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Abstract

This paper generalizes the nonparametric approach to option pricing of Stutzer (1996) by demonstrating that the canonical valuation methodology introduced therein is one member of the Cressie-Read family of divergence measures. While the limiting distribution of the alternative measures is identical to the canonical measure, the finite sample properties are quite different. We assess the ability of the alternative divergence measures to price European call options by approximating the risk-neutral, equivalent martingale measure from an empirical distribution of the underlying asset. A simulation study of the finite sample properties of the alternative measure changes reveals that the optimal divergence measure depends upon how accurately the empirical distribution of the underlying asset is estimated. In a simple Black-Scholes model, the optimal measure change is contingent upon the number of outliers observed, whereas the optimal measure change is a function of time to expiration in the stochastic volatility model of Heston (1993). Our extension of Stutzer's technique preserves the clean analytic structure of imposing moment restrictions to price options, yet demonstrates that the nonparametric approach is even more general in pricing options than originally believed.

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1 Introduction

Due to the poor empirical performance of parametric models, nonparametric option pricing techniques have expanded rapidly in recent years [Hutchinson, Lo, and Poggio (1994), Rubenstein (1994), Aït-Sahalia and Lo (1998), (Broadie, Detemple, Ghysels, and Torres, 2000), Garcia and Gençay (2000)]. Duan (2002) lays out two important criticisms of these nonparametric methods. First, many of these techniques, such as neural networks and kernel regressions, suffer from the curse of dimensionality; i.e., they require large amounts of option pricing data to perform well. Second, many of these techniques are unable to price options of different maturities and therefore do not exploit all of the available cross-sectional information [e.g., Buchen and Kelly (1996)]. This weakness arises because the nonparametric risk-neutral distributions are identified separately according to contract maturity. Therefore, neither the statistical properties of the underlying asset nor the properties of the option prices at different maturities can be used to price options of a specific maturity. This also implies that these techniques cannot be used to price path-dependent derivatives (e.g., barrier options). When pricing redundant securities (as is our focus here), these shortcomings are potentially severe.

In contrast, the nonparametric method of Stutzer (1996) (referred to as canonical valuation) does not require any option pricing data and takes full advantage of the available cross-sectional information. The defining feature of Stutzer's approach is the maximum cross-entropy (or minimum Kullback-Leibler) technique, which minimizes the divergence between the actual probability distribution governing the underlying asset and its risk-neutral counterpart needed to price the derivative security. This minimization is subject to the constraint that the underlying asset price follow a martingale, thus ensuring that the risk-neutral density is in fact of the correct form. Cross-sectional information is imbedded into the estimation process through moment restrictions by imposing that the risk-neutral density correctly price options of the same maturity by different strikes.

Recently, several papers have extended Stutzer's original work and demonstrate that the methodology is flexible and performs very well in the presence of realistic financial time series. Gray and Newman (2005) show that when the underlying asset is generated by a stochastic volatility process, canonical valuation outperforms traditional Black-Scholes estimates. The disparity in the performance increases as the data generating process moves further away from a constant volatility model. We too test our methodology using the more realistic Heston (1993) model and find similar results. Gray, Edwards, and Kalotay (2007) price index options and find that the canonical estimator that incorporates a small amount of cross-sectional information outperforms the Black-Scholes model and generates more effective hedging ratios. Alcock and Carmichael (2008) demonstrate the flexibility of the approach by showing how the canonical estimator can be used to price American options. Like Gray and Newman, the authors find that the performance of the canonical estimator improves dramatically when pricing options in a stochastic volatility world.

The goal of this paper is to test alternatives to the canonical valuation of Stutzer (1996) by generalizing the problem of finding a minimum divergence between the actual and risk-neutral distribution. This generalization is possible because the cross-entropy between two distributions is a special case of the Cressie-Read divergence family. We examine how well other members of the Cressie-Read family (e.g., Euclidean divergence and empirical likelihood divergence) price a European call option in a simulated Black-Scholes environment and stochastic volatility environment. We find that in certain situations, the alternative measure changes outperform the canonical estimator. More specifically, in the Black-Scholes environment we find that the number of outliers observed plays a crucial role in determining the accuracy of the nonparametric method. For reasons described below, the empirical likelihood estimator does a better job of handling outliers and thus outperforms the canonical estimator. This result is robust to different maturities and across several different types of moneyness. This result is also robust to different types of pricing errors (mean-percentage and mean-absolute pricing errors).

In the stochastic volatility simulation, we show that the optimal measure change depends critically on the time to expiration; this result is robust to different levels of moneyness but is not robust across the different types of pricing errors. Mean pricing errors are dramatically reduced by generalizing the option pricing method. Absolute pricing errors, however, are only significantly different for specific maturity structures and moneyness. Moreover, we simulate from the stochastic volatility model using the method of Broadie and Kaya (2006), which drastically reduces the discretization bias hence making it possible to distinguish between discretization error and pricing error.

This paper also contributes to the applied econometrics literature by examining the finite sample properties of various nonparametric estimation methods. Nonparametric methods, such as empirical likelihood, have become increasingly popular among economists and statisticians [Kitamura (2005)] but relatively little is known about the finite sample properties of these alternative estimators.[1] The pricing of options is an ideal environment to study these finite sample properties because changes of measure (from actual to risk-neutral probabilities) are fundamental to pricing derivative securities and therefore, the properties of these measure changes are well known. While we are not performing "full-blown" estimation (i.e., optimizing over a parameter set) as is typically done in econometric applications, we are able to accurately assess how alternative measure changes behave in small samples.

[1]What is known, however, is that the Cressie-Read divergence family becomes degenerate [Cressie and Read (1984)] in the limit; i.e., the limiting values of cross-entropy, Euclidean and empirical likelihood divergences are identical.

2 Nonparametric Pricing of Options

Consider pricing a European call option with expiration date T and strike price X. In the absence of arbitrage, the price of the European call option C discounted at the risk-free rate of interest r is given by

$$C = \mathbb{E}_t^Q\left\{\frac{\max[P_T - X, 0]}{(1+r)^T}\right\}, \tag{2.1}$$

where P_T is the price of the underlying asset at date T, and $\mathbb{E}_t^Q$ implies that the expectation is taken with respect to the risk-neutral (equivalent-martingale) measure. Suppose one had on hand a time series of underlying stock prices, denoted by p_t, of length $t = 1, ..., T-h$, where h is the number of days to expiration. In lieu of imposing a specific functional form on the price process of the underlying asset, Stutzer (1996) advocates forming the empirical distribution of time T asset returns (assuming the stock does not pay a dividend) by forming

$$R_t = \left(\frac{p_{t+h}}{p_t}\right) \qquad t = 1, ..., N \tag{2.2}$$

and weighting each draw equally, $\pi_t = N^{-1}$ for all t, where $N = T - 2h$.[2]

Of course, the weights associated with the empirical distribution ($\pi_t = N^{-1}$) are *not* the risk-neutral weights needed to price the option (2.1). Thus, we seek a transformation from the "real-world" probabilities π_t to their risk-neutral counterpart π_t^Q, where the risk-neutral weights satisfy

$$\sum_{t=1}^{N} \pi_t^Q = 1, \tag{2.3}$$

$$1 = \sum_{t=1}^{N} \pi_t^Q \left(\frac{R_t}{(1+r)^T}\right). \tag{2.4}$$

Equation (2.3) requires the risk-neutral weights sum to one, and more importantly, (2.4) forces the risk-neutral weights to satisfy the martingale property. Risk-neutral weights that satisfy the above restrictions can then be used to price the option according to

$$C = \mathbb{E}_t^Q\left\{\frac{\max[P_T - X, 0]}{(1+r)^T}\right\} = \sum_{t=1}^{N} \pi_t^Q \left\{\frac{\max[P_T - X, 0]}{(1+r)^T}\right\} \tag{2.5}$$

Therefore, we seek a change of measure from $\boldsymbol{\pi} = [\pi_1, ..., \pi_N]'$ to $\boldsymbol{\pi}^Q = [\pi_1^Q, ..., \pi_N^Q]'$ that satisfies (2.3) and (2.4) but does not diverge too far (as per some measure-change metric) from the underlying asset's empirical distribution.

[2] It is interesting to note that this empirical distribution maximizes the entropy of the available information. Alternatively, one could specify a functional form for the distribution of returns and employ Bayesian forecasting techniques to produce a posterior distribution for returns at date T (see, Robertson, Tallman, and Whiteman (2005)).

2.1 Canonical Valuation

Stutzer (1996) controls the divergence between the two measures by minimizing their cross-entropy or Kullback-Leibler divergence. Define the cross-entropy as

$$CE(\pi_t^Q, \pi_t) = \sum_{t=1}^{N} \pi_t^Q \log\left(\frac{\pi_t^Q}{\pi_t}\right). \tag{2.6}$$

Minimizing (2.6) subject to (2.3) and (2.4) is a well-defined convex minimization problem. Stutzer refers to this nonparmetric option pricing technique as canonical valuation (CAN) because the solution takes the form of the Gibbs canonical distribution

$$\pi_{t,CAN}^Q = \frac{\exp\left(\gamma^Q R_t (1+r)^{-T}\right)}{\sum_{t=1}^{N} \exp\left(\gamma^Q R_t (1+r)^{-T}\right)}$$

where γ^Q is the Lagrange multiplier satisfying, $\gamma^Q = \arg\min_\gamma \sum_{t=1}^{N} \exp\left(\gamma \frac{R_t}{(1+r)^T} - 1\right)$.

As mentioned in the introduction, this approach can also easily incorporate cross-sectional option pricing data, if it is available. For example, incorporating an option that matures at the same date but has a different strike X_2 and option price C_2 can be achieved by simply adding the constraint

$$C_2 = \sum_{t=1}^{N} \pi_t^Q \left\{ \frac{\max[P_T - X_2, 0]}{(1+r)^T} \right\} \tag{2.7}$$

to (2.4) and (2.3), and minimizing (2.6).

Given the nice theoretical structure of the problem, it is computationally inexpensive to price options employing this method; simple Excel computations are sufficient. Moreover, the CAN estimator has been shown to accurately (vis-a-vis alternative option pricing methods) price options in realistic settings [Gray and Newman (2005)].

2.2 Alternative Measure Changes

The CAN estimator is one method for deriving the risk-neutral distribution from an estimate of the actual distribution. However, the problem of finding an equivalent martingale measure may be generalized by defining a convex function Ψ that measures the divergence between two probability measures P (actual probabilities) and Q (risk-neutral probabilities):

$$D(Q, P) = \int \Psi\left(\frac{dQ}{dP}\right) dP. \tag{2.8}$$

Given an appropriate choice of Ψ, we seek minimization of (2.8) subject to the constraints (2.3) and (2.4).[3]

Specifically, we examine the Cressie-Read (CR) divergence family as a choice for the convex function $\Psi(x)$ [Cressie and Read (1984), Baggerly (1998)]. The CR divergence between the actual and risk-neutral probability measure is defined by

$$CR_\lambda(\pi_t^Q, \pi_t) = \frac{2}{\lambda(1+\lambda)} \sum_{t=1}^{N} \pi_t \left[\left(\frac{\pi_t^Q}{\pi_t} \right)^{-\lambda} - 1 \right],$$

for a fixed scalar parameter λ.

The choice of the CR divergence stems from the fact that it generalizes several well-known divergence measures, including the cross-entropy measure. For example, $\lambda = -2$ yields the Euclidean divergence, $\lambda = 1$ gives Pearson's Chi-Square, and $\lambda = -1/2$ is the squared Hellinger divergence. Two limiting distributions which are also encountered frequently are empirical likelihood ($\lambda \to 0$) and the cross-entropy measure ($\lambda \to -1$) (see, Bera and Bilias (2002) for a nice review).

Our motivation behind examining alternative measure changes can be clearly seen by factoring the CR objective function according to Basu and Lindsay (1994);[4]

$$\begin{aligned} CR_\lambda(\pi_t^Q, \pi_t) &= \frac{2}{\lambda(\lambda+1)} \sum_{t=1}^{N} \left[\pi_t \left\{ \left(\frac{\pi_t}{\pi_t^Q} \right)^{\lambda} - 1 \right\} + \lambda(\pi_t^Q - \pi_t) \right] \\ &= 2 \sum_{t=1}^{N} \mathcal{D}(\delta_t, \lambda) \pi_t^Q \end{aligned} \tag{2.9}$$

where

$$\mathcal{D}(\delta, \lambda) = \frac{(\delta+1)^{\lambda+1} - (\delta+1)}{\lambda(\lambda+1)} - \frac{\delta}{\lambda+1}, \quad \delta_t = \left(\frac{\pi_t}{\pi_t^Q} - 1 \right).$$

Thus, the CR divergence may be interpreted as a weighted function ($\mathcal{D}$) of disparity measures (δ) between the actual and risk-neutral probability measures. The function $\mathcal{D}(\cdot)$ is non-negative, defined on $[-1, \infty)$ and equals zero if and only if the disparity between the two measures is also zero. Figure 1 plots this disparity measure for $\lambda = [-2, -0.5, 0, 2, 0.5]$.[5]

Note that for positive (negative) values of δ, positive (negative) λ lead to higher values for $\mathcal{D}$. Thus CR divergence measures with positive (negative) λ restrict the

[3] This type of estimation is often referred to as Generalized Minimum Contrast (GMC). The obvious benefit of GMC estimation is the lack of distributional assumption. Moreover, the GMC estimator is shown to possess properties similar to that of parametric likelihood estimators [Kitamura and Stutzer (1997), Kitamura (2005)].

[4] Note that the term $\lambda(\pi_t^Q - \pi_t)$ does not contribute to the disparity.

[5] The measure becomes $(1+\delta)\ln(\delta+1) - \delta$ when $\lambda \to 0$.

degree to which the actual (risk-neutral) probability can exceed the risk-neutral (actual) probability. For the option pricing problem at hand, this implies that if the empirical distribution has fatter tails than the actual distribution, then CR measures with negative lambda will, on average, be more accurate in pricing the option. Conversely, if the empirical distribution has thinner tails than the actual distribution, then CR measures with positive lambda will be more accurate in pricing the option. This is precisely how finite samples lead to different performance metrics for different CR measures.

An alternative, and perhaps more powerful, motivating factor for our approach is to consider the risk appetite of the investor. Haley, McGee, and Walker (2009) show that the alternative tilts of the CR metric correspond to different HARA utility functions. The implication for option pricing is that if the investor is concerned with minimizing tail risk, then selecting a CR measure with a positive lambda will mitigate pricing errors. Conversely, if the investor believes the market will be less volatile over the life of the option, then a negative lambda will more accurately price the option.

Within the context of option pricing, this interpretation speaks specifically to the persistent negative bias of the CAN estimator documented by Gray and Newman (2005). Our interpretation of this negative bias is that the CAN estimator ($\lambda \to -1$) is not symmetric about zero and therefore it weights outliers and inliers non-uniformly (down-weighting outliers disproportionately). In order for the CAN estimator to accurately price the option, the empirical distribution must have fatter tails (significant number of outliers) relative to the the actual distribution. The simulation results reported below average over thousands of repetitions, and document the persistent negative bias in the CAN estimator. Conversely, note that $\lambda \to 0$ is more symmetric about zero than CR divergence measures with negative λ, implying a more equal weight is given to both inliers and outliers. This suggests that as the number of draws from the empirical distribution increases (and assuming one is able to sample from the entire support of the distribution), the empirical likelihood ($\lambda \to 0$) divergence measure should lead to the smaller mean pricing errors due to its more symmetric divergence shape. The upshot is that if the empirical distribution is not representative of the actual distribution needed to *forecast* future values of the underlying asset, then depending upon the bias, alternative CR divergence measures will outperform the CAN estimator.

This paper will focus on three members of the CR family–the canonical estimator ($\lambda \to -1$), the empirical likelihood estimator ($\lambda \to 0$), and the Euclidean divergence ($\lambda = -2$). There are two motivating factors for the selection of these three estimators. First, these estimators have recently been advocated in various econometric settings and the properties of these estimators are becoming well known.[6] Second, the compu-

[6]This literature has become too voluminous to accurately cite all the works that should be given credit. Imbens, Spady, and Johnson (1998), Hansen, Heaton, and Yaron (1996), Kitamura and Stutzer (1997), andNewey and Smith (2004) are among the important recent contributions. Both Maasoumi (1993) and Bera and Bilias (2002) provide excellent reviews.

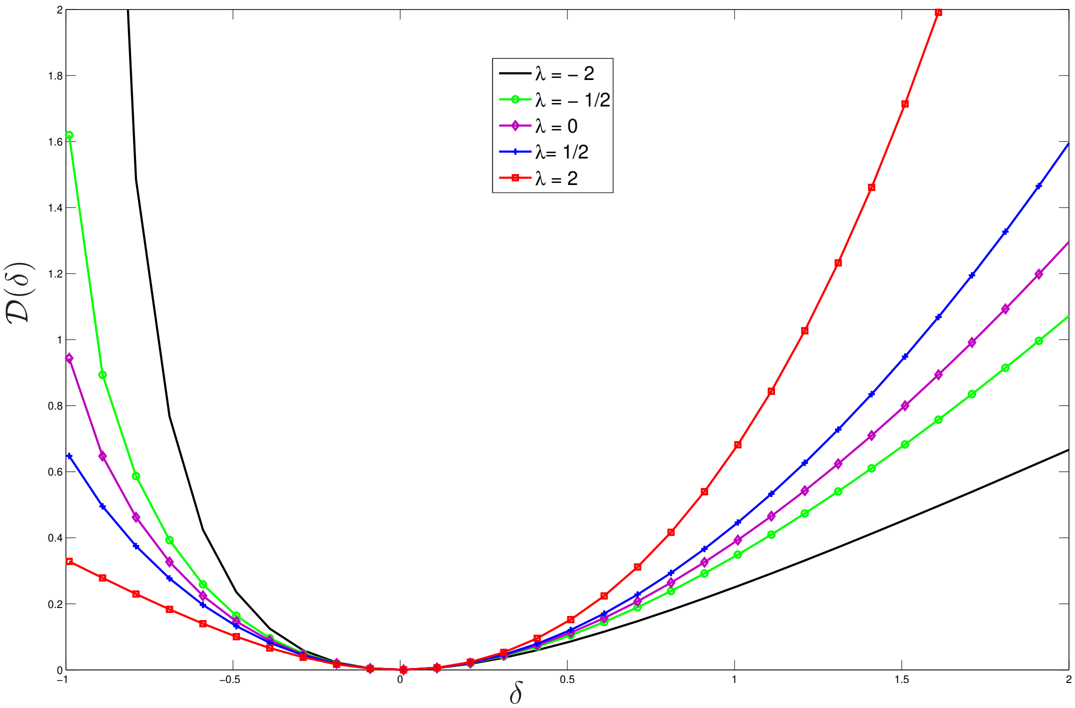


Figure 1: Disparity measures for various λ

tational cost to calculate these three estimators is minimal. In the financial industry, where millions of repetitions like those performed below are executed daily, it is important to keep relative computational cost low. Stutzer (1996) demonstrates the ease of computation associated with the the canonical estimator. The total computational time in calculating the Euclidean divergence and empirical likelihood estimators is even *less* than the canonical estimator.

The formal derivation for these estimators can be found by forming the Lagrangian that minimizes the CR divergence and solves (2.1) subject to (2.3), (2.4) and (2.7). This is given by

$$\mathcal{L}(\boldsymbol{\pi}^Q, k_1, \boldsymbol{k_2}) = \frac{2}{\lambda N(\lambda+1)} \sum_{t=1}^{N} \{(N\pi_t^Q)^{-\lambda} - 1\} + k_1(1 - \sum_{t=1}^{N} \pi_t^Q) + \boldsymbol{k_2}'\left(\boldsymbol{\eta} - \sum_{t=1}^{N} \pi_t^Q \mathbf{f}_t\right)$$

where $\mathbf{f}_t = [R_t/(1+r)^T, \max\{P_J - X, 0\}/(1+r)^T]'$ and $\boldsymbol{\eta} = [1, C_2]$. Setting $\partial\mathcal{L}/\partial\pi_t^Q$ to zero yields extremum of the form

$$\pi_t^Q = \begin{cases} \frac{1}{N}\{1 + c_1 + \mathbf{c_2}'(\mathbf{f}_t - \boldsymbol{\eta})\}^{-1/(\lambda+1)} & \lambda \neq -1 \\ c_1 \exp\{\mathbf{c_2}'(\mathbf{f}_t - \boldsymbol{\eta})\} & \lambda = -1 \end{cases}$$

where c_1 and $\mathbf{c_2}$ are functions of the constraints. This equation makes clear how the CR divergence measure nests the canonical estimator. Using the fact that $\lim_{h\to 0}(t^h -$

$1)/h = \log(t)$, one can also show

$$\lim_{\lambda\to 0} CR_\lambda(\pi_t^Q, N^{-1}) = -2\log(\pi_t^Q N), \quad \lim_{\lambda\to -1} CR_\lambda(\pi_t^Q, N^{-1}) = 2\pi_t^Q \log(N\pi_t^Q)$$

which is minus twice the empirical log likelihood, and twice the cross-entropy. This shows the close relationship between empirical likelihood and cross-entropy and also makes clear why the empirical likelihood treats outliers uniformly while the canonical estimator does not.

The following proposition derives the risk-neutral weights associated with the empirical likelihood and Euclidean estimators.

Proposition 2.1. *The equivalent-martingale measures for the empirical likelihood and Euclidean estimators that price* (2.1) *subject to* (2.3)*,* (2.4) *and* (2.7) *are given by*

$$\pi_t^{Q,EL} = \frac{1}{N}\frac{1}{1+\boldsymbol{k}_2'(\mathbf{f}_t - \boldsymbol{\eta})} \tag{2.10}$$

$$\pi_t^{Q,EU} = \frac{1}{N}[1+\boldsymbol{k}_2'(\mathbf{f}_t - \bar{\mathbf{f}}_t)] \tag{2.11}$$

Proof. Employing Lagrange multipliers, the constrained optimization problem for minimizing the empirical likelihood is

$$\mathcal{L}_{EL} = -\frac{1}{N}\sum_{t=1}^{N}\log(\pi_t^Q N) + k_1(1-\sum_{t=1}^{N}\pi_t^Q) + \boldsymbol{k_2}'\left(\boldsymbol{\eta} - \sum_{t=1}^{N}\pi_t^Q\mathbf{f}_t\right)$$

The first-order conditions are

$$\mathcal{L}_{\pi_t^Q}: \quad -\frac{1}{\pi_t^Q N} + k_1 + \boldsymbol{k}_2'\boldsymbol{f}_t = 0 \;\rightarrow\; \pi_t^Q k_1 + \pi_t^Q \boldsymbol{k}_2'\boldsymbol{f}_t = \frac{1}{N} \tag{2.12}$$

$$\mathcal{L}_{\boldsymbol{k}_2}: \quad \boldsymbol{\eta} - \sum_{t=1}^{N}\pi_t^Q\mathbf{f}_t = 0$$

$$\mathcal{L}_{k_1}: \quad 1 - \sum_{t=1}^{N}\pi_t^Q = 0$$

Summing (2.12) over the N periods gives, $k_1\sum_{t=1}^{N}\pi_t^Q + \boldsymbol{k}_2'\sum_{t=1}^{T}\pi_t^Q\boldsymbol{f}_t = 1$, and the other first-order conditions imply $k_1 = 2 - \boldsymbol{k}_2'\eta$. Concentrating out k_1 from (2.12) gives

$$\pi_t^Q = \frac{1}{N}\frac{1}{1+\boldsymbol{k}_2'(\mathbf{f}_t - \boldsymbol{\eta})}$$

Checking second-order conditions is not necessary since the objective function is strictly concave on a convex set of weights; hence a unique global minimum exists.

Setting $\lambda = -2$ corresponds to the Euclidean divergence, also known as Neyman's χ^2. Form the Lagrangian according to

$$\mathcal{L}_{EU} = \frac{1}{2N}\sum_{t=1}^{N}(\pi_t^Q N - 1)^2 + k_1(1 - \sum_{t=1}^{N}\pi_t^Q) + \boldsymbol{k_2}'\left(\boldsymbol{\eta} - \sum_{t=1}^{N}\pi_t^Q \mathbf{f}_t\right)$$

$$\mathcal{L}_{\pi_t^Q}: \quad N\pi_t^Q - 1 + k_1 - \boldsymbol{k}_2'\mathbf{f}_t = 0 \rightarrow \quad k_1 = 1 + \boldsymbol{k}_2'\mathbf{f}_t - N\pi_t^Q$$

Averaging over t gives, $k_1 = \boldsymbol{k}_2'\bar{\mathbf{f}}_t$, and therefore

$$\pi_t^Q = \frac{1}{N}[1 + \boldsymbol{k}_2'(\mathbf{f}_t - \bar{\mathbf{f}}_t)]$$

□

The numerical speed and precision of these estimators comes from the nice functional forms of the Lagrangian multipliers. For example, Owen (2001) shows the multiplier for the Euclidean estimator simplifies to $\boldsymbol{k}_2 = S^{-1}(\bar{\mathbf{f}}_t - \boldsymbol{\eta})$ where $S = N^{-1}\sum_{t=1}^{N}(\mathbf{f}_t - \bar{\mathbf{f}}_t)(\mathbf{f}_t - \bar{\mathbf{f}}_t)'$.

3 Assessing Alternative Tilts

This section assesses the fit of alternative measures of divergence numerically by assuming the stock process follows specific functional forms that allow for analytical solutions to the option pricing equation (2.1). With analytical solutions in hand, one may examine the pricing errors over a wide range of maturity and moneyness. Pricing errors are calculated following Stutzer (1996) in a Black-Scholes environment. We also follow Gray and Newman (2005) and examine the pricing errors in the more realist stochastic volatility model of Heston (1993).

3.1 Alternative Tilts in a Black-Scholes Environment

Both Stutzer and Gray and Newman compare the CAN estimator with implied-volatility estimators by assuming the underlying stock price follows geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

which gives stock returns as

$$\ln(R_t) \sim N[(\mu - \sigma^2/2)T,\ \sigma^2 T]. \tag{3.1}$$

We assign the same parameter values as Stutzer and Gray and Newman; the drift μ and annual volatility σ are assumed to be 10% and 20% respectively, with a constant riskless rate of interest 5%. For each time to expiration T, 200 returns are drawn from (3.1) and the empirical "predictive" distribution for P_T is formed according to (2.2).

Also, as a basis for comparison, a Black Scholes implied volatility (HBS) estimate is calculated using the sample volatility in the usual manner. Mean percentage (MPE) and mean absolute percentage (MAPE) pricing errors are then calculated based upon 5,000 repetitions of the experiment.

Table 1 reports the MPE results when only one moment restriction (2.4) is employed in the constrained optimization. The table reports results for various time-to-maturities–(assuming 252 trading days) from 6 trading days up to 1 year–and various levels of moneyness–deep out of the money ($S/B = 0.9$) to deep in the money ($S/B = 1.125$). The table reports the results listed in order from top to bottom–Black-Scholes, canonical, Euclidean, and empirical likelihood.

Notice that while the HBS estimator consistently prices options across different time to maturities, the nonparametric methods perform worse as the time to expiration increases. This is because the HBS estimator needs only to accurately estimate the second moment of the risk-neutral distribution because the parametric assumption is correct (that is, the Black-Scholes pricing formula is the correct one here). Conversely, the nonparametric approach re-weights the entire distribution. As the distribution becomes more dispersed, the re-weighting becomes less precise in the nonparametric case, but does not affect the precision of the HBS estimator. The tradeoff is that the HBS estimator makes a dogmatic assumption about the data generating process. If that assumption is correct (as it is here), one only has to match second moments to accurately price options but, as is well documented, the assumption of lognormally distributed returns is empirically implausible.

Second, in almost every case, the EL estimator soundly outperforms the CAN and EU estimators. In many cases, the difference is quite large from an option pricing standpoint and statistically significant. For example, in almost every scenario, the difference in MPE between the EL and the CAN is a factor of 2, and the extreme cases, a factor of 5. In *all cases*, paired t-tests reveal the difference between the CAN and EL estimators to be statistically significant at the 99% level. Conversely, the EU estimator performed much worse than the CAN estimator in every scenario with MPEs almost 10 times that of the EL estimators (this difference is also significant at the 99% level). An explanation for this result can be found by examining the number of outliers of the empirical distribution.

As mentioned above, the motivation behind examining alternative measure changes is that they weight draws differently. Figure 2 plots the number of realizations outside of the 2.5th–97.5th percentiles against the average MPE (5,000 simulations) for the EL, EU and CAN estimators for $T = 1/4$, across all levels of moneyness. Given the empirical distribution has 200 draws, one would expect 10 outliers per sample. Figure 2 shows that if the realization has 11 or fewer outliers, the EL estimator will, on average, outperform the EU and CAN estimators. The intuition for this result is easily seen by returning to Figure 1. If the empirical distribution has thinner (thicker)

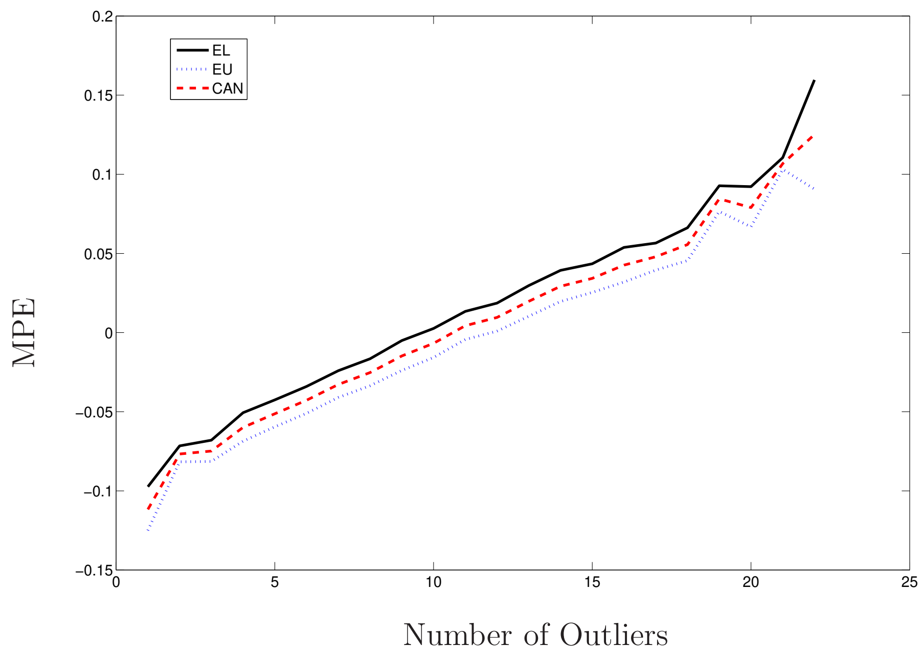


Figure 2: Number of Outliers and MPE in Black-Scholes Model Figure 2 plots the average MPE (5,000 simulations) against the number of realizations in 200 draws that were outside of the 2.5th–97.5th percentile range for the given distribution. The average MPE is reported for all moneyness levels and for time to expiration equal to 1/4.

tails than the actual distribution being sampled, then measure changes with positive (negative) lambda will favor larger (smaller) risk-neutral weights relative to negative (positive) lambda measure changes. In other words, alternative measure changes will correct the bias in small samples. This result is important because when applying this nonparametric technique to real-world data, one does not have the luxury of repeated draws from the known distribution. The performance of the nonparametric estimators depends upon the small sample properties of the empirical distribution. As the empirical distribution diverges from the actual distribution, re-weighting the draws may be an important money saving measure. And as table 1 makes clear, these differences can be quite large. This important point can easily be overlooked when conducting Monte Carlo type numerical simulation.

Third, the EL estimator does not suffer from the persistent negative bias associated with the CAN estimator [documented by Gray and Newman (2005)]. This result is due to the more symmetric nature of the EL estimator relative to the CAN and EU estimators. Returning to Figure 1, the EL estimator is more symmetric about zero than the EL or EU estimators, suggesting that it more uniformly distributes weight from the actual to the risk-neutral distribution. Hence, negative pricing errors will be offset by positive pricing errors. Conversely, the CAN and EU estimators penalize negative delta ($\pi_t < \pi_t^Q$) more heavily, which leads to too low of a price for the option and hence negative pricing errors.

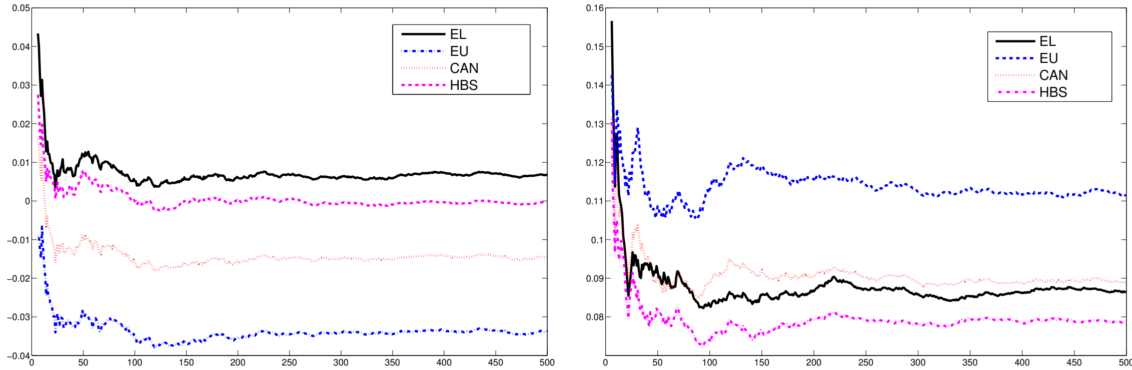
However, the symmetric properties possessed by the EL estimator may not lead to improved performance in pricing options. Table 2 provides the mean absolute pricing errors, which weight negative and positive errors equally. While the superior performance of the EL estimator is again evident, the degree of improvement over the EU and CAN estimators is somewhat tempered when examining the absolute value of the errors–EL outperforms CAN 21 out of the 30 scenarios, a majority of which are statistically significant at the 99% level.

Fourth, Stutzer (1996) argues that in order for the nonparametric method to fairly compared to the HBS estimator an additional moment restriction is necessary. More specifically, imposing that the at-the-money option is correctly priced forces the additional moment restriction

$$C_2 = \sum_{t=1}^{N} \pi_t^Q \left\{ \frac{\max[P_T - X_2, 0]}{(1+r)^T} \right\}$$

on the optimization problem. This additional moment restriction is easily incorporated into the problem and doing so alleviates the bias seen in the nonparametric estimators associated with time to expiration. Table 1 documented the increase in MPE for the nonparametric estimators as time to expiration increased. Tables 3 and 4 show that adding the additional constraint effectively eliminates this bias by placing much more structure on the problem. That is, the previous constraint simply imposed the martingale restriction on returns but did not take into account the specific functional form of the asset pricing equation, whereas the HBS estimator always takes into account the particular asset pricing equation. Adding the additional constraint provides a specific functional form that sufficiently restricts the feasible set of measures such that the corresponding risk-neutral measure is sufficiently close to the actual risk-neutral measure [Stutzer (1996)]. Tables 3 and 4 document that adding the additional constraint dramatically improves the performance of the nonparametric methods. In several instances, the nonparametric methods outperformed the HBS estimator with the exception of deep in-the-money and deep out-of-the-money calls.

Finally, Figure 3 documents the stability and robustness of the results as the number of simulations is increased. Since we are interested in the small sample properties of the estimators, it is important to document how the results change as the number of simulations are increased. Notice that there is an important distinction between repetitions of the experiment and number of draws from the empirical distribution. The convergence of the CR divergence measures occur when the empirical distribution is discrete and by taking the limit as the number of draws increases. For example, if we assume a binomial tree model where the stock price can either move up or down with certain probabilities, then repeating this experiment several thousand times will lead to pricing errors across all CR divergence measures that are statistically indistinguishable. The exercise here is to check the robustness of the results as the number



3 (a): MPE, T=3/4, S/B=1 3 (b): MAPE, T=1/2, S/B=1.125

Figure 3: Stability of Results. This figure plots the MPE and MAPE for different time-to-expiration and levels of moneyness against the number of simulations.

of simulations increases to 500.

Figure 3(a) plots the MPE for time-to-expiration of 3/4 and moneyness of 1, while 3(b) plots the MAPE for time-to-expiration 1/2 and moneyness 1.125 as the number of simulations are increased to 500.[7] As the figure indicates, the ordering for the estimators occurs quite quickly. After 20 simulations for the MPE and 90 simulations for MAPE, the orderings are established for all estimators. Given that a practitioner would perform these operations over several assets and over a long period of time, the excess profitability of the empirical likelihood estimator vis-a-vis the canonical estimator would be nontrivial. The superior performance of the Black-Scholes estimator and inferior performance of the Euclidean estimator occurs much sooner (roughly 10 simulations). Moreover, this graph is another illustration of the negative bias associated with the canonical estimator (and to a greater extent, the Euclidean estimator), as opposed to the much smaller positive bias in the EL estimator.

3.2 Alternative Tilts in a Stochastic Volatility Environment

Gray and Newman (2005) persuasively argued that a more realistic test of the canonical estimator would be to simulate data using Heston's stochastic volatility model, where the stock price follows

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dz_{1,t}$$

and the variance of the return follows a Ornstein-Uhlenbeck process

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dz_{2,t}$$

[7]Stability checks were undertaken for all levels of moneyness and time to expiration; Figure 3 is representative of the degree of stability.

where κ is the speed of mean reversion, θ the long-run variance, ξ is the volatility of the volatility generating process, and $dz_{1,t}$, and $dz_{2,t}$ are Wiener processes with correlation ρ. The appeal of this setup is that the model retains a closed form solution while providing more realistic simulated data.

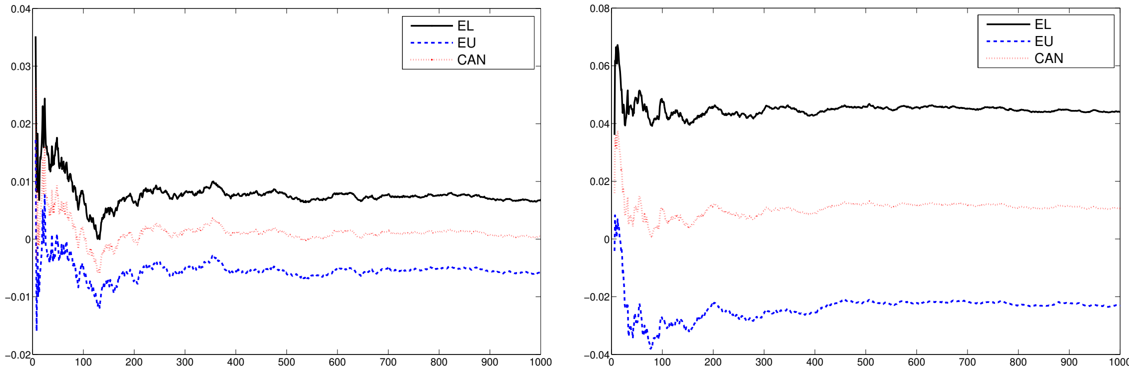
To reduce discretization bias in generating the data, we employ the method of Broadie and Kaya (2006).[8] The conventional way to generate data from the stochastic volatility model is to use Euler discretization, but Broadie and Kaya show that this method may introduce substantial bias into the simulated results. They then show how to simulate data from the *exact* distribution, effectively reducing discretization bias. The data are generated using 1 day time steps and with parameter values given by: stock drift, μ, 10%; long-run mean, θ, 4%; mean reversion, κ, 3; volatility, ξ, 0.4; and correlation, ρ, -0.5. The parameter values follow Gray and Newman, and represent typical estimates from market data.

Tables 5 and 6 give the MPE and MAPE, respectively, for the HBS, CAN, EL, and EU estimators across several maturities and moneyness, assuming 200 draws from the empirical distribution and 5,000 simulations. The obvious disadvantage of the HBS estimator relative to the nonparametric approach is the assumption of an explicit functional form for returns. If the parametric assumption does not fit the data well (as is the case here), the HBS estimator will consistently misprice the option. As indicated in Tables 5 and 6, the HBS estimator overprices out-of-the money options and underprices in-the-money options–a well-known empirical finding. This is primarily due to the fact that the Gaussian distribution, assumed by HBS, has tails that are too thin to adequately capture the dynamics of the SV model. Hence, the HBS estimator is only competitive with the nonparametric estimators if the option has a very short maturity and is at-the-money options.

When comparing across the different measure changes in Table 5, an interesting result emerges. Namely, the optimal measure change is a function of the time to expiration. The EL estimator outperforms the other estimators at short time horizons (1/42, 1/12), the CAN estimator outperforms at medium interval (1/4) and the EU estimator outperforms when the time to expiration is greater than 1/2. This result is robust to all levels of moneyness. The intuition for this result is the following; as time to expiration increases, the probability of outliers influencing the returns becomes more likely. The difference here is that the probability of outliers in the SV model is much greater than the Black-Scholes model. Hence the measure change that handles outliers (inliers) the best, EU (EL), will outperform at longer (shorter) horizons. This result is robust to alternative formulations for the stochastic volatility model (not reported) and the differences in pricing errors are statistically significant at the 99% level.[9] This result suggests that an optimal portfolio would consist of a

[8]We thank Mark Broadie and Ozgur Kaya for permission to use their simulator.

[9]The paired t-tests reported in Tables 1-8 were performed on all pairs of nonparametric estimators. The significance results reported in the tables are of the two nonparametric estimators that had the smallest pricing errors.



4(a): MPE, T=1/4, S/B=0.97, $\xi = 0.1$, $\kappa = 2$. 4(b): MPE, T=1/4, S/B=0.97, $\xi = 0.5$, $\kappa = 10$

Figure 4: Stability of Results Figure 4 plots the MPE for different parameterizations of the SV model with time-to-expiration of 1/4 and levels of moneyness against the number of simulations.

combination of measure changes contingent upon time to maturity.

However, Table 6 shows that the absolute errors across the alternative measure changes are roughly constant. Unlike the Black-Scholes case, choosing the optimal divergence measure to minimize the *mean* percentage error does *not* necessarily minimize the mean absolution percentage errors. Along this dimension, the canonical estimator performed quite well at longer horizons for out-of-the-money and at-the-money options. At shorter maturities, the Euclidean and empirical likelihood estimator outperformed. This pattern again suggests a combination strategy that is a function of maturity and moneyness. Many of these differences at the shorter and longer maturities are statistically significant, while at medium maturity there is little difference across the nonparametric estimators.

It should also be noted that adding the additional constraint, (2.7), to the optimization dramatically improves the performance of all the nonparametric estimators. Tables 7 and 8 provide the MPE and MAPE when the additional constraint is applied, and indicates, yet again, that the nonparametric method is a viable option pricing strategy. Note also that adding the additional constraint makes the nonparametric estimators nearly identical in performance in terms of absolute pricing errors. This is intuitive because the nonparametric estimators converge as the number of moment restrictions imposed increases. As Table 8 shows, there are still statistically significant differences across the estimators with respect to shorter maturities, and only for deep-in-the-money calls at longer maturities.

Figure 4 demonstrates the robustness of the results as the parameters of the SV model change. Figure 4(a) gives the MPE for time-to-expiration of 1/4 and moneyness 0.97 with mean reversion, κ, equal to 2 and volatility, ξ, of 0.1, while 4(b) graphs the MPE with the same moneyness and time to expiration with κ equal to 10, and ξ equal to 0.5. Clearly at the time horizon $T = 1/4$, the canonical estimator dominates the others. This figure is representative of the more general result–the appropriate measure change is a function of time to expiration regardless of moneyness and parameters of the SV model. Moreover, as documented by Figure 4, this result is exacerbated as volatility is increased. That is, the more dispersed the data, the more the optimal measure change will outperform the others. This convergence happens very quickly (less than 50 iterations).

4 Conclusion

In this paper we have examined a generalized version of Stutzer's (1996) canonical valuation option pricing estimator. We framed our analysis around the Cressie-Read family of divergence measures, which captures Stutzer's cross-entropy as a special case. Simulations in both Black-Scholes and stochastic volatility environments suggest that the canonical estimator can be significantly improved upon in finite sample scenarios. Of the Cressie-Read divergences we considered, the empirical likelihood divergence demonstrated itself to be an extremely viable alternative to Stutzer's cross entropy. We trace this advantage back to how each divergence weighs values; we find that the symmetry of the empirical likelihood measure appears to drive its desirable performance. This feature also sheds additional light on the negative bias associated with applications of the canonical estimator as in Gray and Newman (2005). In the stochastic volatility environment, the optimal choice of measure change depended upon the time to expiration. These results suggest that an optimal portfolio approach would advocate for inclusion of *all* measures of the Cressie-Read divergence family when constructing a nonparametric option portfolio.

We believe this paper extends Stutzer's work in an interesting and new direction and compliments the results in Gray, Edwards, and Kalotay (2007) and Alcock and Carmichael (2008), who show that the nonparametric approach is both flexible and performs well when taken to real data. These papers provide obvious extensions to the current paper: How well do the alternative measure changes perform when taken to actual data? Could the results in Alcock and Carmichael be improved upon if alternative tilts are examine? We believe the results derived here suggest that the alternative measures would lead to substantial improvements. However, we leave these questions to future research as they are beyond the scope of the current paper.

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Table 1: MPE in Black-Scholes World with 1 Moment Restriction*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	n/a	0.01576	0.00047	-0.00109	-0.00116	-0.00117
(0.90)	n/a	-0.01040	-0.02217	-0.02874	-0.03375	-0.03801
	n/a	-0.04531	-0.05700	-0.07011	-0.08114	-0.09053
	n/a	0.01578***	0.00620***	0.00676***	0.00830***	0.00982***
Out-of-the-money	0.00060	-0.00112	-0.00111	-0.00098	-0.00089	-0.00082
(0.97)	-0.00643	-0.00767	-0.01093	-0.01568	-0.01997	-0.02381
	-0.01622	-0.01708	-0.02454	-0.03520	-0.04450	-0.05259
	0.00246***	0.00114***	0.00232***	0.00393***	0.00526***	0.00646***
At-the-money	-0.00088	-0.00084	-0.00078	-0.00071	-0.00067	-0.00062
(1.00)	-0.00281	-0.00410	-0.00747	-0.01195	-0.01589	-0.01940
	-0.00589	-0.00877	-0.01617	-0.02584	-0.03421	-0.04153
	0.00035***	0.00073***	0.00166***	0.00294***	0.00413***	0.00526***
In-the-money	0.00007	-0.00022	-0.00040	-0.00045	-0.00045	-0.00044
(1.03)	-0.00080	-0.00215	-0.00509	-0.00900	-0.01252	-0.01571
	-0.00145	-0.00424	-0.01050	-0.01878	-0.02611	-0.03265
	-0.00004***	0.00023***	0.00101***	0.00216***	0.00325***	0.00429***
Deep In-the-money	0.00000	0.00009	0.00015	0.00008	0.00002	-0.00002
(1.125)	-0.00000	-0.00011	-0.00120	-0.00334	-0.00560	-0.00781
	-0.00000	-0.00018	-0.00217	-0.00630	-0.01068	-0.01493
	-0.00000***	0.00000***	0.00016***	0.00074***	0.00142***	-0.00216***

*This table reports the mean pricing error for several estimators assuming a Black-Scholes environment and imposing the risk-neutral constraint. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 2: MAPE in Black-Scholes World with 1 Moment Restriction*

Moneyness	Time to Expiration (Years)					
S/B	1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	n/a	0.21493	0.10956	0.07716	0.06399	0.05629
(0.90)	n/a	0.33035***	0.12508	0.08575	0.07329	0.06729
	n/a	0.33042	0.13600	0.10514	0.10009	0.10155
	n/a	0.33594	0.12525	0.08318*	0.06858***	0.06057***
Out-of-the-money	0.11140	0.06841	0.05023	0.04241	0.03841	0.03571
(0.97)	0.13193	0.07456	0.05466	0.04751	0.04487	0.04368
	0.13311	0.07660	0.05890	0.05577	0.05758	0.06096
	0.13214	0.07427	0.05394***	0.04602***	0.04235***	0.03997***
At-the-money	0.03840	0.03682	0.03442	0.03217	0.03051	0.02915
(1.00)	0.04160	0.04011	0.03812	0.03676	0.03615	0.03610
	0.04211	0.04105	0.04059	0.04225	0.04522	0.04888
	0.04148	0.03994	0.03773***	0.03577***	0.03437***	0.03334***
In-the-money	0.00904	0.01791	0.02290	0.02409	0.02405	0.02369
(1.03)	0.01115***	0.02045	0.02603	0.02804	0.02899	0.02976
	0.01119	0.02073	0.02734	0.03164	0.03534	0.03908
	0.01120	0.02046	0.02590	0.02750***	0.02782***	0.02782***
Deep In-the-money	0.00000	0.00082	0.00510	0.00886	0.01080	0.01191
(1.125)	0.00000***	0.00171***	0.00729***	0.01178***	0.01434	0.01612
	0.00001	0.00168	0.00732	0.01242	0.01603	0.01920
	0.00001	0.00177	0.00749	0.01198	0.01438	0.01588*

*This table reports the mean absolute pricing error for several estimators assuming a Black-Scholes environment and imposing the risk-neutral constraint. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 3: MPE in Black-Scholes World with 2 Moment Restrictions*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	n/a	0.01576	0.00047	-0.00109	-0.00116	-0.00117
(0.9)	n/a	-0.00477***	-0.00443	-0.00353	-0.00297	-0.00263
	n/a	-0.02539	-0.01228	-0.00835	-0.00680	-0.00594
	n/a	0.01078	0.00214***	0.00114***	0.00120***	0.00136***
Out-of-the-money	0.00060	-0.00112	-0.00111	-0.00098	-0.00089	-0.00082
(0.97)	-0.00257	-0.00110	-0.00033	-0.00004	0.00001	-0.00002
	-0.00672	-0.00213	-0.00065	-0.00021	-0.00012	-0.00013
	0.00126***	-0.00011***	0.00005***	0.00026***	0.00031***	0.00031***
In-the-money	0.00007	-0.00022	-0.00040	-0.00045	-0.00045	-0.00044
(1.03)	-0.00059	-0.00061	-0.00051	-0.00041	-0.00033	-0.00028
	-0.00094	-0.00102	-0.00086	-0.00073	-0.00063	-0.00057
	-0.00018***	-0.00017***	-0.00016***	-0.00014***	-0.00013***	-0.00013***
Deep in-the-money	0.00000	0.00009***	0.00015***	0.00008***	0.00002***	-0.00002
(1.125)	0.00000	-0.00014	-0.00071	-0.00104	-0.00117	-0.00121
	0.00000	-0.00020	-0.00109	-0.00172	-0.00201	-0.00216
	0.00000***	-0.00006***	-0.00019***	-0.00021***	-0.00024***	-0.00027***

*This table reports the mean pricing error for several estimators assuming a Black-Scholes environment, and imposing the risk-neutral and exact pricing constraints. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 4: MAPE in Black-Scholes World with 2 Moment Restrictions*

Moneyness	Time to Expiration (Years)					
S/B	1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	n/a	0.21493	0.10956	0.07716	0.06399	0.05629
(0.9)	n/a	0.30452***	0.08720	0.04575	0.03228	0.02537
	n/a	0.30622	0.09050	0.04838	0.03453	0.02745
	n/a	0.30698	0.08636***	0.04507***	0.03180***	0.02493***
Out-of-the-money	0.11140	0.06841	0.05023	0.04241	0.03841	0.03571
(0.97)	0.08905	0.03160	0.01429	0.00891	0.00677	0.00555
	0.09011	0.03213	0.01460	0.00919	0.00704	0.00582
	0.08861***	0.03136***	0.01420***	0.00884***	0.00671***	0.00552
In-the-money	0.00904	0.01791	0.02290	0.02409	0.02405	0.02369
(1.03)	0.00740	0.00859	0.00677	0.00524	0.00439	0.00384
	0.00750	0.00872	0.00692	0.00542	0.00458	0.00404
	0.00735***	0.00853***	0.00671***	0.00458***	0.00435***	0.00379***
Deep in-the-money	0.00000	0.00082	0.00510	0.00142	0.01080	0.01191
(1.125)	0.00001	0.00162***	0.00554	0.01080	0.00751	0.00744
	0.00001	0.00161	0.00563	0.00751	0.00787	0.00789
	0.00001***	0.00166	0.00552	0.00787***	0.00737***	0.00727***

*This table reports the mean absolute pricing error for several estimators assuming a Black-Scholes environment, and imposing the risk-neutral constraint and exact pricing constraint. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 5: MPE in stochastic volatility World with 1 Moment Restriction*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money (0.9)	0.20721	0.11549	0.10484	0.10681	0.10182	0.09291
	-0.02446	-0.00475	0.00426***	0.01287	0.01732	0.01994
	-0.03986	-0.01465	-0.00643	-0.00294***	-0.00440***	-0.00797***
	-0.01052***	0.00415***	0.01399	0.02744	0.03760	0.04630
Out-of-the-money (0.97)	0.01598	0.01997	0.03163	0.04094	0.04306	0.04148
	-0.00779	-0.00255	0.00235***	0.00840	0.01182	0.01402
	-0.01261	-0.00701	-0.00365	-0.00133***	-0.00209***	-0.00427***
	-0.00306***	0.00183***	0.00830	0.01812	0.02589	0.03280
At-the-money (1.00)	-0.00100	0.00437	0.01520	0.02428	0.02741	0.02735
	-0.00376	-0.00197	0.00176***	0.00681	0.00991	0.01192
	-0.00637	-0.00504	-0.00293	-0.00113***	-0.00164***	-0.00344***
	-0.00110***	0.00118***	0.00656	0.01500	0.02199	0.02822
In-the-money (1.03)	-0.00522	-0.00356	0.00431	0.01222	0.01567	0.01652
	-0.00197	-0.00144	0.00124***	0.00549	0.00830	0.01007
	-0.00327	-0.00354	-0.00241	-0.00100***	-0.00133***	-0.00288***
	-0.00060***	0.00078***	0.00510	0.01240	0.01868	0.02425
Deep in-the-money (1.125)	-0.00132	-0.00666	-0.00917	-0.00683	-0.00455	-0.00306
	-0.00033	-0.00096	0.00014***	0.00253	0.00454	0.00567
	-0.00041	-0.00152	-0.00152	-0.00097***	-0.00099***	-0.00203***
	-0.00024***	-0.00030***	0.00206	0.00658	0.01104	0.01491

*This table reports the mean pricing error for several estimators assuming a Stochastic volatility environment and imposing the risk-neutral constraint. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 6: MAPE in stochastic volatility World with 1 Moment Restriction*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	0.29049	0.15405	0.12293	0.11839	0.11157	0.10251
(0.90)	0.34605	0.13175	0.08378***	0.06947***	0.06247***	0.05858***
	0.34470***	0.13335	0.08550	0.07050	0.06421	0.06039
	0.34947	0.13199	0.08428	0.07229	0.06803	0.06776
Out-of-the-money	0.07237	0.05794	0.05642	0.05870	0.05838	0.05612
(0.97)	0.07649	0.05758	0.04951	0.04607***	0.04347***	0.04234**
	0.07742	0.05838	0.05043	0.04664	0.04428	0.04290
	0.07615***	0.05745	0.04970	0.04796	0.04733	0.04884
At-the-money	0.03943	0.03934	0.04184	0.04457	0.04512	0.04416
(1.00)	0.04205	0.04114	0.04004	0.03891***	0.03761***	0.03712
	0.04251	0.04171	0.04072	0.03933	0.03817	0.03739
	0.04182***	0.04104	0.04015	0.04042	0.04088	0.04272
In-the-money	0.02069	0.02741	0.03208	0.03471	0.03560	0.03540
(1.03)	0.02262	0.02931	0.03252	0.03297***	0.03266*	0.03267
	0.02280	0.02966	0.03299	0.03329	0.03297	0.03275
	0.02257	0.02926	0.03258	0.03419	0.03539	0.03746
Deep in-the-money	0.00172	0.00950	0.01708	0.01963	0.02054	0.02095
(1.125)	0.00258	0.01001*	0.01717	0.02002*	0.02117	0.02214
	0.00256***	0.01004	0.01733	0.02013	0.02126	0.02210
	0.00261	0.01004	0.01723	0.02073	0.02282	0.02502

*This table reports the mean absolute pricing error for several estimators assuming a stochastic volatility environment and imposing the risk-neutral constraint. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 7: MPE in stochastic volatility World with 2 Moment Restrictions*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	0.20721	0.11549	0.10484	0.10681	0.10182	0.09291
(0.90)	-0.02111	-0.00274	0.00003***	0.00096***	0.00113***	0.00118***
	-0.03544	-0.00696	-0.00192	-0.00115	-0.00138	-0.00182
	-0.00751***	0.00116***	0.00183	0.00300	0.00364	0.00426
Out-of-the-money	0.01598	0.01997	0.03163	0.04094	0.04306	0.04148
(0.97)	-0.00208	-0.00001***	0.00014	0.00035	0.00031	0.00034
	-0.00299	-0.00023	0.00002***	0.00015***	0.00001***	-0.00006***
	-0.00120***	0.00022	0.00026	0.00058	0.00064	0.00079
In-the-money	-0.00522	-0.00356	0.00431	0.01222	0.01567	0.01652
(1.03)	-0.00039	-0.00024	-0.00023	-0.00027	-0.00022	-0.00029
	-0.00069	-0.00042	-0.00032	-0.00027	-0.00016***	-0.00015***
	-0.00008***	-0.00005***	-0.00014***	-0.00028	-0.00032	-0.00047
Deep in-the-money	-0.00132	-0.00666	-0.00917	-0.00683	-0.00455	-0.00306
(1.125)	-0.00034	-0.00093	-0.00079	-0.00089	-0.00076	-0.00098
	-0.00043	-0.00126	-0.00116	-0.00112	-0.00088	-0.00097**
	-0.00023***	-0.00053***	-0.00038***	-0.00063***	-0.00065***	-0.00105

*This table reports the mean pricing error for several estimators assuming a stochastic volatility environment and imposing the risk-neutral and exact pricing constraints. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.

Table 8: MAPE in stochastic volatility World with 2 Moment Restrictions*

Moneyness S/B	Time to Expiration (Years) 1/42	1/12	1/4	1/2	3/4	1
Deep out-of-the-money	0.29049	0.15405	0.12293	0.11839	0.11157	0.10251
(0.90)	0.32174	0.09107	0.04042	0.02728	0.02215	0.01866
	0.32166	0.09227	0.04099	0.02784	0.02284	0.01949
	0.32442	0.09069***	0.04022***	0.02725	0.02217	0.01869
Out-of-the-money	0.07237	0.05794	0.05642	0.05870	0.05838	0.05612
(0.97)	0.03229	0.01454	0.00795	0.00582	0.00488	0.00424
	0.03251	0.01465	0.00802	0.00588	0.00497	0.00435
	0.03221**	0.01450**	0.00792**	0.00583	0.00489	0.00426
In-the-money	0.02069	0.02741	0.03208	0.03471	0.03560	0.03540
(1.03)	0.00897	0.00698	0.00503	0.00400	0.00348	0.00317
	0.00905	0.00704	0.00509	0.00404	0.00354	0.00325
	0.00893***	0.00697	0.00501**	0.00400	0.00348	0.00317
Deep in-the-money	0.00172	0.00950	0.01708	0.01963	0.02054	0.02095
(1.125)	0.00241	0.00697	0.00868	0.00822	0.00773	0.00747
	0.00240**	0.00704	0.00880	0.00836	0.00789	0.00766
	0.00244	0.00695	0.00862***	0.00817***	0.00768***	0.00741***

*This table reports the mean pricing error for several estimators assuming a stochastic volatility environment and imposing the risk-neutral and exact pricing constraints. Pricing errors are based on 200 draws from the appropriate distribution and 5,000 repetitions. Listed in order from top to bottom: historical Black-Scholes, Canonical estimator, Euclidean estimator, and the empirical likelihood estimator. * indicate significance level of paired t-tests of the two smallest (in an absolute value sense) nonparametric estimators to to test the difference in the mean pricing errors. The null is that the mean pricing errors are not statistically different. *, **, and *** denotes significance at the 95, 97.5 and 99% level, respectively.