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Mino, Kazuo

Institute of Economic Research, Kyoto University

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Kazuo Mino[2]

Faculty of Economics, Kobe University

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[2]Faculty of Economics, Kobe University, Rokkodai, Nada-ku, Kobe 657-8501, Japan. (e-mail: mino@rose.rokkodai.kobe-u.ac.jp)

Abstract

This paper demonstrates that preference structure may play a pivotal role in generating indeterminacy in the stylized model of endogenous growth. By examining two-sector models of endogenous growth with human capital formation, we show that if the utility function of the representative family is not additively separable between consumption and pure leisure time, indeterminacy may hold even if production technologies satisfy social constant returns. We also examine models with quality leisure in which leisure activities require human capital as well as time. In contrast to the pure-leisure time model, we find that the quality-leisure time model generally needs increasing returns to scale technologies to generate indeterminacy. It is also shown that nonseparability of utility function is crucial for generating indeterminacy in the quality leisure model as well.

1 Introduction

The last several years have seen extensive investigations on indeterminacy of equilibrium in the representative agent models of economic growth. Most studies on this issue have examined models with external increasing returns. Early studies such as Benhabib and Farmer (1994) and Boldrin and Rustichini (1994) reveal that the degree of increasing returns should be sufficiently large to produce indeterminacy. The real business cycle theorists criticize this result and they claim that empirical validity of the business cycle theory based on indeterminacy and sunspots is dubious.[1] To cope with the criticism, the recent literature intends to find out the conditions under which indeterminacy emerges without assuming strong degree of increasing returns to scale: see, for example, Benhabib and Farmer (1996), Harrison (2001), Perli (1998) and Wen (1998).

The purpose of this paper is to make a contribution to such a research endeavour. In finding indeterminacy conditions, we put more emphasis on the role of preference structure rather than on that of production technologies. More specifically, we introduce sector-specific externalities into the two-sector endogenous growth models à la Lucas (1988, 1990). It is demonstrated that if the utility function of the representative family is not additively separable between consumption and pure leisure time, then indeterminacy may hold rather easily even if technologies of the final good and the new human capital production sectors satisfy social constant returns. We also explore the models with quality leisure time in which effective leisure units are defined as the amount of time spent for leisure activities augmented by the level of human capital. In this formulation, we verify that at least small degree of increasing returns is necessary to yield indeterminacy. We find that non-separability of the utility function is also crucial for generating indeterminacy in the quality leisure model.

In the existing literature, Benhabib and Perli (1998) and Xie (1994) explore indeterminacy in the Lucas model. Xie (1994) analyzes transitional dynamics of the Lucas model with multiple equilibria. Since the model he examines does not allow labor-leisure choice, indeterminacy condition needs the presence of strong increasing returns. Benhabib and Perli (1994) introduce endogenous labor supply and demonstrate that indeterminacy may emerge under relatively small degree of increasing returns. They assume that the instantaneous utility function is additively separable between consumption and leisure, and thus indeterminacy is mainly generated by the production structure specified in their model. In contrast to those

[1] Schmitt-Grohé (1997) presents a detailed examination of empirical plausibility of those studies.

contributions, the main discussion of this paper focuses on the role of non-separable utility function.[2]

The central concern of this paper is also closely related to two recent developments in the literature on growth models with human capital accumulation under constant returns. The first is the investigation on the global dynamics of the Lucas model in the absence of market distortion conducted by Ladrón-de-Guevara, Ortigueira and Santos (1997, 1999) and Ortigueira (2000). Ladrón-de-Guevara, Ortigueira and Santos (1997, 1999) give detailed studies of a pure-leisure-time version of the Lucas model. They reveal that in the pure leisure time setting the balanced-growth equilibrium may be multiple so that the destiny of the economy may hinge on the initial levels of physical and human capital. Ortigueira (2000), on the other hand, shows that the Lucas model with quality leisure displays well behaved dynamics in the sense that the balanced-growth equilibrium is uniquely determined and it satisfies global saddle stability. Since the models discussed by those authors do not involve any market distortion, there always exists a unique perfect-foresight competitive equilibrium even though there are multiple long-run equilibria. Therefore, indeterminacy of the converging path towards the balanced-growth equilibrium is not the issue in their studies.

The other development that is closely related to our analysis is made by Benhabib and Nishimura (1998, 1999). These authors reveal that indeterminacy may hold in the neoclassical multi-sector growth models with externalities and social constant returns. The key condition for indeterminacy in their finding is that the capital good sectors use more capital intensive technologies than the consumption good sector from the social perspective but they use more labor intensive technologies from the private perspective. Benhabib, Meng and Nishimura (2000) and Mino (2001) confirm that this conclusion also holds in the two-sector endogenous growth models with externalities in which both final good and education sectors use human as well as physical capital under social constant returns.[3] Since the Lucas model used in this paper assumes that the education sector employs human capital alone, there is no factor intensity reversal between the social and the private technologies. Therefore, the cause of indeterminacy with social constant returns in our discussion mainly comes from the preference side rather than from the production side emphasized by Benhabib and Nishimura (1998, 1999).

[2] See also Mitra (1998) for indeterminacy of equilibrium in a discrete-time version of the Lucas model.

[3] In the absence of externalities, the two-sector endogenous growth model of this type generally has a unique equilibrium: see Bond, Wang and Yip (1996) and Mino (1996).

The paper is organized as follows. Section 2 sets up the base model with pure leisure time. Section 3 characterizes the dynamics of the model and presents indeterminacy results. Section 4 re-examines the base model by using an alternative specification of leisure activities. Section 5 explores models without physical capital and finds the global indeterminacy conditions. Section 6 concludes the paper.

2 The Base Model

The analytical framework of this paper is essentially the same as that of Lucas (1988, 1990). We introduce sector-specific externalities into the original model. Production side of the economy consists of two sectors. The first sector produces a final good that can be used either for consumption or for investment on physical capital. The production technology is given by

$$Y_1 = K^{\alpha} H_1^{\beta_1} \bar{K}^{\varepsilon} \bar{H}_1^{\phi_1}, \quad \alpha,\ \beta_1 > 0,\ \alpha + \beta_1 + \varepsilon + \phi_1 = 1, \tag{1}$$

where Y_1 denotes the final good, K is stock of physical capital and H_1 is human capital devoted to the final good production. $\bar{K}^{\varepsilon}$ and $\bar{H}_1^{\phi_1}$ represent sector-specific externalities associated with physical and human capital employed in this sector. The key assumption in (1) is that the production technology satisfies social constant returns to scale.

Following the Uzawa-Lucas formulation, we assume that new human capital production does not employ physical capital and its technology is specified as

$$Y_2 = \gamma H_2^{\beta_2} \bar{H}_2^{\phi_2}, \quad \gamma, \beta_2, \phi_2 > 0, \quad \beta_2 + \phi_2 = 1. \tag{2}$$

Here, H_2 is human capital used in the education sector, $\bar{H}_2^{\phi_2}$ stands for sector specific externalities. Again, the production technology of new human capital exhibits social constant returns.

We assume that the total time available to the representative household is unity in each moment. Thus denoting the time length devoted to leisure activities by $l \in [0, 1]$, the full employment condition for human capital is

$$H_1 + H_2 = (1 - l) H,$$

where H is the total stock of human capital. As a result, if we define $v = H_1/H$, accumulation of physical and human capital respectively given by

$$\dot{K} = K^{\alpha} (vH)^{\beta_1} \bar{K}^{\varepsilon} \bar{H}_1^{\phi_1} - C - \delta K, \quad 0 < \delta < 1, \tag{3}$$

$$\dot{H}=\gamma\left[\left(1-v-l\right)H\right]^{\beta_2}\bar{H}_2^{\phi_2}-\eta H,\quad 0<\eta<1. \tag{4}$$

In the above, C denotes consumption, and δ and η are the depreciation rates of physical and human capital.

The objective function of the representative household is

$$U=\int_0^{\infty}u\left(C,l\right)e^{-\rho t}dt,\quad \rho>0,$$

where the instantaneous utility function is given by the following:[4]

$$u\left(C,l\right)=\begin{cases}\dfrac{\left[C\Lambda\left(l\right)\right]^{1-\sigma}-1}{1-\sigma}, & \sigma>0,\ \sigma\neq 1,\\ \ln C+\ln\Lambda\left(l\right), & \text{for } \sigma=1.\end{cases} \tag{5}$$

Function $\Lambda\left(l\right)$ is assumed to be monotonically increasing and strictly concave in l. We also assume that

$$\sigma\Lambda\left(l\right)\Lambda''\left(l\right)+\left(1-2\sigma\right)\Lambda'\left(l\right)^2<0. \tag{6}$$

This assumption, along with strict concavity of $\Lambda\left(l\right)$, ensures that $u\left(C,l\right)$ is strictly concave in C and l. Since we assume that leisure needs pure time alone, $l\times 100\%$ of human capital is not used for any activity.

The representative household maximizes U subject to (3), (4) and given initial levels of K and H by controlling C, v and l. In so doing, the household takes sequences of external effects, $\left\{\bar{K}\left(t\right),\bar{H}_1^{\phi_1}\left(t\right),\bar{H}_2^{\phi_2}\left(t\right)\right\}_{t=o}^{\infty}$, as given.[5]. The current value Hamiltonian for the optimization problem can be set as

$$\begin{aligned}\mathcal{H} \;=\;& \frac{\left[C\Lambda\left(l\right)\right]^{1-\sigma}-1}{1-\sigma}+p_1\left[K^{\alpha}\left(vH\right)^{\beta_1}\bar{K}^{\varepsilon}\bar{H}_1^{\phi_1},-C-\delta K\right]\\ &+p_2\left[\gamma\left(1-v-l\right)^{\beta_2}H^{\beta_2}\bar{H}_2^{\phi_2}-\eta H\right],\end{aligned}$$

where p_1 and p_2 are respectively denote the prices of consumption good and new human capital. Under given sequences of external effects, the necessary conditions for an optimum

[4] As is well known, if the utility function involves pure leisure time as an argument, the functional form should be (5) in order to define feasible balanced-growth equilibrium.

[5] In this formulation we set up a pseudo-planning problem in which the planner (household) seeks to maximize the discounted sum of utility under given sequences of external effects. Alternatively, we may find the perfect-foresight competitive equilibrium by analyzing a decentralized economy. In such a setting, the household maximizes U subject to the flow budget constraint under given sequences of prices and distributed profits and the firms maximize their profits by taking the externalities as given. See Mino (2000) for the details.

are the following:

$$C^{-\sigma}\Lambda(l)^{1-\sigma} = p_1, \tag{7}$$

$$C^{1-\sigma}\Lambda'(l)\Lambda(l)^{-\sigma} = \gamma p_2\beta_2(1-v-l)^{\beta_2-1}H^{\beta_2}\bar{H}_2^{\phi_2}, \tag{8}$$

$$p_1\beta_1K^{\alpha}v^{\beta_1-1}H^{\beta_1}K_E^{\varepsilon}\bar{H}_1^{\phi_1} = \gamma p_2\beta_2(1-v-l)^{\beta_2-1}H^{\beta_2}\bar{H}_2^{\phi_2}, \tag{9}$$

$$\dot{p}_1 = p_1\left[\rho+\delta-\alpha K^{\alpha-1}(vH)^{\beta_1}\bar{K}^{\varepsilon}\bar{H}_1^{\phi_1}\right], \tag{10}$$

$$\begin{aligned}\dot{p}_2 \;\; = \;\; & p_2\left[\rho+\eta-\gamma\beta_2(1-v-l)^{\beta_2}H^{\beta_2-1}\bar{H}_2^{\phi_2}\right] \\ & -p_1\left[\beta_1K^{\alpha-1}v^{\beta_1}H^{\beta_1-1}\bar{K}^{\varepsilon}\bar{H}_1^{\phi_1}\right],\end{aligned} \tag{11}$$

together with the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}p_1K = 0; \quad \lim_{t\to\infty} e^{-\rho t}p_2H = 0. \tag{12}$$

3 Equilibrium Dynamics and Indeterminacy Conditions

3.1 The Dynamical System

For analytical simplicity, in the following we specify $\Lambda(l)$ as

$$\Lambda(l) = \exp\left(\frac{l^{1-\theta}-1}{1-\theta}\right), \quad \theta > 0, \;\; \theta \neq 1, \tag{13}$$

where $\Lambda(l) = l$ for $\theta = 1$. Given this specification, when $\sigma = 1$, the instantaneous utility function becomes

$$u(C,l) = \ln C + \frac{l^{1-\theta}-1}{1-\theta},$$

which has been frequently used in the real business cycle literature. It is to be noted that, under this specification, condition (6) reduces to

$$(1-\sigma)l^{1-\theta} - \sigma\theta < 0. \tag{14}$$

If we assume that the number of firms is normalized to one, in equilibrium it holds that $\bar{K}(t) = K(t)$ and $\bar{H}_i(t) = H_i(t)$ for all $t \geq 0$. Thus, keeping in mind that $\alpha+\beta_1+\varepsilon+\phi_1 = 1$ and $\beta_2+\phi_2 = 1$, from (7) and (8) we obtain

$$\frac{C\Lambda'(l)}{\Lambda(l)} = \frac{p_2\gamma\beta_2H}{p_1}.$$

Given (13), the above becomes

$$C = (p_2/p_1)\gamma\beta_2l^{\theta}H. \tag{15}$$

Letting $x = K/vH$, (9) is written as

$$\frac{p_2}{p_1} = \frac{\beta_1}{\gamma\beta_2} x^{\alpha+\varepsilon}. \tag{16}$$

Equations (15) and (16) give $C = \beta_1 l^\theta x^{\alpha+\varepsilon} H$. Hence, using $x = K/vH$, the commodity market equilibrium conditions (3) and (4) yield the following growth equations of capital stocks:

$$\frac{\dot{K}}{K} = x^{\alpha+\varepsilon-1} - \frac{\beta_1 l^\theta x^{\alpha+\varepsilon}}{k} - \delta, \tag{3'}$$

$$\frac{\dot{H}}{H} = \gamma\left(1 - l - \frac{k}{x}\right) - \eta. \tag{4'}$$

On the other hand, (10) gives the following:

$$\dot{p}_1/p_1 = \rho + \delta - \alpha x^{\alpha+\varepsilon-1}, \tag{10'}$$

Additionally, in view of (9), equation (11) becomes

$$\dot{p}_2/p_2 = \rho + \eta - \gamma\beta_2 (1 - l). \tag{11'}$$

As a result, by use of (10'), (11') and (16), x changes according to

$$\frac{\dot{x}}{x} = \frac{1}{\alpha+\varepsilon}\left[\eta - \delta + \alpha x^{\alpha+\varepsilon-1} - \beta_2\gamma(1-l)\right]. \tag{17}$$

Under (13), equation (7) is expressed as

$$C^{-\sigma} l^{-\theta} \exp\left((1-\sigma)\frac{l^{1-\theta}-1}{1-\theta}\right) = p_1.$$

Thus combining (15) with the above equation and conducting logarithmic differentiation with respect to time, we obtain

$$\left[(1-\sigma) l^{1-\theta} - \sigma\theta\right]\frac{\dot{l}}{i} = (1-\sigma)\frac{\dot{p}_1}{p_1} + \sigma\left(\frac{\dot{p}_2}{p_2} + \frac{\dot{H}}{H}\right). \tag{18}$$

Note that if the utility function is additively separable ($\sigma = 1$), the above presents

$$\frac{\dot{l}}{l} = -\frac{1}{\theta}\left(\frac{\dot{p}_2}{p_2} + \frac{\dot{H}}{H}\right).$$

Namely, the optimal change in leisure time is negatively proportional to the change in aggregate value of human capital.

Using (4'), (10') and (11'), equation (18) yields the dynamic equation of leisure time:

$$\frac{\dot{l}}{l} = \Delta(l)\left\{\alpha(1-\sigma) x^{\alpha+\varepsilon-1} + \sigma\gamma\frac{k}{x} - \sigma\gamma(1-\beta_2)(1-l) - \rho - (1-\sigma)\delta\right\}, \tag{19}$$

where $\Delta(l) = \left[\sigma\theta - (1-\sigma) l^{1-\theta}\right]^{-1}$, which has a positive value under the assumption of (14). Finally, (3') and (4') mean that the dynamic equations for the behavior of k $(= K/H)$ is shown by

$$\frac{\dot{k}}{k} = x^{\alpha+\varepsilon-1} - \frac{\beta_1 l^{\theta} x^{\alpha+\varepsilon}}{k} - \delta + \eta - \gamma\left(1 - l - \frac{k}{x}\right). \tag{20}$$

Consequently, we find that (17), (19) and (20) constitute a complete dynamic system with respect to $k\,(= K/H)$, $x\,(= K/vH)$ and l.

3.2 Non-Separable Utility

Since the nonlinear dynamic system derived above is three dimensional, the precise analytical conditions for generating indeterminacy are hard to obtain. The common strategy to deal with such a situation is to find out numerical examples exhibiting indeterminacy by setting parameter values at empirically plausible magnitudes. In the following, rather than depending entirely on the numerical experiments, we impose some restrictions on the parameter values in order to obtain analytical conditions for indeterminacy. Following Xie's [30] idea, we focus on the special case where $\sigma = \alpha$ and $\theta = 1$.[6] As shown below, these conditions enable us to reduce the three-dimensional dynamic system to a two-dimensional one. Additionally, we also assume that $\delta = \eta$, that is, physical and human capital depreciate at the identical rate. This assumption is made only for notational simplicity and the main results obtained below are not altered when $\delta \neq \eta$.

The assumptions that $\sigma = \alpha$ and $\theta = 1$ simplify the dynamical system in the following sense:

Lemma 1 *If $\sigma = \alpha$ and $\theta = 1$, the consumption-physical capital ratio, C/K, stays constant over time.*

Proof. Let us define $z = \beta_1 x^{\alpha+\varepsilon} l/k\,(= C/K)$. If $\sigma = \alpha$, $\theta = 1$ and $\delta = \eta$, then (19) and (20) respectively become

$$\frac{\dot{l}}{l} = (1-\alpha)\,x^{\alpha+\varepsilon-1} + \gamma\frac{k}{x} - \gamma(1-\beta_2)(1-l) - \frac{\alpha + (1-\alpha)\delta}{\alpha}, \tag{19$'$}$$

$$\frac{\dot{k}}{k} = x^{\alpha+\varepsilon-1} - z - \gamma(1-l) + \gamma\frac{k}{x}. \tag{20$'$}$$

[6] Xie [30] examines a model with fixed labor supply in which the consumption-capital ratio stays constant over time if $\sigma = \alpha$.

Therefore, by (17), (19′) and (20′) we obtain:

$$\begin{aligned}\frac{\dot{z}}{z} &= (\alpha+\varepsilon)\frac{\dot{x}}{x}+\frac{\dot{l}}{l}-\frac{\dot{k}}{k}\\ &= z-\frac{\alpha+(1-\alpha)\delta}{\alpha}.\end{aligned}$$

Since this system is completely unstable, on the perfect-foresight competitive equilibrium path the following should hold for all $t \geq 0$:

$$z\left(=\frac{C}{K}\right)=\frac{\rho+(1-\alpha)\delta}{\alpha}.$$

Hence, consumption and physical capital change at the same rate even in the transition process. ■

The above result means that on the equilibrium path x is related to k and l in such a way that

$$x=\left(\left(\frac{\rho+(1-\alpha)\delta}{\alpha}\right)\frac{k}{l}\right)^{\frac{1}{\alpha+\varepsilon}}.$$

Substituting this into (19′) and (20′), we obtain the following set of differential equations:

$$\frac{\dot{k}}{k}=\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}}+\frac{\gamma}{\lambda}\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}}l-\gamma(1-l)-\lambda,$$

$$\frac{\dot{l}}{l}=(1-\alpha)\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}}+\frac{\gamma}{\lambda}\left(\lambda\frac{k}{l}\right)^{1-\frac{1}{\alpha+\varepsilon}}l-\gamma(1-\beta_2)(1-l)-\lambda,$$

where $\lambda=[\rho+(1-\alpha)\delta]/\alpha$. To simplify further, denote $q=(\lambda k/l)^{1-\frac{1}{\alpha+\varepsilon}}$. Then the above system may be rewritten as follows:

$$\frac{\dot{q}}{q}=\left(\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}\right)[\gamma\beta_2(1-l)-\alpha q], \tag{21}$$

$$\frac{\dot{l}}{l}=\left(1-\alpha+\frac{\gamma}{\lambda}l\right)q-\gamma(1-\beta_2)(1-l)-\lambda. \tag{22}$$

Under the conditions where $\sigma=\alpha$ and $\theta=1$, this system is equivalent to the original dynamic equations given by (17), (19) and (20).

By inspection of (21) and (22), we find the following results:

Lemma 2 *If the dynamic system consisting of (21) and (22) has a stationary point with a saddle-point property, then the original dynamic system exhibits local determinacy. If a stationary point of (21) and (22) is a sink, then the original system holds local indeterminacy.*

Proof. If the dynamic system consisting of (21) and (22) has a saddle-point property, there (at least) locally exists a one-dimensional stable manifold around the steady state. Hence, the relation between q and l on the stable manifold can be expressed as $q = q(l)$. By displaying phase diagrams of (21) and (22), it is easy to confirm that if the stationary point is saddle, the stable arms has negative slopes. By definition of q, it holds that

$$k = lq(l)^{\frac{\alpha+\varepsilon}{\alpha+\varepsilon-1}} \tag{23}$$

Since on the saddle path q is negatively related to l, the right hand side of the above monotonically increases with l. This implies that under a given initial level of k, the initial value of l is uniquely determined to satisfy (23). Thus converging path in the original system with respect to (k, x, l) is uniquely given as well. In contrast, if the steady state of (21) and (22) is a sink, there locally exist an infinite number of converging paths in (q, l) space Thus we cannot specify a unique set of initial values of l and x under a given initial level of k. ■

As for existence of the balanced-growth equilibrium, we find the following conditions:

Lemma 3 *(i) There is a unique, feasible balanced growth equilibrium, if and only if*

$$\gamma(\beta_2 - \alpha) > \rho + (1-\alpha)\delta. \tag{i}$$

(ii) There may exist dual balanced-growth equilibria, if

$$\gamma(\beta_2 - \alpha) < \rho + (1-\alpha)\delta. \tag{ii}$$

Proof. Condition $\dot{q} = 0$ in (21) yields $q = (\gamma\beta_2/\alpha)(1-l)$. Thus conditions $\dot{l} = \dot{q} = 0$ are established if the following equation holds:

$$\Gamma(l) = \frac{\gamma\beta_2}{\alpha}\left(1 - \alpha + \frac{\gamma}{\lambda}l\right)(1-l) - \gamma(1-\beta_2)(1-l) - \lambda = 0. \tag{24}$$

Note that

$$\begin{aligned}
\Gamma(0) &= (\gamma\beta_2/\alpha)(1-\alpha) - \gamma(1-\beta_2) - \lambda \\
&= (1/\alpha)[\gamma(\beta_2 - \alpha) - \rho - (1-\alpha)\delta] \\
\Gamma(1) &= -\lambda = -(1/\alpha)[\rho + (1-\alpha)\delta] < 0
\end{aligned}$$

If condition (i) is met, $\Gamma(0) > 0$ and $\Gamma(l)$ is monotonically decreasing with l for $l \in [0, 1]$. Hence, $\Gamma(l) = 0$ has a unique solution in between 0 and 1. If (ii) is satisfied, then $\Gamma(0) < 0$.

Since $\Gamma(l) = 0$ is a quadratic equation, if $\Gamma(l) = 0$ has solutions for $l \in [0, 1]$, there are two roots. ■

Using the results shown above, we obtain the indeterminacy results for the special case of $\sigma = \alpha$ and $\theta = 1$:

Proposition 1 *Suppose that $\sigma = \alpha$ and $\theta = 1$. Then the balanced-growth equilibrium is locally indeterminate, if and only if the following conditions are satisfied:*

$$\left(1 - \beta_2 - \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon}\right)\bar{l} + \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon} + \frac{\rho + (1 - \alpha)\delta}{\alpha} < 0, \tag{25}$$

$$\beta_2 - \alpha + \frac{\alpha\gamma\beta_2}{\rho + (1 - \alpha)\delta}\left(2\bar{l} - 1\right) > 0, \tag{26}$$

where $\bar{l}$ denotes the steady-state value of leisure time.

Proof. Linearizing (21) and (22) at the stationary point and using the steady-state conditions that satisfy $\dot{l} = \dot{q} = 0$, we find that signs of the trace and the determinant of the coefficient matrix of the linearized system fulfill the following:

$$\begin{aligned} & \operatorname{sign}\ \{\text{trace}\} \\ = \ & \operatorname{sign}\ \left\{\left(1 - \beta_2 - \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon}\right)\bar{l} + \frac{\beta_2(\alpha + \varepsilon - 1)}{\alpha + \varepsilon} + \frac{\rho + (1 - \alpha)\delta}{\alpha}\right\}, \end{aligned}$$

$$\operatorname{sign}\ \{\det\} = \operatorname{sign}\ \left\{\beta_2 - \alpha + \frac{\alpha\gamma\beta_2}{\rho + (1 - \alpha)\delta}\left(2\bar{l} - 1\right)\right\}.$$

Therefore, if (25) and (26) hold, then the trace and the determinant respectively have negative and positive values. This means that the linearized system has two eigenvalues with negative real parts, and thus in view of Lemma 2, the balanced-growth equilibrium is locally indeterminate. ■

This proposition is useful to characterize patterns of the local dynamics around the balanced-growth equilibrium. The results may be summarized by the following proposition:

Proposition 2 *If there is a unique balanced-growth equilibrium, it is either locally indeterminate or totally unstable. If there are dual balanced-growth equilibria, the one with a higher growth rate is locally determinate, while the other with a lower growth rate is either locally indeterminate or totally unstable.*

Proof. If the system involves a unique steady state, Lemma 1 shows that $\Gamma'\left(\bar{l}\right) < 0$, where $\bar{l}$ satisfies $\Gamma\left(\bar{l}\right) = 0$. It is easy to verify that this holds if and only if (26) is satisfied at $l = \bar{l}$. This means that the steady state is not a saddle point, which shows that it is a sink if (25) holds and it is a source if (25) is violated. Similarly, if the system has two stationary points, from Lemma 1 we see that $\Gamma'\left(\bar{l}_1\right) > 0$ and $\Gamma'\left(\bar{l}_2\right) < 0$, where $\bar{l}_1$ and $\bar{l}_2$ denote two roots of $\Gamma(l) = 0$ with $\bar{l}_1 < \bar{l}_2$. Hence, (26) does not hold at $l = \bar{l}_1$. As a result, the steady state with a lower l (so a higher growth rate) is a saddle point, while the steady state with a higher l (a lower growth rate) is a sink if (25) holds or it is a source if (25) does not hold. Consequently, in the case of dual steady states, the balanced-growth path with a higher growth rate is locally determinate and the one with a lower growth rate is either locally intermediate or totally unstable. ■

Since the indeterminacy conditions (25) and (26) contain an endogenous variable, $\bar{l}$, we consider some numerical examples. In the first example, we assume that $\alpha = \sigma = 0.3$, $\varepsilon = 0.1$, $\beta_2 = 0.7$, $\rho = 0.03$, $\delta = 0.04$ and $\gamma = 0.15$. Given those values, equation (24) has a unique, positive solution, $\bar{l} = 0.37317$. Substituting this and the parameter values specified above into (25) and (26), we find that both (25) and (26) are met. Thus the unique balanced-growth equilibrium is locally indeterminate. If we raise the value of β_2 up to 0.8, we obtain $\bar{l} = 0.55233$. Similarly, when $\beta_2 = 1$, (24) has $\bar{l} = 0.69564$. In both cases, (25) and (26) are still satisfied, so that indeterminacy emerges around the balanced-growth equilibrium. Note that when $\beta_2 = 1$, there is no external effect in the new human capital producing sector. Thus when the utility function is not separable between consumption and pure leisure time, the Lucas model with social constant returns may produce indeterminacy if there is small external effect in the final good producing sector.

To present examples of dual balanced-growth equilibria, let us now assume that $\alpha = \sigma = 0.3$, $\varepsilon = 0.1$, $\rho = 0.03$, $\delta = 0.04$, $\gamma = 0.2$ and $\beta_2 = 0.6$. Except for γ and β_2, the parameters have the same magnitudes as before. In this example, (24) has two positive solutions: $\bar{l} = 0.118$ and 0.608. It is confirmed that in the balanced-growth equilibrium associated with $\bar{l} = 0.118$ does not satisfy (26). Hence, the balanced-growth equilibrium with a lower leisure time establishes well behaved saddle-point stability so that local determinacy holds. On the other hand, the balanced-growth equilibrium with $\bar{l} = 0.608$ fulfills both (25) and (26). Thus the low-growth steady state yields indeterminacy of equilibrium. If we set $\alpha = 0.4$ and do not change other parameter values, we obtain $\bar{l} = 0.12028$ and 0.65472. In this case, the steady state with a lower $\bar{l}$ holds determinacy. However, the steady state with a higher $\bar{l}$ violates (25) so that it is totally unstable: we cannot find any converging path out of the balanced growth equilibrium.[7]

3.3 Separable Utility

In this section, we examine the model with a separable utility function that has been frequently used in the literature. To simplify the algebra, in what follows we assume that the production technology of education sector is not associated with externalities, that is, $\beta_2 = 1$. Although the magnitude of β_2 would be relevant for evaluating the dynamic behaviors of the model quantitatively, it can be shown that the value of β_2 does not yield essential effects on the analytical results.

Let us define $y = x^{\alpha+\varepsilon-1}$ and remember that $k/x = (K/H)(vH/K) = v$. Then assuming that $\sigma = \theta = \beta_2 = 1$, in the case of separable utility the dynamic system consisting of (17), (19) and (20) can be written as the following:

$$\frac{\dot{y}}{y} = \frac{1-(\alpha+\varepsilon)}{\alpha+\varepsilon}\left[\gamma(1-l) - \alpha y\right], \tag{27}$$

$$\frac{\dot{v}}{v} = y\left(1 - \frac{\beta_1 l}{v}\right) - \gamma(1-l-v) - \frac{1}{\alpha+\varepsilon}\left[\alpha y - \gamma(1-l)\right], \tag{28}$$

$$\frac{\dot{l}}{l} = \gamma v - \rho. \tag{29}$$

It is to be noted that by definition, it holds that $k = vy^{\frac{1}{\alpha+\varepsilon-1}}$, and thus that the initial values of v and y must satisfy this equation under a given initial level of k. Hence, as before, if the linearized system of (27), (28) and (29) has more than two stable roots, the balanced growth equilibrium is locally indeterminate.

We first examine existence of the balanced-growth equilibrium:

[7] Recently, several authors have examined the role of nonseparable utility in growth models. Bennett and Farmer [8] introduce a non-separable utility function into the model in Benhabib and Farmer [2] and show that indeterminacy may emerge under relatively small degree of increasing returns. Pelloni and Waldmann [24 and 25] examine Romer's [27] model with labor-leisure choice. They reveal that indeterminacy may hold one-sector endogenous growth model if the utility function is nonseparable. Using a one-sector, exogenous growth model with perfect markets, de Hek [13] points out that the nonseparable utility would yield multiple steady states.

Lemma 4 *Suppose that* $\sigma = \theta = \beta_2 = 1$ *and* $\rho < \gamma$. *Then the economy involves a unique, feasible balanced-growth equilibrium if*

$$\frac{\rho}{\gamma} - \beta_1 \left(1 - \frac{\rho}{\gamma}\right) < 0, \tag{i}$$

while there may exist dual balanced-growth equilibria if

$$\frac{\rho}{\gamma} - \beta_1 \left(1 - \frac{\rho}{\gamma}\right) > 0. \tag{ii}$$

Proof. Since the steady-state value of v is uniquely given in such a way that $v = \rho/\gamma$, the feasible steady-state level of l must satisfy $l \in [0, 1 - \rho/\gamma.]$. Hence, using the steady-state conditions that satisfy $\dot{y} = \dot{w} = \dot{l} = 0$, we find that if

$$(1 - l)\left[\frac{1}{\alpha} - 1 - \frac{\gamma\beta_1}{\alpha\rho} l\right] + \frac{\rho}{\gamma} = 0 \tag{30}$$

has a solution for $l \epsilon [0, 1 - \gamma]$, a feasible balanced-growth equilibrium may exist. Define

$$\Omega(l) = \frac{\gamma\beta_1}{\alpha\rho} l^2 + \left(1 - \frac{1}{\alpha} - \frac{\gamma\beta_1}{\alpha\rho}\right) l + \frac{1}{\alpha} - 1 + \frac{\rho}{\gamma}. \tag{31}$$

Then we see that

$$\Omega(0) = \frac{1}{\alpha} - 1 + \frac{\rho}{\gamma} > 0,$$
$$\Omega\left(1 - \frac{\rho}{\gamma}\right) = \frac{1}{\alpha}\left[\frac{\rho}{\gamma} - \beta_1\left(1 - \frac{\rho}{\gamma}\right)\right].$$

Therefore, if $\Omega(1 - \rho/\gamma) < 0$, equation $\Omega(l) = 0$ has a root in between 0 and $1 - \rho/\gamma$. If $\Omega(1 - \rho/\gamma) > 0$, then either $\Omega(l) = 0$ has no root or it has two roots for $l \in [0, 1 - \rho/\gamma]$. ■

As for the stability of balanced-growth equilibrium, we can establish the following results:

Proposition 3 *Suppose that* $\sigma = \theta = \beta_2 = 1$ and $\rho < \gamma$. *Then if there is a unique balanced-growth equilibrium, it is locally determinate. If there are dual balanced-growth equilibria, one with a lower leisure time (a higher growth rate) is locally determinate, while the other with a higher leisure time (a lower growth rate) is either locally indeterminate or totally unstable.*

Proof. The coefficient matrix of the dynamic system linearized at the steady state is given by

$$J = \begin{bmatrix} \gamma\left(1 - \frac{1}{\alpha+\varepsilon}\right)\left(1 - \bar{l}\right) & 0 & \frac{\gamma^2}{\alpha}\left(1 - \frac{1}{\alpha+\varepsilon}\right)\left(1 - \bar{l}\right) \\ \left(\frac{\varepsilon}{\alpha+\varepsilon}\right)\frac{\rho}{\gamma} - \beta_1 \bar{l} & \frac{\beta_1\gamma^2}{\alpha\rho}\left(1 - \bar{l}\right)\bar{l} + \rho & -\frac{\gamma\beta_1}{\alpha}\left(1 - \bar{l}\right) + \left(1 - \frac{1}{\alpha+\varepsilon}\right)\rho \\ 0 & \gamma\bar{l} & 0 \end{bmatrix},$$

where l satisfies (30). By (30) it holds that

$$\begin{aligned}\text{trace } J &= \gamma\left\{\left(1-\bar{l}\right)\left[1+\frac{\gamma\beta_1}{\alpha\rho}\bar{l}-\frac{1}{\alpha+\varepsilon}\right]+\frac{\rho}{\gamma}\right\}\\ &> \gamma\left\{\left(1-\bar{l}\right)\left[1+\frac{\gamma\beta_1}{\alpha\rho}\bar{l}-\frac{1}{\alpha}\right]+\frac{\rho}{\gamma}\right\}=2\rho>0\end{aligned}$$

On the other hand, the determinant of J is

$$\det J=\gamma^2\bar{l}\left(\frac{1}{\alpha+\varepsilon}-1\right)\left(1-\bar{l}\right)\left[\frac{\gamma\beta_1}{\alpha}\left(2\bar{l}-1\right)+\rho-\frac{\rho}{\alpha}\right]$$

From (31) we obtain

$$\Omega'\left(l\right)=\frac{1}{\rho}\left[\frac{\gamma\beta_1}{\alpha}\left(2\bar{l}-1\right)+\rho-\frac{\rho}{\alpha}\right].$$

If the economy involves a unique steady-state values of $\bar{l}$, it satisfies $\Omega'\left(\bar{l}\right)<0$. This means that $\det J<0$. Since trace $J>0$, the system has one negative and two positive (possibly complex) eigenvalues, which shows that the balanced-growth equilibrium is locally determinate. On the other hand, if there are dual steady-state levels of $\bar{l}_1$ and $\bar{l}_2$ $\left(\bar{l}_1<\bar{l}_2\right)$, we have $\Omega'\left(\bar{l}_1\right)<0$ and $\Omega'\left(\bar{l}_2\right)>0$. Thus, as well as the case of unique steady state, the equilibrium path is determinate at $l=\bar{l}_1$. When $l=\bar{l}_2$, it holds that $\det J>0$. The sum of the second-order principal minors of J is given by

$$J_2=\gamma\left(1-\frac{1}{\alpha+\varepsilon}\right)\left(1-\bar{l}\right)\left[\frac{\beta_1\gamma^2}{\alpha\rho}\left(1-\bar{l}\right)\bar{l}+\rho\right]+\gamma\bar{l}\left[\frac{\gamma\beta_1}{\alpha}\left(1-\bar{l}\right)-\left(1-\frac{1}{\alpha+\varepsilon}\right)\rho\right]$$

The Routh theorem states that the number of eigenvalues of J with positive real parts equal to the number of changes in signs of the following sequence:[8]

$$-1,\quad \text{trace } J,\quad -J_2+\frac{\det J}{\text{trace } J},\quad \det J$$

Since $\det J>0$ and trace $J>0$, the above sequence changes sign once if $J_2<0$. This means that J has two eigenvalues with negative real parts, so that local indeterminacy holds around $l=\bar{l}_2$. In contrast, if $-J_2+\frac{\det J}{\text{trace } J}<0$, the sign changes three times and hence all of the eigenvalues of J have positive real parts. If this is the case, the economy is totally unstable at $l=\bar{l}_2$. ■

A relevant difference between the model with nonseparable and one with separable utility is that indeterminacy will not emerge if the model with a separable utility function involves a unique balanced-growth equilibrium. If the separable utility model has a dual balanced-growth equilibria, there still remains the possibility of indeterminacy in the low-growth steady

[8] See p.180 in Gantmacher [11].

state. However, since the dual long-run equilibria condition (ii) would not hold under the empirically plausible parameter values, it is safe to say that indeterminacy of equilibrium is hard to observed in the Lucas model with a separable utility function and social constant returns. In section 5.1, we see this point more clearly in the context of a simpler model.

4 Quality Leisure Time

So far, we have assumed that utility of the household depends upon consumption and pure leisure time. An alternative formulation suggested by Becker [1] is to assume that leisure activities need human capital as well as time. The simplest form of utility function capturing this idea is[9]

$$u(C, lH) = \begin{cases} \dfrac{\left[C^{\psi} (lH)^{\zeta}\right]^{1-\sigma} - 1}{1-\sigma}, & \sigma > 0, \quad \sigma \neq 1, \ \ \psi, \ \zeta \in (0.1), \\ \psi \log C + \zeta \log (lH), & \text{for } \sigma = 1. \end{cases} \tag{32}$$

Given this specification, the necessary conditions for an optimum for the base model (7), (8) and (11) are respectively replaced with

$$\psi C^{\psi(1-\sigma)-1} (lH)^{\zeta} = p_1, \tag{33}$$

$$\zeta C^{\psi(1-\sigma)} l^{\zeta(1-\sigma)-1} H^{\zeta(1-\sigma)} = p_2 \beta_2 \gamma (1 - v - l)^{\beta_2 - 1} H^{\beta_2} \bar{H}_2^{\phi_2}, \tag{34}$$

$$\begin{aligned} \dot{p}_2 &= p_2 (\rho + \eta - \beta_2 \gamma (1 - v - l)) - p_1 \beta_1 v^{\beta_1 + \phi_1} H^{\beta_1 + \phi_1 - 1} \\ & \quad - \zeta C^{\psi(1-\sigma)} l^{\zeta(1-\sigma)} H^{\zeta(1-\sigma)-1}. \end{aligned} \tag{35}$$

The other conditions, (9), (10) and (12), are also necessary for an optimum.

Noting that $\alpha + \beta_1 + \varepsilon + \phi_1 = 1$ and $\beta_2 + \phi_2 = 1$, equations (33) and (34) present

$$\frac{C}{lH} = \frac{\psi}{\zeta} \left(\frac{p_2}{p_1} \right) = \frac{\beta_1 \psi}{\gamma \beta_2 \zeta} x^{\alpha + \varepsilon}. \tag{36}$$

Using (33) and (34), equation (35) reduces to

$$\dot{p}_2 / p_2 = \rho + \eta - \beta_2 \gamma. \tag{37}$$

[9] In order to establish the balanced-growth equilibrium, the utility function should have the particular form given by (32). Ladròn-de-Guevara et al. [14 and15] compare dynamic property of a two-sector endogenous growth model under (5) with that under (32).

Thus the price of new human capital changes at a constant rate. Compared to the pure leisure time model, this property simplifies the analysis of the quality leisure time model. By (34) and (36), we obtain:

$$\zeta\,(lH)^{(\psi+\zeta)(1-\sigma)-1}\left(\frac{\beta_1\psi}{\gamma\beta_2\zeta}\right)^{\psi(1-\sigma)}x^{\psi(\alpha+\varepsilon)(1-\sigma)}=p_2\gamma\beta_2.$$

This shows that the dynamics of l is described by

$$\frac{\dot{l}}{l}=\frac{1}{\sigma}\left(\beta_2\gamma-\rho-\eta\right)-\frac{\dot{H}}{H}+\frac{\psi\left(\alpha+\varepsilon\right)\left(1-\sigma\right)}{\sigma\left[1-\left(\psi+\zeta\right)\left(1-\sigma\right)\right]}\frac{\dot{x}}{x}.$$

Based on the conditions obtained above, if we assume that $\delta=\eta$, the complete dynamic system is given by the following set of differential equations:

$$\frac{\dot{k}}{k}=x^{\alpha+\varepsilon-1}-\gamma\left(1-l-\frac{k}{x}\right)-\frac{\beta_1\psi}{\gamma\beta_2\zeta}\frac{x^{\alpha+\varepsilon}l}{k}, \tag{38}$$

$$\frac{\dot{x}}{x}=\frac{1}{\alpha+\varepsilon}\left(\alpha x^{\alpha+\varepsilon-1}-\beta_2\gamma\right), \tag{39}$$

$$\begin{aligned}\frac{\dot{l}}{l} &= \frac{\beta_2\gamma}{\sigma}-\gamma\left(1-l-\frac{k}{x}\right)+\frac{\psi\left(\alpha+\varepsilon\right)\left(1-\sigma\right)}{\sigma\left[1-\left(\psi+\zeta\right)\left(1-\sigma\right)\right]}\left(\alpha x^{\alpha+\varepsilon-1}-\beta_2\gamma\right)\\ &\quad-\frac{\rho+\left(1-\rho\right)\delta}{\sigma}.\end{aligned} \tag{40}$$

Observe that since (39) exhibits self stabilizing behavior, the system has at least one stable root. Thus if the balanced growth equilibrium exists, the local stability of the economy is ensured.

Unlike the pure leisure time model, the quality leisure time model with social constant returns has simpler properties. The results may be summarized in the following manner:

Lemma 5 *If the quality leisure time model with social constant returns involves a feasible balanced-growth equilibrium, it is uniquely given.*

Proof. In (39) $\dot{x}=0$ yields a unique steady-state value of x such that $\bar{x}=\left(\gamma\beta_2/\alpha\right)^{1/(\alpha+\varepsilon-1)}$. By substituting $\bar{x}$ into $\dot{l}=0$ condition in (40), we see that $\gamma\left(1-l-k/\bar{x}\right)=$ a constant. Therefore $\dot{k}=0$ condition in (38) shows that $l/k=$ a constant. Since these two conditions are linear functions of k and l, the steady-state values of k and l are uniquely determined as well. ■

If a feasible balanced-growth exists, we can easily verify the following:

Proposition 4 *If the quality leisure time model with social constant returns involves the balanced-growth equilibrium, it is locally determinate.*

Proof. Letting ξ be the eigenvalue of the coefficient matrix of the linearized system of (38), (39) and (40) evaluated the steady state, we find that one of the characteristic root of the linearized system is

$$\xi = \left(\frac{\alpha+\varepsilon-1}{\alpha+\varepsilon}\right)\beta_2\gamma,$$

which has a negative value. The other two eigenvalues satisfy the following equation:

$$\left\{\xi^2 - \left(\frac{\gamma\bar{k}}{\bar{x}} + \pi\frac{\bar{l}}{\bar{k}} + \frac{\gamma\bar{l}}{\bar{x}}\right)\xi + \gamma\bar{l}\left(\frac{\gamma\bar{k}}{\bar{x}} + \pi\frac{\bar{k}}{\bar{k}} + \pi\frac{\gamma\bar{l}}{\bar{x}}\right)\right\} = 0,$$

where $\pi = \beta_1\psi\bar{x}^{\alpha+\varepsilon}\bar{l}/\gamma\beta_2\bar{k} > 0$. Since both roots of the above equation have positive real parts, the balanced-growth equilibrium is locally determinate. ■

As a result, the simple formulation of quality leisure excludes indeterminacy under socially constant returns technologies. Indeterminacy in this model thus requires that production technology exhibits some degree of increasing returns. For example, suppose that external effects in the final good sector depend on the aggregate level of human capital in such a way that

$$Y_1 = K^\alpha H_1^{\beta_1}\bar{K}^\varepsilon\bar{H}^{\phi_1},$$

where $\bar{H} = H$ in equilibrium. In this case, we can verify that the model with a nonseparable utility function may yield indeterminacy under a small degree of increasing returns, that is, $\alpha+\beta_1+\varepsilon+\phi_1$ is close to one. A simpler model discussed in the next section confirms this result.

5 Global Indeterminacy in Models without Physical Capital

In this section we briefly examine models without physical capital. Although the endogenous growth model that does not involve physical capital may lack reality, it is helpful for analyzing the global behavior of the economy without imposing any restrictions on the parameter values involved in the model. The production and preference structure are the same as before. Only difference is that there is no physical capital: both final good and new human capital producing sectors use human capital alone. Since in this setting the final good is used only for consumption, the market equilibrium condition for the first good is

$$C = (vH)^{\beta_1}\,\bar{H}_1^{\phi_1}, \quad \beta_1 \in (0,1), \quad \phi_1 > 0. \tag{41}$$

The production function of new human capital is (2) in the base model.

5.1 Pure Leisure Time

We first consider a model with pure leisure time where the utility function is given by (5).[10] Again, we assume that the consumption good sector has a socially constant returns to scale technology so that $\beta_1 + \phi_1 = 1$. The Hamiltonian function for the household's optimization problem is

$$\mathcal{H} = \frac{\left[C\Lambda\left(l\right)\right]^{1-\sigma} - 1}{1-\sigma} + p_1 \left[\left(vH\right)^{\beta_1} \bar{H}_1^{\phi_1} - C\right] + p_2 \left[\gamma\left(1-v-l\right)^{\beta_2} H^{\beta_2} \bar{H}_2^{\phi_2} - \eta H\right],$$

where p_1 is the price of the consumption good. Keeping in mind that $\bar{H}_1 = vH$ and $\bar{H}_2 = \left(1-l-v\right)H$ when $\beta_1 + \phi_1 = \beta_2 + \phi_2 = 1$, the necessary conditions for optimization are:

$$C^{-\sigma} \exp\left(\left(1-\sigma\right)\frac{l^{1-\theta}-1}{1-\theta}\right) = p_1, \tag{42}$$

$$C^{1-\sigma} l^{-\theta} \exp\left(\left(1-\sigma\right)\frac{l^{1-\theta}-1}{1-\theta}\right) = \gamma p_2 \beta_2 H, \tag{43}$$

$$\beta_1 p_1 = \gamma \beta_2 p_2, \tag{44}$$

$$\dot{p}_2 = p_2\left[\rho + \eta - \gamma\beta_2\left(1-l\right)\right]. \tag{45}$$

In addition, the transversality condition is given by $\lim_{t\to\infty} p_2 e^{-\rho t} H = 0$.

Using (42), (43) and (44), we obtain

$$C = \beta_1 l^{\theta} H. \tag{46}$$

On the other hand, in the presence of socially constant returns to scale technologies, (41) becomes $C = vH$. Thus (46) gives the relation between l and v:

$$v = \beta_1 l^{\theta}. \tag{47}$$

Substituting (46) into (43) and taking logarithmic differentiation with respect to time, we obtain

$$-\sigma\theta\frac{\dot{l}}{l} - \sigma\frac{\dot{H}}{H} + \left(1-\sigma\right)l^{1-\theta}\frac{\dot{l}}{l} = \frac{\dot{p}_2}{p_2}.$$

Accordingly, from (4'), (45) and (47), the above yields a complete dynamic equation of leisure time l:

$$\dot{l} = l\Delta\left(l\right)\left[\gamma\left(\beta_2 - \delta\right)\left(1-l\right) + \sigma\gamma\beta_1 l^{\theta} - \rho - \left(1-\sigma\right)\eta\right], \tag{48}$$

[10] Mino (1999) considers this type of model that assumes a different form of nonseparable utility function.

where $\Delta(l) = \sigma\theta - (1-\sigma) l^{1-\theta} > 0$ by the concavity assumption (14). Equation (48) summarizes the entire model. Since the initial level of l is not specified, if (48) is stable around the stationary point, local indeterminacy emerges.

Inspection of (48) reveals the following results:

Lemma 6 *(i) There is a unique balanced-growth equilibrium, if either (i-a) or (i-b) below is satisfied:*

$$\sigma > \max\left\{\beta_2, \frac{\rho+\eta}{\gamma\beta_1+\eta}\right\}, \tag{i-a}$$

$$\sigma < \min\left\{\beta_2, \frac{\rho+\eta}{\gamma\beta_1+\eta}\right\}. \tag{i-b}$$

(ii) There may exist dual balanced-growth equilibria, if either (ii-a) or (ii-b) below holds:

$$\frac{\rho+\eta}{\gamma\beta_2+\eta} < \sigma < \beta_2,\ \gamma(\beta_2-\sigma) > \rho + (1-\sigma)\eta \text{ and } \theta < 1, \tag{ii-a}$$

$$\frac{\rho+\eta}{\gamma\beta_2+\eta} < \sigma < \beta_2,\ \gamma(\beta_2-\sigma) < \rho + (1-\sigma)\eta \text{ and } \theta \geq 1. \tag{ii-b}$$

Proof. Define

$$\Phi(l) = \gamma(\beta_2-\sigma)(1-l) + \sigma\beta_1 l^\theta - [\rho + (1-\sigma)\eta].$$

The balanced growth equilibrium level of l is a solution of $\Phi(l) = 0$. Note that

$$\begin{aligned} \Phi(0) &= \gamma(\beta_2-\sigma) - [\rho + (1-\sigma)\eta], \\ \Phi(1) &= (\gamma\beta_1+\eta)\sigma - (\rho+\eta). \end{aligned}$$

If condition (i-a) is held, it is easy to see that $\Phi(l)$ is monotonically increasing and $\Phi(1) > 0 > \Phi(0)$. Thus $\Phi(l) = 0$ has a unique solution $l \in (0,1)$. In the case of condition (i-b), we see that $\Phi(0) > 0 > \Phi(1)$ and $\Phi(l)$ is monotonically decreasing. Hence, $\Phi(l) = 0$ has only one solution in between 0 and 1. If $(\rho+\eta)/(\gamma\beta_2+\eta) < \sigma < \beta_2$, then $\Phi(0)$ and $\Phi(1)$ have the same sign. This means that if the balanced-growth path exists, there are at least two equilibria. Under conditions (ii-a), $\Phi(0) < 0$, $\Phi(1) < 0$ and $\Phi(l)$ is strictly convex in l. Therefore, if $\Phi(l) = 0$ has solutions, there are two solutions in between 0 and 1. Conversely, under conditions (ii-b), we find that $\Phi(0) > 0$, $\Phi(1) > 0$ and $\Phi(l)$ is strictly concave, and hence $\Phi(l) = 0$ also have dual solutions for $l \in (0,1)$. ■

Those results immediately yield the following proposition:

Proposition 5 *Given condition (i-a), the balanced-growth equilibrium is globally determinate, while it is globally indeterminate if condition (i-b) holds. If conditions in (ii-a) are satisfied, the balanced-growth equilibrium with a lower level of l is locally indeterminate, while the other with a higher level of l is locally determinate. In case of (ii-b), the opposite results hold.*

Proof. Since condition (i-a) ensures that $d\dot{l}/dl > 0$ for all $l \in [0,1]$, the balanced-growth equilibrium is globally determinate. Given condition (i-b), $d\dot{l}/dl < 0$ for all $l \in [0,1]$, so that global indeterminacy is established. In the similar way, it is easy to see that results for the cases of (ii-a) and (ii-b) can be held. ■

We should note that if the utility function is additively separable between consumption and leisure ($\sigma = 1$) and $\beta_2 < 1$, then only condition (i-a) can be satisfied because it should hold $\rho < \gamma\beta_1$[11]. Therefore, we never observe indeterminacy if we assume a separable utility function. In addition, if $\theta = 1$ but $\sigma \neq 1$, then $\Phi(l)$ function in the proof of Lemma 5 becomes $\Phi(l) = \gamma(\sigma\beta_1 + \delta - \beta_2)\, l + \gamma(\beta_2 - \delta) - \rho - (1-\sigma)\eta$. Therefore, if $\sigma < (\beta_2 - \delta)/\beta_1$, then the balanced-growth equilibrium is globally indeterminate. Otherwise, it is globally determinate.

5.2 Quality Leisure Time

As suggested by the model in Section 4, we can verify that the quality leisure time model without physical capital will not yield indeterminacy if the production technologies satisfy social constant returns. Furthermore, it is shown that, given our specification, if externalities in the consumption good sector are sector specific, indeterminacy does not exist regardless of the degree of returns to scale. We, therefore, assume that production function of the consumption good sector is

$$C = (vH)^{\beta_1}\, \bar{H}^{\phi_1}, \quad \beta_1 \in (0,1), \quad \phi_1 > 0. \tag{49}$$

In equilibrium, it holds that $\bar{H} = H$. That is, external effects for the consumption good sector are associated with the aggregate human capital rather than the sector-specific human

[11] If $\sigma = 1$, the steady-state conditions mean that $\dot{H}/H = \gamma(1-l-v) - \eta = -\dot{p}_2/p_2 = \gamma\beta_2(1-l) - \rho$. Thus noting that $\beta_2 < 1$ and $l^\theta < 1$, from (47) we obtain

$$\rho = \gamma\beta_1 l^\theta + \gamma(\beta_2 - 1) < \gamma\beta,$$

implying that $(\rho + \eta)/(\gamma\beta_1 + \eta) < 1$.

capital. In this case, optimization with respect to C, l and v yield:

$$\psi C^{\psi(1-\sigma)-1}(lH)^{\zeta(1-\sigma)} = p_1, \tag{50}$$

$$\zeta C^{\psi(1-\sigma)}(lH)^{\zeta(1-\sigma)-1} = p_2\gamma\beta_2, \tag{51}$$

$$p_1\beta_1 v^{\beta_1-1}H^{\beta_1+\phi_1-1} = p_2\gamma\beta_2. \tag{52}$$

In view of (51), changes in the price of new human capital is

$$\dot{p}_2 = p_2(\rho+\eta-\gamma\beta_2). \tag{53}$$

By use of (36), (50), and (51), we find:

$$\frac{\zeta C}{\psi lH} = \frac{\gamma\beta_2 p_2}{p_1} = \beta_1 v^{\beta_1-1}H^{\beta_1+\phi_1-1}. \tag{54}$$

Substituting (49) into (54)the above gives

$$v = \frac{\beta_1\psi}{\zeta}l. \tag{55}$$

From (51), (54) and (55), we obtain the following equation:

$$\left(\frac{\beta_1\psi}{\zeta}\right)^{\beta_1\psi(1-\sigma)} l^{(1-\sigma)\beta_1\psi+\zeta(1-\sigma)-1}H^{(\beta_1+\phi_1)(1-\sigma)+\zeta(1-\sigma)-1} = \frac{\beta_2\gamma p_2}{\zeta}.$$

Consequently, taking logarithmic differentiation of both sides of the above with respect to time and using (4') and (52), we obtain a complete dynamic equation of l as follows:

$$\begin{array}{c}\frac{\dot{l}}{l} = \frac{1-\psi(\beta_1+\phi_1)(1-\sigma)-\zeta(1-\sigma)}{\psi\beta_2(1-\sigma)+\zeta(1-\sigma)-1}\left\{\gamma\left(1-\left(1+\frac{\beta_1\psi}{\zeta}\right)l\right)-\eta\right\} \\ +\frac{\rho+\eta-\beta_2\gamma}{(1-\sigma)\beta_1\psi+\zeta(1-\sigma)-1}\end{array}. \tag{56}$$

Inspection of this equation gives the following results:

Proposition 6 *The quality leisure time model without physical capital is globally indeterminate, if and only if*

$$1-\frac{1}{\psi(\beta_1+\phi_1)} < \sigma < 1-\frac{1}{\psi(\beta_1+\phi_1)+\zeta}. \tag{57}$$

Proof. Since the right hand side of (56) is a linear function of l, if the system has a stationary point in between $l=0$ and 1, it should be uniquely given. Thus if $d\dot{l}/dl<0$ holds, global indeterminacy is established. We see that $d\dot{l}/dl<0$ for all $l\in[0,1]$, if and only if $(1-\sigma)\beta_1\psi+\zeta(1-\sigma)-1$ and $1-\psi(\beta_1+\phi_1)(1-\sigma)-\zeta(1-\sigma)$ have the same sign.

If both of them are positive, it should hold that $\phi_1\beta_1(1-\sigma) < 0$ so that $\sigma > 1$. However, $(1-\sigma)\beta_1\psi + \zeta(1-\sigma) - 1 > 0$ cannot be satisfied for $\sigma > 1$. In contrast, if

$$(1-\sigma)\beta_1\psi + \zeta(1-\sigma) - 1 < 0 \;\text{ and }\; 1 - \psi(\beta_1 + \phi_1)(1-\sigma) - \zeta(1-\sigma) < 0,$$

then $\psi\phi_1(1-\sigma) > 0$ and thus $\sigma \in (0,1)$. The above conditions can be expressed as in the proposition statement. ■

Condition (57) makes three points. First, if the utility function is separable ($\sigma = 1$), then (57) cannot be met and indeterminacy will not emerge. Second, indeterminacy needs social increasing returns in the consumption good sector, that is, $\beta_1 + \phi_1 > 1$. Third, magnitude of external effects in the new human represented by ϕ_2 does not affect indeterminacy condition, which means that the model may exhibit indeterminacy when the new human capital producing sector has no external effects. Finally, it is seen that, when indeterminacy holds, there is a trade-off between magnitude of returns to scale, $\beta_1 + \phi_1$, and the value of σ: the smaller the degree of returns to scale, $\beta_1 + \phi_1$, is, the larger the intertemporal substitutability in felicity, $1/\sigma$, should be.

6 Conclusion

This paper has demonstrated that preference structure may play a pivotal role in generating indeterminacy of equilibrium in the Lucas model, which is one of the prototype models of endogenous growth. Unlike the existing studies on indeterminacy in the Lucas model that empathize the role of external increasing returns, we have demonstrated that the Lucas model with nonseparable utility between consumption and leisure time may yield indeterminacy even in the absence of social increasing returns. Since our model precludes the possibility of reversal of social and private factor intensity conditions emphasized by Benhabib and Nishimura (1998, 1999), indeterminacy in our setting mainly stems from the preference structure.

We have also shown that indeterminacy results depend upon specification of leisure. If effective leisure is defined as the length of time spent for leisure activities, the economy may involve multiple balanced-growth paths and indeterminacy tends to emerge rather easily under socially constant returns to scale technologies. If we assume that effective leisure depends on the level of human capital as well as on time, the economy has a unique balanced-growth equilibrium. In this setting indeterminacy will not emerge under social constant returns. Additionally, it is shown that nonseparable utility may also be relevant for generating indeterminacy in the quality leisure model with social increasing returns. These results suggest that if we consider leisure as a home good produced by a more general technology than that we assume in the paper, the possibility of emergence of indeterminacy would increase even in the absence of increasing returns to scale technologies.

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