Sector-Specific Externalities and Endogenous Growth under Social Constant Returns

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Abstract

By examining two-sector models of endogenous growth with physical and human capital, this paper demonstrates that indeterminacy of equilibrium may emerge even in the absence of social increasing returns. The first model we examine assumes that both final good and new human capital production sectors employ physical as well as human capital under social constant returns but private decreasing returns due to the presence of sector-specific externalities. It is shown that a small divergence between private and social factor intensity conditions generates indeterminacy of equilibrium rather easily even under constant returns. I addition, we show that introducing endogenous labor supply may enhance the possibility of indeterminacy. Some extensions and intuitive interpretation of the indeterminacy conditions are also presented.

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1 Introduction

The problem of indeterminacy of equilibrium in growth models has been discussed extensively, but mainly under the assumption of increasing returns. Boldrin and Rustichini (1994), Benhabib and Farmer (1994), and Farmer and Guo (1994), for example, revealed indeterminacy in one-sector growth models with external increasing returns, while Benhabib and Perli (1994), Xie (1994) and Mitra (1997) examined indeterminacy in two-sector endogenous growth models à la Lucas (1988) where external increasing returns are associated with human capital formation. One of the common features of earlier studies is that degree of increasing returns should be sufficiently large to produce indeterminacy. This conclusion has been criticized by the empirically oriented, real business cycle theorists. More recent literature, on the other hand, suggests that small degree of aggregate increasing returns would be enough to hold indeterminacy if we consider multi-sector models with endogenous labor supply: see Benhabib and Farmer (1996).

Judging from the existing investigations, one may conjecture that assumption of increasing returns is indispensable for establishing indeterminacy in infinite-horizon models of economic growth. However, recent studies by Bebnhabib and Nishimura (1996 and 1998) demonstrated that in multi-sector growth models indeterminacy may emerge even if the aggregate technology satisfies constant returns. Benhabib and Nishimura (1998) constructed a three-sector model with a concave utility function in which aggregate technology of each production sector satisfies constant returns but technology of an individual firm exhibits decreasing returns. The divergence between private and social returns to scale is due to the presence of sector-specific externalities. Using numerical examples, they show that if there exist small degree of externalities, equilibrium path can be intermediate around the steady state under plausible parameter values of technology and preferences. Additionally, Benhabib and Nishimura (1996) derived analytical conditions for generating indeterminacy in a multi-capital good model with Cobb-Douglas technologies and a linear utility function.

This paper also considers indeterminacy of equilibrium in growth models with social constant returns. The key difference between Benhabib and Nishimura (1996 and 1998) and the present study is that their papers use exogenous growth models where labor grows at a given constant rate, while our paper analyzes endogenous growth models where labor force is reproducible due to human capital investment. More specifically, we analyze two-sector models of endogenous growth in which both final good and new human capital production sectors employ human as well as physical capital. This type of model without externalities has been analyzed extensively and applied to a variety of issues. Among others, Bond et al. (1996), Mino (1996) and Ladron-de-Guevara et al. (1997) showed that the two-sector model without externalities satisfies uniqueness and stability of equilibrium under weak restrictions. Mulligan and Sala-i-Martin (1993) introduced external increasing returns into the base model and analyzed transition dynamics based on numerical experiments. In their study, however, the issue of indeterminacy was out of touch. In this paper, we show that the presence of small degree of externalities would be enough to yield indeterminacy rather easily even without assuming social increasing returns.

The main part of this paper assumes that labor supply is fixed. Given this assumption, it is shown that the divergence between social and private capital intensity conditions plays a pivotal role in generating indeterminacy. In addition to the case of fixed labor supply, we briefly examine how the results would be modified if labor supply is variable. We demonstrate that the introduction of labor-leisure choice benhances the possibility of indeterminacy under social constant returns. We also present some extensions of the basic model as well as intuitive implication of the main results.

This paper is organized as follows. The next section sets up the basic model. Section 3 displays the conditions for indeterminacy and present their economic intuitions. Section 4 re-examine the basic model under more general form of production functions. Section 5 introduces variable labor supply and analyzes how the indeterminacy conditions derived in Section 3 will be modified. Finally, Section 6 concludes the paper.

2 A Two-Sector Model with Physical and Human Capital

In this section we introduce sector-specific externalities into the two-sector endogenous growth with physical and human capital analyzed by Bond et al. (1996), Mino (1996) and Ladron-de-Guevara (1997).[1]

[1] This model was first analyzed by King, Plosser and Rebelo (1988) in the context of real business cycle theory with human capital formation. Bond, Yip and Wang (1996) and Mino (1996) analyzed the local uniqueness and stability of equilibrium of the base model,

2.1 The Base Model

Let us assume that the first sector produces a final good that can be used either for consumption or investment on physical capital. The second sector produces new human capital. Both sectors produce by use of physical as well as human capital. Production technology of each sector is specified as:

$$Y_i = K_i^{\alpha_i} H_i^{\beta_i} K_{iE}^{\varepsilon_i} H_{iE}^{\phi_i}, \qquad i = 1, 2, \tag{1}$$

where parameters involved satisfy the following:

$$\alpha_i,\ \beta_i,\ \varepsilon_i,\ \phi_i \in (0,1), \qquad \alpha_i + \beta_i + \varepsilon_i + \phi_i = 1, \quad i = 1, 2.$$

In the above, $K_{iE}^{\varepsilon_i}$ and $H_{iE}^{\phi_i}$ denote sector specific externalities associated with physical and human capital employed by the i-th sector. The key assumption here is that production technology of each sector exhibits social constant returns to scale when it includes external effects.

The market equilibrium conditions for the first and the second goods are

$$Y_1 = C + \dot{K} + \delta K, \tag{2}$$

$$Y_2 = \dot{H} + \eta H. \tag{3}$$

where δ and η respectively denote depreciation rate of physical and human capital. We focus on the interior equilibrium, so that it is assumed that both physical and human capital are fully employed in each moment of time:

$$K = K_1 + K_2, \tag{4}$$

$$H = H_1 + H_2. \tag{5}$$

Since the private technology exhibits decreasing returns to scale, the competitive firms may earn positive profits. Provided that that neither entrance nor exhit of the firms are possible, we assume that the profits are distributed back to the hoseholds who own phsical and human capital. In the base model, it is assmed that moment utility of the representative household depends on consumption alone. The objective function of the household is to maximize a discounted sum of utilities

$$U = \int_0^\infty \frac{C^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \qquad \sigma > 0,\ \sigma \neq 1.$$

while Ladron-de-Guevara, Oritegura and Santos (1997) proved the global stability.

subject to a flow budget constraint,

$$\dot{A} = (r - p_1\delta - \dot{p}_1) K + (w - p_2\eta - \dot{p}_2) H + \pi - C$$

and a wealth constraint, $A = p_1K + p_2H$, where p_1 and p_2 are the prices of final good and new human capital, r and w are nominal rates of return to physical and human capitl, and π denotes profits distributed to the household. When selecting optimal consumption-saving plan, the houseold is assumed to take sequences of prices and profits, $\{p_i(t), r(t), w(t), \pi(t)\}_{t=0}^{\infty}$,as given. Notice that substituting (,), (.), (.), (,) and definition of the total wealth, A, into the flow budget constraint given above, we obtain the definition of distributed profits: $\pi = p_1Y_1 + p_2Y_2 - rK - wH$.

2.2 The Dynamic System

It is easy to confirm that the market equilibrium can characterized directly by solving a pseudo-planing problem in which the planner maximizes (.) under the constraints (.),(.), (.) and (.). In so doing, the planner is assumed to take the sequences of external effects, $\{K_{iE}(t), H_{iE}(t)\}_{t=0}^{\infty}$, as given. To solve this optimization problem, set up the Hamiltonian function in the following manner.

$$\begin{aligned}\mathcal{H} &= \frac{C^{1-\sigma}-1}{1-\sigma} + p_1\left(K_1^{\alpha_1}H_1^{\beta_1}K_{1E}^{\varepsilon_1}H_{1E}^{\phi_1} - C - \delta K\right) \\ &\quad + p_2\left(K_2^{\alpha_2}H_2^{\beta_2}K_{2E}^{\varepsilon_2}H_{2E}^{\phi_2} - \eta H\right) \\ &\quad + r(K - K_1 - K_2) + w(H - H_1 - H_2).\end{aligned}$$

The costate variables in the Hamiltonian function correspond to th market prices defined above. The necessary conditions for an optimum include the following conditions:

$$r = p_i\alpha_i K_i^{\alpha_i-1}H_i^{\beta_i}K_{iE}^{\varepsilon_i}H_{iE}^{\phi_i}, \qquad i = 1,2 \tag{6}$$

$$w = p_i\beta_i K_i^{\alpha_i}H_i^{\beta_i-1}K_{iE}^{\varepsilon_i}H_{iE}^{\phi_i} \qquad i = 1,2$$

$$C^{-\sigma} = p_1 \tag{7}$$

$$\dot{p}_1 = (\rho + \delta) p_1 - r \tag{8}$$

$$\dot{p}_2 = (\rho + \eta) p_2 - w \tag{9}$$

$$\lim_{t\to\infty} p_1 e^{-\rho t} K = \lim_{t\to\infty} p_2 e^{-\rho t} H = 0. \tag{10}$$

Suppose that the number of private agents is normalized to one. Then the market equilibrium requires that the external effects satisfy $K_{iE} = K_i$ and $H_{iE} = H_i$ for all $t \geq 0$. Taking externalities into account, conditions (6) and (7) are respectively written as

$$r = p_i \alpha_i k_i^{\alpha_i + \varepsilon_i - 1}, \quad i = 1, 2 \tag{6'}$$

$$w = p_i \beta_i k_i^{\alpha_i + \varepsilon_i}, \quad i = 1, 2, \tag{7'}$$

where $k_i = K_i / H_i$.

Denoting the rental ratio $w/r = \omega$, (6′) and (7′) give:

$$k_i = (\alpha_i / \beta_i)\, \omega, \quad i = 1, 2 \tag{11}$$

Thus if we express the price of new human capital in terms of the final good, it is given by

$$p = \frac{p_2}{p_1} = \pi \omega^{\alpha_1 + \varepsilon_1 - (\alpha_2 + \varepsilon_2)} \tag{12}$$

where

$$\pi = \frac{\alpha_1^{\alpha_1 + \varepsilon_1} \beta_1^{1 - (\alpha_1 + \varepsilon_1)}}{\alpha_2^{\alpha_2 + \varepsilon_2} \beta_2^{1 - (\alpha_2 + \varepsilon_2)}}$$

By use of (9), (10), (6′) and (7′), the relative price changes according to

$$\begin{aligned} \frac{\dot{p}}{p} &= \alpha_1 k_1^{\alpha_1 + \varepsilon_1 - 1} - \beta_2 k_2^{\alpha_2 + \varepsilon_2} + \eta - \delta \\ &= \alpha_1 \left(\frac{\alpha_1 \omega}{\beta_1} \right)^{\alpha_1 + \varepsilon_1 - 1} - \beta_2 \left(\frac{\alpha_2 \omega}{\beta_2} \right)^{\alpha_2 + \varepsilon_2} + \eta - \delta. \end{aligned}$$

Therefore, dynamic behavior of the rental ratio ω is given by

$$\dot{\omega} = \frac{\omega}{\alpha_1 + \varepsilon_1 - (\alpha_2 + \varepsilon_2)} \left[\alpha_1 \left(\frac{\alpha_1 \omega}{\beta_1} \right)^{\alpha_1 + \varepsilon_1 - 1} - \beta_2 \left(\frac{\alpha_2 \omega}{\beta_2} \right)^{\alpha_2 + \varepsilon_2} + \eta - \delta \right] \tag{13}$$

On the other hand, the full employment conditions for physical and human capital yield:

$$\frac{H_1}{H} = \frac{k - k_2}{k_1 - k_2}, \quad \frac{H_2}{H} = \frac{k_1 - k}{k_1 - k_2},$$

where $k = K/H$. Using these expressions, (2) and (3) present growth rate of physical and human capital in such a way that

$$\frac{\dot{K}}{K} = \frac{1}{k}\left(\frac{k - k_2}{k_1 - k_2} k_1^{\alpha_1 + \varepsilon_1}\right) - \frac{C}{K} - \delta, \tag{14}$$

$$\frac{\dot{H}}{H} = \frac{k_1 - k}{k_1 - k_2} k_2^{\alpha_2 + \varepsilon_2} - \eta. \tag{15}$$

In addition, by use of (12), (15) and (6′), we find that the optimal consumption dynamics is as follows:

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left(\alpha_1 k_1^{\alpha_1 + \varepsilon_1 - 1} - \rho - \delta\right). \tag{16}$$

Now define

$$k = K/H, \quad c = C/H.$$

Then, from (12), (15), (16) and (17) motions of k and c are given by (18) and (19) below, respectively:

$$\begin{aligned}\dot{k} = &\left(k - \frac{\alpha_2}{\beta_2}\omega\right)\frac{(\alpha_1/\beta_1)^{\alpha_1+\varepsilon_1}}{\Delta}\omega^{\alpha_1+\varepsilon_1-1} - c - k(\delta - \eta) \\ &-k\left(\frac{\alpha_1}{\beta_1}\omega - k\right)\frac{(\alpha_2/\beta_2)^{\alpha_2+\varepsilon_2}}{\Delta}\omega^{\alpha_2+\varepsilon_2-1},\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\dot{c}}{c} = &\frac{1}{\sigma}\left[\alpha_1\left(\frac{\beta_1}{\alpha_2}\right)^{\alpha_1+\varepsilon_1}\omega^{\alpha_1+\varepsilon_1-1} - \delta - \rho\right] \\ &-\frac{1}{\Delta}\left(\frac{\alpha_1}{\beta_1}\omega - k\right)\left(\frac{\beta_2}{\alpha_2}\right)^{\alpha_2+\varepsilon_2}\omega^{\alpha_2+\varepsilon_2-1} + \eta,\end{aligned} \tag{18}$$

where

$$\Delta = \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}.$$

The sign of Δ expresses the relative magnitudes of private capital intensities.

Consequently, we obtain a complete dynamic system constituted by (14), (18) and (19) that describes behaviors of k, c and ω.

3 Indeterminacy of Equilibrium

3.1 Local Dynamics

Denote the right-hand sides in (14), (18) and (19) as $\Omega(\omega)$, $\Lambda(k, c, \omega)$ and $\Gamma(k, \omega)$, respectively. Then the dynamic system is expressed as

$$\begin{aligned} \dot{k} &= \Lambda(k, c, \omega), \\ \dot{c} &= c\Gamma(k, \omega), \\ \dot{\omega} &= \Omega(\omega). \end{aligned} \tag{19}$$

The balanced-growth equilibrium can be defined recursively. First, $\dot{\omega} = \Omega(\omega) = 0$ yields

$$\alpha_1 \left(\frac{\alpha_1 \omega}{\beta_1}\right)^{\alpha_1+\varepsilon_1-1} - \delta = \beta_2 \left(\frac{\alpha_2 \omega}{\beta_2}\right)^{\alpha_2+\varepsilon_2} - \eta \tag{20}$$

This gives the steady state value of ω. Equation (21) is the non-arbitrage condition between holding physical and human capital in the steady state. The left hand side of the above decreases monotonically with ω, while the right hand side is a monotonic increasing function of ω. Therefore, the steady-state value of $\bar{\omega}$ is uniquely determined. Given $\bar{\omega}$, the long-run equilibrium level of k satisfies $\dot{c} = c\Gamma(k, \bar{\omega}) = 0$, that is

$$\alpha_1 \left(\frac{\beta_1}{\alpha_1}\right)^{\alpha_1+\varepsilon_1} \bar{\omega}^{\alpha_1+\varepsilon_1-1} - \delta - \rho = \frac{1}{\Delta}\left(\frac{\alpha_1}{\beta_1}\bar{\omega} - k\right)\left(\frac{\beta_2}{\alpha_2}\right)^{\alpha_2+\varepsilon_2} \bar{\omega}^{\alpha_2+\varepsilon_2-1} - \eta, \tag{21}$$

Thus the steady-state value of k is also unique. Finally, condition $\dot{k} = \Lambda\left(\bar{k}, c, \bar{\omega}\right) = 0$ presents which shows that $\bar{c}$ is uniquely determinate as well.

Note that in the balanced growth equilibrium, C, K and H grow at a common such that

$$g = \frac{1}{\sigma}\left[\alpha_1 \left(\frac{\beta_1 \bar{\omega}}{\alpha_2}\right)^{\alpha_1+\varepsilon_1-1} - \delta - \rho\right].$$

We should assume that $\bar{\omega}$ fulfills $g(1-\sigma) < \rho$, which ensures the transversality conditions (11) in the balanced growth equilibrium.

The coefficient matrix of the dynamic system (20) linearized around the steady state equilibrium is

$$J = \begin{bmatrix} \Lambda_k & -1 & \Lambda_\omega \\ \bar{c}\Gamma_k & 0 & \bar{c}\Gamma_\omega \\ 0 & 0 & \Omega' \end{bmatrix},$$

where

$$\Lambda_k = \frac{1}{\Delta}\left[\left(\frac{\alpha_1}{\beta_1}\right)^{\alpha_1+\varepsilon_1}\bar{\omega}^{\alpha_1+\varepsilon_1-1} - \left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{\alpha_2\bar{\omega}}{\beta_2}\right)^{\alpha_2+\varepsilon_2} + 2\bar{k}\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{\alpha_2}{\beta_2}\right)^{\alpha_2+\varepsilon_2}\bar{\omega}^{\alpha_2+\varepsilon_2-1}\right],$$

$$\Gamma_k = \frac{1}{\Delta}\left(\frac{\alpha_2}{\beta_2}\right)^{\alpha_2+\varepsilon_2}\bar{\omega}^{\alpha_2+\varepsilon_2-1},$$

$$\begin{aligned}\Omega' &= \frac{\bar{\omega}}{\alpha_1+\varepsilon_1-(\alpha_2+\varepsilon_2)}\left[\alpha_1(\alpha_1+\varepsilon_1-1)\left(\frac{\alpha_1}{\beta_1}\right)^{\alpha_1+\varepsilon_1-1}\bar{\omega}^{\alpha_2+\varepsilon_2-2}\right. \\ &\left.-\beta_2(\alpha_2+\beta_2)\left(\frac{\alpha_2}{\beta_2}\right)^{\alpha_2+\varepsilon_2}\bar{\omega}^{\alpha_2+\varepsilon_2-1}\right].\end{aligned}$$

Letting λ be the eigenvalue of J, the characteristic equation is

$$(\lambda-\Omega')\left[\lambda^2-\Lambda_k+\bar{c}\Gamma_k\right]=0.$$

The eigenvalues are thus given by

$$\lambda = \Omega'(\bar{\omega}) \text{ and } (1/2)\left[\Lambda_k \pm \left(\Lambda_k^2 - 4\bar{c}\Gamma_k\right)^{1/2}\right]. \tag{22}$$

3.2 Conditions for Local Indeterminacy

As well as the two-sector model of exogenous growth with constant returns, the dynamic behavior of this model depends upon the relative factor intensity conditions.

Case (i): $\alpha_2+\varepsilon_2 > \alpha_1+\varepsilon_1$

In this case, the aggregate technology of the new human capital producing sector has a more physical capital intensive than that of the final good sector. Using the above conditions, it is easy to see that the following holds:

Proposition 1 *Suppose that the social technology of the new human capital producing sector is more physical capital intensive than that of the final good sector. Then if there exists a feasible balanced-growth equilibrium and if the balanced-growth rate is positive, then the economy exhibits local determinacy around the steady state.*

Proof. Notice that

$$\text{sign } \Omega'(\omega) = \text{sign } [\alpha_2 + \varepsilon_2 - \alpha_1 - \varepsilon_1],$$

and hence the dynamic system involves one negative eigenvalue if and only if $\Gamma_k < 0$. This is equivalent to the condition that $\Delta = (\alpha_1/\beta_1) - (\alpha_2/\beta_2) < 0$. If $\Delta > 0$ (so that $\Gamma_k > 0$), then the system has two stable roots under $\Lambda_k < 0$. From the steady-state condition (22), it holds that

$$\bar{k} = \frac{\alpha_1}{\beta_1}\bar{\omega} - \frac{\Delta(g+\eta)}{(\alpha_2/\beta_2)^{\alpha_2+\varepsilon_2}\bar{\omega}^{\alpha_2+\varepsilon_2-1}}$$

Therefore, we obtain

$$\Lambda_k = \frac{1}{\Delta}\left[\left(\frac{\alpha_1}{\beta_1}\right)^{\alpha_1+\varepsilon_1}\bar{\omega}^{\alpha_1+\beta_1-1} + \left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{\alpha_2\bar{\omega}}{\beta_2}\right)^{\alpha_2+\varepsilon_2} - 2\Delta(g+\eta)\right]$$

It can be shown that if g, $\bar{k}$ and $\bar{c}$ have positive values, then $\Lambda_k > 0$ for $\Delta > 0$. Therefore, given the assumptions, indeterminacy may not be observed around the steady state.□

It is to be noted that when the balanced-growth rate, g, is negative under rather extreme values of parameters, we still have the possibility of indeterminacy, that is, there are two stable roots. However, it is safe to state that under plausible conditions the system will not show indeterminacy when the technology of new human capital producing sector's technology is more physical capital intensive than that of the final good sector from the social perspective.

Case (ii): $\alpha_1 + \varepsilon_1 > \alpha_2 + \varepsilon_2$

If the aggregate technology of the final good sector is more physical capital intensive than the new human capital producing sector, indeterminacy may exist in the following manner:

Proposition 2 *Suppose that the social technology of the final good sector is more physical capital intensive than that of the new human capital production sector and that the balanced growth equilibrium is feasible with a positive growth rate. Then indeterminacy emerges, if and only the private technology is more physical capital intensive in the new human capital sector than the final good sector.*

Proof. This case ensures that $\Omega'(\bar{\omega}) < 0$. Hence, the system has at least one stable root. If $\Delta < 0$, then $\Gamma_k < 0$. Thus the subsystem has one stable and one unstable roots. This shows that there are two stable roots, and therefore the system displays local indeterminacy. In contrast, if $\Delta > 0$, according to the feasibility conditions mentioned in Proposition 1, it is easy to see that $\Lambda_k > 0$ and $\Gamma_k > 0$. Accordingly, the system has only one stable root, $\Omega'(\bar{\omega})$, under which uniqueness of equilibrium path is established.□

This result is parallel to a proposition shown in Benhabib and Nishimura (1998) in the context of an exogenous growth model. This proposition makes two points. First, the magnitudes of externalities associated with human capital (i.e. values of ϕ_i) do not appear in the indeterminacy condition. Second, the above result demonstrates that indeterminacy will emerge even if external effects are sufficiently small. For example, if $\alpha_1 < \alpha_2$ but they are close enough each other, then small externality effect, ε_1, would produce indeterminacy for $\varepsilon_2 = 0$.

3.3 Implication

In the dynamic general equilibrium frameworks, it is generally difficult to obtain clear intuition behind the presence of indeterminacy. It is, however, rather easy to give economic implication of the indeterminacy conditions derived above. First, remember that in our economy if the social planner controls the economy and internalizes externalities, the model coincides with the standard two sector endogenous growth economy examined by Bond, Yip and Wang (1996), Mino (1996) and others. Thus the capital intensity chosen by the producers is written as $k_i = \frac{\alpha_1+\varepsilon_1}{1-\alpha_1-\omega_1}\omega$. In this standard case, if $k_1 > k_2$ (i.e. $\alpha_1 + \varepsilon_1 > \alpha_2 + \varepsilon_2$), Rybczynskyi's theorem gives

$$\frac{\partial (Y_1/H)}{\partial k} > 0, \quad \frac{\partial (Y_2/H)}{\partial k} < 0.$$

Now suppose that the economy initially stays in the balanced-growth equilibrium, and the aggregate capital intensity, k, increases unanticipatedly. If ω is fixed at the original level $\bar{\omega}$, then $\dot{C}/C$ stays constant and $\dot{K}/K$ and $\dot{H}/H$ respectively increases and decrease. Therefore, both $k\,(= K/H)$ and $c\,(= C/H)$ continue to increase. To recover the stability, the value of c should be re-selected to make k decreasing However, since the dynamic behavior of c is independent of c, it is generally impossible to determine c that may attain the balanced growth, if ω is fixed and the initial values of k diverges from its steady state level of $\bar{k}$. Accordingly, in order to make the economy back to the balanced growth equilibrium, the factor price ratio (so the relative price) should be adjusted to keep the economy on the one dimensional stable manifold converging to the steady state. This means that the standard model ensures determinacy of equilibrium around the steady state.

In contrast, when the market economy does not attain the social planning due to the presence of externalities, the social and private capital intensities diverge. If $\alpha_1 + \varepsilon_1 > \alpha_2 + \varepsilon_2$ but $\alpha_1/\beta_1 < \alpha_2/\beta_2$, then the Rybczynski condition becomes:

$$\frac{\partial (Y_1/H)}{\partial k} < 0, \quad \frac{\partial (Y_2/H)}{\partial k} > 0.$$

Again, assume that the initial position of the economy is in the balanced growth equilibrium and that there is an unanticipated rise in k. In contrast to the social planning economy, $\dot{K}/K$ starts to decrease and $\dot{H}/H$ starts to increase. Hence, if ω is stays at $\bar{\omega}$, then both k and c are lowered. In this case, behavior of k exhibits self stabilizing behavior, so that the appropriate choice of the initial value of c may attain the balanced-growth equilibrium eventually without adjusting the level of ω. This property can be obtained regardless of the fixed level of ω. In other words, the saddle point stability in c-k space can be established for any fixed value of ω. Since the initial value of ω is not predetermined, this result means that we may find a continuum of converging equilibrium around the balanced growth equilibrium.

The above intuition may be confirmed in a more formal manner.

4 General Technology and Factor Income Taxation

4.1 A Generalization

For analytical simplicity, the foregoing discussion has assumed Cobb-Douglas production functions. In this section, we show that the main results derived above may hold for a more general class of production technology. Suppose that production function of each sector is specified as

$$Y_i = F_i (K_i, H_i) \Theta_i (K_{iE}, H_{iE}), \quad i = 1, 2$$

where private technology is presented by $F_i(K_i, H_i)$ that is increasing, strictly quasi-concave and homogenous of degree $\gamma \in (0,1)$ in K_i and H_i. Sector specific externality is expressed by a function, $\Theta_i(K_{iE}, H_{iE})$, which is assumed to be increasing and homogenous of degree $1-\gamma$ in K_{iE} and H_{iE}. Due to homogeneity assumptions, the above can be written as

$$Y_i = H_i^{\gamma} H_{iE}^{1-\gamma} f_i(k_i)\, \theta_i(k_{iE}), \tag{23}$$

where $f_i(k_i) = F_i(K_i/H_i, 1)$ and $\theta(k_{iE}) = \Theta(K_{iE}/H_{iE}, 1)$.

Considering that in equilibrium $H_{iE} = H_i$ and $K_{iE} = K_i$, the profit maximization conditions for the firm yield:

$$\begin{aligned} r &= p_i f_i'(k_i)\, \theta_i(k_i), \\ w &= p_i \left[\gamma f_i(k_i) - k_i f'(k_i)\right] \theta_i(k_i). \end{aligned}$$

Hence, the factor price ratio is related to capital intensity of each production sector in such a way that

$$k_i + \omega = \frac{\gamma f_i(k_i)}{f_i'(k_i)}. \tag{24}$$

This equation gives

$$k_i = k_i(\omega), \quad k_i'(\omega) = -f_i'^2/\gamma f_i f_i'' > 0, \quad i = 1, 2.. \tag{25}$$

The relative price $p\,(= p_2/p_1)$ is thus expressed by

$$p(\omega) = \frac{\theta_1(k_1(\omega))\, f_1'(k_1(\omega))}{\theta_2(k_2(\omega))\, f_2'(k_2(\omega))}. \tag{26}$$

Consequently, in view of (.) and (.), it is easy to see that logarithmic differentiation of (.) yields the following:

$$p'(\omega) = p\left[\left(1 - \frac{\hat{\theta}_2}{\hat{f}_2'}\right)\frac{1}{k_2+\omega} - \left(1 - \frac{\hat{\theta}_1}{\hat{f}_1'}\right)\frac{1}{k_1+\omega}\right],$$

where $\hat{\theta}_i \left(= \hat{\theta}_i k_i/\theta_i\right)$ and $\hat{f}_i'\,(= -f_i'' k_i/f_i')$ denote the elasticities of θ_i and f_i' functions.

Using the relations derived above, we can show the following:

Proposition 3 *Suppose that the social productivity function* $f_i(k_i)\,\theta_i(k_i)$ *satisfies strict concavity. Then a feasible balanced-growth equilibrium with a positive growth rate exhibits local indeterminacy, if and only if* $k_2(\omega) > k_1(\omega)$ *and*

$$\left(1 - \frac{\hat{\theta}_2}{\hat{f}_2'}\right)\frac{1}{k_2+\omega} > \left(1 - \frac{\hat{\theta}_1}{\hat{f}_1'}\right)\frac{1}{k_1+\omega}.$$

Proof. Given the production technologies specified above, growth rates of consumption, physical and human capital are respectively given by,

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[f_1'(k_1(\omega))\,\theta_1(k_1(\omega)) - \rho - \delta\right]$$

$$\frac{\dot{K}}{K} = \frac{k - k_1(\omega)}{k_1(\omega) - k_2(\omega)} f_1(k_1(\omega))\,\theta(k_1(\omega)) - \frac{c}{k} - \delta,$$

$$\frac{\dot{H}}{H} = \frac{k_2(\omega) - k}{k_1(\omega) - k_2(\omega)} f_2(k_2(\omega))\,\theta_2(k_2(\omega)) - \eta.$$

In addition, the factor price ratio changes in accordance with

$$\begin{aligned}\dot{\omega} &= \frac{p'(\omega)\,\omega}{p(\omega)}\left\{f_1'(k_1(\omega))\,\theta_1(k_1(\omega))\right.\\ &\quad \left.-\gamma f_2(k_2(\omega)) - k_2(\omega)\,f_2'(k_2(\omega))\,\theta_2(k_2(\omega)) + \eta - \delta\right\}.\end{aligned}$$

Accordingly, the dynamics of the economy can be summarized as a set of differential equations with respect to k, c and ω. Given the concavity assumption on $f_i(k_i)\,\theta_i(k_i)$, there exists a unique balanced growth equilibrium. Inspecting the characteristic roots of the linearized dynamic system, it is easy to confirm that indeterminacy may emerge if $p'(\omega)\left[k_1(\omega) - k_2(\omega)\right] < 0$. Again, if a feasible balanced growth rate is positive, the case where $p'(\omega) < 0$ and $k_1(\omega) > k_2(\omega)$ cannot hold. Therefore, local indeterminacy is observable, if and only if $k_2(\omega) > k_1(\omega)$ and $p'(\omega) > 0$ around the balanced growth equilibrium.□

In the case of Cobb-Douglas technologies, $f_i(k_i) = k_i^{\alpha_i}$, $\theta_i(k_i) = k_i^{\varepsilon_i}$and $\gamma_i = \alpha_i + \beta_i$.. As a result, $k_i = \alpha_i\omega/\beta_i$, .$\hat{\theta}_i = \varepsilon_i$ and $\hat{f}_i'$ =$1 - \alpha_i$, .and hence, sign $(dp/d\omega)$ =sign $\left[\alpha_1 + \varepsilon_1 - (\alpha_2 + \varepsilon_2)\right]$.

4.2 Factor Income Taxation

Now assume that the government levies sector-specific factor income. Letting τ_{K_i} and τ_{H_i} respectively denote rate of tax (subisdies if they have positive values)on physical and human capital employed in sector i. We assume that the government neither consumes nor invests and that the government budget is balanced by adjusting lumpsum transfer or taxes for the households. If we denote after-tax rate of return to capital by r and w, profit maximization conditions () is replaced with

$$\begin{aligned} r &= (1-\tau_{Ki})\, p_i f_i'(k_i)\,\theta_i(k_i)\,, \\ w &= (1-\tau_{H_i})\, p_i\left[f_i(k_i) - k_i f_i'(k_i)\right]\theta_i(k_i)\,. \end{aligned} \tag{27}$$

These equations present the relation between k_i and ω as follows:

$$k_i + \lambda_i\omega = \frac{f_i(k_i)}{f'(k_i)},\quad \lambda_i = \frac{1-\tau_{K_i}}{1-\tau_{H_i}}, \tag{28}$$

which yields a function $k_i = k_i(\lambda_i\omega)$. It is easy to see that in this case condition () is written as

$$p'(\omega) = p\left[\left(1-\frac{\hat{\theta}_2}{\hat{f}_2'}\right)\frac{\lambda_2}{k_2+\lambda_2\omega} - \left(1-\frac{\hat{\theta}_1}{\hat{f}_1'}\right)\frac{\lambda_1}{k_1+\lambda_1\omega}\right]. \tag{29}$$

The dynamics of ω is given by

$$\begin{aligned} \dot{\omega} &= \frac{p'(\omega)\,\omega}{p(\omega)}\left\{(1-\tau_{K_1})\, f_1'(k_1)\,\theta_1(k_1) - (1-\tau_{H_2})\left[f_2(k_2) - k_2 f_2'(k_2)\right]\theta_2(k_2) + \eta - \delta\right\} \\ &= \Omega\left(\omega;\tau_{K_1},\tau_{K_2},\tau_{H_1},\tau_{H_2}\right), \end{aligned}$$

where $k_i = k_i(\lambda_i\omega)$ and $k_i' > 0$ $(i = 1, 2)$.

As pointed out by Bond, Yip and Wang (1996), even in the absence of externalities, indeterminacy may emerge if distortionary taxation the relation between the relative price and capital intensity in each production sector. In our formulation, if there is no externality, (,) gives

$$\operatorname{sign} p'(\omega) = \operatorname{sign}\ \left[\lambda_1 k_2 - \lambda_2 k_1\right].$$

Thus if $k_1 > k_2$ and $\lambda_1 k_2 > \lambda_2 k_1$, there is local indeterminacy around the balanced growth equilibrium. Obviously, if taxation is symmetric between the two sectors (i.e. $\lambda_1 = \lambda_2$), the rates of tax will not affect indeterminacy condition. It is also to be noted that, if the production technologies are of Cobb-Douglas type, sector-specific distortioary taxation do not generate indeterminacy either. In fact, if $f_i(k_i) = k_i^{\alpha_i}$ and $\theta_i(k_i) = k_i^{\varepsilon_i}$, then $k_i = \lambda_i\left(\frac{1}{\alpha_i} - 1\right)\omega$, .so that λ_1 and λ_2 do not appear in (.). To sum up, we obtain:

Proposition 4 *Unless the production technology in each sector is of Cobb-Douglas type, sector specific factor income taxation may generate indeterminacy, if and only if $k_2 > k_1$ and*

$$\left(1 - \frac{\hat{\theta}_2}{\hat{f}_2'}\right)\frac{1 - \tau_{K_2}}{(1 - \tau_{H_2})k_2 + (1 - \tau_{K_2})\omega} > \left(1 - \frac{\hat{\theta}_1}{\hat{f}_1'}\right)\frac{1 - \tau_{K_1}}{(1 - \tau_{H_1})k_1 + (1 - \tau_{K_1})\omega}.$$

For example, suppose that $F_i(K_i, H_i) = \left(K_i^{\nu_i} + a_i H_i^{\nu_i}\right)^{\frac{1-\varepsilon_i-\phi_i}{\nu_i}}$ and $\Theta_i(K_i, H_i) = K_i^{\varepsilon_i} H_i^{\phi_i}$. This means that

$$f_i(k_i) = \left(k_i^{\nu_i} + a_i\right)^{\frac{1-\varepsilon_i-\phi_i}{\nu_i}}, \quad \theta_i(k_i) = k_i^{\varepsilon_i}.$$

Using these functions, we find that $k_i = (\lambda_i\omega)^{\frac{1}{\nu_i - 1}}$. Thus condition () is expressed by

$$\frac{p'(\omega)\omega}{p(\omega)} = \left(1 - \frac{\varepsilon_2}{1 - \alpha_2}\right)\frac{1}{(\lambda_2\omega)^{\frac{\nu_2}{\nu_2 - 1}} + 1} - \left(1 - \frac{\varepsilon_1}{1 - \alpha_1}\right)\frac{1}{(\lambda_1\omega)^{\frac{\nu_1}{\nu_1 - 1}} + 1}.$$

(NUMERICAL EXAMPLE)

5 Home Production

In this section, we examine the effects of introducing labor-leisure choice into the base model uded in Section 2. Recently, several authrs have demonstrated that introducing labor-leisure choice into the standard growth models may produce complex dynamics such as multiple steady states and cycles: see, for example, Ladron-de- Guevara et al.(1997) and Hek (1998).[2] Since those

[2] Ladron-de-Guevara et al.(1997) treats a two-sector endogenous growth model, while de Hek (1998) examines an exogenous growth model. Both studies demonstrate that introducing pure leisure time in utility function may produce multiple steady-state equilibria and hence a stable low-development trap.

authors have not assumed market distortions, the competitive equilibrium attains the social optimum. Therefore, indeterminacay Thoes studies do not assume , Mino (1998), ...and (1998). . Possible complexity under variable labor supply stems from two sources: forms of utility function and the treatment of leisure activity. In the former, it is often critical wether or not the utility function is additively separable between leisure and consumption. In the latter, it tends to yield complex dynamics when one assumes that the utility function involves pure leisure time rather than 'quality leiusre' that depends on human capital as well as on time length spent for leisure activities.[3] In what follows, we assume that the utility function is non separable but leisure depends on quality time. It is shown that variable labor supply considerablly enhances the possibility of indeterminacy.The production side of the economy consists of three sectors. There are no externalities in production technology of each sector.

5.1 A Model with Home Production

Suppose that there is a non-market consumption good that is prouced and consumed within the household. In parallel to the market goods, this home good is produced according to:

$$Q = K_3^{\alpha_3} H_3^{\beta_3} K_{E3}^{\varepsilon_3} H_{E3}^{\phi 3},$$

where Q denotes the home good, and K_3 and H_3 are physical and human capital devoted to home production activities. Again, K_{3E} and H_{3E} expresse external effects and the magnitudes of the prameters, $\alpha_3, \beta_3, \varepsilon_3$ and ϕ_3 satisfy the same conditions assumed in (.). Due to the intoduction of the third sector, the full employment conditions of each capital are now replaced with the folloing:

$$K = K_1 + K_2 + K_3, \tag{30}$$

$$H = H_1 + H_2 + H_3. \tag{31}$$

The objective function of the planner is given by

$$U = \int_0^\infty \frac{\left(C^\gamma Q^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} \exp(-\rho t)\, dt, \quad \gamma \in [0,1), \quad \sigma > 0, \quad \rho > 0.$$

[3] The way how to formulate labor-leisure choice also affects long-run effects of fiscal policy in endogenous growth models with human capital formation. Milesi-Ferretti and Roubini (1998a and b) present detail analyses on this issue.

When $\sigma = 1$, the instantaneous utility function becomes $\gamma \log C + (1-\gamma) \log Q$. The flow budget constraint for the household is

$$\dot{A} = (r - p_1 \delta - \dot{p}_1)(K - K_3) + (w - p_2 \eta - \dot{p}_2)(H - H_3) - p_2 C.$$

The shousehold maximizes U subject to the flow budget constraint together with the asset constraint, $A = p_1 (K - K_3) + p_2 (H - H_3)$, and the home production technology (.).

Notice that the above specification of home production involves the model with labor-leisure choice in which leisure activities depend upon 'quality time'. In this case, the home prodution technology is simply specified by $Q = H_3$. This means that, if the time length for each hosehold is fixed at unity and leisure time denoted by $l \in (0, 1)$, then $Q = H_3 = lH$. Namely, leisure is 'produced' by the household using human caital as well as time.

A psued-planning problem corresponding to the competitive economy characterized by the foregoing assumptions has a Hamiltonian function such that

$$\begin{aligned} \mathcal{H} \;=\; & \frac{\left(C^{\gamma} Q^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} + p_1 \left(K_1^{\alpha_1} H_1^{\beta_1} K_{1E}^{\varepsilon_1} H_{1E}^{\phi_1} - C - \delta K \right) \\ & + p_2 \left(K_2^{\alpha_2} H_2^{\beta_2} K_{2E}^{\varepsilon_2} H_{2E}^{\phi_2} - \eta H \right) + p_3 \left(K_3^{\alpha_3} H_3^{\beta_3} K_{E3}^{\varepsilon_3} H_{E3}^{\phi 3} - Q \right) \\ & + r \left(K - \sum\nolimits_{i=1}^{3} K_i \right) + w \left(H - \sum\nolimits_{i=1}^{3} H_i \right), \end{aligned}$$

where p_3 represents the iplicit price of the home good. In addition to the conditions (.)-(,), the first-oder conditions for an optimum of the above problem include the following:

$$\gamma C^{\gamma(1-\sigma)-1} Q^{(1-\sigma)(1-\gamma)} = p_1, \tag{32}$$

$$(1-\gamma) C^{(1-\sigma)\gamma} Q^{-\gamma-\sigma+\gamma\sigma} = p_3, \tag{33}$$

$$p_3 \alpha_3 K_3^{\alpha_3 - 1} H_3^{\beta_3} K_{E3}^{\varepsilon_3} H_{E3}^{\phi 3} = r, \tag{34}$$

$$p_3 \beta_3 K_3^{\alpha_3} H_3^{\beta_3 - 1} K_{E3}^{\varepsilon_3} H_{E3}^{\phi 3} = w \tag{35}$$

together with conditions from (16) to (20).

From (.) and (.) the private capital intensity of the home production is written as $k_3 = (\alpha_3 / \beta_3)\, \omega$, while the (implicit) price of home good in terms of the final good is

$$q = \frac{p_3}{p_1} = \tilde{\pi} \omega^{\alpha_1 + \varepsilon_1 - (\alpha_3 + \varepsilon_3)}$$

where $\tilde{\pi} = \left[\alpha_1^{\alpha_1 + \varepsilon_1} \beta_1^{1 - (\alpha_1 + \varepsilon_1)} \right] / \left[\alpha_2^{\alpha_3 + \varepsilon_3} \beta_2^{1 - (\alpha_3 + \varepsilon_3)} \right]$.

5.2 Dynamic System

Define the total consumption (in terms of the final good) as $Z = C + qQ$. Then, from (21) and (22) we obtain:

$$C = \alpha Z, \quad Q = (1-\alpha) Z/p. \tag{36}$$

Substituting the above into (21) yields

$$\gamma^{\gamma(1-\alpha)} (1-\gamma)^{(1-\gamma)(1-\sigma)} q^{-(1-\gamma)(1-\sigma)} Z^{-\sigma} = p_1.$$

Differentiating both sides of the above with respect to time and using (24), it holds that

$$\frac{\dot{Z}}{Z} = \frac{1}{\sigma}\left[r - \rho - \delta - (1-\gamma)(1-\sigma)\frac{\dot{q}}{q}\right]. \tag{37}$$

Denoting $K/H = k$ and $\lambda_i = H_i/H$ $(i = 1, 2, 3)$, full-employment conditions (19) and (20) give the following:

$$\lambda_1 k_1 + \lambda_2 k_2 + \lambda_3 k_3 = k,$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_i \in [0, 1].$$

Equations (17) and (28) present

$$Q/H = (1-\gamma) z/q = \lambda_3 k_3^{\alpha_3+\varepsilon_3},$$

where $z = Z/H$. Hence, the allocation rate of human capital to the pure consumption good sector is

$$\lambda_3 = \frac{(1-\gamma)z}{\tilde{\pi}(\alpha_3/\beta_3)\,\omega^{\alpha_1+\varepsilon_1}} \equiv \lambda^3(z,\omega).$$

Thus the rates of allocation of human capital to the first and the third sectors may be expressed in the following manner:

$$\lambda_1 = \frac{k-k_2}{k_1-k2} + \frac{k_2-k_3}{k_1-k_2}\lambda^3(z,\omega) \equiv \lambda^1(k,z,\omega),$$

$$\lambda_2 = \frac{k_1-k}{k_1-k_2} + \frac{k_3-k_1}{k_1-k_2}\lambda^3(z,\omega) \equiv \lambda^2(k,z,\omega),$$

Thus Y_1/H and Y_2/H are expressed as

$$Y_i/H = \lambda^i(k,z,\omega)\,(\alpha_i\omega/\beta_i)^{\alpha_i+\varepsilon_i} = y^i(k,z,\omega), \quad i = 1, 2$$

Using the above equations and noting that $\dot{q}/q = [q'(\omega)/q(\omega)]\dot{\omega}$ (by (27)), we find that k $(= K/H)$ and $z\,(= Z/H)$ respectively change according to the differential equations given below:

$$\dot{k} = y^1(k,z,\omega) + k[\eta - \delta - y^2(k,z,\omega)] - \gamma z, \tag{38}$$

$$\dot{z}/z = (1/\sigma)\left[\alpha_1(\alpha_1\omega/\beta_1)^{\alpha_1+\varepsilon_1-1} - \rho - \delta - (1-\gamma)(1-\sigma)(q'(\omega)/q(\omega))\Omega(\omega)\right] \\ -y^2(k,z,\omega) + \eta \tag{39}$$

Consequently, a complete dynamic system that describes growth process of the economy is given by a set of dynamic equations (32), (33) and (34) with respect to k, z and ω.

It is easy to check that if the model involves a feasible balanced-grwoth equilibrium, it must be uniquely determined. Again, notice that $\dot{\omega} = \Omega(\bar{\omega}) = 0$ yields a unique level of $\bar{\omega}$. Thus conditions $\dot{k} = \dot{z} = 0$ respectively become:

$$y^1(k,z,\bar{\omega}) + k[\eta - \delta - y^2(k,z,\bar{\omega})] - \gamma z = 0,$$

$$(1/\sigma)\left[\alpha_1(\alpha_1\bar{\omega}/\beta_1)^{\alpha_1+\varepsilon_1-1} - \rho - \delta\right] - y^2(k,z,\omega) + \eta = 0.$$

Given $\bar{\omega}$, we can find that the above eauations are linear functions of k and z, so that the steady-state levels of k and z are, if they exisit, uniqurely determined as well.[4]

5.3 Conditions for Indeterminacy

Denoting the common, balanced growth rate of C, K, H and Y_i by $\bar{g}$, the set of dynamic equations consisting of (32), (33) and (34) can be approximated at the steady state by the following linearized system:

$$\begin{bmatrix} \dot{k} \\ \dot{z} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} y_k^1 - (\bar{g}+\delta) - \bar{k}y_k^2 & y_z^1 - \bar{k}y_z^2 - \gamma & y_\omega^1 - \bar{k}y_\omega^2 \\ -\bar{z}y_k^2 & -\bar{z}y_z^2 & B \\ 0 & 0 & \Omega'(\bar{\omega}) \end{bmatrix} \begin{bmatrix} k - \bar{k} \\ z - \bar{z} \\ \omega - \bar{\omega} \end{bmatrix},$$

where $B = \bar{z}\left[(f_1''/\sigma) - (1-\gamma)(1-\sigma)\frac{p'}{p}\Omega' - y_\omega^2\right]$ and each partial derivative is evaluated at $(\bar{k}, \bar{z}, \bar{\omega})$. As well as in the model without home sector, it is

[4] Note that, unlike the model discussed previously, the determination of the steady-state levels of endogenous variables are not recuresive. The values of $\bar{k}$ and $\bar{z}$ are determined simultaneously.

clusial for checking indeterminacy to focus of the submatrix such that

$$J = \begin{bmatrix} y_k^1 - (\bar{g}+\delta) - \bar{k}y_k^2 & y_z^1 - \bar{k}y_z^2 - \gamma \\ -\bar{z}y_k^2 & -\bar{z}y_z^2 \end{bmatrix}.$$

which gives the characteristic roots such that trace $J \pm \left[(\text{trace } J)^2 - 4 \det J\right]^{1/2}$, where

$$\text{trace } J = y_k^1 - (\bar{g}+\delta) - \bar{k}y_k^2 - \bar{z}y_z^2,$$
$$\det J = \bar{z}\left(\bar{g}+\delta - y_k^1\right) y_z^2 + \bar{z}\left(y_z^1 - \gamma\right) y_k^2.$$

The conditions for indeterminacy again higes upon the relative magnitude of social capital intensities between the final good and th new human capital production sectors.

Case (i): $\alpha_2 + \varepsilon_2 > \alpha_1 + \varepsilon_1$
In tis case, we can show the following:

Proposition 5 *Suppose that the technology of the new human capoital producing sector is physical capital intensive than that of the final good sector. Then if the private capital intensities satisfy $k_2 > k_3 > k_1$, indeterminacy may hold around the balanced-growth equilibrium.*

Proof. Since in this case one of the characteristic roots, $\Omega(\bar{\omega})$, has a positive value, the presence of indeterminacy requires that the submatrix J has two stable roots. Notice that the partial derivatives of $y^1(k, z, \omega)$ and $y^2(k, z, \omega)$ satisfy

$$\begin{aligned} y_k^1 &= \text{sign } (k_1 - k_2) = \text{sign } \Delta, \\ y_z^1 &= \text{sign } (k_1 - k_2)(k_2 - k_3) = \text{sign } \Delta (k_2 - k_3), \\ y_k^2 &= \text{sign } (k_2 - k_1) = \text{sign } -\Delta, \\ y_z^2 &= \text{sign } (k_2 - k_1)(k_3 - k_1) = \text{sign } -\Delta (k_3 - k_1). \end{aligned}$$

Hence, provided that the gross rate of equilibrium, $\bar{g} + \delta$, is positive, the condition $k_2 > k_3 > k_1$ ensures that $y_k^1 < 0$, $y_k^2 > 0$ and $y_z^2 > 0$ so that trace $J < 0$. Furthermore, by use of (.), it is shown that

$$\det J = \frac{\bar{z}}{k_1 - k_2}\left\{(1-\gamma)\lambda_z^3\left(\frac{\alpha_2\bar{\omega}}{\beta_2}\right)^{\alpha_2+\varepsilon_2}\left[(\bar{g}+\delta)(k_1 - k_3) + \left(\frac{\alpha_1\bar{\omega}}{\beta_1}\right)^{\alpha_1+\varepsilon_1}\right] + \gamma\right\}.$$

Therefore, if $k_3 - k_1$, it is possible to hold that $\det J > 0,$. and thus indeterminacy may be present around the balanced growth equilibrium.□

As an example, consider the following neumerical example.

$$\begin{aligned} \delta &= \eta = 0.05,\ \sigma = 1, \\ \alpha_1 &= 0.3,\ \alpha_2 = 0.4,\ \ \alpha_3 = 0.35,\ \ \beta_1 = 0.5,\ \ \beta_2 = 0.5,\ \ \beta_3 = 0.35 \end{aligned}$$

Given these magnitudes, from (.) the steady state level of rental ratio is $\bar{\omega} =$ .. Thus (.) shows that the balanced growth rate is $\bar{g} =$

Case (ii): $\alpha_1 + \varepsilon_1 > \alpha_2 + \varepsilon_2$
The result of this case is summrized as follows:

Proposition 6 *Suppose that the social capitl intensity is larger in the final good sector than in the new human capitl producing sector. Then indeterminacy may emerge either when* $k_2 > k_1$ *or* $k_1 > k_2 > k_3,$ *.if the feasible steady state has a positive balanced-grwoth rate.*

Proof. In this case, the system contains at least one stable root, $\Omega'(\bar{\omega})\,(< 0)$. First, consider the case of $k_2 > k_1$. If $k_1 > k_3$, then $\det J$ is negative, so that there are two stable roots. It is to be noted that if $k_1 > k_3$, it is possible to make $\det J > 0$. Addiationaly, if it is the case, we see that trace J is strictly negative. Therefore, when $k_2 > k_1 > k_3$, the system may involves three stable roots. In the case where $k_1 > k_2$, as well as in Proposition 5, $\det J$ may has a negative value if $k_3 > k_1$. Since neumerical example can be found as shown above, it is possible to have inderterminacy even though the relative magnigudes of capital intensity between the final good and new human capital production satisfy the same condition.□

Propositions 5 and 6 demonstrate that introducing the home production sector enhances the possibility of indeterminacy. When the model does not involve the home production sector, indeterminay never emerges if $\alpha_2 + \varepsilon_2 > \alpha_1 + \varepsilon_1$ of if $\alpha_1 + \varepsilon_1 > \alpha_2 + \varepsilon_2$ and $k_1 > k_2$. The above results show that indeterminacy may hold even though those conditions are met. To obtain intuition, consider the case of $\alpha_2 + \varepsilon_2 > \alpha_1 + \varepsilon_1$ and $k_2 > k_3 > k_1$ as an example.

5.4 Alternative Fourmulations

As pointed out before, Let us assume that in each moment the representative family is endowed with one unite of available time. The houseolds devotes $l \in [0,1]$ unite of time to leisure activities and $1-\ l$ to production activities. According to Becker (1975), it is assumed that the leisure activities needs human capital as well as time. The 'qyality time' for leisure is detnoted by $H_3 = lH$.[5] The instanteneous utility function is thus specified as

$$U(C, H_3) = \begin{cases} \dfrac{\left(C^{\gamma} H_3^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}, & \text{for } \sigma > 0,\ \sigma \neq 1, \\ \gamma \ln C + (1-\gamma) \ln H_3, & \text{for } \sigma = 1. \end{cases}$$

$$H_3/H = \psi c \omega^{-(\alpha_1+\varepsilon_1)}, \tag{40}$$

where $\psi = (1-\gamma)\, \alpha_1^{\alpha_1+\varepsilon_1} / \gamma \beta_1^{\alpha_1+\varepsilon_1-1}$ and $c = C/H$.

$$\begin{aligned} \frac{H_1}{H} &= \frac{k-k_2}{k_1-k_2} + \frac{k_2}{k_1-k_2} \psi c \omega^{-(\alpha_1+\varepsilon_1)} \\ &= \frac{k/\omega - (\alpha_2/\beta_2)}{\Delta} + \frac{(\alpha_2/\beta_2)\,\psi}{\Delta} c \omega^{-(\alpha_1+\varepsilon_1)}, \end{aligned}$$

$$\begin{aligned} \frac{H_2}{H} &= \frac{k_1-k}{k_1-k_2} + \frac{k_1}{k_1-k_2} \psi c \omega^{-(\alpha_1+\varepsilon_1)} \\ &= \frac{(\alpha_1/\beta 1) - k/\omega}{\Delta} + \frac{(\alpha_1/\beta_1)\,\psi}{\Delta} c \omega^{-(\alpha_1+\varepsilon_1)}, \end{aligned}$$

An lternative formulation of labor-leisure choice is to assume that leisure activities need pure time so that they are independent of stock of human capital. In this case, the instanteneous utility function is written as $U = U(C, l)$,where l denotes the time spent for leisure. The condition for human capital allocation is thus given by $H_1 + H_2 = (1-l)\, H$. As shown by

[5] A more general formulation is to assume that leisure activity, denoted by L, depends on human and physical capital as well as on and pure leisure time: $L = L(K_L, H_L, l)$, . where K_L and H_L denote physical and human capital devoted to leisure. In the endogenous growth literature, this kind of home production approach has been adopted by ....(1998). We may anticipate that such a general formulation tends to generate indeterminacy more easily.

Oriteguera et al. (1997), ...(1998), ...(1998) and Mino (1998), this formulation tend to yield more complex results than the model in which human capital as well as time are necessary for leisure activities.

$$u(C,l) = \begin{cases} \dfrac{C^{1-\sigma}}{1-\sigma} h(l), & \sigma \in (0,1), \\ \log C + h(l), & \text{for } \sigma = 1. \end{cases}$$

The full employment condition of human capital is

$$H_1 + H_2 = (1-l) H.$$

It is easy to see that the necessary conditions for dynamic optimiztion include

$$C^{-\sigma} h(l) = p_1,$$
$$\frac{C^{1-\sigma}}{1-\sigma} h'(l) = wH,$$
$$\dot{p}_2 = p_2 (\rho + \delta) - w(1-l),$$

together with (6), (7), ... and ( ). By use of .(.) and (.), we obtain

$$\frac{w}{p_1} = \frac{h'(l)\,c}{(1-\sigma)\,h(l)}.$$

Thus from (.) , $\beta_1 (\alpha_1 \omega / \beta_1)^{\alpha_1 + \varepsilon_1} = h'(l)\,c/(1-\sigma)\,h(l)$, and hence l is written as a function of c and ω in such a way that $l = l(c, \omega)$.

On the other hand, since $\dot{p}_2 / p_2 = \rho + \delta - \beta_2 (\alpha_2 \omega / \beta_2)^{\alpha_1 + e_1} [1 - l(c, \omega)]$. Consequently, the rental ratio, ω, changes accordingt to

$$\dot{\omega} = \frac{\omega}{\alpha_1 + \varepsilon_1 - \alpha_2 - \varepsilon_2} \left[\alpha_1 \left(\frac{\alpha_1 \omega}{\beta_1} \right)^{\alpha_1 + \varepsilon_1 - 1} - \beta_2 \left(\frac{\alpha_2 \omega}{\beta_2} \right)^{\alpha_1 + e_1} [1 - l(c, \omega)] + \eta - \delta \right].$$

The key difference between this formulation and the previous one is that in the pure laisure time model price system is no more independent of quqntity system, because consumption-hman capital ratio, c, affects the behavior of ω. Complexty arizing from this fact is the main source of the presence of multiple balanced-growth equilibra (Oritegura et al. 1997, 1998). The model studied by Oritgura et al. (1997) does not assume externalities, the competitive equilibrium coincides with the soluion for the social otimum. Thus even in the presnce of multiple steady states, one may determine an optimal path converging to one the steady states that exhibit saddle point stability. In our case, however, indeterminafy may emerg rather easily around the one of steady states.

6 Conclusion

This paper has demonstrated that social increasing returns is not indispensable for establishing indeterminacy in the standard models of endogenous growth. In the first model, we emphasize that a small divergence between the private and social capital intensity conditions would be enough to produce indeterminacy even under the assumption of social constant returns. On the other hand, by assuming endogenous labor supply, the second model demonstrated that a non-separable utility function may play a relevant role for establishing indeterminacy.

In this paper, we have concentrated to show that finding indeterminacy conditions does not necessarily require extreme assumptions on production technologies, so that we have not touched upon characterization of transition processes under indeterminacy. The issues concerning transitional dynamics under indeterminacy may deserve further investigation.

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