set language which generalizes the classical maximin theorem but unlike it defines a coalition imputation also.

The theorem about maximin random coalitions has been introduced as a random set form of classical maximin theorem. This interpretation of the maximin theorem indicate the characteristic function of the game and its close connection with optimal random coalitions. So we can write the apparent natural formula of coalition imputation generalizing the strained formulas of imputation have been in the game theory till now. Those formulas of imputation we call the strained formulas because it is unknown from where the characteristic function of the game appears and because it is necessary to make additional suppositions about a type of distributions of random coalitions. The reformulated maximin theorem has both as its corollaries. The main outputs are two results of the games theory were united and the type of characteristic function of game defined by the game matrix was discovered.

Key words: games theory, random coalition, coalition imputation.

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Slow and steady wins the race.
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Introduction

Metaphysical systems. Neither economic, social nor political systems can be imagined without a man. More over every such system can be defined as a set of "small" individuals whose interactions are realizing in forming, existence and disintegration of social, economical or political coalitions. These systems resemble systems of physical particles in statistic mechanics excepting one: interactions of metaphysical particles is defined by their intellect, feelings, social, economical and political state, never by physical laws. So although metaphysical particles exist in usual physical space they are showing themselves as elements of some virtual abstract metaphysical space essentially. Great Von Neumann was the fist who noticed [1] that the classic mathematics had been created mainly for solving problems of physics cannot be used for description of these systems. A new theory is necessary to make a correct description of interactions of particles in *metaphysical systems*. It is similar with statistic mechanics but quite different from it. This new theory can be called the *mathematical metaphysics*.

Metaphysical particles are individuals or some material or non-material elements that sets arise, support and disintegrate by participate of individuals. For example, sets of everyday events, sets of financial instruments on financial markets, sets of goods on good markets and etc. The metaphysical system can be consider as a set of interacting metaphysical particles interaction of that shows by forming, supporting and disintegrating random sets consist of these particles.

We are elements of metaphysical systems. The principle difference metaphysics from physics is the possibility to imagine oneself as a metaphysical particle in the true without any stretch, because a subject of metaphysics is statistical mutual dependencies between individuals that are fastened them by random coalitions. This is impossible chance in physics to look at the situations from inside to imagine oneself as an element of random coalition with own variance and covariance with neighbours. This possibility allows us to make new interpretation of many usual statistic ideas of metaphysics.

Being among sets of metaphysical systems. Our being is a being inside and among sets of metaphysical systems. The individual being combined with others always recreates a diversity of metaphysical systems involving us into life process. We always take part, enter and exit from set of different mutual dependent coalitions that raised a metaphysics interaction between them on different levels and spheres. Levels of metaphysical interactions: nation, ethnos, mass, society, crowd, group and others. Spheres of metaphysical interactions: politics, economics, finance, insurance, science, education, history, psychology, sociology, philosophy and others.

Bases of mathematical metaphysics. There are a lot of reasons to think that modern theory of games and utilities transformed to the theory of random coalition games and the mathematical theory of needs and risks with the random set theory, statistic mechanics of metaphysics, will found a theory never called the mathematical metaphysics, perhaps. But its mathematical idea will be just defined by these three mathematical bases.

Mathematical theory of needs, utilities and risks. The leading idea of the utility and making decision theory is an individual coming to a decision ruled by his own preferences. The aim of mathematical analysis of making decision is to choose such behavior of the individual that consequences will be the most prefer and the probabilities of states of metaphysical system will be the most acceptable for himself. Even the utility theory formed in XVIII and XIX its main development had happened in the second part of the XX century. The main contribution to that development made economics, statistics, mathematics, psychology and cybernetics. The next development of the utility theory is connected with the random set theory and especially with the integer random set theory, that will be consolidated with the theory of risks to make a new mathematical theory of needs based on the random set approach of description of a structure of preferences of the individual as a metaphysical particle.

Random set theory. The theory of random finite abstract sets is the theory of random elements with values of the set of all subsets of the finite abstract set. The main idea of the modern random set theory is that structure of statistic mutual dependencies of subsets of the finite set defined fully by random set distribution has been given on the set of all its subsets. Distribution of a random set is a useful mathematical instrument for description of all possible ways to combine elements into coalitions other words all ways of interaction elements between each other. Although the random set theory is connected with the multivariate statistical analysis but its subject for examine is a random finite abstract set that differs from the random vector because it belongs to abstract spaces had no usual linear structures. The random set theory has been developed last years. This development was promoted by its use for description of metaphysical systems.

Dependencies of events are the base of probability theory pyramid. Random sets keep on our attention again and again to the intent examination of dependencies, a single important subject of the probability theory. It differs the probability theory from the general measure theory and turns up the pyramid of the probability theory, put it in natural position. Therefore instead of examination of rare statistic independency on the boundless underground of dependencies we will examine a diversity of statistic dependencies on the pure independent underground.

Metric covariance and covariance analysis of events. Covariance analysis of events is a part of the probability theory formed its base as a science has been destined to study structures of dependencies between events. Now in covariance analysis one can formulate more problems than their decisions.

A notion of the covariance between random sets is introduced which didn't exist up to now but which is analogous to the usual notion of the covariance between random variables. Seldom in the modern probability theory it is called an attention to the metric interpretation of the covariance which is lied in a base of the new notion.

Game is a model of conflict. We are surrounded by world of conflicted random coalitions, all of us participate in their game directly. Game is thought as the best model of conflict and the game theory is thought as the best mathematical theory of conflicts since Von Neumann times till now [1].

Random set coalitions theory. Forming, existence and disintegration of coalitions of individuals is a metaphysical phenomenon. It is clear that the classical description of game in characteristic form does not take into account any state for metaphysical limitations that promote for development of coalitions. An absence of adequate mathematical instruments describing forming, existence and desintegration of coalitions as consequence of metaphysical reasons is a main theoretical and practical gap in modern theory of games. This theoretical gap can be supplied by the random set theory that gives new ideas a random coalition and coalition imputation. These main ideas of the random set games theory help to formulate and to prove a random set analogy of the maximin theorem, that not only guarantees existence of equilibrium random strategies (random coalitions) but find value of game for every gambler (coalition imputation).

Maximin theorem by Von Neumann in random set language. If in reasons by Von Neumann brought him to the proof of the maximin theorem [1] we will consider random coalitions of gamblers instead of separate gamblers then for this game we proved generalized maximin theorem, the theorem of existence of number of equilibrium random coalitions.

So given the set simple defines characteristic function of every random coalition as so called equilibrium coalition imputation generalizing value of characteristic game by Shapley [3] and index of Banzaf [2] and probabilistic value of characteristic game by Vilkas [5].

1 Theorem about maximin random coalitions

Let consider a finite set of gamblers

$$\mathfrak{X} = \{x_1, \dots, x_N\}$$

under which *two independent random coalitions of gamblers* are defined as two independent *random finite abstract sets (RFASs)*

$$K : (\Omega, \mathcal{F}, \mathbf{P}) \to \left(2^{\mathfrak{X}}, 2^{2^{\mathfrak{X}}}\right), \quad K' : (\Omega, \mathcal{F}, \mathbf{P}) \to \left(2^{\mathfrak{X}}, 2^{2^{\mathfrak{X}}}\right).$$

Other words arbitrary subsets $X \subseteq \mathfrak{X}$ are the values of these random coalitions, i.e. each coalition has the same set $2^{\mathfrak{X}}$ of pure strategies. A *payoff function of the first coalition* K is defined on the set of all possible outcomes of game $2^{\mathfrak{X}} \times 2^{\mathfrak{X}}$ (the cartesian product of sets of pure strategies)

$$g : 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \to R,$$

by a *matrix of game*

$$\mathbf{g} = \left\{ g(X, X') : \ (X, X') \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \right\}.$$

Definition (game of two random coalitions with zero sum). A triple of objects

$$(K, \mathbf{g}, K'),$$

are forming by *two random coalitions* K and K' and a *matrix of game* $\mathbf{g}$ is called *a game of two random coalitions with zero sum.*

It has no trouble to reformulate the main theorem of games theory (the maximin theorem) for the game of two random coalitions with zero sum.

Theorem (about maximin for the game of two random coalitions with zero sum). *For any game* $(K, \mathbf{g}, K')$ *there are such random coalitions* K *and* K' *with distributions*

$$\mathbf{p}(K) = \left\{ p(X) : \ X \in 2^{\mathfrak{X}} \right\}, \quad \mathbf{p}'(K') = \left\{ p'(X) : \ X \in 2^{\mathfrak{X}} \right\}, \tag{1}$$

that

$$\max_{K} \min_{K'} \mathbf{E} g(K, K') = \min_{K'} \max_{K} \mathbf{E} g(K, K'),$$

where

$$\mathbf{E} g(K, K') = \sum_{X \in 2^{\mathfrak{X}}} \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p(X) p'(X')$$

is an expectation of prize for the first random coalition. Other words it is an expectation of random variable $g(K, K')$*, which values are corresponded to elements* $g(X, X')$ *of the matrix* $\mathbf{g}$ *with probabilities*

$$p(X)p'(X') = \mathbf{P}(K = X, \ K' = X'), \ (X, X') \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}}.$$

Before proving the theorem let introduce some notation and formulate auxiliary statements. Let

$$\mathcal{P}_{\mathfrak{X}} = \left\{ \mathbf{p} = \left(p(X), \ X \in 2^{\mathfrak{X}} \right) : \ p(X) \geq 0, \ \sum_{X \in 2^{\mathfrak{X}}}^{m} p(X) = 1 \right\} =$$

$$= \Big\{ K : \quad K \text{ is a RFAS under } \mathfrak{X} \Big\}$$

be a $2^{|\mathfrak{X}|}$-simplex or a space of all probabilistic distributions under $\mathfrak{X}$, or a *space of all random coalitions* under $\mathfrak{X}$. Let introduce notation to the values

$$V(\mathbf{g}) = \max_K \min_{K'} \mathbf{E} g(K, K'), \quad V'(\mathbf{g}) = \min_K \max_{K'} \mathbf{E} g(K, K'),$$

describing the matrix of game $\mathbf{g}$.

Statement 1. *For a square matrix*

$$\mathbf{g} = \big\{ g(X, X') : \ (X, X') \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \big\}$$

there is the RFAS K with distribution $\mathbf{p}(K) \in \mathcal{P}_{\mathfrak{X}}$ where[4]

$$\mathbf{g} \cdot \mathbf{p}^T(K) \leq \mathbf{0}, \tag{2}$$

or there is the RFAS K' with distribution $\mathbf{p}'(K') \in \mathcal{P}_{\mathfrak{X}}$ where

$$\mathbf{p}'(K') \cdot \mathbf{g} \geq \mathbf{0}. \tag{3}$$

The vector form has been used in Statement 1 is equivalent to the next

$$\mathbf{g} \cdot \mathbf{p}^T(K) \leq \mathbf{0} \iff \sum_{X \in 2^{\mathfrak{X}}} g(X, X') p(X) \leq 0, \quad X' \in 2^{\mathfrak{X}},$$

$$\mathbf{p}'(K') \cdot \mathbf{g} \geq \mathbf{0} \iff \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p'(X') \geq 0, \quad X \in 2^{\mathfrak{X}}.$$

Statement 2. *For any function φ on $2^{\mathfrak{X}} \times 2^{\mathfrak{X}}$*

$$\max_X \min_{X'} \varphi(X, X') \leq \min_{X'} \max_X \varphi(X, X').$$

Lemma 1. *For any random coalition K with distribution $\mathbf{p}(K) \in \mathcal{P}_{\mathfrak{X}}$*

$$\min_{K'} \mathbf{E} g(K, K') = \min_{K'} \sum_{X \in 2^{\mathfrak{X}}} \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p(X) p'(X') = \min_{X' \in 2^{\mathfrak{X}}} \sum_{X \in 2^{\mathfrak{X}}} g(X, X') p(X), \tag{4}$$

[4]Here $\mathbf{p}^T(K)$ is the transposed vector of distribution vector $\mathbf{p}(K)$.

for any random coalition K' with distribution $\mathbf{p}'(K') \in \mathcal{P}_{\mathfrak{X}}$

$$\max_K \mathbf{E} g(K, K') = \max_K \sum_{X \in 2^{\mathfrak{X}}} \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p(X) p'(X') = \max_{X \in 2^{\mathfrak{X}}} \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p'(X'). \tag{5}$$

Proof of theorem. Firstly

$$\max_K \min_{K'} \mathbf{E} g(K, K') \leq \min_{K'} \max_K \mathbf{E} g(K, K'),$$

is going from the Statement 2. Or in our notation

$$V(\mathbf{g}) \leq V'(\mathbf{g}). \tag{6}$$

If (3) of Statement 1 is true the random coalition K with distribution $\mathbf{p}_K \in \mathcal{P}_{\mathfrak{X}}$ will exist. And for this coalition

$$\sum_{X \in 2^{\mathfrak{X}}} g(X, X') p(X) \geq 0, \quad X' \in 2^{\mathfrak{X}},$$

therefore

$$\min_{X'} \sum_{X \in 2^{\mathfrak{X}}} g(X, X') p(X) \geq 0.$$

Hence $V(\mathbf{g}) \geq 0$ because (4) and the definition of value $V(\mathbf{g})$ take places.

If (2) of Statement 1 is true the random coalition K' with distribution $\mathbf{p}_{K'} \in \mathcal{P}_{\mathfrak{X}}$ will exist. And for it

$$\sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p'(X') \leq 0, \quad X \in 2^{\mathfrak{X}},$$

so

$$\min_{X} \sum_{X' \in 2^{\mathfrak{X}}} g(X, X') p'(X') \leq 0.$$

Therefore $V'(\mathbf{g}) \leq 0$ because (5) and the definition of value $V'(\mathbf{g})$ take places.

Evidently $V(\mathbf{g}) \geq 0$ or $V'(\mathbf{g}) \leq 0$ has to be. Other words

$$V(\mathbf{g}) < 0 < V'(\mathbf{g}). \tag{7}$$

is impossible.

Let choose a value z and instead of the matrix $\mathbf{g}$ use the matrix

$$\mathbf{g}_z = \left\{ g_z(X, X') : \ (X, X') \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \right\} = \left\{ g(X, X') - z : \ (X, X') \in 2^{\mathfrak{X}} \times 2^{\mathfrak{X}} \right\}.$$

So there is

$$\mathbf{E}g_z(K,K') = \mathbf{E}g(K,K') - z \sum_{X\in 2^{\mathfrak{X}}} \sum_{X'\in 2^{\mathfrak{X}}} p(X)p'(X') = \mathbf{E}g(K,K') - z.$$

So

$$V(\mathbf{g}_z) = V(\mathbf{g}) - z, \quad V'(\mathbf{g}_z) = V'(\mathbf{g}) - z.$$

can be written in analogy. If we apply the same algorithm has been put to (7) to $V(\mathbf{g}_z)$ and to $V'(\mathbf{g}_z)$ we can see

$$V(\mathbf{g}) < z < V'(\mathbf{g}) \tag{8}$$

is right for any value z. Let $V(\mathbf{g}) < V'(\mathbf{g})$ be true. We can find value z then (8) will be true. But it is impossible. So only one possibility is going from (6) for the values $V(\mathbf{g})$ and $V'(\mathbf{g})$

$$V(\mathbf{g}) = V'(\mathbf{g}).$$

The theorem is proved.

Definition (optimal random coalitions). Random coalitions K and K' are called *optimal or maximin relatively game matrix* $\mathbf{g}$ whenever they are satisfied to

$$\max_K \min_{K'} \mathbf{E}g(K,K') = \min_{K'} \max_K \mathbf{E}g(K,K').$$

Remark 1. If the game matrix $\mathbf{g}$ have no a saddle point, i.e. a strict inequality

$$\max_{X\in 2^{\mathfrak{X}}} \min_{X'\in 2^{\mathfrak{X}}} g(X,X') < \min_{X'\in 2^{\mathfrak{X}}} \max_{X\in 2^{\mathfrak{X}}} g(X,X')$$

is true then the optimal random coalitions exist and they are unique according the maximin theorem.

In result according to the random set reformulation of maximin theorem for the saddleless matrix of game of two coalitions with zero sum the two unique optimal random coalitions K and K' provide an optimal game. A deviation of the optimal game will result losses in prize of the coalition which behavior is not optimal.

2 Coalition imputation as corollary of the random set reformulation of maximin theorem

Let consider a game of two random coalitions under the same finite set of gamblers $\mathfrak{X}$. Each gambler $x \in \mathfrak{X}$ can choose a one of four ways to enter into two random coalitions:

$$\{x \in K, x \in K'\}, \ \{x \in K, x \notin K'\}, \ \{x \notin K, x \in K'\}, \ \{x \notin K, x \notin K'\}.$$

Each coalition of gamblers $X \in 2^{\mathfrak{X}}$ has two mean prizes as outputs of games of two random coalitions K and K'. The first mean prize

$$w(X) = \mathbf{E}g(X, K') = \sum_{X' \in 2^{\mathfrak{X}}} g(X, X')p'(X')$$

is for the participation in game on side of the first random coalition and the second mean prize

$$w'(X') = \mathbf{E}g(K, X') = \sum_{X \in 2^{\mathfrak{X}}} g(X, X')p(X)$$

is for the participation in game on side of the second coalition.

It is taking a curious situation when the maximin theorem only reformulated for games of random coalitions not only asserts the existence of optimal coalitions but also defines two functions

$$\mathbf{w}(K) = \left\{w(X),\ X \in 2^{\mathfrak{X}}\right\}, \quad \mathbf{w}(K') = \left\{w'(X),\ X \in 2^{\mathfrak{X}}\right\}, \tag{9}$$

where $w(X)$ and $w'(X)$ are the mean prizes of the coalition $X \in 2^{\mathfrak{X}}$ for its participation on side of the first and the second random coalitions. These functions in the modern games theory are called *characteristic functions of game* [4]. Basing ourselves on the characteristic functions of game the existence of which follows from the random set reformulation of maximin theorem we can make a formula of coalition imputation for each gambler as a sum of coalition imputations for his participation in the first and in the second random coalition:

$$\begin{gathered} \varphi_x = \varphi_x(K) + \varphi_x(K') = \\ = \mathbf{E}\Big(w(K') - w(K' \setminus \{x\})\Big) + \mathbf{E}\Big(w'(K) - w'(K \setminus \{x\})\Big) \end{gathered} \tag{10}$$

This formula generalizes the formula of imputation by Shapley [**?**]:

$$\varphi_x[w] = \sum_{\{x\} \subseteq X \subseteq \mathfrak{X}} \frac{(|X| - 1)!(N - |X|)!}{N!} \Big(w(X) - w(X \setminus \{x\})\Big), \quad x \in \mathfrak{X}, \tag{11}$$

and the index of Banzhaf [2]:

$$\varphi_x[w] = \sum_{\{x\} \subseteq X \subseteq \mathfrak{X}} \frac{1}{2^{N-1}} \Big(w(X) - w(X \setminus \{x\})\Big), \quad x \in \mathfrak{X}, \tag{12}$$

and the formula by Vilkas [5]:

$$\varphi_x[w] = \sum_{\{x\} \subseteq X \subseteq \mathfrak{X}} \pi_x(X) \Big(w(X) - w(X \setminus \{x\})\Big), \quad x \in \mathfrak{X}. \tag{13}$$

3 Comparison of coalition imputation with formulas by Shapley, Banzhaf and Vilkas

Let compare formulas of imputation by Shapley (11), Banzhaf (12) and Vilkas (13) with the formula of coalition imputation (10) and let only indicate their suppositons. In all formulas the authors suppose evident or non evident the existence of probabilistic distribution of some random coalition we shall mark $\mathcal{K}$. Vilkas makes an apparent supposition of the existence of some random coalition. Shapley and Banzhaf make the same but tacitly. Apparently they didn't realize that the existence of some random coalition is supposed by them really. In reality the random coalition itself is not needed for them. Since they used instead of it its "projections", i.e. random coalitions like

$$\mathcal{K}_x = \mathcal{K} \setminus \{x\}, \quad x \in \mathfrak{X},$$

the distribution of which is defined by distribution of the random coalition $\mathcal{K}$ and looks like

$$\mathbf{p}_x = \Big\{ p_x(X) = p_{\mathcal{K}}(X) + p_{\mathcal{K}}(X + x), \quad X \subseteq \{x\}^c \Big\}, \quad x \in \mathfrak{X},$$

where

$$\mathbf{p}_{\mathcal{K}} = \left\{ p_{\mathcal{K}}(X), \quad X \in 2^{\mathfrak{X}} \right\}$$

is the distribution of random coalition $\mathcal{K}$ and $\{x\}^c = \mathfrak{X} \setminus \{x\}$ is a complement of $\{x\}$. So axioms by Shapley are equivalent to an idea that "projections" $\mathcal{K}_x$ of the random coalition $\mathcal{K}$ have the same distributions for any $x \in \mathfrak{X}$:

$$p_x(X) = \frac{|X|!(N - |X| - 1)!}{N!}, \quad X \subseteq \{x\}^c.$$

Suppositions by Banzhaf are equivalent to an idea that "projections" have the uniform distribution

$$p_x(X) = \frac{1}{2^{N-1}}, \quad X \subseteq \{x\}^c.$$

And only Vilkas say about arbitrary probabilistic distributions of "projections" $\mathcal{K}_x$:

$$p_x(X) = \pi_x(X), \quad X \subseteq \{x\}^c,$$

but he entails the random coalition $\mathcal{K}$ generating "projections" $\mathcal{K}_x$ and about types of distributions of these "projections".

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